MÉCANIQUE CÉLESTE.

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BY THE

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TRANSLATED, WITH A COMMENTARY,

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ADVERTISEMENT.

THIS second volume completes the first part of the work. Considerable additions have been made to it in the notes, for the purpose of introducing the late improvements in the calculation of the attractions of spheroids, and in the determination of the figure of the earth. The third volume, being the commencement of the second part of the work, will be put to press immediately, and will require for its completion, about the same time as the preceding. The two remaining volumes will follow at similar intervals.

CONTENTS OF THE SECOND VOLUME.

THIRD BOOK.

ON THE FIGURES OF THE HEAVENLY BODIES.

FOURTH BOOK.

ON THE OSCILLATIONS OF THE SEA AND ATMOSPHERE.

FIFTH BOOK.

ON THE MOTIONS OF THE HEAVENLY BODIES ABOUT THEIR OWN CENTRES OF GRAVITY.

IN THE COMMENTARY.

Among the subjects treated of in the notes, we may particularly notice the following :

E

THIRD BOOK.

ON THE FIGURES OF THE HEAVENLY BODIES.

THE figures of the heavenly bodies depend on the law of gravity at their surfaces, and as this gravity is the resultant of the attraction of all their particles, it must also depend on their figures; therefore the law of gravity, at the surfaces of the heavenly bodies, and their figures, have a mutual connexion, which renders the knowledge of the one necessary for the determination of the other. In consequence of this, the investigation becomes very difficult, and it seems to require an analysis specially adapted to the subject. If the planets were entirely solid, they might be of any figures whatever; but if they be, like the earth, covered with a fluid, all the particles of this fluid must be so situated as to be in equilibrium; and the figure of the external surface will depend on the form of the included nucleus, and on the forces which act upon the fluid. We shall suppose generally, that all the heavenly bodies are covered by a fluid; and in this hypothesis, which actually takes place on the earth, and may very naturally be supposed to take place in the other bodies of the system of the world, we shall determine the law of gravity at their surfaces. The analysis we shall use is a singular application of the calculus of partial differentials; in which, by simply taking the differentials, we shall obtain some very general results, that could not have been found except with much difficulty by the usual methods of integration.

CHAPTER I.

ON THE ATTRACTIONS OF HOMOGENEOUS SPHEROIDS TERMINATED BY SURFACES OF THE SECOND ORDER.

1. WE shall, in the first place, determine the attraction of a body of a given figure. We have already found in § 11 of the second book [469, &c.] the attractions of a sphere, and of a spherical stratum; we shall now consider the attraction of a spheroid, terminated by a surface of the second order.

Putting x, y, z, for the three rectangular co-ordinates of a particle dM of the spheroid, and supposing the whole spheroid to be homogeneous, *and*
[1346] *its density equal to unity*, we shall have*

[1347]
$$dM = dx \, . \, dy \, . \, dz.$$

[1347′] Let a, b, c, be the rectangular co-ordinates of the point attracted by the spheroid, and A, B, C, the attractions of the spheroid upon this point, resolved parallel to the axes of x, y, z, respectively, and *directed towards the*
[1347″] *origin of the co-ordinates.* It is evident from § 11 [453] of the second book, that we shall have†

Attraction of a spheroid of any form upon any given point.

[1348]
$$A = \int\!\!\int\!\!\int \frac{(a-x) \, . \, dx \, . \, dy \, . \, dz}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{3}{2}}};$$
$$B = \int\!\!\int\!\!\int \frac{(b-y) \, . \, dx \, . \, dy \, . \, dz}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{3}{2}}};$$
$$C = \int\!\!\int\!\!\int \frac{(c-z) \, . \, dx \, . \, dy \, . \, dz}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{3}{2}}};$$

all these triple integrals must be extended to the whole mass of the spheroid. Under this form the integrations are very difficult; they may however be often rendered more simple, by an appropriate transformation of the differentials. The general principle of making such transformations is as follows.

[1348′] We shall take into consideration the differential function $P \, . \, dx \, . \, dy \, . \, dz$, P being any function of x, y, z. We may suppose x to be a function of the variable quantities y, z, and another variable quantity p. Let this function
[1348″] be $x = \varphi . (y, z, p)$; then we shall have, by supposing y and z to be constant,
[1348‴] $dx = \beta . dp$, β being a function of y, z, and p; and the preceding differential

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[1346a] * (918) This value of dM is easily deduced from [452], by writing x, y, z, for x', y', z', respectively, and putting $\rho = 1$.

† (919) Making in [453] the change of x', y', z', into x, y, z, as in the preceding note; putting also a, b, c, for x, y, z, respectively, we shall get the value of A [1348]. The values
[1346b] B, C, are easily deduced from A, by changing a, x, into b, y, and c, z, respectively, and the contrary. The same changes being made in $\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}$, [455a],
[1348a] it becomes $\sqrt{\{(a-x)^2+(b-y)^2+(c-z)^2\}}$, representing the distance of the particle dM from the attracted point, which will be used hereafter, particularly in [1448a], where it is called f.

will become $\beta . P . dp . dy . dz$. To integrate it, we must substitute in P the value of $x = \varphi . (y, z, p)$ [1348″].

In like manner we may suppose, in this last differential, $y = \varphi' . (z, p, q)$, [1348⁗]
q being another variable quantity, and $\varphi' . (z, p, q)$ being any function whatever of the three variable quantities z, p, and q. We shall have, by supposing z and p to be constant, $dy = \beta' . dq$, β' being a function of z, p, q. The [1348ᵛ]
preceding differential will by this means become of the form $\beta \beta' . P . dp . dq . dz$, [1348ᵛⁱ]
and to integrate it, we must substitute in βP, for y, its value $\varphi' . (z, p, q)$ [1348⁗].

Lastly, we may suppose z equal to $\varphi'' . (p, q, r)$, r being another variable [1348ᵛⁱⁱ]
quantity, and $\varphi'' . (p, q, r)$ being any function whatever of p, q, r. We shall have, by supposing p and q to be constant, $dz = \beta'' . dr$, β'' being a [1348ᵛⁱⁱⁱ]
function of p, q, r; the preceding differential [1348ᵛⁱ] will thus become $\beta . \beta' . \beta'' . P . dp . dq . dr$, and to integrate it we must substitute in $\beta . \beta' . P$, [1348ⁱˣ]
for z, its value $\varphi'' . (p, q, r)$ [1348ᵛⁱⁱ]. The proposed differential function will be, in this manner, transformed into another, relative to three different variable quantities p, q, r, which will be connected with the preceding, by the equations

Change from rectangular to other co-ordinates.

$$x = \varphi . (y, z, p) \, ; \qquad y = \varphi' . (z, p, q) \, ; \qquad z = \varphi'' . (p, q, r). \qquad [1349]$$

We must now deduce from these equations the values of β, β', β''.

For this purpose we shall observe, that they give x, y, z, in functions of the variable quantities p, q, r; we shall therefore consider the three first variable quantities as functions of the three last. β'' being the coefficient of dr, in the differential of z, taken in the hypothesis that p and q are constant, [1348ᵛⁱⁱⁱ], we shall have

$$\beta'' = \left(\frac{dz}{dr}\right). \qquad [1350]$$

β' is the coefficient of dq, in the differential of y, taken upon the supposition that p and z are constant [1348ᵛ]. We shall therefore obtain β', by taking [1350′]
the differential of y, supposing p to be constant; and eliminating dr, by means of the differential of z, taken upon the supposition that p is constant, and putting $dz = 0$. We shall thus get these two equations,*

* (920) The calculations in this section are exactly like those in [297—303″], changing a, b, c, β, into p, q, r, ε, respectively, and using d', $d_{,}$, as in [299a, b, &c.] By this means, [299] becomes as in [1351], and [300] becomes as in [1352].

$$dy = \left(\frac{dy}{dq}\right) . dq + \left(\frac{dy}{dr}\right) . d'r ;$$

[1351]

$$0 = \left(\frac{dz}{dq}\right) . dq + \left(\frac{dz}{dr}\right) . d'r ;$$

hence

[1352]

$$dy = dq \, . \, \frac{\left\{ \left(\frac{dy}{dq}\right) . \left(\frac{dz}{dr}\right) - \left(\frac{dy}{dr}\right) . \left(\frac{dz}{dq}\right) \right\}}{\left(\frac{dz}{dr}\right)} ;$$

therefore

[1353]

$$\beta' = \frac{\left(\frac{dy}{dq}\right) . \left(\frac{dz}{dr}\right) - \left(\frac{dy}{dr}\right) . \left(\frac{dz}{dq}\right)}{\left(\frac{dz}{dr}\right)} .$$

[1353'] Lastly, β is the coefficient of dp, in the differential of x, supposing y and z to be constant, [1348″]; hence we get the three following equations,

$$dx = \left(\frac{dx}{dp}\right) . dp + \left(\frac{dx}{dq}\right) . d_{,}q + \left(\frac{dx}{dr}\right) . d_{,}r ;$$

[1354]

$$0 = \left(\frac{dy}{dp}\right) . dp + \left(\frac{dy}{dq}\right) . d_{,}q + \left(\frac{dy}{dr}\right) . d_{,}r ;$$

$$0 = \left(\frac{dz}{dp}\right) . dp + \left(\frac{dz}{dq}\right) . d_{,}q + \left(\frac{dz}{dr}\right) . d_{,}r .$$

If we put

[1355]

$$\begin{aligned} \varepsilon = {} & \left(\frac{dx}{dp}\right) . \left(\frac{dy}{dq}\right) . \left(\frac{dz}{dr}\right) - \left(\frac{dx}{dp}\right) . \left(\frac{dy}{dr}\right) . \left(\frac{dz}{dq}\right) \\ & + \left(\frac{dx}{dq}\right) . \left(\frac{dy}{dr}\right) . \left(\frac{dz}{dp}\right) - \left(\frac{dx}{dq}\right) . \left(\frac{dy}{dp}\right) . \left(\frac{dz}{dr}\right) \\ & + \left(\frac{dx}{dr}\right) . \left(\frac{dy}{dp}\right) . \left(\frac{dz}{dq}\right) - \left(\frac{dx}{dr}\right) . \left(\frac{dy}{dq}\right) . \left(\frac{dz}{dp}\right) , \end{aligned}$$

The value of dy thus obtained being put equal to its assumed value $\beta' dq$, gives β' as in [1353]. Again, the same changes being made in [301], we shall get [1354], and [302] changes into [1355], also [303] into [1356]. Putting the value of dx [1356] equal to its assumed value [1348‴], $\beta \, dp$, we find

$$\beta = \frac{\varepsilon}{\left(\frac{dy}{dq}\right) . \left(\frac{dz}{dr}\right) - \left(\frac{dy}{dr}\right) . \left(\frac{dz}{dq}\right)} .$$

This being multiplied by β', [1353], and by β'', [1350], we get $\beta . \beta' . \beta'' = \varepsilon$, as in [1356″].
[1355a] Substituting this in $P . dx . dy . dz = \beta . \beta' . \beta'' . P . dp . dq . dr$, [1348ix], it becomes $\varepsilon . P . dp . dq . dr$, as in [1356‴]

we shall have

$$d x = \frac{\varepsilon \,.\, d p}{\left(\frac{d y}{d q}\right) . \left(\frac{d z}{d r}\right) - \left(\frac{d y}{d r}\right) . \left(\frac{d z}{d q}\right)} ; \qquad [1356]$$

which gives

$$\beta = \frac{\varepsilon}{\left(\frac{d y}{d q}\right) . \left(\frac{d z}{d r}\right) - \left(\frac{d y}{d r}\right) . \left(\frac{d z}{d q}\right)} ; \qquad [1356']$$

therefore $\beta \,.\, \beta' \,.\, \beta'' = \varepsilon$, and the differential $P \,.\, d x \,.\, d y \,.\, d z$ will be [1356'']
transformed into $\varepsilon \,.\, P \,.\, d p \,.\, d q \,.\, d r$; P being here what P becomes, when [1356''']
we substitute for x, y, z, their values in p, q, r. All that is now necessary, is to select such variable quantities p, q, r, as will render the integrations possible.

We shall transform the co-ordinates x, y, z, into the radius drawn from
the attracted point to the particle, and the angles which this radius makes [1356'''']
with given right lines or planes. Let r be this radius ;* p the angle which it

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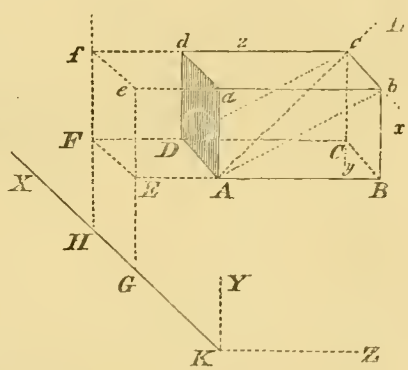

* (921) In the adjoined figure, which is the same as that in Vol. I, page 8, A is the attracting point, c the attracted point, K the origin ; so that the co-ordinates of the point A are $KG = x$ $GE = y$, $EA = z$, and those of the point c are $KH = a$, $Hf = b$, $fc = c$; hence

$$cb = ef = GH = a - x ;$$
$$Bb = Cc = Ff = Hf - HF = b - y ;$$
$$AB = CD = cd = fc - EA = c - z. \qquad [1355b]$$

Then by [1356'''', v], we have $Ac = r$, the angle $Acb = p$; $CcD = q$; and the rectangular triangle Abc gives

$$cb = Ac \,.\, \cos. Acb = r \,.\, \cos p ; \qquad Ab = cD = Ac \,.\, \sin. Acb = r \,.\, \sin. p ; \qquad [1355b']$$

and in the rectangular triangle cCD, we have

$$Cc = cD \,.\, \cos. CcD = cD \,.\, \cos. q ; \qquad CD = cD \,.\, \sin. CcD = cD \,.\, \sin. q ;$$

substituting the value of cD, they become $Cc = r \,.\, \sin. p \,.\, \cos. q$; $CD = r \,.\, \sin. p \,.\, \sin. q$. Putting these values of cb, Cc, CD, equal to those in [1355b], we get

$$a - x = r \,.\, \cos. p, \qquad b - y = r \,.\, \sin. p \,.\, \cos. q, \qquad c - z = r \,.\, \sin. p \,.\, \sin. q, \qquad [1355c]$$

as in [1357]. The sum of the squares of the two last is

$$(b - y)^2 + (c - z)^2 = r^2 \,.\, \sin.^2 p \,.\, \{\cos.^2 q + \sin.^2 q\} = r^2 \,.\, \sin.^2 p. \qquad [1355d]$$

makes with a right line, drawn through the attracted point, parallel to the
[1356v] axis of x; and q the angle which the projection of this radius upon the plane of y, z, makes with the axis of y, we shall have

[1357] $x=a-r.\cos.p\,;\qquad y=b-r.\sin.p.\cos.q\,;\qquad z=c-r.\sin.p.\sin.q.$

[1357′] This being supposed, we shall find* $\varepsilon=-r^2.\sin.p\,;$ and the differential

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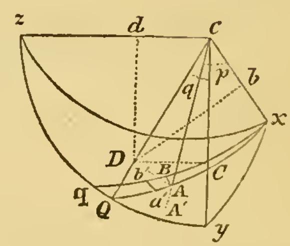
Adding this to $(a-x)^2=r^2.\cos.^2p$, and putting $\cos.^2p+\sin.^2p=1$, we get
[1355e] $r^2=(a-x)^2+(b-y)^2+(c-z)^2\,;$ hence we find

[1355f] $$a-x=\sqrt{\{r^2-(b-y)^2-(c-z)^2\}},\qquad b-y=\sqrt{\{r^2.\sin.^2p-(c-z)^2\}},$$
$$c-z=r.\sin.p.\sin.q\,;$$

which will be of use hereafter, [1357a].



To obtain a correct idea of the limits of the values of r, p, q, we shall suppose a spherical surface $yxzQ$ to be described about the attracted point c as a centre, with a radius equal to cA. To prevent confusion, this is drawn separately in the annexed figure; in which the points x, y, z, are those where this surface intersects the lines cb, cC, cd, drawn through c parallel to the axes of x, y, z, respectively; the letters of reference being the same in both figures. Then x is the pole of the great circle yQz; and if through x and A, we draw the quadrantal arc xAQ, it is evident that the angle $Acx=p$,
[1355g] and $ycQ=q$, $cA=r$. If there be an attracting mass placed near A, we may determine the attraction of one of its elements in the following manner. Draw the quadrantal arc $xBb'q$ infinitely near to $xAa'Q$, and perpendicular to the arc yQz, also the arcs AB, $a'b'$, parallel to Qq, and infinitely near each other. Then we may suppose a particle of the body placed at A to be of the form of a parallelopiped, whose base is $ABb'a'$, and its height, measured on the continuation of cA, $AA'=dr$. Now as in note 168, Vol. I, page 181, we have $Qq=rdq$, $AB=Qq.\sin.Acx=rdq.\sin.p$, and $Aa'=rdp$. Hence the sides of this parallelopiped AB, Aa', and the height of it, are
[1355h] respectively $rdq.\sin.p$, rdp, and dr; the product of these three gives the mass
[1355i] of the particle $r^2.\sin.p.dp.dq.dr$ [1357″].

* (922) The values of x, y, z, [1357], give

$$\left(\frac{dx}{dp}\right)=r.\sin.p\,;\qquad \left(\frac{dx}{dq}\right)=0\,;\qquad \left(\frac{dx}{dr}\right)=-\cos.p\,;$$

[1355k] $$\left(\frac{dy}{dp}\right)=-r.\cos.p.\cos.q\,;\qquad \left(\frac{dy}{dq}\right)=r.\sin.p.\sin.q\,;\qquad \left(\frac{dy}{dr}\right)=-\sin.p.\cos.q\,;$$

$$\left(\frac{dz}{dp}\right)=-r.\cos.p.\sin.q\,;\qquad \left(\frac{dz}{dq}\right)=-r.\sin.p.\cos.q\,;\qquad \left(\frac{dz}{dr}\right)=-\sin.p.\sin.q.$$

$dx \,.\, dy \,.\, dz$ [1356‴] will, by this means, be transformed into*

$$-r^2 \,.\, \sin.\, p \,.\, dp \,.\, dq \,.\, dr\,;$$ [1357″]

this is the expression of the particle dM, and as dM is positive, we must, if we suppose $\sin.p$, dp, dq, dr, to be positive, change the sign of the

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Hence we obtain

$$\left(\frac{dy}{dq}\right).\left(\frac{dz}{dr}\right)-\left(\frac{dy}{dr}\right).\left(\frac{dz}{dq}\right)=-r\,.\sin.^2 p\,.\sin.^2 q-r\,.\sin.^2 p\,.\cos.^2 q$$

$$=-r\,.\sin.^2 p\,.(\sin.^2 q+\cos.^2 q)=-r\,.\sin.^2 p\,;$$

$$\left(\frac{dy}{dr}\right).\left(\frac{dz}{dp}\right)-\left(\frac{dy}{dp}\right).\left(\frac{dz}{dr}\right)=\left\{\begin{array}{c} r\,.\sin.p\,.\cos.p\,.\sin.q\,.\cos.q \\ -r\,.\sin.p\,.\cos.p\,.\sin.q\,.\cos.q \end{array}\right\}=0\,;$$ [1356a]

$$\left(\frac{dy}{dp}\right).\left(\frac{dz}{dq}\right)-\left(\frac{dy}{dq}\right).\left(\frac{dz}{dp}\right)=r^2.\sin.p\,.\cos.p\,.\cos.^2 q+r^2.\sin.p\,.\cos.p\,.\sin.^2 q$$

$$=r^2.\sin.p\,.\cos.p\,.(\cos.^2 q+\sin.^2 q)=r^2.\sin.p\,.\cos.p.$$

Multiplying the preceding expressions respectively by

$$\left(\frac{dx}{dp}\right)=r\,.\sin.p, \qquad \left(\frac{dx}{dq}\right)=0, \qquad \left(\frac{dx}{dr}\right)=-\cos.p,$$

we obtain the three following expressions,

$$\left(\frac{dx}{dp}\right).\left(\frac{dy}{dq}\right).\left(\frac{dz}{dr}\right)-\left(\frac{dx}{dp}\right).\left(\frac{dy}{dr}\right).\left(\frac{dz}{dq}\right)=-r^2.\sin.p\,.\sin.^2 p\,;$$

$$\left(\frac{dx}{dq}\right).\left(\frac{dy}{dr}\right).\left(\frac{dz}{dp}\right)-\left(\frac{dx}{dq}\right).\left(\frac{dy}{dp}\right).\left(\frac{dz}{dr}\right)=0\,;$$ [1356b]

$$\left(\frac{dx}{dr}\right).\left(\frac{dy}{dp}\right).\left(\frac{dz}{dq}\right)-\left(\frac{dx}{dr}\right).\left(\frac{dy}{dq}\right).\left(\frac{dz}{dp}\right)=-r^2.\sin.p\,.\cos.^2 p.$$

Adding these products together, the first member becomes equal to the value of ε [1355], and the second member, by putting $\cos.^2 p+\sin.^2 p=1$, becomes $-r^2.\sin.p$; hence [1356c]
$\varepsilon=-r^2.\sin.p$, as in [1357′].

* (923) This expression of $dx.dy.dz=-r^2.\sin.p\,.dp\,.dq.dr$, has also been obtained, in a very simple manner, from values similar to [1355f]; taking the differentials so as to make x vary with r, y with p, and z with q; which will leave dx, dy, dz, as well as dp, dq, dr, independent of each other. The differentials [1355f] are very easily taken [1356d]
in noticing these conditions, because these expressions are designedly so arranged that in the second members only one quantity occurs which is considered as variable. Thus in finding $-dx$ from the first of the equations [1355f], the quantity r of the second member is the only one which is considered as variable, y, z, being independent, and therefore constant. In

preceding expression; which is the same thing as to change the sign of ε,
[1357‴] and to suppose $\varepsilon = r^2 . \sin. p$. The preceding values of A, B, C, will therefore become*

Attraction of a spheroid upon any given point. Second form.

$$A = \int\int\int dr . dp . dq . \sin. p . \cos. p ;$$
$$B = \int\int\int dr . dp . dq . \sin.^2 p . \cos. q ;$$
[1358]
$$C = \int\int\int dr . dp . dq . \sin.^2 p . \sin. q .$$

It is easy to obtain these expressions in another manner, by observing that
[1358′] the particle dM may be supposed equal to a rectangular parallelopiped† whose

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finding dy from the second, p is considered variable, and r, z, independent or constant; lastly, in finding $-dy$ from the third equation [1355f], q is considered variable, and r, p, independent or constant; hence we get

$$-dx = \frac{r dr}{\sqrt{[r^2-(b-y)^2-(c-z)^2]}} = \frac{r dr}{a-x} = \frac{r dr}{r . \cos. p} = \frac{dr}{\cos. p} ;$$

[1357a]
$$-dy = \frac{r^2 dp . \sin. p . \cos. p}{\sqrt{[r^2 . \sin.^2 p-(c-z)^2]}} = \frac{r^2 dp . \sin. p . \cos. p}{b-y} = \frac{r^2 dp . \sin. p . \cos. p}{r . \sin. p . \cos. q} = \frac{r dp . \cos. p}{\cos. q} ;$$

$$-dz = r dq . \sin. p . \cos. q .$$

The product of these three expressions is

$$-dx . dy . dz = \frac{dr}{\cos. p} . \frac{r dp . \cos. p}{\cos. q} . r dq . \sin. p . \cos. q = r^2 . \sin. p . dp . dq . dr ,$$

[1357b] as in [1357″]. Now from [1347], $dM = dx . dy . dz$ represents a particle of the mass of the spheroid, which must be a positive quantity; and if we take, as usual, the differentials dp, dq, dr, positive, we must change the sign of the preceding expression, and we shall
[1357c] have $dM = dx . dy . dz = r^2 . \sin. p . dp . dq . dr$, as in [1357″, &c.]

* (923a) From [1355e] it appears that the denominators of A, B, C, [1347], are r^3; also $dx . dy . dz = r^2 . \sin. p . dp . dq . dr$, [1357$c$]; substituting these in [1348], and putting in the numerators the values of $a-x$, $b-y$, $c-z$, [1355c], they become, by a very easy reduction, of the same forms as in [1358].

[1359a] † (924) This is proved in [1355h]. Now if the particle $dM = r^2 . \sin. p . dp . dq . dr$ be placed at the point A, fig. 1, page 5, its attraction on the point c, in the direction cA, will be $\frac{dM}{Ac^2}$, or $\frac{dM}{r^2}$. This, resolved in the directions cb, cC, cd, parallel to the axes x, y, z, will be represented by $\frac{dM}{r^2} . \frac{cb}{Ac}$; $\frac{dM}{r^2} . \frac{Cc}{Ac}$; $\frac{dM}{r^2} . \frac{cd}{Ac}$; and since by [1355$b$, c]
[1359a'] $cb = a-x = r . \cos. p$; $Cc = b-y = r . \sin. p . \cos. q$; $cd = c-z = r . \sin. p . \sin. q$; $Ac = r$, these forces will become $\frac{dM}{r^2} . \cos. p$; $\frac{dM}{r^2} . \sin. p . \cos. q$; $\frac{dM}{r^2} . \sin. p . \sin. q$; as in [1359].

three sides are dr, $r\,dp$, and $r\,dq\,.\,\sin.p$, and that the attractions of the particle, parallel to the three axes x, y, z, are respectively

$$\frac{dM}{r^2}\,.\cos.p\;;\qquad \frac{dM}{r^2}\,.\sin.p\,.\cos.q\;;\qquad \frac{dM}{r^2}\,.\sin.p\,.\sin.q. \qquad [1359]$$

The triple integrals of the expressions of A, B, C, must be taken within such limits as to include the whole mass of the spheroid. The integrations relative to r are easily found; but they are of different forms, according as the attracted point is situated within or without the spheroid. In the first [1359′] case, the right line passing through the attracted point, and terminating at the surface of the spheroid, is divided into two parts by that point; and if we call these parts r, and r', we shall find*

Attraction of a spheroid upon a point within its surface. Third form.

$$A=\int\int(r+r')\,.\,dp\,.\,dq\,.\sin.p\,.\cos.q\;;$$
$$B=\int\int(r+r')\,.\,dp\,.\,dq\,.\sin.^2p\,.\cos.q\;; \qquad [1360]$$
$$C=\int\int(r+r')\,.\,dp\,.\,dq\,.\sin.^2p\,.\sin.q\;;$$

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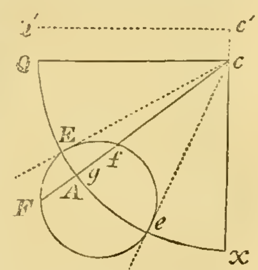
* (925) To show more sensibly the limits of these attractions, we shall suppose the plane xcQ of the figure, page 6, to form, by its intersection with the surface of the proposed spheroid, a curve $efEF$, as in the annexed figure, which is drawn separately to avoid any confusion in the lines. Then for all the points situated in this curve, the angle q is constant; and if we draw any line cfF intersecting the surface in the points f, F, the value of p for all points situated on the line fF is constant. Hence if we have any integral of the form $\int\int\int P\,.\,dp\,.\,dq\,.\,dr$, in which P is a function of p, q, r, and the integral is to include the whole mass of the spheroid, we may first consider p, q, as constant, and take the integral $\int P\,.\,dr$, supposing r only to be variable, the limits of this integral being between the values $cf=r_{,}$, and $cF=r'$. The values $r_{,}$, r', depend on the angle $xcA=p$, and on the situation of the plane xcQ, corresponding to the angle q; therefore $r_{,}$ and r' will be functions of p and q. Substituting these values of p, q, in the preceding integral $\int P\,.\,dr$, it will become a function of p, q, which we shall denote by P', and then the preceding integral will become $\int\int P'\,.\,dp\,.\,dq$. If we now draw ce, cE, tangents to the [1359b] curve $efEF$, we may take the preceding integral relative to p, or $\int P'\,.\,dp$, considering [1359c] q as constant, and limiting the integral by these tangents; observing that these limits are determined by the situation of the plane xcQ, or by the value of the angle q, so that $\int P'\,.\,dp$, after the integration and substitution of these limits, will become a function of q, which we shall denote by Q, and then the integral $\int\int\int P\,.\,dp\,.\,dq\,.\,dr=\int Q\,.\,dq$. Now [1359d]



[1360'] the integrals relative to p and q being taken from p and q equal to nothing, to p and q equal two right angles.

[1360''] In the second case, if we put r for the radius at the ingress into the spheroid, and r' for the same radius at the egress, we shall have

Attraction of a spheroid upon an external point. Fourth form.

[1361]
$$A = \int\int (r'-r) \,.\, dp \,.\, dq \,.\, \sin. p \,.\, \cos. p\,;$$
$$B = \int\int (r'-r) \,.\, dp \,.\, dq \,.\, \sin.^2 p \,.\, \cos. q\,;$$
$$C = \int\int (r'-r) \,.\, dp \,.\, dq \,.\, \sin.^2 p \,.\, \sin. q\,;$$

the limits of the integrals relative to p and q ought to be fixed at the points

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taking this integral, from the least to the greatest value of q, we shall obtain the required value of $\int\int\int P \,.\, dp \,.\, dq \,.\, dr$.

If this reasoning be applied to the equations [1358], the quantity which we have called P will be independent of r. Thus for the first of these equations, $P = \sin. p \,.\, \cos. p$, and $\int P \,.\, dr = P \,.\, \int dr = P \,.\, (r + \text{constant})$. This integral, commencing at the point f, where $r = r_{,}$, is $\int P \,.\, dr = P \,.\, (r - r_{,})$; and if it terminate at the point F, where $r = r'$, it becomes $\int P \,.\, dr = P \,.\, (r' - r_{,})$. By this means, the equations [1358] become as in [1361]; the point c being supposed to fall without the body $efEF$. The integrals of
[1359e] [1361], relative to p, must be made from the angle $p = xce$ to $p = xcE$, and since at the point e, or E, the values $r_{,}$, r', are equal, or $r' - r_{,} = 0$, these limits must be determined by the values of p corresponding to $r' - r_{,} = 0$. Again, a little consideration will show, that if the plane xcQ fig. p. 6, 9, revolve about the axis xc, this plane will be tangent to the spheroid at the points corresponding to the least and greatest values of q; and on each of these planes a line may be drawn similar to ce, or cE, touching the spheroid, and making $r' - r_{,} = 0$; consequently the limits of the integrals of [1361] relative to p, q, are determined by the
[1359f] equation $r' - r_{,} = 0$, as in [1361'], the mark on $r_{,}$ being rejected, since the letter r in [1360, 1361] is the same as that we have denoted by $r_{,}$. In all the preceding calculations it has been supposed that the point c falls without the body, so that the least value of r is positive. Supposing now the point c to approach towards f, fig. p. 9, till it becomes 0 at f, and negative as at c fig. 4; we must then make r negative in [1361], and it will become as in [1360]. In this case the limits will be changed; for while Fcf revolves in the plane of fig. 4, from any line QcQ', through a semicircle, that line will pass over the whole plane $FcfE$, and if we suppose a semi-revolution to be made about the axis QQ', the figures similar to $FcfE$ corresponding to all these points of revolution, will include the whole spheroid; consequently in this case the limits must be taken, from p and q equal to nothing to p and q equal to two right angles, as in [1360'].

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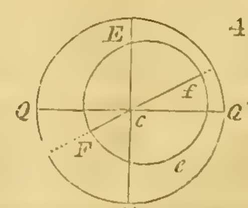

where $r' - r = 0$, that is where the radius r is a tangent to the surface of the spheroid. [1361']

2. We shall apply these results to spheroids terminated by surfaces of the second order.* The general equation of these surfaces, referred to three rectangular co-ordinates x, y, z, is

Surface of the second order.

$$0 = A + B.x + C.y + E.z + F.x^2 + H.xy + L.y^2 + M.xz + N.yz + O.z^2. \quad [1362]$$

The change of the origin of the co-ordinates introduces three arbitrary quantities, because the position of this new origin, referred to the first, depends upon three arbitrary co-ordinates. The change of the position of the co-ordinates, about their origin, introduces three arbitrary angles.† [1362']
Therefore if we make, at once, both these changes in the preceding equation, we shall obtain another equation of the second degree, whose coefficients will be functions of the preceding coefficients and of these six arbitrary quantities. If we then put the coefficients of the first power of the co-ordinates equal to nothing; and also those of the products of each two of [1362'']
the co-ordinates, we shall determine these arbitrary quantities; and the

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* (926) The computation of the attraction of an ellipsoid, partially treated of by Newton, in his Principia, was extended by Maclaurin, in a geometrical solution, remarkable for its elegance and simplicity, to the determination of the whole attraction upon any point, within or upon the surface of the ellipsoid of revolution. The same result was afterwards obtained by La Grange, in an analytical form, by a change of the co-ordinates, upon the principles explained in [1348', &c.] Le Gendre extended the investigation so as to embrace all points, whether within or without the surface of the ellipsoid. Finally La Place obtained the general attraction of any ellipsoid, in all cases, whether the principal axes were equal or unequal, and upon any point without or within the surface; by the method given in this chapter. Since the publication of this method, a great improvement has been made by Mr. Ivory, in which the computation of the attraction of any ellipsoid upon an external point is reduced, by his analysis, to the much more simple case, of finding the attraction of another ellipsoid, upon an internal point, or upon a point situated in the surface of the ellipsoid. Mr. Ivory has also treated the subject in a very elegant geometrical manner, in the Encyclopedia Britannica, under the article Attraction. In conformity to the plan adopted in these notes, we shall give the necessary illustrations of the calculations of the author, and shall afterwards show how the [1360a]
same results can be obtained by Mr. Ivory's improved method.

† (927) An example of this is given in [171], where the change of the axes x, y, z, into $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, depends on the three angles θ, ψ, φ.

general equation of surfaces of the second order will become of this very simple form,*

[1363] $$x^2+my^2+nz^2=k^2;$$

Surface of the second order, or ellipsoid.

under which form we shall hereafter consider it.

In these researches, we shall notice only bodies terminated by finite
[1363'] surfaces; and then m and n must be positive. In this case the body will be an ellipsoid, whose three semi-axes are what the variable quantities x, y, z, respectively become, when the other two are equal to nothing. We shall

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* (928) Changing the origin of the co-ordinates, we may put in [1362] $x=\alpha+x'$, $y=\beta+y'$, $z=\gamma+y'$, and it becomes of this form

[1362a] $$0=A'+B'x'+C'y'+E'z'+F'x'^2+H'x'y'+L'y'^2+M'x'z'+N'y'z'+O'z'^2;$$

in which A', B', C', &c., are functions of the *ten* coefficients A, B, C, &c., and the *three* quantities α, β, γ. If we change the positions of the axes x', y', z', into three other rectangular axes $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$, we shall have, as in [172a], $x'=A_0x_{\prime\prime\prime}+B_0y_{\prime\prime\prime}+C_0z_{\prime\prime\prime}$; $y'=A_1x_{\prime\prime\prime}+B_1y_{\prime\prime\prime}+C_1z_{\prime\prime\prime}$; $z'=A_2x_{\prime\prime\prime}+B_2y_{\prime\prime\prime}+C_2z_{\prime\prime\prime}$; the coefficients A_0, B_0, C_0, A_1, &c., [171a], being functions of the three arbitrary angles θ, ψ, φ. Substituting these in [1362a], it becomes of the form

[1362b] $$0=A''+B''x_{\prime\prime\prime}+C''y_{\prime\prime\prime}+E''z_{\prime\prime\prime}+F''x_{\prime\prime\prime}^2+H''x_{\prime\prime\prime}y_{\prime\prime\prime}+L''y_{\prime\prime\prime}^2+M''x_{\prime\prime\prime}z_{\prime\prime\prime}+N''y_{\prime\prime\prime}z_{\prime\prime\prime}+O''z_{\prime\prime\prime}^2;$$

in which the *ten* coefficients A'', B'', &c., are functions of the *ten* coefficients A, B, C, &c., and the *six* arbitrary quantities α, β, γ, θ, ψ, φ; and we may generally use these six arbitrary quantities to make *six* of the coefficients of [1362b] equal to nothing. Thus if we determine α, β, γ, θ, ψ, φ, by means of the six equations $B''=0$, $C''=0$, $E''=0$, $H''=0$, $M''=0$, $N''=0$, it will become $0=A''+F''x_{\prime\prime\prime}^2+L''y_{\prime\prime\prime}^2+O''z_{\prime\prime\prime}^2$; which, by neglecting the accents on $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$, may be easily reduced to the form [1363] $x^2+my^2+nz^2=k^2$. If the coefficients A'', F'', L'', O'', were each equal to nothing, or of the form $\frac{0}{0}$, we must retain one of the coefficients, B'', C'', &c., which was put equal to nothing, and use instead of it one of the retained terms A'', F'', &c. Thus if instead of putting $B''=0$, we put $A''=0$, the equation will become

$$0=B''x_{\prime\prime\prime}+F''x_{\prime\prime\prime}^2+L''y_{\prime\prime\prime}^2+O''z_{\prime\prime\prime}^2;$$

but it is not necessary to notice particularly these cases, because the form [1363] corresponds very nearly to the figures of the heavenly bodies.

thus have* k, $\frac{k}{\sqrt{m}}$, $\frac{k}{\sqrt{n}}$, for these three semi-axes respectively, parallel [1363″]
to the axes of x, y, z. The solidity of the ellipsoid will be†

$$M = \frac{4\pi . k^3}{3 . \sqrt{mn}} ; \qquad [1363'''']$$

supposing always that π represents the ratio of the semi-circumference of a circle to its radius. [1363″″]

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* (929) If m were negative, the equation [1363] would become $x^2 - my^2 + nz^2 = k^2$. Now if we suppose z to be a given quantity, and $k^2 - nz^2 = G$, this will become $x^2 - my^2 = G$, or $x^2 = my^2 + G$, and by putting $y = \infty$, x would also become infinite. The same result would be found by putting n negative. Again, when m and n are positive, it is evident from the equation [1363] that $x = \pm\sqrt{\{k^2 - my^2 - nz^2\}}$; therefore the greatest value of x is found by putting y, z, equal to nothing; and if we put this value of x equal to x', the preceding equation will become $x' = k$. In like manner from [1363] we obtain $y = \pm\left(\frac{k^2}{m} - \frac{x^2}{m} - \frac{nz^2}{m}\right)^{\frac{1}{2}}$; $z = \pm\left(\frac{k^2}{n} - \frac{x^2}{n} - \frac{my^2}{n}\right)^{\frac{1}{2}}$;
from which it is evident that the greatest value of y is found by putting $x = 0$, $z = 0$; and the greatest value of z, by putting $x = 0$, $y = 0$; and if we put y', z', for these greatest values, we shall get

$$y' = \left(\frac{k^2}{m}\right)^{\frac{1}{2}} = \frac{k}{\sqrt{m}}; \qquad z' = \left(\frac{k^2}{n}\right)^{\frac{1}{2}} = \frac{k}{\sqrt{n}}; \qquad [1363a]$$

as in [1363″]. It may be observed, that the double sign of the radicals, in the values of x, y, z, just given, proves that if any two of the co-ordinates, as y, z, are given, the other, $x = \pm\sqrt{\{k^2 - my^2 - nz^2\}}$, will have two values, equal to each other, but of different signs; which indicates, as in the ellipsis, that the origin of the co-ordinates is at the centre of the ellipsis or ellipsoid. [1363b]

† (930) From [1347] we get $M = \int dz . \int dy . \int dx$; and $\int dx$, integrated from $-x$ to $+x$, is $2x = 2 . \sqrt{\{k^2 - my^2 - nz^2\}}$; hence $M = 2 . \int dz . \int dy . \sqrt{\{k^2 - my^2 - nz^2\}}$. If we put $R^2 = \frac{k^2 - nz^2}{m}$, it becomes $M = 2 . \sqrt{m} . \int dz . \int dy . \sqrt{\{R^2 - y^2\}}$. But

$$\int dy . \sqrt{\{R^2 - y^2\}} = \tfrac{1}{2} y . \sqrt{\{R^2 - y^2\}} + \tfrac{1}{2} R^2 . \left(\text{arc. sin. } \frac{y}{R}\right) + \text{constant},$$

as is easily proved by differentiation. This integral is to be taken from the least to the greatest value of y, corresponding to any given value of z or R. Now if we divide [1363] by m, and

Now if in the preceding equation we substitute for x, y, z, their values in p, q, r, given in the preceding article, we shall get*

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substitute the preceding values of R, we shall get $y^2 = R^2 - \frac{x^2}{m}$; and it is evident from this, that the least value of y is $-R$, and the greatest is $+R$, corresponding to $x = 0$. At both these limits, the term $\frac{1}{2} y . \sqrt{\{R^2 - y^2\}}$ of the preceding integral is 0,

[1363c] and it becomes $\int dy . \sqrt{\{R^2 - y^2\}} = R^2 . \text{arc. sin.} \frac{R}{R} = \frac{1}{2} R^2 . \pi$; hence

$$M = \pi . \sqrt{m} . \int dz . R^2 = \pi . \sqrt{m} . \int dz . \frac{k^2 - n z^2}{m} = \frac{\pi}{\sqrt{m}} . \int (k^2 . dz - n z^2 . dz)$$

$$= \frac{\pi}{\sqrt{m}} . \{k^2 z - \tfrac{1}{3} n z^3 + \text{constant}\}.$$

This integral, taken from the least to the greatest value of z, that is from $-\frac{k}{\sqrt{n}}$ to

[1363d] $\frac{k}{\sqrt{n}}$, is $M = \frac{4\pi}{3 . \sqrt{mn}} . k^3$, as above. The same might have been obtained from geometrical considerations, by observing that if planes be drawn perpendicular to the axis of x, the sections formed by the planes and ellipsoid will be similar *ellipses;* and the areas of these ellipses will evidently be in a constant ratio to the circles, formed by the intersection of the same planes with the surface of a sphere, whose radius is k, and centre the same as that of the ellipsoid. Now the semi-axes of the ellipsis, thus formed, by the plane passing through the centre of the ellipsoid, are $\frac{k}{\sqrt{m}}$, $\frac{k}{\sqrt{n}}$, [1363a], and the radius of the corresponding circle is k. The area of the circle is to that of the ellipsis as $k^2 : \frac{k}{\sqrt{m}} . \frac{k}{\sqrt{n}}$, or as $1 : \frac{1}{\sqrt{mn}}$, [378u], and the solidity of the sphere must be to that of the ellipsoid in the

[1363e] same ratio. But the solidity of a sphere, *whose radius is* k, is $\frac{4}{3} \pi . k^3$, [275b, &c.]; consequently that of the ellipsoid must be $\frac{4\pi . k^3}{3 . \sqrt{mn}}$, as in [1363″]. If the body be formed by the revolution of an ellipsis about the axis of x, so that $m = n$, its solidity will be $\frac{4}{3} \pi . \frac{k^3}{m}$; and if we put $k = \sqrt{m} = 1 - \alpha$, it will

[1369a] become $\frac{4}{3} \pi . (1 - \alpha)$. If we put, for the sake of symmetry,

Axes of an ellipsoid.

[1369b]

$$k = \alpha, \qquad \frac{k}{\sqrt{m}} = \beta, \qquad \frac{k}{\sqrt{n}} = \gamma,$$

the preceding value of the mass of the ellipsoid M will become

Mass of an ellipsoid.

[1369c]

$$M = \tfrac{4}{3} \pi . \alpha \beta \gamma.$$

* (931) Substituting the values of x, y, z, [1357] in [1363], and arranging its terms according to the powers of r^2, it becomes as in [1364], and by using the abridged symbols

$$r^2.\{\cos.^2 p+m.\sin.^2 p.\cos.^2 q+n.\sin.^2 p.\sin.^2 q\}$$ [1364]
$$-2r.\{a.\cos.p+mb.\sin.p.\cos.q+nc.\sin.p.\sin.q\}=k^2-a^2-mb^2-nc^2;$$

[1365], we get $L r^2-2Ir=k^2-a^2-mb^2-nc^2$; hence

$$r=\frac{I\pm\sqrt{[I^2+(k^2-a^2-mb^2-nc^2).L]}}{L}.$$

This, by means of the value of R [1365], changes into [1366]. Hence $r=\frac{I-\sqrt{R}}{L}$; $r'=\frac{I+\sqrt{R}}{L}$; whose sum and difference are as in [1367]. The values of $r+r'$ and $r'-r$, thus found, being substituted in [1360, 1361], give respectively [1368, 1369]. It was observed in [1361′], that the limits of the integrals of [1361] must be where $r'-r=0$, which corresponds to $R=0$ [1367]; therefore the limits of the integrals of [1369] must be found by putting $R=0$, as in [1369′]. [1370a]

If we divide the values of A, B, C, [1368], by a, b, c, respectively, and add the quotients, we shall get

$$\frac{A}{a}+\frac{B}{b}+\frac{C}{c}=2.\int\int dp.dq.\frac{\sin.p}{L}.\left\{\frac{I}{a}.\cos.p+\frac{I}{b}.\sin.p.\cos.q+\frac{I}{c}.\sin.p.\sin.q\right\}.$$ [1370b]

The integrals are to be taken from $p=0$ to $p=\pi$, and from $q=0$ to $q=\pi$, [1360′]; now since $\cos.(\pi-p)=-\cos.p$, $\cos.(\pi-q)=-\cos.q$, we may, in [1370c] the terms multiplied by I, neglect those containing only the first power of $\cos.p$ or $\cos.q$, because the positive values are destroyed by the negative ones, as is done in finding A [1370‴]; and we shall thus have

$$\frac{I}{a}.\cos.p=\cos.^2 p;\qquad \frac{I}{b}.\sin.p.\cos.q=m.\sin.^2 p.\cos.^2 q;\qquad \frac{I}{c}.\sin.p.\sin.q=n.\sin.^2 p.\sin.^2 q;$$ [1370d]

The sum of these three expressions is equal to the value of L [1365]; substituting it in [1370b], the quantity L will vanish from the equation, and its second member will be reduced to $2.\int\int dp.dq.\sin.p$; but $\int dq=\pi$, between the proposed limits $q=0$ and [1370e] $q=\pi$; hence the preceding expression becomes $2\pi.\int dp.\sin.p=4\pi$, because [1370f] $\int dp.\sin.p=1-\cos.p$, taken so as to vanish when $p=0$ is equal to 2, when $p=\pi$. Hence the equation [1370b] changes into

$$\frac{A}{a}+\frac{B}{b}+\frac{C}{c}=4\pi.$$ [1370g]

Theorem by Le Gendre.

This theorem was first discovered by Le Gendre.

therefore if we suppose

$$I = a \,.\, \cos. p + m b \,.\, \sin. p \,.\, \cos. q + n c \,.\, \sin. p \,.\, \sin. q,$$

[1365] $$L = \cos.^2 p + m \,.\, \sin.^2 p \,.\, \cos.^2 q + n \,.\, \sin.^2 p \,.\, \sin.^2 q,$$

$$R = I^2 + \{k^2 - a^2 - m b^2 - n c^2\} \,.\, L,$$

we shall have

[1366] $$r = \frac{I \pm \sqrt{R}}{L};$$

hence we deduce r', by using the positive radical, and r by using the negative; we shall therefore find

[1367] $$r + r' = \frac{2 I}{L}; \qquad r' - r = \frac{2 \,.\, \sqrt{R}}{L};$$

and we shall get, for points within the ellipsoid,

Attraction of an ellipsoid upon an internal point.

$$A = 2 \,.\, \int\!\!\int \frac{dp \,.\, dq \,.\, I \,.\, \sin. p \,.\, \cos. p}{L};$$

$$B = 2 \,.\, \int\!\!\int \frac{dp \,.\, dq \,.\, I \,.\, \sin.^2 p \,.\, \cos. q}{L};$$

[1368] $$C = 2 \,.\, \int\!\!\int \frac{dp \,.\, dq \,.\, I \,.\, \sin.^2 p \,.\, \sin. q}{L};$$

and for points without the ellipsoid,

Attraction of an ellipsoid upon an external point.

$$A = 2 \,.\, \int\!\!\int \frac{dp \,.\, dq \,.\, \sin. p \,.\, \cos. p \,.\, \sqrt{R}}{L};$$

$$B = 2 \,.\, \int\!\!\int \frac{dp \,.\, dq \,.\, \sin.^2 p \,.\, \cos. q \,.\, \sqrt{R}}{L};$$

[1369] $$C = 2 \,.\, \int\!\!\int \frac{dp \,.\, dq \,.\, \sin.^2 p \,.\, \sin. q \,.\, \sqrt{R}}{L};$$

these three last integrals ought to be taken between the two limits
[1369'] corresponding to $R = 0$.

3. The expressions relative to points situated within the ellipsoid being
[1369''] the most simple, we shall begin by examining them. We shall observe, in the first place, that the semi-axis of the ellipsoid k does not enter into the values of I and L [1365]; consequently the values of A, B, C, [1368], are
[1369'''] independent of k; whence it follows, that we may increase at pleasure the strata of the ellipsoid above the attracted point, without changing the

attraction of the spheroid upon that point, provided that the values of m and n remain constant.* Hence we obtain the following theorem. [1369'''']

A point placed within an elliptical stratum, whose internal and external surfaces are similar and similarly placed, is equally attracted in every direction. [1369v]

Attraction on a point within an elliptical stratum.

This theorem is an extension of that we have demonstrated in § 12 [469'''] of the second book, for a spherical stratum.

We shall resume the value of A [1368]. If we substitute the values of I and L, [1365], it becomes

$$A = 2.\int\int \frac{dp\,.\,dq\,.\sin.p\,.\cos.p\,.\{a\,.\cos.p + mb\sin.p\,.\cos.q + nc\,.\sin.p\,.\sin.q\}}{\cos.^2 p + m\,.\sin.^2 p\,.\cos.^2 q + n\,.\sin.^2 p\,.\sin.^2 q}\,; \qquad [1370]$$

the integrals relative to p and q, must be taken from p and q equal to nothing to p and q equal to two right angles, [1360']. It is evident that we have in general [1370']

$$\int P\,.\,dp\,.\cos.p = 0\,; \qquad [1370'']$$

P being a rational function of $\sin.p$ *and* $\cos.^2 p$; because the value of p being taken at equal distances above and below a right angle, the corresponding values of $P\,.\cos.p$ will be equal and of contrary signs.† therefore we shall have [1370''']

$$A = 2a\,.\int\int \frac{dp\,.dq\,.\sin.p\,.\cos.^2 p}{\cos.^2 p + m\,.\sin.^2 p\,.\cos.^2 q + n\,.\sin.^2 p\,.\sin.^2 q}\,. \qquad [1371]$$

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* (932) If m, n, remain constant, the expressions I, L, [1365], corresponding to the same attracted point, will be constant; consequently A, B, C, [1368], will not be varied by increasing k; the limits of p and q being 0 and two right angles, which are not affected by these changes. The theorem [1369v] corresponds to any ellipsoid, whose three principal semi-axes may be unequal, and is an extension of the theorem of Newton for ellipsoids of revolution. Princip. Lib. I, Prop. xci. This theorem is demonstrated in another manner in [1503].

† (933) $\sin.p = \sin.(\pi - p)$, and as $\cos.p = -\cos.(\pi - p)$, we have $\cos.^2 p = \cos.^2(\pi - p)$. Hence if $P = \text{func.}(\sin.p\,.\cos.^2 p)$, it will remain unchanged by writing $\pi - p$ for p, and the two elements of the integral $\int P\,.\cos.p\,.\,dp$ corresponding to the angles p and $\pi - p$, will be represented by

$$P\,.\,dp\,.\cos.p + P\,.\,dp\,.\cos.(\pi - p) = 0\,;$$

If we integrate this relative to q, from $q=0$ to q equal to two right angles, we shall have*

[1372] $$A=\frac{2a\pi}{\sqrt{mn}}\cdot\int\frac{dp\,.\,\sin.p\,.\,\cos.^2p}{\sqrt{\left\{1+\left(\frac{1-m}{m}\right)\cdot\cos.^2p\right\}\cdot\left\{1+\left(\frac{1-n}{n}\right)\cdot\cos.^2p\right\}}};$$

---

because $\cos.p+\cos.(\pi-p)=0$. Now the limits of this integral being $p=0$ and $p=\pi$, there must be the same number of elements depending on $\cos.p$ as on $\cos.(\pi-p)$,
[1370h] consequently the whole integral $\int P.dp.\cos.p=0$. The value of A, [1370], by altering the arrangement of the terms, may be put under the form

$$A=2a.\iint\frac{dp.dq.\sin.p.\cos.^2p}{\cos.^2p+m.\sin.^2p.\cos.^2q+n.\sin.^2p.\sin.^2q}$$
$$+2.\iint\frac{(mb.\sin.^2p.\cos.q+nc.\sin.^2p.\sin.q).\cos.p}{\cos.^2p+m.\sin.^2p.\cos.^2q+n.\sin.^2p.\sin.^2q}\cdot dp.dq.$$

The coefficient of $2.\cos.p.dp.dq$, in this last term, considered only as it respects the variable quantity p, is a function of $\sin.p$ and $\cos.^2p$. Putting this, as above, equal to P, it becomes $2.\iint P.\cos.p.dp.dq$, or $2.\int dq.\int P.dp.\cos.p$; and as $\int P.dp.\cos.p$ was shown in [1370h] to be equal to nothing, this term must vanish, and the remaining term of A will become identical with [1371].

* (934) Putting $\sin.^2q=1-\cos.^2q$, in [1371], it becomes

[1371a] $$A=2a.\iint\frac{dp.dq.\sin.p.\cos.^2p}{\cos.^2p+n.\sin.^2p-(n-m).\sin.^2p.\cos.^2q}=2a.\int\frac{dp.\sin.p.\cos.^2p}{(n-m).\sin.^2p}\cdot\int\frac{dq}{D^2-\cos.^2q},$$

[1372a] using for brevity $D^2=\frac{\cos.^2p+n.\sin.^2p}{(n-m).\sin.^2p}$; the part of this integral relative to q, which we shall call W, is $W=\int\frac{dq}{D^2-\cos.^2q}$. If we put $\cos.q=v$, which gives $dq=\frac{-dv}{\sqrt{(1-v^2)}}$, we shall get $W=\int\frac{-dv}{(D^2-v^2).\sqrt{(1-v^2)}}$. To avoid the radicals, we shall put $v=\frac{1-z^2}{1+z^2}$, whence

$\sqrt{(1-v^2)}=\frac{2z}{1+z^2}$, $\quad -dv=\frac{4z\,dz}{(1+z^2)^2}$, $\quad \frac{-dv}{\sqrt{(1-v^2)}}=\frac{2dz}{1+z^2}$, and

$$W=\int\frac{2dz}{1+z^2}\cdot\frac{1}{D^2-\frac{(1-z^2)^2}{(1+z^2)^2}}=\int\frac{2dz.(1+z^2)}{D^2.(1+z^2)^2-(1-z^2)^2},$$

the integral being taken from $\cos. p = 1$ to $\cos. p = -1$. Put $\cos. p = x$, and let M be the whole mass of the spheroid, we shall have, by [1372']

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which is free from radicals ; and as the denominator of this last expression is divisible into the two factors $D.(1+z^2)-(1-z^2)$, $D.(1+z^2)+(1-z^2)$, the expression of W may be put under the form

$$W = \frac{1}{D} \cdot \int \left\{ \frac{dz}{D.(1+z^2)-1+z^2} + \frac{dz}{D.(1+z^2)+1-z^2} \right\}$$

$$= \frac{1}{D} \cdot \int \frac{dz}{D-1+(D+1).z^2} + \frac{1}{D} \cdot \int \frac{dz}{D+1+(D-1).z^2}$$

$$= \frac{1}{D.(D+1)} \cdot \int \frac{dz}{\frac{D-1}{D+1}+z^2} + \frac{1}{D.(D-1)} \cdot \int \frac{dz}{\frac{D+1}{D-1}+z^2}. \qquad [1372b]$$

Now $$\int \frac{dz}{\frac{D-1}{D+1}+z^2} = \left(\frac{D+1}{D-1}\right)^{\frac{1}{2}} . \text{arc. tang.}\, z . \left(\frac{D+1}{D-1}\right)^{\frac{1}{2}},$$

and $$\int \frac{dz}{\frac{D+1}{D-1}+z^2} = \left(\frac{D-1}{D+1}\right)^{\frac{1}{2}} . \text{arc. tang.}\, z . \left(\frac{D-1}{D+1}\right)^{\frac{1}{2}}, \qquad [51] \text{ Int.}$$

The limits of the integral [1370'] relative to q, are $q = 0$, $q = \pi$; and as $v = \cos. q$, the limits of v must be 1 and -1 ; but $v = \frac{1-z^2}{1+z^2}$ gives $z^2 = \frac{1-v}{1+v}$; therefore at the first limit, where $v = 1$, $z = 0$; and at the last limit, where $v = -1$, $z = \infty$; consequently the arc whose tangent is $z . \left(\frac{D-1}{D+1}\right)^{\frac{1}{2}}$, or $z . \left(\frac{D+1}{D-1}\right)^{\frac{1}{2}}$, must be a right angle, or $\frac{1}{2}\pi$, and

$$\int \frac{dz}{\frac{D-1}{D+1}+z^2} = \left(\frac{D+1}{D-1}\right)^{\frac{1}{2}} . \tfrac{1}{2}\pi, \qquad \int \frac{dz}{\frac{D+1}{D-1}+z^2} = \left(\frac{D-1}{D+1}\right)^{\frac{1}{2}} . \tfrac{1}{2}\pi.$$

Substituting these in W, [1372b], it becomes

$$W = \frac{1}{D.(D+1)} \cdot \left(\frac{D+1}{D-1}\right)^{\frac{1}{2}} . \tfrac{1}{2}\pi + \frac{1}{D.(D-1)} \cdot \left(\frac{D-1}{D+1}\right)^{\frac{1}{2}} . \tfrac{1}{2}\pi$$

$$= \frac{\pi}{D.\sqrt{[(D+1).(D-1)]}} = \frac{\pi}{D.\sqrt{[D^2-1]}};$$

and by resubstituting the value of D [1372a], which gives $D^2 - 1 = \frac{\cos.^2 p + m . \sin.^2 p}{(n-m) . \sin.^2 p}$, we get $W = \frac{\pi . (n-m) . \sin.^2 p}{\sqrt{[(\cos.^2 p + n . \sin.^2 p) . (\cos.^2 p + m . \sin.^2 p)]}}$, hence A [1371a] becomes

$A = 2a\pi . \int \frac{dp . \sin. p . \cos.^2 p}{\sqrt{[(\cos.^2 p + n . \sin.^2 p) . (\cos.^2 p + m . \sin.^2 p)]}}$. Substituting for $\cos.^2 p + n . \sin.^2 p$

[1372″] § 2 [1363‴], $M=\frac{4\pi.k^3}{3.\sqrt{mn}}$, therefore $\frac{4\pi}{\sqrt{mn}}=\frac{3M}{k^3}$; hence we shall have

[1373] $$A=\frac{3aM}{k^3}.\int_0^1\frac{x^2.dx}{\sqrt{\left\{1+\left(\frac{1-m}{m}\right).x^2\right\}.\left\{1+\left(\frac{1-n}{n}\right).x^2\right\}}};$$

the integral being taken from $x=0$ to $x=1$.*

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its value $\cos.^2p+n.(1-\cos.^2p)=n.\left\{1+\left(\frac{1-n}{n}\right).\cos.^2p\right\}$,

also $\cos.^2p+m.\sin.^2p=m.\left\{1+\left(\frac{1-m}{m}\right).\cos.^2p\right\}$,

[1372c] it becomes as in [1372]; and as the limits of p are $p=0$, and $p=\pi$, [1370′], these limits must correspond to $\cos.p=1$, and $\cos.p=-1$, as above.

* (934) The value $\cos.p=x$ gives $dp.\sin.p=-dx$, and from [1372″] $\frac{2\pi}{\sqrt{mn}}=\frac{3M}{2k^3}$. Substituting these in [1372], we get

$$A=\frac{3aM}{2k^3}.\int\frac{-x^2.dx}{\left[\left\{1+\left(\frac{1-m}{m}\right).x^2\right\}.\left\{1+\left(\frac{1-n}{n}\right).x^2\right\}\right]^{\frac{1}{2}}};$$

the limits of the integral being $x=1$ and $x=-1$, [1372c]. But as the term x^2 only occurs in the value of A, its sign must be the same from $x=1$ to $x=0$, as from $x=0$ to $x=-1$. Therefore we may take the integral between the limits $x=1$ and $x=0$, and double it. If we change the sign of $-dx$ in A, we may change the limits so as to be from $x=0$ to $x=1$; and then we shall have the same value of A as in [1373].

In the formula [1373], and in other parts of this work, the limits of the integral are connected with the sign $\int$, according to the notation proposed by the Baron Fourier, and now generally used. These limits were not given in this manner in the original work. According

[1373a] to this notation, $\int_{x'}^{x''}P.dx$ denotes the integral of $P.dx$ from $x=x'$ to $x=x''$;

Signs $\int_{x'}^{x''}$, $\int_{y'}^{y''}$.

the *lowest* term x' corresponds to the *commencement* of the integral, the *upper* term to the *end.* If there be more than one sign $\int$ corresponding to different elements dx, dy, dz, &c., the limits connected with the signs $\int$ *are to be taken in the same order as the elements are arranged in that formula.* Thus $\int_{x'}^{x''}\int_{y'}^{y''}P.dx.dy$ denotes, that the first $\int_{x'}^{x''}$

[1373b] corresponds to the integration relative to the first element dx, and the second sign $\int_{y'}^{y''}$

If we integrate the expressions of B, C, [1368] in the same manner, we may reduce them to simple integrals, but it is easier to deduce these integrals from the preceding value of A. For this purpose we shall observe, that this expression may be considered as a function of a, and of the squares of the semi-axes of the ellipsoid, k^2, $\frac{k^2}{m}$, $\frac{k^2}{n}$, [1363″], parallel to the co-ordinates [1373′] of the attracted point a, b, c. Putting therefore k'^2 for the square of the semi-axis parallel to b, consequently $k'^2 . m$ and $k'^2 . \frac{m}{n}$, for the square of [1373″] the other two semi-axes,* B will be a similar function of b, k'^2, $k'^2 . m$, and $k'^2 . \frac{m}{n}$. Therefore to obtain B, we must change, in the preceding [1373‴] expression of A, a into b, k into k', or $\frac{k}{\sqrt{m}}$, m into $\frac{1}{m}$, and n into $\frac{n}{m}$; [1373⁗]

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corresponds to the second element dy. I had used in these notes a somewhat similar method [1373c] of notation, previous to the publication of that by the Baron Fourier. A like notation is used Sign with the sign Σ of finite integrals, thus $\Sigma_0^i = A_0 + A_1 + A_2 \ldots + A_n \ldots + A_i$, represents Σ_0^i. the sum of $i+1$ terms of the general form A_n, commencing with A_0 and ending with A_i. [1373d]

* (935) The square of the semi-axis parallel to b, or $\frac{k^2}{m}$, [1363″], being put equal to k'^2, gives $k^2 = k'^2 . m$. This value of k^2 being substituted in the expressions of the squares of the semi-axes parallel to b, a, c, [1363″], namely $\frac{k^2}{m}$, k^2, $\frac{k^2}{n}$, they become respectively k'^2, $k'^2 . m$, $k'^2 . \frac{m}{n}$. Comparing these with the values before used for the semi-axes parallel to a, b, c, namely, k^2, $\frac{k^2}{m}$, $\frac{k^2}{n}$, it appears that the former may be derived from the latter, by writing k' for k, m for $\frac{1}{m}$, and $\frac{n}{m}$ for n. Making these changes in [1373], and writing also b for a, we obtain this value of

$$B = \frac{3b . M}{k'^3} . \int \frac{x^2\,dx}{\sqrt{\left\{1 + \frac{\left(1 - \frac{1}{m}\right)}{\frac{1}{m}} . x^2\right\} . \left\{1 + \frac{\left(1 - \frac{n}{m}\right)}{\frac{n}{m}} . x^2\right\}}};$$

and as $\frac{1 - \frac{1}{m}}{\frac{1}{m}} = m - 1$, $\frac{1 - \frac{n}{m}}{\frac{n}{m}} = \frac{m - n}{n}$, $\frac{1}{k'^3} = \frac{m^{\frac{3}{2}}}{k^3}$, this expression becomes identical with [1374].

hence we get

[1374] $$B=\frac{3\,b\,M}{k^3}\cdot\int_0^1\frac{m^{\frac{3}{2}}\,.\,x^2\,d\,x}{\sqrt{\left\{1+(m-1)\,.\,x^2\right\}\,.\,\left\{1+\left(\frac{m-n}{n}\right)\,.\,x^2\right\}}}\,.$$

Putting

[1375] $$x=\frac{t}{\sqrt{m+(1-m)\,.\,t^2}},$$

we shall obtain*

[1376] $$B=\frac{3\,b\,M}{k^3}\cdot\int_0^1\frac{t^2\,d\,t}{\left\{1+\left(\frac{1-m}{m}\right)\,.\,t^2\right\}^{\frac{3}{2}}\,.\,\left\{1+\left(\frac{1-n}{n}\right)\,.\,t^2\right\}^{\frac{1}{2}}}\,;$$

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* (936) The value of x [1375] being substituted in the factors of the denominator of B [1374], putting also for brevity $M=1+\left(\frac{1-m}{m}\right)\,.\,t^2$, $N=1+\left(\frac{1-n}{n}\right)\,.\,t^2$, we get

$$1+(m-1)\,.\,x^2=1+\frac{(m-1)\,.\,t^2}{m+(1-m)\,.\,t^2}=\frac{m}{m+(1-m)\,.\,t^2}=\frac{1}{1+\left(\frac{1-m}{m}\right)\,.\,t^2}=\frac{1}{M}\,;$$

$$1+\left(\frac{m-n}{n}\right)\,.\,x^2=1+\frac{\left(\frac{m-n}{n}\right)\,.\,t^2}{m+(1-m)\,.\,t^2}=\frac{m+\left(\frac{m}{n}-m\right)\,.\,t^2}{m+(1-m)\,.\,t^2}=\frac{1+\left(\frac{1-n}{n}\right)\,.\,t^2}{1+\left(\frac{1-m}{m}\right)\,.\,t^2}=\frac{N}{M}.$$

[1376a] Hence $\left[\left\{1+(m-1)\,.\,x^2\right\}\,.\,\left\{1+\left(\frac{m-n}{n}\right)\,.\,x^2\right\}\right]^{\frac{1}{2}}=\left(\frac{1}{M}\,.\,\frac{N}{M}\right)^{\frac{1}{2}}=\frac{N^{\frac{1}{2}}}{M}$. Moreover the differential of $x=\{m\,t^{-2}+(1-m)\}^{-\frac{1}{2}}$ [1375] is

$$d\,x=\frac{m\,t^{-3}\,d\,t}{\{m\,t^{-2}+(1-m)\}^{\frac{3}{2}}}=\frac{d\,t}{m^{\frac{1}{2}}\,.\left\{1+\left(\frac{1-m}{m}\right)\,.\,t^2\right\}^{\frac{3}{2}}}=\frac{d\,t}{m^{\frac{1}{2}}\,.\,M^{\frac{3}{2}}}\,;$$

and $$m^{\frac{3}{2}}\,.\,x^2=\frac{m^{\frac{3}{2}}\,.\,t^2}{m+(1-m)\,.\,t^2}=\frac{m^{\frac{1}{2}}\,.\,t^2}{1+\left(\frac{1-m}{m}\right)\,.\,t^2}=\frac{m^{\frac{1}{2}}\,.\,t^2}{M}\,;$$ hence

$m^{\frac{3}{2}}\,.\,x^2\,d\,x=\frac{m^{\frac{1}{2}}\,.\,t^2}{M}\,.\,\frac{dt}{m^{\frac{1}{2}}.M^{\frac{3}{2}}}=\frac{t^2\,d\,t}{M^{\frac{5}{2}}}$. Dividing this by the denominator $\frac{N^{\frac{1}{2}}}{M}$, [1376a], we get $\frac{t^2\,d\,t}{M^{\frac{3}{2}}\,.\,N^{\frac{1}{2}}}$, for the terms under the sign $\int$ in [1374], which agrees with [1376]. The limits of t are obtained from [1375], which gives $t^2=\frac{m\,x^2}{1+(m-1)\,.\,x^2}$. This, when $x=0$, becomes $t=0$, and when $x=1$, becomes $t^2=\frac{m}{1+m-1}=1$, or $t=1$. These limits are as in [1376'].

the integral relative to t, must be taken between the same limits as that relative to x, namely, from $t=0$ to $t=1$; because $x=0$ gives $t=0$, and $x=1$ gives $t=1$. [1376']

Hence it follows, that if we put

$$\frac{1-m}{m}=\lambda^2; \qquad \frac{1-n}{n}=\lambda'^2; \qquad F=\int_0^1 \frac{x^2\,dx}{\sqrt{(1+\lambda^2.x^2).(1+\lambda'^2.x^2)}}; \qquad [1377]$$

we shall have*

$$B=\frac{3\,b\,M}{k^3}.\left(\frac{d.\lambda F}{d\lambda}\right). \qquad [1378]$$

If in this expression, we change b into c, λ into λ', and the contrary, we shall obtain the value of C. The attractions of the ellipsoid A, B, C, parallel to its three axes, will therefore be given by the following formulas, [1378']

Attraction of an ellipsoid upon a point within its surface.

$$A=\frac{3\,a\,M}{k^3}.F; \qquad B=\frac{3\,b\,M}{k^3}.\left(\frac{d.\lambda F}{d\lambda}\right); \qquad C=\frac{3\,c\,M}{k^3}.\left(\frac{d.\lambda' F}{d\lambda'}\right). \qquad [1379]$$

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* (937) The value of F [1377] gives $\lambda F=\int_0^1 (\lambda^{-2}+x^2)^{-\frac{1}{2}}.(1+\lambda'^2.x^2)^{-\frac{1}{2}}.x^2dx$, [1378a]
whose differential, relative to λ, is

$$\left(\frac{d.\lambda F}{d\lambda}\right)=\int_0^1 (\lambda^{-2}+x^2)^{-\frac{3}{2}}.(1+\lambda'^2.x^2)^{-\frac{1}{2}}.\lambda^{-3}.x^2dx=\int_0^1\frac{x^2\,dx}{(1+\lambda^2.x^2)^{\frac{3}{2}}.(1+\lambda'^2.x^2)^{\frac{1}{2}}}.$$

Now this definite integral is taken between the same limits, $x=0$, $x=1$, as those of t [1376]; so that we may change x into t, and the preceding expression will become the same as that in B [1376]. Hence B will become as in [1378]. Changing in this b into c, λ into λ', and the contrary, it is evident that B will be changed into C, [1379]. Lastly, A [1379] is as in [1373, 1377.]

The values A, B, C, [1379], represent the attraction of the ellipsoid, upon a point whose co-ordinates are a, b, c, situated upon its surface or within it, and resolved in directions parallel to the axes of the spheroid, α, β, γ, [1369b]. If we denote by $A_{,}$, $B_{,}$, $C_{,}$, the values of these attractions at the extremities of these axes α, β, γ, respectively, we shall obtain $A_{,}$, $B_{,}$, $C_{,}$, from A, B, C, [1379], by putting α, β, γ, for a, b, c, respectively; whence we shall get

$$A_{,}=\frac{3\,\alpha M}{k^3}.F, \qquad B=\frac{3\,\beta M}{k^3}.\left(\frac{d.\lambda F}{d\lambda}\right), \qquad C=\frac{3\gamma M}{k^3}.\left(\frac{d.\lambda' F}{d\lambda'}\right). \qquad [1379a]$$

Comparing these with [1379], we get

Maclaurin's Formulas.

$$A=\frac{a}{\alpha}.A_{,}, \qquad B=\frac{b}{\beta}.B_{,}, \qquad C=\frac{c}{\gamma}.C_{,}, \qquad [1379b]$$

which were first demonstrated by Maclaurin.

We may observe that these formulas take place for all points within the
[1379'] ellipsoid, and therefore for points infinitely near to its surface ; hence we may infer that they must take place also for points situated on the surface itself.

Hence it appears that the determination of the attractions of an ellipsoid,
[1379''] depends only upon the value of F. This is a definite integral ; but it has all the difficulties of an indefinite integral, when λ and λ' are indeterminate.
[1379'''] For if we represent this definite integral, taken from $x=0$ to $x=1$, by
[1379''''] $\varphi.(\lambda^2,\lambda'^2)$, it is evident that the indefinite integral will be $x^3.\varphi.(\lambda^2x^2,\lambda'^2x^2)$;* so that the first being given, the second will also be known. The indefinite
[1379v] integral is not in itself possible,† except one of the quantities λ or λ' is nothing, or $\lambda=\lambda'$; in these two cases the body is an ellipsoid of revolution,
[1379vi] and k will be its semi-axis of revolution, if λ and λ' are equal.‡ Now in this last case

[1380] $$F=\int_0^1\frac{x^2\,dx}{1+\lambda^2x^2}=\frac{1}{\lambda^3}\,.\,\{\lambda-\text{arc. tang. }\lambda\}.$$

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* (938) The function F, [1377], by developing the denominators according to the powers of $\lambda^2x^2,\lambda'^2x^2$, becomes

[1379c] $$F=\int x^2\,dx\,.\,(1-\tfrac{1}{2}\lambda^2x^2-\tfrac{1}{2}\lambda'^2x^2+\&c.)=x^3\,.\,(\tfrac{1}{3}-\tfrac{1}{10}\lambda^2x^2-\tfrac{1}{10}\lambda'^2x^2+\&c.).$$

The part within the parenthesis is a function of $\lambda^2x^2,\lambda'^2x^2$, which we may represent by $\varphi\,.\,(\lambda^2x^2,\lambda'^2x^2)$, and then we shall have $F=x^3\,.\,\varphi\,.\,(\lambda^2x^2,\lambda'^2x^2)$, which, on account of the factor x^3, is 0 when $x=0$, and when $x=1$, it becomes $F=\varphi\,.\,(\lambda^2,\lambda'^2)$; so that this last being known, we may easily deduce from it the indefinite integral, by writing $\lambda^2x^2,\lambda'^2x^2$ for λ^2,λ'^2, and multiplying by x^3.

[1379d] † (938a) The meaning of the author is, that this integral cannot be obtained by circular arcs and logarithms. This integral can be easily obtained by means of elliptical functions of the second and third form, as Le Gendre has shown in Vol. XI of the Memoirs of the Academy of Arts and Sciences of Paris, and in Vol. II, p. 525 of his Exercices de Calcul Integral. The form of the integration, in the case of the attracted point being without the surface, is also the same as for the internal attracted point, as Le Gendre has shown in the same volume, p. 529; where he has treated the whole subject in his usual elegant manner, and has given some new theorems on these attractions.

‡ (939) The three semi-axes of the spheroid parallel to the axes x, y, z, are in [1363''], represented by k, $\frac{k}{\sqrt{m}}$, $\frac{k}{\sqrt{n}}$; and the two first equations [1377] give

To deduce from this the partial differentials $\left(\frac{d\,.\,\lambda F}{d\lambda}\right)$, and $\left(\frac{d\,.\,\lambda' F}{d\lambda'}\right)$, which enter into the expressions of B and C, we shall observe that*

$$dF = \frac{d\lambda}{\lambda}\,.\left(\frac{d\,.\,\lambda F}{d\lambda}\right) + \frac{d\lambda'}{\lambda'}\,.\left(\frac{d\,.\,\lambda' F}{d\lambda'}\right) - F\,.\left(\frac{d\lambda}{\lambda} + \frac{d\lambda'}{\lambda'}\right); \qquad [1381]$$

now we have, when λ is equal to λ',

$$\left(\frac{d\,.\,\lambda F}{d\lambda}\right) = \left(\frac{d\,.\,\lambda' F}{d\lambda'}\right); \qquad \frac{d\lambda}{\lambda} = \frac{d\lambda'}{\lambda'}; \qquad [1382]$$

therefore†

$$\left(\frac{d\,.\,\lambda F}{d\lambda}\right).\,d\lambda = \tfrac{1}{2}\,.\,\lambda\,dF + F\,.\,d\lambda = \frac{1}{2\lambda}\,.\,d\,.\,\lambda^2 F. \qquad [1383]$$

$\frac{1}{m} = 1 + \lambda^2$, $\frac{1}{n} = 1 + \lambda'^2$, therefore these three semi-axes become respectively k, [1380a] $k\,.\sqrt{(1+\lambda^2)}$, $k\,.\sqrt{(1+\lambda'^2)}$. Now when λ, or λ', is nothing, two of these semi-axes become equal to k, and the surface must then be an ellipsoid of revolution about the other axis; and if $\lambda = \lambda'$, the two semi-axes parallel to y, z, must be equal to $k\,.\sqrt{(1+\lambda^2)}$, and the surface must then be an ellipsoid of revolution about the axis of x. When $\lambda = \lambda'$, we shall have, from [1377], $F = \int \frac{x^2\,dx}{1+\lambda^2 x^2}$; and if we put $\lambda x = x'$, or $x = \frac{x'}{\lambda}$, it becomes

$$F = \frac{1}{\lambda^3}\,.\int \frac{x'^2\,dx'}{1+x'^2} = \frac{1}{\lambda^3}\,.\int\left(dx' - \frac{dx'}{1+x'^2}\right) = \frac{1}{\lambda^3}\,.\,(x' - \text{arc.tang.}\,x') = \frac{1}{\lambda^3}\,.\,(\lambda x - \text{arc.tang.}\,\lambda x). \qquad [1380b]$$

This vanishes when $x = 0$, and when $x = 1$, it becomes $F = \frac{1}{\lambda^3}\,.\,(\lambda - \text{arc.tang.}\,\lambda)$, as in [1380], which corresponds to an oblate ellipsoid.

* (940) F being a function of λ, λ', we have the complete differential of F, or

$$dF = \left(\frac{dF}{d\lambda}\right).\,d\lambda + \left(\frac{dF}{d\lambda'}\right).\,d\lambda'; \qquad \left(\frac{d\,.\,\lambda F}{d\lambda}\right) = F + \lambda\,.\left(\frac{dF}{d\lambda}\right); \qquad \text{hence}$$

$\left(\frac{dF}{d\lambda}\right) = \frac{1}{\lambda}\,.\left(\frac{d\,.\,\lambda F}{d\lambda}\right) - \frac{1}{\lambda}\,.\,F.$ In like manner we have $\left(\frac{dF}{d\lambda'}\right) = \frac{1}{\lambda'}\,.\left(\frac{d\,.\,\lambda' F}{d\lambda'}\right) - \frac{1}{\lambda'}\,.\,F.$ These being substituted in the preceding value of dF, it becomes as in [1381].

† (941) Substituting in [1381] the values [1382], it becomes

$$dF = \frac{2\,d\lambda}{\lambda}\,.\left(\frac{d\,.\,\lambda F}{d\lambda}\right) - F\,.\,\frac{2\,d\lambda}{\lambda}.$$

Substituting the value of F, we shall get

[1384] $$\left(\frac{d\,.\,\lambda F}{d\lambda}\right)=\frac{1}{2\lambda^3}\,.\left\{\text{arc. tang.}\,\lambda-\frac{\lambda}{1+\lambda^2}\right\}.$$

[1384'] Hence we shall have, for an ellipsoid of revolution, whose semi-axis of revolution is k,*

Attraction of an oblate ellipsoid of revolution, on a point situated in, or within its surface.

$$A=\frac{3a\,.\,M}{k^3\,.\,\lambda^3}\,.\,\{\lambda-\text{arc. tang.}\,\lambda\};$$

$$B=\frac{3b\,.\,M}{2k^3\,.\,\lambda^3}\,.\left\{\text{arc. tang.}\,\lambda-\frac{\lambda}{1+\lambda^2}\right\};$$

[1385] $$C=\frac{3c\,.\,M}{2k^3\,.\,\lambda^3}\,.\left\{\text{arc. tang.}\,\lambda-\frac{\lambda}{1+\lambda^2}\right\}.$$

Multiplying this by $\frac{1}{2}\lambda$, and transposing $Fd\lambda$, we get $\left(\frac{d\,.\,\lambda F}{d\lambda}\right).\,d\lambda=\frac{1}{2}\lambda dF+Fd\lambda$, and the second member may be put under the form $\frac{1}{2\lambda}\,.\,d\,.\,\lambda^2 F$, [1383], as is easily proved by development. If we substitute the value of F [1380], we shall find

$$\left(\frac{d\,.\,\lambda F}{d\lambda}\right)=\frac{1}{2\lambda}\,.\,d\,.\,\frac{\left\{1-\frac{\text{arc. tang.}\,\lambda}{\lambda}\right\}}{d\lambda}=\frac{1}{2\lambda}\,.\left\{\frac{\text{arc. tang.}\,\lambda}{\lambda^2}-\frac{d\,.(\text{arc. tang.}\,\lambda)}{\lambda d\lambda}\right\}$$

$$=\frac{1}{2\lambda}\,.\left\{\frac{\text{arc. tang.}\,\lambda}{\lambda^2}-\frac{1}{\lambda.(1+\lambda^2)}\right\}=\frac{1}{2\lambda^3}\,.\left\{\text{arc. tang.}\,\lambda-\frac{\lambda}{1+\lambda^2}\right\},$$

as in [1384].

* (942) Substituting the value of F [1380], and $\left(\frac{d\,.\,\lambda F}{d\lambda}\right)$ or $\left(\frac{d\,.\,\lambda' F}{d\lambda'}\right)$ [1384], in [1379], we shall obtain the values of A, B, C, [1385].

When the ellipsoid of revolution, about the axis of x, is prolate, $\lambda^2=\lambda'^2$ [1377] becomes negative, and we must change λ into $\lambda\,.\sqrt{-1}$ in [1380—1385]; by which means $\text{arc. tang.}\,\lambda=\int\frac{d\lambda}{1+\lambda^2}$ [51] Int. becomes

$$\text{arc. tang.}\,(\sqrt{-1}\,.\,\lambda)=\sqrt{-1}\,.\int\frac{d\lambda}{1-\lambda^2}=\tfrac{1}{2}\,.\sqrt{-1}\,.\int\left\{\frac{d\lambda}{1-\lambda}+\frac{d\lambda}{1+\lambda}\right\}$$

$$=\tfrac{1}{2}\,.\sqrt{-1}\,.\,\{-\log.(1-\lambda)+\log.(1+\lambda)\}=\tfrac{1}{2}\,.\sqrt{-1}\,.\log.\left(\frac{1+\lambda}{1-\lambda}\right);$$

4. We shall now consider the attraction of spheroids upon an external point. This investigation is much more difficult than the preceding, on [1385′] account of the radical $\sqrt{R}$ in the differentials, which makes it impossible to find the integrals under that form. We may render the integrations [1385″] possible, by a suitable transformation of the variable quantities of which the integrals are composed; but instead of this method, we shall use the following, which depends wholly on the differentiation of functions.*

If we put V *equal to the sum of all the particles of the spheroid, divided by* [1385‴] *their respective distances from the attracted point*; $x, y, z,$ being the co-ordinates of the particle of the spheroid dM, and $a, b, c,$ those of the [1385⁗] attracted point, we shall have†

Function V.

$$V = \int \frac{dM}{\sqrt{(a-x)^2+(b-y)^2+(c-z)^2}}. \qquad [1386]$$

---

and the values [1385], corresponding to a prolate spheroid, become

Attraction of a prolate ellipsoid of revolution upon a point situated in, or within, the surface.

$$A = \frac{3a.M}{k^3.\lambda^3}.\left\{\tfrac{1}{2}.\log.\left(\frac{1+\lambda}{1-\lambda}\right)-\lambda\right\};$$

$$B = \frac{3b.M}{2k^3.\lambda^3}.\left\{\frac{\lambda}{1-\lambda^2}-\tfrac{1}{2}.\log.\left(\frac{1+\lambda}{1-\lambda}\right)\right\};$$

$$C = \frac{3c.M}{2k^3.\lambda^3}.\left\{\frac{\lambda}{1-\lambda^2}-\tfrac{1}{2}.\log.\left(\frac{1+\lambda}{1-\lambda}\right)\right\}. \qquad [1385a]$$

The semi-axis of revolution being k, the equatorial semi-axis $k.\sqrt{1-\lambda^2}$.

* (943) This method of finding the attraction of an ellipsoid upon an external point has been greatly simplified by Mr. Ivory, as will be seen hereafter in [1428a—σ].

† [944] This expression of V is similar to that in [455], changing $x, y, z, x', y', z', \rho,$ [1386a] into $a, b, c, x, y, z, 1,$ respectively, as in [1346a, b], and using dM [1347]. The same changes being made in [453], will produce the first value of A [1387], and this is evidently equal to $-\left(\frac{dV}{da}\right)$, deduced from [1386], and agrees with what has been proved in [455′, 453], changing as before x into a. Again, as V [1386] is symmetrical in $a, b, c,$ and by [1387] $-\left(\frac{dV}{da}\right)$ represents the attraction of the spheroid in the direction parallel to a, the attractions in the directions parallel to $b, c,$ must evidently be represented by $-\left(\frac{dV}{db}\right)$, $-\left(\frac{dV}{dc}\right)$, respectively, as in [1388]; these forces tend to decrease the co-ordinates, so that the forces $\left(\frac{dV}{da}\right)$, $\left(\frac{dV}{db}\right)$, $\left(\frac{dV}{dc}\right)$, *tend to increase them*, as [1387a] evidently appears from the notation and figure used in note 921, page 5.

[1386′] Then putting, as before, A, B, C, equal to the attractions of the spheroid, parallel to the axes x, y, z, and *directed towards the origin of the co-ordinates*, we shall have

[1387]
$$A=\int\frac{(a-x)\,.\,dM}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{3}{2}}}=-\left(\frac{dV}{da}\right).$$

In like manner we shall find

[1388]
$$B=-\left(\frac{dV}{db}\right);\qquad C=-\left(\frac{dV}{dc}\right).$$

Hence it follows, that *if V be known, we may deduce, by differentiation, the*
[1388′] *attraction of the spheroid parallel to any right line, supposing that line to be one of the rectangular co-ordinates of the attracted point*, as we have already remarked in the second book, § 11 [455‴].

The preceding value of V, developed in a series, becomes*

Value of V in a series, for any spheroid.

[1389]
$$V=\int\frac{dM}{\sqrt{a^2+b^2+c^2}}\,.\left\{\begin{array}{l}1+\frac{1}{2}\,.\,\dfrac{\{2ax+2by+2cz-x^2-y^2-z^2\}}{a^2+b^2+c^2}\\[2mm] +\frac{3}{8}\,.\,\dfrac{\{2ax+2by+2cz-x^2-y^2-z^2\}^2}{(a^2+b^2+c^2)^2}+\&c.\end{array}\right\}.$$

[1389′] This series is ascending, relative to the dimensions of the spheroid; and
[1389″] descending, relative to the powers and products of the co-ordinates of the attracted point. If we notice only the first term, which is sufficient, when the attracted point is at a very great distance, we shall find,

Value of V when the attracted point is at a very great distance.

[1390]
$$V=\frac{M}{\sqrt{a^2+b^2+c^2}};$$

---

* (945) The term under the radical in [1386], namely, $(a-x)^2+(b-y)^2+(c-z)^2$, being developed, may be put under the form

$$a^2+b^2+c^2-(2ax+2by+2cz-x^2-y^2-z^2),$$

and if for brevity we put $a^2+b^2+c^2=a'^2$, $2ax+2by+2cz-x^2-y^2-z^2=b'$, the expression [1386] will become

$$V=\int\frac{dM}{\sqrt{(a'^2-b')}}=\int\frac{dM}{a'}\,.\left(1-\frac{b'}{a'^2}\right)^{-\frac{1}{2}}=\int\frac{dM}{a'}\,.\left(1+\tfrac{1}{2}\,.\,\frac{b'}{a'^2}+\tfrac{3}{8}\,.\,\frac{b'^2}{a'^4}+\&c.\right),$$

which, by resubstituting the values of a'^2, b', becomes as in [1389].

M being the whole mass of the spheroid. This expression will be more [1390′]
accurate, if we place the origin of the co-ordinates in the centre of gravity of the spheroid; for we have, by the property of this centre,*

$$\int x \,.\, dM = 0\,; \qquad \int y \,.\, dM = 0\,; \qquad \int z \,.\, dM = 0\,; \tag{1391}$$

so that if we consider the ratio, of the dimensions of the spheroid to its [1391′]
distance from the attracted point, as an infinitely small quantity of the first order, the equation

$$V = \frac{M}{\sqrt{a^2+b^2+c^2}} \tag{1392}$$

will be exact, except in quantities of the second and higher orders. We [1392′]
shall now investigate the accurate values of V, corresponding to an ellipsoid.

5. If we use the symbols of § 1, we shall have†

$$V = \int \frac{dM}{r} = \iiint r\,dr\,.\,dp\,.\,dq\,.\sin. p = \tfrac{1}{2}\,.\iint (r'^2 - r^2)\,.\,dp\,.\,dq\,.\sin. p. \tag{1393}$$

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* (946) The expressions [1391] are the same as in [124], changing m into dM, and Σ into $\int$. Now the terms of V [1389], depending on the first power of x, y, z, using a' as in the last note, are

$$\int \frac{dM}{a'^3}\,.\,\tfrac{1}{2}\,.\,(2ax+2by+2cz) = \frac{a}{a'^3}\,.\int x\,.\,dM + \frac{b}{a'^3}\,.\int y\,.\,dM + \frac{c}{a'^3}\,.\int z\,.\,dM\,;$$

and these vanish by means of the formulas [1391]. Therefore if we neglect the squares and higher powers of x, y, z, in [1389], we shall have $V = \int \frac{dM}{\sqrt{(a^2+b^2+c^2)}} = \frac{M}{\sqrt{(a^2+b^2+c^2)}}$, as in [1392].

† (947) Substituting $\sqrt{\{(a-x)^2+(b-y)^2+(c-z)^2\}} = r$ [1355e] in [1386], [1393a]
we get $V = \int \frac{dM}{r}$, which is also easily deduced from the definition of r, V, [1356′′′′, 1385′′′]; and if we use the value of dM [1357c], it becomes

$$V = \iiint r\,dr\,.\,dp\,.\,dq\,.\sin. p = \iint dp\,.\,dq\,.\sin. p\,.\int r\,dr.$$

Now $\int r\,dr = \tfrac{1}{2}r^2 + \text{const.} = \tfrac{1}{2}r^2 - \tfrac{1}{2}r_{\prime}^2$, supposing the integral to commence with the least value of r, denoted by $r_{\prime}$. If the greatest value of r be r', the complete integral will

Substituting the values of r and r' found in § 2 [1367], we shall find

[1394] $$V = 2 . \int\!\!\int \frac{dp . dq . \sin. p . I . \sqrt{R}}{L^2}.$$

We shall resume the values of A, B, C, corresponding to an external attracted point, given in § 2, [1369],

[1395] $$A = 2 . \int\!\!\int \frac{dp . dq . \sin. p . \cos. p . \sqrt{R}}{L};$$
$$B = 2 . \int\!\!\int \frac{dp . dq . \sin.^2 p . \cos. q . \sqrt{R}}{L};$$
$$C = 2 . \int\!\!\int \frac{dp . dq . \sin.^2 p . \sin. q . \sqrt{R}}{L}.$$

[1395'] Since at the limits of these integrals we have $\sqrt{R} = 0$ [1369'], it is evident, that if we take the first differentials of V, A, B, C, relative to any
[1395''] one of the six quantities a, b, c, k, m, and n, we may neglect the effect of the variations of the limits; so that we shall have, for example,

[1396] $$\left(\frac{dV}{da}\right) = 2 . \int\!\!\int dp . dq . \sin. p . \left\{ \frac{d . \frac{I . \sqrt{R}}{L^2}}{da} \right\}.$$

For the integral $\int \frac{dp . \sin. p . I . \sqrt{R}}{L^2}$, towards these limits, is nearly
[1396'] proportional to $R^{\frac{3}{2}}$;* which renders its differential nothing, at these

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be $\frac{1}{2} r'^2 - \frac{1}{2} r_{\prime}^2$. Changing $r_{\prime}$ into r, to conform to the notation [1360''], we shall get $\int r\,dr = \frac{1}{2} . (r'^2 - r^2)$, and the preceding value of V will become

$$V = \tfrac{1}{2} . \int (r'^2 - r^2) . dp . dq . \sin. p,$$

as in [1393].

[1393b] Multiplying the values of $r' + r$, $r' - r$, [1367], we get $r'^2 - r^2 = \frac{4 I . \sqrt{R}}{L^2}$; hence we obtain the expression of V [1394].

* (948) The expression of V [1394] is similar to that of $\int\!\!\int P' . dp . dq$ [1359b], putting $P' = \frac{\sin. p . I . \sqrt{R}}{L^2}$, which makes $V = \int dq . \int P'\,dp$. The integral
[1397a] $\int P'\,dp$ is to be found upon the supposition that q is constant, [1359c], and the limits

limits. This being supposed, it is easy to prove by differentiation, that if we put for brevity

$$a \,.\, A + b \,.\, B + c \,.\, C = F,$$ [1397]

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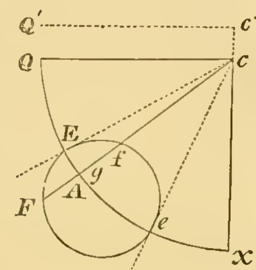
of the values of p are the angles $x\,c\,e = p_{\prime}$, and $x\,c\,E = p'$, in the annexed figure, which is the same as in page 9; these limits being the values of p, to be found by putting $r' - r_{\prime} = 0$, or $R = 0$; as will appear from [1359c, e, f]. The values of $p_{\prime}$, p', being functions of q, the integral $\int P'\,dp$ will become a function of q, which in [1359d] is called Q, and then $V = \int Q\,dq$. The equation $R = 0$, from which $p_{\prime}$, p', are determined, becomes, by means of R, I, L, [1365],



$$0 = \{a \,.\, \cos. p + m\,b \,.\, \sin. p \,.\, \cos. q + n\,c \,.\, \sin. p \,.\, \sin. q\}^2$$
$$+ \{k^2 - a^2 - m\,b^2 - n\,c^2\} \,.\, \{\cos.^2 p + m \,.\, \sin.^2 p \,.\, \cos.^2 q + n \,.\, \sin.^2 p \,.\, \sin.^2 q\}.$$ [1397b]

This equation will give $p_{\prime}$ and p' in terms of a, b, c, m, n, k, q; and if any one of these last quantities should be varied, the angles $p_{\prime}$ and p' would also vary. Thus if a be changed into $a + d\,a$, the limiting angles $p_{\prime}$ and p' will become $p_{\prime} + \left(\frac{dp_{\prime}}{da}\right) . d\,a$, and $p' + \left(\frac{dp'}{d\,a}\right) . d\,a$. [1397$b'$] This is also evident by geometrical considerations. For if we continue the line $x\,c$ to c', making $c\,c' = d\,a$, the lines drawn from c', tangents to the curve $e\,f\,E\,F$, will not in general be parallel to the tangents $c\,e$, $c\,E$; consequently the angles, which these tangents make with the line $x\,c\,c'$, will be varied by this increase of the value of a. Therefore, by this increment of the value of a, the quantity which we have above denoted by Q, will be varied in two different manners. *First*, by the variation of Q, on account of the value of a contained in it *explicitly*; this variation being usually expressed by $\left(\frac{d\,Q}{d\,a}\right) . d\,a$. *Secondly*, [1397$c$] by the variation of Q, on account of the value of a contained *implicitly* in the limiting angles $p_{\prime}$ and p'; which angles vary by $\left(\frac{d\,p_{\prime}}{d\,a}\right) . d\,a$, and $\left(\frac{dp'}{da}\right) . da$, [1397$b'$]. The effect of this second cause of variation may be computed in the following manner. In the annexed figure, which is similar to the above, but is drawn separately for distinctness, the line $c\,f\,F$ is supposed to be drawn infinitely near to $c\,E$, the limit of p'. The integral $\int P'\,dp$ will be affected by the action of all the particles contained in the space $f\,g\,F\,E$; but this mass being called m', and its mean distance from the point c being $c\,E = r''$, its effect on V, or on $\int \frac{dM}{r}$, [1393], will be nearly $\frac{m'}{r''}$. Now if from the point E we let fall upon $F\,f$ the perpendicular $E\,g = t$, it will nearly bisect this line $F\,f = r' - r = \rho$, in g; ρ being [1397d]

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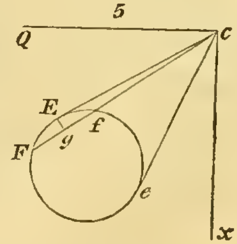

we shall have, between the four quantities B, C, F, V, the following equation of partial differentials,*

[1397d] used for brevity. This small arc FEf may be considered as a circular or a parabolic arc, on account of its being infinitely small; and the area $m'=fgFE$, considered as parabolic, is by a well known rule equal to $\frac{2}{3}.Eg.Ff=\frac{2}{3}t.\rho$. If we suppose D to be the diameter of the circle of curvature corresponding to the point E, Eg will be its versed sine, and $Fg=\frac{1}{2}\rho=$ sine of the arc FE; thus, by the nature of a circle, we shall have nearly, $D.Eg=Fg^2=\frac{1}{4}\rho^2$; hence $Eg=t=\frac{\rho^2}{4D}$. Substituting this in $m'=\frac{2}{3}t.\rho$, it becomes $m'=\frac{1}{6D}.\rho^3$; but $\rho=r'-r=\frac{2.\sqrt{R}}{L}$, [1367]; hence the value of m' becomes $m'=\frac{4}{3DL^3}.R^{\frac{3}{2}}$. Therefore this variation of V, or

[1397e] $\frac{m'}{r''}$ is $\frac{4}{3DL^3.r''}.R^{\frac{3}{2}}$. Now at the point E the quantity R is nothing, and in general, for the points near to E, the values D, L r'', do not vary much, for all the particles of the mass m'; therefore the variation of V is nearly proportional to $R^{\frac{3}{2}}$. This quantity and its differential both vanish when $R=0$; therefore we may neglect the second cause of variation in the value of V, and it is evident that what we have said relative to V [1394] will also apply to the values of A, B, C, [1395]. For if we compare the elements of V [1394] with those of A [1395], we shall find that they are in the ratio of $\frac{1}{L}:\cos.p$, or, by [1367], $\frac{r+r'}{2}:\cos.p$, which is nearly as $r'':\cos.p$; and as this is nearly constant for all those points m' of the spheroid which are near the limits, we shall get the required part of A, by multiplying that of V [1397e], by $\frac{\cos.p}{r''}$ nearly. Hence it is evident that this part of A will be of the order $R^{\frac{3}{2}}$; and in the same way we may prove that the corresponding parts of B, C, are of the order $R^{\frac{3}{2}}$; and these expressions, as well as their differentials, will vanish when $R=0$. Hence in all cases, when finding the partial differentials of V, A, B, C, [1394, 1395], relative to a, b, c, m, n, k, we may neglect the

[1397f] variations of the limits of $p_{,}$, p'. The limits $q_{,}$, q', of q, depend on the same principles; therefore we may also neglect the consideration of these limits, in finding the partial differentials of V, A, B, C. Hence it follows, that in finding the partial differentials of V, A, B, C, it is only necessary to notice the first cause of variation, mentioned in [1397c], namely, that which arises from the *explicit* values of a, b, c, m, n, k, contained in these quantities.

* (949) To investigate the equation [1398] *a priori*, we may find the partial differentials of V, A, B, C, relative to a, b, c, m, n, k, and connect them by constant coefficients

Equation of partial differentials for finding the attraction of an ellipsoid upon an external point.

$$0 = \left\{ \frac{a^2+b^2+c^2-k^2}{2} \right\} . k . \left\{ \left(\frac{dV}{dk}\right) - \left(\frac{dF}{dk}\right) \right\} + k^2 . (V-F)$$
$$+ k^2 . \left(\frac{m-1}{m}\right) . b . \left\{ \left(\frac{dF}{db}\right) - \tfrac{1}{2} . \left(\frac{dV}{db}\right) - B \right\}$$
$$+ k^2 . \left(\frac{n-1}{n}\right) . c . \left\{ \left(\frac{dF}{dc}\right) - \tfrac{1}{2} . \left(\frac{dV}{dc}\right) - C \right\} \qquad (1)$$ [1398]
$$- k^2 . (m-1) . \left(\frac{dF}{dm}\right) - k^2 . (n-1) . \left(\frac{dF}{dn}\right).$$

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g, h, i, &c. Then take these constant quantities so as to make the terms depending on the same cosines and sines disappear, and we may obtain the proposed formula. We shall not, however, use this method, but shall proceed according to the directions of the author, by means of the differentiation of the quantities V, A, B, C, F.

We shall use, for brevity, the following abridged values of x, y, z, k', S, G; observing, however, that these values of x, y, z, differ from those used in § 1, 2. We shall then, from [1365, 1395, 1397], get the following values of I, L, R, F,

$$x = \cos. p, \qquad y = \sin. p . \cos. q, \qquad z = \sin. p . \sin. q,$$
$$k' = k^2 - a^2 - m b^2 - n c^2,$$
$$S = R + I^2,$$
$$G = a . \cos. p + b . \sin. p . \cos. q + c . \sin. p . \sin. q = ax + by + cz,$$ [1398a]
$$I = ax + mby + ncz, \quad L = x^2 + my^2 + nz^2, \quad R = I^2 + (k^2 - a^2 - mb^2 - nc^2) . L = I^2 + k' . L,$$
$$F = 2 . \textstyle\iint dp . dq . \sin. p . R^{\frac{1}{2}} . L^{-1} . G.$$
$$x^2 + y^2 + z^2 = \cos.^2 p + \sin.^2 p . (\cos.^2 q + \sin.^2 q) = \cos.^2 p + \sin.^2 p = 1.$$

Comparing together the values [1394, 1395] and that of F [1398a], we shall find that they all contain the factor $2\, dp . dq . \sin. p$, under the sign $\iint$; and if the equation [1398] become identically nothing, by means of these values of V, A, B, C; it must also be identically nothing, if we neglect $2 . \iint dp . dq . \sin. p$, in all these quantities. Therefore we may put, instead of [1394, 1395], [1398b]

$$V = I R^{\frac{1}{2}} . L^{-2}, \qquad A = R^{\frac{1}{2}} . L^{-1} x, \qquad B = R^{\frac{1}{2}} . L^{-1} y,$$
$$C = R^{\frac{1}{2}} . L^{-1} z, \qquad F = R^{\frac{1}{2}} . L^{-1} G.$$ [1398c]

We may eliminate B, C, and F from this equation, by means of their values [1387, 1388, 1397],

[1398'] $$-\left(\frac{dV}{db}\right), \qquad -\left(\frac{dV}{dc}\right), \qquad -a.\left(\frac{dV}{da}\right)-b.\left(\frac{dV}{db}\right)-c.\left(\frac{dV}{dc}\right),$$

we shall thus have an equation of partial differentials in V only. We shall now put

[1399] $$V=\frac{4\pi.k^3}{3.\sqrt{mn}}.v=M.v,$$

Now collecting together all the terms of the second member of [1398], depending *explicitly* on V, and calling the same k^2u'; putting also k^2u'' for the rest of those terms, the values u', u'', will be as in the following expressions, and the equation to be demonstrated

[1398d] will be $k^2u'+k^2u''=0$, or $u'+u''=0$, and we shall have

$$u'=\tfrac{1}{2}.(a^2+b^2+c^2-k^2).\frac{1}{k}.\left(\frac{dV}{dk}\right)+V-\frac{(m-1)}{2m}.b.\left(\frac{dV}{db}\right)-\frac{(n-1)}{2n}.c.\left(\frac{dV}{dc}\right);$$

[1398e] $$u''=\tfrac{1}{2}.(-a^2-b^2-c^2+k^2).\frac{1}{k}.\left(\frac{dF}{dk}\right)-F+\frac{(m-1)}{m}.\left\{b.\left(\frac{dF}{db}\right)-bB-m.\left(\frac{dF}{dm}\right)\right\}$$
$$+\frac{(n-1)}{n}.\left\{c.\left(\frac{dF}{dc}\right)-cC-n.\left(\frac{dF}{dn}\right)\right\}.$$

From V, F, [1398c], we obtain the following partial differentials; observing that I, L, G, [1398a], do not contain k; that L does not contain b, k; and G does not contain m, k:

$$\left(\frac{dV}{dk}\right)=\tfrac{1}{2}IR^{-\frac{1}{2}}.L^{-2}.\left(\frac{dR}{dk}\right), \qquad \left(\frac{dV}{db}\right)=R^{\frac{1}{2}}.L^{-2}.\left(\frac{dI}{db}\right)+\tfrac{1}{2}IR^{-\frac{1}{2}}.L^{-2}.\left(\frac{dR}{db}\right),$$

[1398f] $$\left(\frac{dF}{dk}\right)=\tfrac{1}{2}R^{-\frac{1}{2}}.GL^{-1}.\left(\frac{dR}{dk}\right), \qquad \left(\frac{dF}{db}\right)=\tfrac{1}{2}R^{-\frac{1}{2}}.GL^{-1}.\left(\frac{dR}{db}\right)+R^{\frac{1}{2}}.L^{-1}.\left(\frac{dG}{db}\right),$$

$$\left(\frac{dF}{dm}\right)=\tfrac{1}{2}R^{-\frac{1}{2}}.GL^{-1}.\left(\frac{dR}{dm}\right)-R^{\frac{1}{2}}.GL^{-2}.\left(\frac{dL}{dm}\right).$$

It is not necessary to write down the values of $\left(\frac{dV}{dc}\right)$, $\left(\frac{dF}{dc}\right)$, $\left(\frac{dF}{dn}\right)$, since they may be derived from $\left(\frac{dV}{db}\right)$, $\left(\frac{dF}{db}\right)$, $\left(\frac{dF}{dm}\right)$, respectively, by changing b, m, y,

M being, by § 1, [1346], the mass of the ellipsoid; and instead of the [1399']
variable quantities m and n, we shall use θ, ϖ, found by putting

$$\theta = \left(\frac{1-m}{m}\right) . k^2 ; \qquad \varpi = \left(\frac{1-n}{n}\right) . k^2 ; \qquad [1400]$$

$B = -\left(\frac{dV}{db}\right)$, into c, n, z, $C = -\left(\frac{dV}{dc}\right)$, and the contrary; as is evident from [1398f']
the values [1398a], observing that these changes do not affect the values of k, I, L, R, F, G, S.

From the values I, L, R, [1398a], we get, by observing that x, y, z, are independent of k, b, m,

$$\left(\frac{dI}{db}\right) = my, \qquad \left(\frac{dI}{dm}\right) = by, \qquad \left(\frac{dL}{dm}\right) = y^2, \qquad \left(\frac{dG}{db}\right) = y,$$

$$\left(\frac{dR}{dk}\right) = 2k.L, \qquad \left(\frac{dR}{db}\right) = 2I.\left(\frac{dI}{db}\right) - 2mb.L = 2my.I - 2mb.L, \qquad [1398g]$$

$$\left(\frac{dR}{dm}\right) = 2I.\left(\frac{dI}{dm}\right) - b^2.L + k'.\left(\frac{dL}{dm}\right) = 2I.by - b^2.L + k'.y^2.$$

Making the changes mentioned in [1398f'], we may obtain $\left(\frac{dI}{dc}\right)$, $\left(\frac{dI}{dn}\right)$, &c.
Substituting these in [1398f], we shall get the values of $\left(\frac{dV}{dk}\right)$, $\left(\frac{dV}{db}\right)$, &c.,
from which we may obtain the values of u', u'', [1398e]. The object being to prove that
$0 = u' + u''$ [1398d] is an identical equation, the demonstration will not be affected, by [1398h]
multiplying all the quantities [1398c, f], by the common factor $2R^{\frac{1}{2}}.L^2$, because all
these quantities $\left(\frac{dV}{dk}\right)$, $\left(\frac{dV}{db}\right)$, &c., occur only in a linear form, or of the first
degree in u', u''; and this multiplication will render the quantities more simple. After
performing this multiplication, we must substitute the values [1398g], and we shall get the [1398i]
following system of equations, neglecting this factor in the first member.

θ is the difference of the squares of the two semi-axes of the spheroid,

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$$V=2IR, \qquad A=2RL.x, \qquad B=2RL.y, \qquad C=2RL.z, \qquad F=2R.GL,$$

$$\left(\frac{dV}{dk}\right)=2k.IL,$$

$$\left(\frac{dV}{db}\right)=2R.my+I.(2my.I-2mb.L)=2m.\{(R+I^2).y-b.IL\}=2m.(S.y-b.IL),$$

[1398k]
$$\left(\frac{dF}{dk}\right)=GL.2k.L=2k.GL^2,$$

$$\left(\frac{dF}{db}\right)=GL.(2my.I-2mb.L)+2RL.y,$$

$$\left(\frac{dF}{dm}\right)=GL.(2I.by-b^2.L+k'y^2)-2RG.y^2.$$

We may obtain $\left(\frac{dV}{dc}\right)$, $\left(\frac{dF}{dc}\right)$, $\left(\frac{dF}{dn}\right)$, by changing b, m, &c., as in [1398f']. Hence we get the four following expressions, of which the fourth may be derived from the third, by making the changes of b into c, m into n, y into z, &c., as above.

$$\tfrac{1}{2}.(a^2+b^2+c^2-k^2).\frac{1}{k}.\left(\frac{dV}{dk}\right)= \qquad (a^2+b^2+c^2-k^2).IL,$$

$$V=2IR,$$

[1398l]
$$-\left(\frac{m-1}{2m}\right).b.\left(\frac{dV}{db}\right)=-(m-1).b.(S.y-b.IL)=(mb^2-b^2).IL-(m-1).by.S,$$

$$-\left(\frac{n-1}{2n}\right).c.\left(\frac{dV}{dc}\right)= \qquad (nc^2-c^2).IL-(n-1).cz.S.$$

Adding these four equations together, the first member becomes equal to the value of u', [1398e]. The coefficient of IL, in the second member, neglecting the terms of the sum which destroy each other, is equal to $a^2+mb^2+nc^2-k^2=-k'$ [1398a]; and the coefficient of $-S$ is $(m-1).by+(n-1).cz$, which, by using the values I, G, [1398a], is evidently equal to $I-G$; hence we get

[1398m]
$$u'=2IR-k'.IL-(I-G).S=SG+I.(2R-k'.L-S)=SG;$$

because the coefficient of I vanishes; since the substitution of $S=R+I^2$ [1398a]

[1398m'] makes it $2R-k'.L-S=R-k'.L-I^2$, which becomes nothing by using the value of $R=I^2+k'.L$ [1398a].

We shall now compute the value of u'' [1398e], by means of [1398k], from which we get

$$b.\left(\frac{dF}{db}\right)=GL.(2mby.I-2mb^2.L)+2b.RL.y,$$

[1398n]
$$-bB= \qquad -2b.RL.y,$$

$$-m.\left(\frac{dF}{dm}\right)=GL.(-2mby.I+mb^2.L-k'my^2)+2RG.my^2,$$

which are drawn parallel to the axes of y and x; and ϖ is the difference

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whose sum is

$$b \cdot \left(\frac{dF}{db}\right) - bB - m \cdot \left(\frac{dF}{dm}\right) = GL \cdot (-mb^2 \cdot L - k' \cdot my^2) + 2RG \cdot my^2$$

$$= mG \cdot \{-b^2 \cdot L^2 + (2R - k' \cdot L) \cdot y^2\}. \qquad [1398o]$$

If we multiply this by $\frac{m-1}{m}$, and substitute $2R - k'.L = S$, deduced from [1398m'], we shall get the second of the three following equations. The third of these equations may be deduced from the second, by changing b, m, y, into c, n, z, respectively, and the contrary, as in [1398f']. The first of these equations is obtained by substituting the values of $\left(\frac{dF}{dk}\right)$, F, [1398k].

$$\tfrac{1}{2} \cdot (-a^2 - b^2 - c^2 + k^2) \cdot \frac{1}{k} \cdot \left(\frac{dF}{dk}\right) - F = (-a^2 - b^2 - c^2 + k^2) . GL^2 - 2R . GL,$$

$$\left(\frac{m-1}{m}\right) \cdot \left\{ b \cdot \left(\frac{dF}{db}\right) - bB - m \cdot \left(\frac{dF}{dm}\right) \right\} = -(m-1) \cdot b^2 \cdot GL^2 + (m-1) \cdot y^2 \cdot SG, \qquad [1398p]$$

$$\left(\frac{n-1}{n}\right) \cdot \left\{ c \cdot \left(\frac{dF}{dc}\right) - cC - n \cdot \left(\frac{dF}{dn}\right) \right\} = -(n-1) \cdot c^2 \cdot GL^2 + (n-1) \cdot z^2 \cdot SG.$$

Adding these three equations together, the first member of the sum is equal to u'' [1398e]; hence, by connecting and reducing the terms of the coefficient of GL^2, we get

$$u'' = (-a^2 - mb^2 - nc^2 + k^2) \cdot GL^2 - 2R \cdot GL + \{(m-1) \cdot y^2 + (n-1) \cdot z^2\} \cdot SG. \qquad [1398q]$$

Now the value of R [1398a] gives

$$(-a^2 - mb^2 - nc^2 + k^2) \cdot L = R - I^2 = 2R - S, \qquad [1398a];$$

and if we subtract $x^2 + y^2 + z^2 = 1$, from L [1398a], we shall get

$$(m-1) \cdot y^2 + (n-1) \cdot z^2 = L - 1;$$

hence, by substitution in this last value of u'', we obtain

$$u'' = (2R - S) \cdot GL - 2R \cdot GL + (L-1) \cdot SG. \qquad [1398r]$$

Rejecting the terms which mutually destroy each other, it becomes $u'' = -SG$. Adding this to $u' = SG$, [1398m], we get $u' + u'' = 0$, which is the equation required to be proved in [1398d], and is equivalent to the equation [1398].

[1400′] of the squares of the semi-axes which are drawn parallel to z and x;* so that if we take for the axis of x, the least of the three axes of the spheroid, $\sqrt{\theta}$ and $\sqrt{\varpi}$ will be the two excentricities. We shall then find†

[1401]

$$k\cdot\left(\frac{dV}{dk}\right)=M\cdot\left\{2\theta\cdot\left(\frac{dv}{d\theta}\right)+2\varpi\cdot\left(\frac{dv}{d\varpi}\right)+k\cdot\left(\frac{dv}{dk}\right)+3v\right\};$$

$$\left(\frac{dV}{dm}\right)=-M\cdot\left\{\frac{k^2}{m^2}\cdot\left(\frac{dv}{d\theta}\right)+\frac{v}{2m}\right\};$$

$$\left(\frac{dV}{dn}\right)=-M\cdot\left\{\frac{k^2}{n^2}\cdot\left(\frac{dv}{d\varpi}\right)+\frac{v}{2n}\right\};$$

[1400a] * (950) The three semi-axes parallel to x, y, z, are $\alpha=k$, $\beta=\frac{k}{\sqrt{m}}$, $\gamma=\frac{k}{\sqrt{n}}$, [1369b], respectively, and α being supposed to be the least of these quantities, the excentricities will be denoted by $\sqrt{(\beta^2-\alpha^2)}$, $\sqrt{(\gamma^2-\alpha^2)}$, [377″, 378m]. Substituting these values of α, β, γ, we get

[1400b]

$$\sqrt{(\beta^2-\alpha^2)}=\left(\frac{k^2}{m}-k^2\right)^{\frac{1}{2}}=k\cdot\left(\frac{1-m}{m}\right)^{\frac{1}{2}}=\sqrt{\theta}, \qquad [1400],$$

$$\sqrt{(\gamma^2-\alpha^2)}=\left(\frac{k^2}{n}-k^2\right)^{\frac{1}{2}}=k\cdot\left(\frac{1-n}{n}\right)^{\frac{1}{2}}=\sqrt{\varpi}, \qquad [1400].$$

† (951) In the first members of [1401], V is a function of k, m, n; in the second members, V is a function of k, θ, ϖ; the constant quantities a, b, c, being common to both hypotheses. Hence in finding $\left(\frac{dV}{dk}\right)$, or $k\cdot\left(\frac{dV}{dk}\right)$, in the first of the equations [1401], we must suppose, in the second member of this equation, k to be contained in V, *explicitly*, also *implicitly*, by means of θ, ϖ, which are in v [1399]. So that from $V=Mv$, [1399], we shall have

[1401a]

$$\left(\frac{dV}{dk}\right)=\left(\frac{dM}{dk}\right)\cdot v+M\cdot\left(\frac{dv}{dk}\right)+M\cdot\left(\frac{dv}{d\theta}\right)\cdot\left(\frac{d\theta}{dk}\right)+M\cdot\left(\frac{dv}{d\varpi}\right)\cdot\left(\frac{d\varpi}{dk}\right);$$

the two first terms arise from the *explicit* values of k, and the two last from the *implicit* values; and we may observe, that in conformity to the usual rules for finding partial differentials, the quantities m, n, are considered as constant, in finding this partial differential $\left(\frac{dV}{dk}\right)$, relative to the independent variable quantity k of the first member. Now from [1363‴] we get

$$\left(\frac{dM}{dk}\right)=\frac{4\pi\cdot k^2}{\sqrt{mn}}=\frac{4\pi\cdot k^3}{3\cdot\sqrt{mn}}\cdot\frac{3}{k}=M\cdot\frac{3}{k};$$

V being considered, in the first members of these equations, as a function of a, b, c, k, m, and n; and v in the second members, as a function of a, b, c, [1401′]
θ, ϖ, and k. If we put

$$Q = a\cdot\left(\frac{dv}{da}\right) + b\cdot\left(\frac{dv}{db}\right) + c\cdot\left(\frac{dv}{dc}\right),$$ [1402]

we shall find*

$$F = -MQ,$$ [1403]

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and from [1400] we find

$$\left(\frac{d\theta}{dk}\right) = \left(\frac{1-m}{m}\right).2k = \frac{2\theta}{k}; \qquad \left(\frac{d\varpi}{dk}\right) = \left(\frac{1-n}{n}\right).2k = \frac{2\varpi}{k};$$ [1401b]

hence [1401a] becomes

$$\left(\frac{dV}{dk}\right) = M.\frac{3}{k}.v + M.\left(\frac{dv}{dk}\right) + M.\left(\frac{dv}{d\theta}\right).\frac{2\theta}{k} + M.\left(\frac{dv}{d\varpi}\right).\frac{2\varpi}{k};$$

multiplying this by k, we easily obtain the first of the equations [1401]. In like manner from

$V = Mv$, [1399], we obtain $\left(\frac{dV}{dm}\right) = \left(\frac{dM}{dm}\right).v + M.\left(\frac{dv}{d\theta}\right).\left(\frac{d\theta}{dm}\right)$, because [1401c]

v contains m implicitly in θ. From $M = \frac{4\pi.k^3}{3.\sqrt{n}}.m^{-\frac{1}{2}}$ [1363‴] we get

$$\left(\frac{dM}{dm}\right) = -\frac{4\pi.k^3}{3.\sqrt{n}}.\tfrac{1}{2}m^{-\frac{3}{2}} = -\frac{4\pi.k^3}{3.\sqrt{mn}}.\frac{1}{2m} = -M.\frac{1}{2m};$$

and from $\theta = (m^{-1}-1).k^2$, [1400], we find $\left(\frac{d\theta}{dm}\right) = -m^{-2}.k^2 = -\frac{k^2}{m^2}$. [1401d]

Substituting these in [1401c], we get the second of the equations [1401], and the third may be found in the same manner; or more simply, by changing m, θ, into n, ϖ, and the contrary.

* (952) In $V = Mv$ [1399], the value of $M = \frac{4\pi.k^3}{3.\sqrt{mn}}$, does not contain a, b, c; so that if we take the partial differentials of $V = Mv$, relative to a, b, c, and use [1387, 1388], we shall get

$$-A = \left(\frac{dV}{da}\right) = M.\left(\frac{dv}{da}\right), \quad -B = \left(\frac{dV}{db}\right) = M.\left(\frac{dv}{db}\right), \quad -C = \left(\frac{dV}{dc}\right) = M.\left(\frac{dv}{dc}\right).$$

Multiplying these respectively by $-a$, $-b$, $-c$, and adding the products, we shall find

$$aA + bB + cC = -M.\left\{a.\left(\frac{dv}{da}\right) + b.\left(\frac{dv}{db}\right) + c.\left(\frac{dv}{dc}\right)\right\},$$

and we shall obtain the values of $k\,.\left(\frac{dF}{dk}\right)$, $\left(\frac{dF}{dm}\right)$, $\left(\frac{dF}{dn}\right)$, by
[1403'] changing, in the preceding values of $k\,.\left(\frac{dV}{dk}\right)$, $\left(\frac{dV}{dm}\right)$, $\left(\frac{dV}{dn}\right)$, v into $-Q$. Moreover, V and F are homogeneous functions of a, b, c, k, $\sqrt{\theta}$, $\sqrt{\varpi}$, of the second degree. For V is the sum of the particles of the spheroid, divided by their distances from the attracted point [1385'''], and each particle
[1403''] having three dimensions, V must be of the second degree, as well as F, which is of the same degree as V.* v and Q are therefore homogeneous
[1403'''] functions of the same quantities, of the degree -1; we shall therefore have, by the nature of homogeneous functions,†

[1404]
$$a\,.\left(\frac{dv}{da}\right)+b\,.\left(\frac{dv}{db}\right)+c\,.\left(\frac{dv}{dc}\right)+2\,\theta\,.\left(\frac{dv}{d\theta}\right)+2\varpi\,.\left(\frac{dv}{d\varpi}\right)+k\,.\left(\frac{dv}{dk}\right)=-v\,;$$

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or $F=-MQ$, [1397, 1402], as in [1403]. Now from the equation [1399], $V=Mv$, we have deduced the equations [1401], and if we compare [1399] with [1403], we shall find that the second may be deduced from the first, by changing V into F, and
[1402a] v into $-Q$. Making the same changes in [1401], we shall obtain the values of $k\,.\left(\frac{dF}{dk}\right)$, $\left(\frac{dF}{dm}\right)$, $\left(\frac{dF}{dn}\right)$, [1403'].

* (954) V being of the second degree in a, b, c, k, the quantities

$$A=-\left(\frac{dV}{da}\right),\qquad B=-\left(\frac{dV}{db}\right),\qquad C=-\left(\frac{dV}{dc}\right),\qquad [1387,\ 1388],$$

must be of the first degree, consequently $F=aA+bB+cC$ [1397] must be of the second degree. Now $M=\frac{4\pi\,.\,k^3}{3\,.\sqrt{mn}}$ is of the third degree in k, and by [1399] $v=\frac{V}{M}$; consequently v must be of the degree -1. In like manner, by [1403], $Q=-\frac{F}{M}$ must be of the degree -1.

† (955) If we for a moment put $\sqrt{\theta}=\theta'$, $\sqrt{\varpi}=\varpi'$, the quantity v will be homogeneous in a, b, c, θ', ϖ', k, and of the degree -1. From the nature of such functions, as expressed by the equation

[1403a]
$$a\,.\left(\frac{dA}{da}\right)+a'\,.\left(\frac{dA}{da'}\right)+a''\,.\left(\frac{dA}{da''}\right)+\&c.=m\,.A,\qquad [1001a],$$

which equation may be put under this form

$$2\theta\cdot\left(\frac{dv}{d\theta}\right)+2\varpi\cdot\left(\frac{dv}{d\varpi}\right)+k\cdot\left(\frac{dv}{dk}\right)=-v-Q. \qquad [1405]$$

We shall in like manner have

$$a\cdot\left(\frac{dQ}{da}\right)+b\cdot\left(\frac{dQ}{db}\right)+c\cdot\left(\frac{dQ}{dc}\right)+2\theta\cdot\left(\frac{dQ}{d\theta}\right)+2\varpi\cdot\left(\frac{dQ}{d\varpi}\right)+k\cdot\left(\frac{dQ}{dk}\right)=-Q. \qquad [1406]$$

This being supposed; if in the equation [1398], we substitute the values of V and F, and their partial differentials given above, and also for m and n their values [1400],

$$m=\frac{k^2}{k^2+\theta}, \qquad n=\frac{k^2}{k^2+\varpi}, \qquad [1406']$$

we shall get*

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in which A is a homogeneous function of the degree m, in a, a', a'', &c, we shall have $A=v$, $a'=b$, $a''=c$, &c., $m=-1$; hence

$$a\cdot\left(\frac{dv}{da}\right)+b\cdot\left(\frac{dv}{db}\right)+c\cdot\left(\frac{dv}{dc}\right)+\theta'\cdot\left(\frac{dv}{d\theta'}\right)+\varpi'\cdot\left(\frac{dv}{d\varpi'}\right)+k\cdot\left(\frac{dv}{dk}\right)=-v. \qquad [1404a]$$

Now as v contains θ', only as it is found in θ, we shall have $\left(\frac{dv}{d\theta'}\right)=\left(\frac{dv}{d\theta}\right)\cdot\left(\frac{d\theta}{d\theta'}\right)$, and the differential of $\theta'^2=\theta$, is $2\theta'.d\theta'=d\theta$, or $\frac{d\theta}{d\theta'}=2\theta'=\frac{2\theta'^2}{\theta'}=\frac{2\theta}{\theta'}$, hence $\left(\frac{dv}{d\theta'}\right)=\frac{2\theta}{\theta'}\cdot\left(\frac{dv}{d\theta}\right)$, or $\theta'\cdot\left(\frac{dv}{d\theta'}\right)=2\theta\cdot\left(\frac{dv}{d\theta}\right)$. In the same manner from $\varpi'^2=\varpi$, we obtain $\varpi'\cdot\left(\frac{dv}{d\varpi'}\right)=2\varpi\cdot\left(\frac{dv}{d\varpi}\right)$. Substituting these in [1404a], we shall get [1404], the three first terms of which are, by [1402], equal to Q. This being substituted, it becomes $Q+2\theta\cdot\left(\frac{dv}{d\theta}\right)+2\varpi\cdot\left(\frac{dv}{d\varpi}\right)+k\cdot\left(\frac{dv}{dk}\right)=-v$, as in [1405]. Again, as v and Q are both homogeneous functions of the same quantities [1403'''] of the degree -1, we may proceed in the same manner with Q, as with v; therefore we may change v into Q, in [1404], and thus obtain [1406].

* (957) From [1406'] we get $m-1=\frac{-\theta}{k^2+\theta}$, $n-1=\frac{-\varpi}{k^2+\varpi}$, also from [1400] $k^2\cdot\left(\frac{m-1}{m}\right)=-\theta$, $k^2\cdot\left(\frac{n-1}{n}\right)=-\varpi$; hence [1398] becomes

$$0=(a^2+b^2+c^2)\cdot\left[v+\tfrac{1}{2}Q-\tfrac{1}{2}\cdot\left\{a\cdot\left(\frac{dQ}{da}\right)+b\cdot\left(\frac{dQ}{db}\right)+c\left(\frac{dQ}{dc}\right)\right\}\right]$$

[1407] $$+\theta^2\cdot\left(\frac{dQ}{d\theta}\right)+\varpi^2\cdot\left(\frac{dQ}{d\varpi}\right)-\frac{k^3}{2}\cdot\left(\frac{dQ}{dk}\right)+\tfrac{1}{2}\cdot(\theta+\varpi)\cdot Q \qquad (2)$$

$$+b\theta\cdot\left(\frac{dQ}{db}\right)+c\varpi\cdot\left(\frac{dQ}{dc}\right)-\tfrac{1}{2}b\theta\cdot\left(\frac{dv}{db}\right)-\tfrac{1}{2}c\varpi\cdot\left(\frac{dv}{dc}\right).$$

6. Suppose the function v to be developed in a series, ascending according
[1407′] to the powers of the dimensions of the ellipsoid, and therefore descending

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$$0=\tfrac{1}{2}\cdot(a^2+b^2+c^2-k^2)\cdot k\cdot\left\{\left(\frac{dV}{dk}\right)-\left(\frac{dF}{dk}\right)\right\}+k^2\cdot(V-F)$$

[1407a] $$-b\theta\cdot\left\{\left(\frac{dF}{db}\right)-\tfrac{1}{2}\cdot\left(\frac{dV}{db}\right)-B\right\}-c\varpi\cdot\left\{\left(\frac{dF}{dc}\right)-\tfrac{1}{2}\cdot\left(\frac{dV}{dc}\right)-C\right\}$$

$$+\frac{k^2\theta}{k^2+\theta}\cdot\left(\frac{dF}{dm}\right)+\frac{k^2\varpi}{k^2+\varpi}\cdot\left(\frac{dF}{dn}\right).$$

The first of the equations [1401], by substituting the value of the first member of [1405], becomes $k\cdot\left(\frac{dV}{dk}\right)=M\cdot(-v-Q+3v)=M\cdot(2v-Q)$. The value of $k\cdot\left(\frac{dF}{dk}\right)$ may be deduced from the first equation [1401], changing v into $-Q$, as in [1403′]; hence we get

$$k\cdot\left(\frac{dF}{dk}\right)=M\cdot\left\{-2\theta\cdot\left(\frac{dQ}{d\theta}\right)-2\varpi\cdot\left(\frac{dQ}{d\varpi}\right)-k\cdot\left(\frac{dQ}{dk}\right)-3Q\right\};$$

but from [1406] we have

$$-2\theta\cdot\left(\frac{dQ}{d\theta}\right)-2\varpi\cdot\left(\frac{dQ}{d\varpi}\right)-k\cdot\left(\frac{dQ}{dk}\right)=Q+a\cdot\left(\frac{dQ}{da}\right)+b\cdot\left(\frac{dQ}{db}\right)+c\cdot\left(\frac{dQ}{dc}\right),$$

hence $$k\cdot\left(\frac{dF}{dk}\right)=M\cdot\left[-2Q+\left\{a\cdot\left(\frac{dQ}{da}\right)+b\cdot\left(\frac{dQ}{db}\right)+c\cdot\left(\frac{dQ}{dc}\right)\right\}\right],$$ and

$$k\cdot\left\{\left(\frac{dV}{dk}\right)-\left(\frac{dF}{dk}\right)\right\}=M\cdot\left[2v+Q-\left\{a\cdot\left(\frac{dQ}{da}\right)+b\cdot\left(\frac{dQ}{db}\right)+c\cdot\left(\frac{dQ}{dc}\right)\right\}\right];$$

lastly, the values of V, F, [1399, 1403], give $V-F=M\cdot(v+Q)$. Substituting these in the top line of [1407a], namely

$$\tfrac{1}{2}\cdot(a^2+b^2+c^2-k^2)\cdot k\cdot\left\{\left(\frac{dV}{dk}\right)-\left(\frac{dF}{dk}\right)\right\}+k^2\cdot(V-F),$$

it becomes equal to

$$(a^2+b^2+c^2-k^2)\cdot M\cdot\left[v+\tfrac{1}{2}Q-\tfrac{1}{2}\cdot\left\{a\cdot\left(\frac{dQ}{da}\right)+b\cdot\left(\frac{dQ}{db}\right)+c\cdot\left(\frac{dQ}{dc}\right)\right\}\right]+k^2M\cdot(v+Q).$$

relative to the quantities a, b, c; then this series will be of the following form,

$$v = U^{(0)} + U^{(1)} + U^{(2)} + U^{(3)} + \&c.\ ; \qquad [1408]$$

The coefficient of $k^2 M$ in this expression, is

$$\tfrac{1}{2}\cdot\left\{Q + a\cdot\left(\frac{dQ}{da}\right) + b\cdot\left(\frac{dQ}{db}\right) + c\cdot\left(\frac{dQ}{dc}\right)\right\} = -\theta\cdot\left(\frac{dQ}{d\theta}\right) - \varpi\cdot\left(\frac{dQ}{d\varpi}\right) - \tfrac{1}{2}k\cdot\left(\frac{dQ}{dk}\right),$$

[1406], and by the substitution, in these first terms, of [1407a], they become

$$\begin{aligned}(a^2+b^2+c^2)\cdot M\cdot\left[v + \tfrac{1}{2}Q - \tfrac{1}{2}\cdot\left\{a\cdot\left(\frac{dQ}{da}\right) + b\cdot\left(\frac{dQ}{db}\right) + c\cdot\left(\frac{dQ}{dc}\right)\right\}\right]& \\ + k^2 M\cdot\left\{-\theta\cdot\left(\frac{dQ}{d\theta}\right) - \varpi\cdot\left(\frac{dQ}{d\varpi}\right) - \tfrac{1}{2}k\cdot\left(\frac{dQ}{dk}\right)\right\}.&\end{aligned} \qquad [1407b]$$

Again, from [1388], $\quad -B = \left(\frac{dV}{db}\right)$, hence

$$\left(\frac{dF}{db}\right) - \tfrac{1}{2}\cdot\left(\frac{dV}{db}\right) - B = \left(\frac{dF}{db}\right) + \tfrac{1}{2}\cdot\left(\frac{dV}{db}\right).$$

But from [1399, 1403], we have $\quad \tfrac{1}{2}\cdot\left(\frac{dV}{db}\right) = \tfrac{1}{2}M\cdot\left(\frac{dv}{db}\right), \quad \left(\frac{dF}{db}\right) = -M\cdot\left(\frac{dQ}{db}\right)$;

therefore $\quad \left(\frac{dF}{db}\right) - \tfrac{1}{2}\cdot\left(\frac{dV}{db}\right) - B = -M\cdot\left\{\left(\frac{dQ}{db}\right) - \tfrac{1}{2}\cdot\left(\frac{dv}{db}\right)\right\}$. Multiplying this by $-b\theta$, we get the next term of [1407a],

$$-b\theta\cdot\left\{\left(\frac{dF}{db}\right) - \tfrac{1}{2}\cdot\left(\frac{dV}{db}\right) - B\right\} = b\theta\cdot M\cdot\left\{\left(\frac{dQ}{db}\right) - \tfrac{1}{2}\cdot\left(\frac{dv}{db}\right)\right\}. \qquad [1407c]$$

The factor of $-c\varpi$, may be computed in the same way, or derived from the preceding, by changing b, B, into c, C, and the contrary; hence

$$-c\varpi\cdot\left\{\left(\frac{dF}{dc}\right) - \tfrac{1}{2}\cdot\left(\frac{dV}{dc}\right) - C\right\} = c\varpi\cdot M\cdot\left\{\left(\frac{dQ}{dc}\right) - \tfrac{1}{2}\cdot\left(\frac{dv}{dc}\right)\right\}. \qquad [1407d]$$

If in $\left(\frac{dV}{dm}\right)$ [1401] we change v into $-Q$, we shall get $\left(\frac{dF}{dm}\right)$ [1403']. Substituting the value of m [1406'], and then multiplying by $\frac{k^2\theta}{k^2+\theta}$, we shall get the next term of [1407a],

$$\frac{k^2\theta}{k^2+\theta}\cdot\left(\frac{dF}{dm}\right) = M\theta\cdot\left\{(k^2+\theta)\cdot\left(\frac{dQ}{d\theta}\right) + \tfrac{1}{2}Q\right\}. \qquad [1407e]$$

Changing in this m, θ, into n, ϖ, we shall obtain the similar part depending on $\left(\frac{dF}{dn}\right)$,

$$\frac{k^2\varpi}{k^2+\varpi}\cdot\left(\frac{dF}{dn}\right) = M\varpi\cdot\left\{(k^2+\varpi)\cdot\left(\frac{dQ}{d\varpi}\right) + \tfrac{1}{2}Q\right\}. \qquad [1407f]$$

[1408'] $U^{(0)}$, $U^{(1)}$, $U^{(2)}$, &c., being homogeneous functions of a, b, c, k, $\sqrt{\theta}$, $\sqrt{\varpi}$, and separately homogeneous relative to the three first, and to the three last, of these six quantities, the dimensions relative to the three first are always decreasing, and the dimensions relative to the three last are always increasing.* These functions are of the same degree as v, being all of the
[1408''] degree -1 [1403'''].

If we substitute the preceding value of v expressed in a series [1408], in the equation [1407], and put s for the degree of $U^{(i)}$ in terms of k,
[1408'''] $\sqrt{\theta}$, $\sqrt{\varpi}$, consequently $-s-1$, for its degree in a, b, c, [1408a] ; in like manner, s' for the degree of $U^{(i+1)}$ in terms of k, $\sqrt{\theta}$, $\sqrt{\varpi}$, consequently

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Adding together these different terms of [1407a], as they are found in [1407b, c, d, e, f], and dividing by the common factor M, we shall get

[1407g]
$$0=(a^2+b^2+c^2)\cdot\left[v+\tfrac{1}{2}Q-\tfrac{1}{2}\left\{a\cdot\left(\frac{dQ}{da}\right)+b\cdot\left(\frac{dQ}{db}\right)+c\cdot\left(\frac{dQ}{dc}\right)\right\}\right]$$
$$+k^2\cdot\left\{-\theta\cdot\left(\frac{dQ}{d\theta}\right)-\varpi\cdot\left(\frac{dQ}{d\varpi}\right)-\tfrac{1}{2}k\cdot\left(\frac{dQ}{dk}\right)\right\}+b\theta\cdot\left\{\left(\frac{dQ}{db}\right)-\tfrac{1}{2}\cdot\left(\frac{dv}{db}\right)\right\}$$
$$+c\varpi\cdot\left\{\left(\frac{dQ}{dc}\right)-\tfrac{1}{2}\cdot\left(\frac{dv}{dc}\right)\right\}+\theta\cdot\left\{(k^2+\theta)\cdot\left(\frac{dQ}{d\theta}\right)+\tfrac{1}{2}Q\right\}+\varpi\cdot\left\{(k^2+\varpi)\cdot\left(\frac{dQ}{d\varpi}\right)+\tfrac{1}{2}Q\right\}.$$

In this the coefficient of k^2, neglecting the terms which mutually destroy each other, is $-\tfrac{1}{2}k\cdot\left(\frac{dQ}{dk}\right)$, making this term equal to $-\tfrac{1}{2}k^3\cdot\left(\frac{dQ}{dk}\right)$; and the expression [1407g], by small alterations in the arrangement of the terms, becomes as in [1407].

* (958) In [1389, &c.], V is developed, in a series ascending relative to the dimensions of the spheroid, and descending relative to the powers and products of the co-ordinates a, b, c, of the attracted point; therefore $v=\frac{V}{M}$ [1399] can be developed in a similar function, corresponding to the form assumed in [1408]. Each term of V [1389, 1403''] is homogeneous and of the degree 2, and each term of v is also homogeneous and of the degree -1, in terms of the co-ordinates of the attracted point a, b, c, and of the dimensions of the spheroid, represented as above by k, $\sqrt{\theta}$, $\sqrt{\varpi}$. Now each term of v being homogeneous in a, b, c, k, $\sqrt{\theta}$, $\sqrt{\varpi}$, and of the degree -1, ascending relative to k, $\sqrt{\theta}$, $\sqrt{\varpi}$, and descending
[1408a] relative to a, b, c; it follows that if any term of v, represented by $U^{(i)}$, be of the order s in terms of k, $\sqrt{\theta}$, $\sqrt{\varpi}$, it must be of the order $-s-1$ in terms of a, b, c, so that the whole term may be of the order -1, in a, b, c, k, $\sqrt{\theta}$, $\sqrt{\varpi}$.

$-s'-1$, for its degree in a, b, c; we shall have, by the nature of homogeneous functions,*

$$a\cdot\left(\frac{d\,U^{(i)}}{d\,a}\right)+b\cdot\left(\frac{d\,U^{(i)}}{d\,b}\right)+c\cdot\left(\frac{d\,U^{(i)}}{d\,c}\right)=-(s+1)\cdot U^{(i)};$$

$$a\cdot\left(\frac{d\,U^{(i+1)}}{d\,a}\right)+b\cdot\left(\frac{d\,U^{(i+1)}}{d\,b}\right)+c\cdot\left(\frac{d\,U^{(i+1)}}{d\,c}\right)=-(s'+1)\cdot U^{(i+1)};$$ [1409]

and we shall get, by rejecting terms of a higher degree in k, $\sqrt{\delta}$, $\sqrt{\varpi}$, [1409′]
than the terms which are here retained,†

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* (959) The equations [1409] are easily deduced from [1403a], putting $a'=b$, [1409a]
$a''=c$, also $A=U^{(i)}$, $m=-(s+1)$ for the first equation [1409], and $A=U^{(i+1)}$, $m=-(s'+1)$, for the second of those equations.

† (960) If we, for brevity, use the sign Σ to denote the sum of the terms of the second member of [1408], so that it may be denoted by $\Sigma\,.\,U^{(i)}$, the expression of Q [1402] will become $Q=\Sigma\,.\left\{a\cdot\left(\frac{d\,U^{(i)}}{d\,a}\right)+b\cdot\left(\frac{d\,U^{(i)}}{d\,b}\right)+c\cdot\left(\frac{d\,U^{(i)}}{d\,c}\right)\right\}$; which, by means of the first of the equations [1409], becomes $-\Sigma\,.\,(s+1)\,.\,U^{(i)}$, so that we shall have

$$v=\Sigma\,.\,U^{(i)},\qquad Q=-\Sigma\,.\,(s+1)\,.\,U^{(i)}.$$ [1410a]

Supposing now the coefficient of $a^2+b^2+c^2$, in the first line of [1407], to be [1410a']
represented by S, and all the other terms in the second and third lines of that equation to be represented by T, we shall get

$$(a^2+b^2+c^2)\,.\,S+T=0;$$ [1410b]

and if we use the value of Q [1410a], we shall find

$$a\cdot\left(\frac{d\,Q}{d\,a}\right)+b\cdot\left(\frac{d\,Q}{d\,b}\right)+c\cdot\left(\frac{d\,Q}{d\,c}\right)$$

$$=-\Sigma\,.\,(s+1)\,.\left\{a\cdot\left(\frac{d\,U^{(i)}}{d\,a}\right)+b\cdot\left(\frac{d\,U^{(i)}}{d\,b}\right)+c\cdot\left(\frac{d\,U^{(i)}}{d\,c}\right)\right\}=\Sigma\,.\{(s+1)^2\,.\,U^{(i)}\},$$

as is evident from the first of the equations [1409]. Substituting this in the preceding value of S [1410a'], it becomes $S=v+\frac{1}{2}\,Q-\frac{1}{2}\,\Sigma\,.\{(s+1)^2\,.\,U^{(i)}\}$, and by using v, Q, [1410a], we get $S=\Sigma\,.\{1-\frac{1}{2}\,.\,(s+1)-\frac{1}{2}\,.\,(s+1)^2\}\,.\,U^{(i)}=-\Sigma\,.\,\frac{1}{2}\,s\,.(s+3)\,.\,U^{(i)}$. [1410$c$]
Again, from [1410a], we get $Q-\frac{1}{2}\,v=-\Sigma\,.\,(s+\frac{3}{2})\,.\,U^{(i)}$, and by taking its partial

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[1410] $$U^{(i+1)} = \frac{\left\{ \begin{array}{l} \tfrac{1}{2} \cdot (s+1) \cdot k^3 \cdot \left(\frac{d\,U^{(i)}}{d\,k}\right) - (s+1) \cdot \theta^2 \cdot \left(\frac{d\,U^{(i)}}{d\,\theta}\right) \\ - (s+1) \cdot \varpi^2 \cdot \left(\frac{d\,U^{(i)}}{d\,\varpi}\right) - \frac{s+1}{2} \cdot (\theta + \varpi) \cdot U^{(i)} \\ - (s+\tfrac{3}{2}) \cdot b\,\theta \cdot \left(\frac{d\,U^{(i)}}{d\,b}\right) - (s+\tfrac{3}{2}) \cdot c\,\varpi \cdot \left(\frac{d\,U^{(i)}}{d\,c}\right) \end{array} \right\}}{s' \cdot \frac{(s'+3)}{2} \cdot (a^2+b^2+c^2)}. \qquad (3)$$

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differentials, we obtain $b\,\theta \cdot \left(\frac{dQ}{db}\right) - \tfrac{1}{2}\,b\,\theta \cdot \left(\frac{dv}{db}\right) = - \Sigma \cdot (s+\tfrac{3}{2}) \cdot b\,\theta \cdot \left(\frac{d\,U^{(i)}}{d\,b}\right)$, also $c\,\varpi \cdot \left(\frac{d\,Q}{d\,c}\right) - \tfrac{1}{2}\,c\,\varpi \cdot \left(\frac{d\,v}{d\,c}\right) = - \Sigma \cdot (s+\tfrac{3}{2}) \cdot c\,\varpi \cdot \left(\frac{d\,U^{(i)}}{d\,c}\right)$; hence the two lower lines of [1407], which constitute the value of T, become, by using in the terms depending on Q, the value of Q [1410a],

[1410d] $$T = -\theta^2 \cdot \Sigma \cdot \left\{(s+1) \cdot \left(\frac{d\,U^{(i)}}{d\,\theta}\right)\right\} - \varpi^2 \cdot \Sigma \cdot \left\{(s+1) \cdot \left(\frac{d\,U^{(i)}}{d\,\varpi}\right)\right\} + \tfrac{1}{2}\,k^3 \cdot \Sigma \cdot \left\{(s+1) \cdot \left(\frac{d\,U^{(i)}}{d\,k}\right)\right\}$$
$$-\tfrac{1}{2} \cdot (\theta+\varpi) \cdot \Sigma \cdot \{(s+1) \cdot U^{(i)}\} - b\,\theta \cdot \Sigma \cdot \left\{(s+\tfrac{3}{2}) \cdot \left(\frac{d\,U^{(i)}}{d\,b}\right)\right\} - c\,\varpi \cdot \Sigma \cdot \left\{(s+\tfrac{3}{2}) \cdot \left(\frac{d\,U^{(i)}}{d\,c}\right)\right\};$$

and if we alter the arrangement of the terms, putting for brevity $W^{(i)}$ equal to the numerator of the second member of [1410], or

[1410e] $$W^{(i)} = \tfrac{1}{2} \cdot (s+1) \cdot k^3 \cdot \left(\frac{d\,U^{(i)}}{dk}\right) - (s+1) \cdot \theta^2 \cdot \left(\frac{d\,U^{(i)}}{d\theta}\right) - (s+1) \cdot \varpi^2 \cdot \left(\frac{d\,U^{(i)}}{d\varpi}\right)$$
$$- \tfrac{1}{2} \cdot (s+1) \cdot (\theta+\varpi) \cdot U^{(i)} - (s+\tfrac{3}{2}) \cdot b\,\theta \cdot \left(\frac{d\,U^{(i)}}{db}\right) - (s+\tfrac{3}{2}) \cdot c\,\varpi \cdot \left(\frac{d\,U^{(i)}}{d\,c}\right);$$

we shall get $T = \Sigma \cdot W^{(i)}$, and then from [1410b, c] we shall have

[1410f] $$0 = -(a^2+b^2+c^2) \cdot \Sigma \cdot \tfrac{1}{2}\,s \cdot (s+3) \cdot U^{(i)} + \Sigma \cdot W^{(i)}.$$

Now each term of $W^{(i)}$ [1410e], is evidently of the same order in a, b, c, as the quantity $U^{(i)}$, and the corresponding term of the same order depending on

$$-(a^2+b^2+c^2) \cdot \Sigma \cdot \tfrac{1}{2}\,s \cdot (s+3) \cdot U^{(i)},$$

must be that arising from the next term $U^{(i+1)}$, which by [1408'''] produces

$$-(a^2+b^2+c^2) \cdot \tfrac{1}{2}\,s' \cdot (s'+3) \cdot U^{(i+1)},$$

so that we shall have, by putting this equal to nothing,

$$0 = -(a^2+b^2+c^2) \cdot \tfrac{1}{2}\,s' \cdot (s'+3) \cdot U^{(i+1)} + W^{(i)},$$

This equation gives the value of $U^{(i+1)}$, by means of $U^{(i)}$ and its partial differentials; now we have

$$U^{(0)} = \frac{1}{(a^2+b^2+c^2)^{\frac{1}{2}}};$$ [1411]

since by noticing only the first term of the series, we have found, in § 4, [1392],*

$$V = \frac{M}{(a^2+b^2+c^2)^{\frac{1}{2}}}.$$ [1412]

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hence

$$U^{(i+1)} = \frac{W^{(i)}}{\frac{1}{2}s'.(s'+3).(a^2+b^2+c^2)},$$ [1410g]

as in [1410]; and $U^{(i+1)}$ will be of the same order in k, $\sqrt{\theta}$, $\sqrt{\varpi}$, as $W^{(i)}$, and will therefore from [1410e] exceed by 2 the order of $U^{(i)}$, so that we shall have $s'=s+2$, [1408‴]. [1410h]

This value of $U^{(i+1)}$ will make every term of the equation [1410f] vanish, from $U^{(0)}$ to the last term $U^{(i'+1)}$, retained in [1408], except the two following

$$-(a^2+b^2+c^2).\tfrac{1}{2}s.(s+3).U^{(0)} + W^{(i'+1)};$$

of which the last is neglected on account of its smallness, in [1409′], and the other vanishes because it is multiplied by the factor s, corresponding to $U^{(0)}$, which is shown in [1412v] to be $s=2i=0$. Therefore the general expression of $U^{(i+1)}$, [1410], being used in [1408], will give the value of v, which will satisfy the equation [1407], neglecting terms of the order mentioned in [1409′].

* (961) Substituting [1408] in [1399], we get $V=M.(U^{(0)}+U^{(1)}+U^{(2)}+\&c.)$, and [1411a] by [1408‴], the functions $U^{(0)}$, $U^{(1)}$, $U^{(2)}$, &c., are arranged according to the order of the powers of k, $\sqrt{\theta}$, $\sqrt{\varpi}$, so that, if these quantities be considered as infinitely small in comparison with a, b, c, the first term $U^{(0)}$, will be much greater than the rest, and by retaining only this term, it will become $V=M.U^{(0)}$; but the value of V corresponding to this case is $V=\frac{M}{\sqrt{(a^2+b^2+c^2)}}$, [1392]; making these equal to each other, we get $U^{(0)}=\frac{1}{\sqrt{(a^2+b^2+c^2)}}$, [1411]. Putting now $i=0$ in [1410], the first member will become $U^{(1)}$, and the second member will contain $U^{(0)}$, and if we substitute the preceding value of $U^{(0)}$, we shall obtain $U^{(1)}$. Then putting $i=1$, in [1410], we shall get $U^{(2)}$ in terms of $U^{(1)}$; and, by proceeding in this way, we may obtain the general value of $v=U^{(0)}+U^{(1)}+U^{(2)}+\&c.$ [1408].

Therefore if we substitute this value of $U^{(0)}$, in the preceding formula [1410], we shall obtain that of $U^{(1)}$; and then, by means of $U^{(1)}$, we shall obtain that of $U^{(2)}$, and so on for others. But it is remarkable that
[1412'] *none of these quantities contain k.* For since $U^{(0)}$ [1411] does not contain k, it is evident from [1410] that $U^{(1)}$ will not contain it;* and not being in $U^{(1)}$, it will not be in $U^{(2)}$, and so on for others; therefore the whole series $v = U^{(0)} + U^{(1)} + U^{(2)} + \&c.$ will be independent of k; or, in other
[1412''] words, $\left(\frac{dv}{dk}\right) = 0$. The values of v, $-\left(\frac{dv}{da}\right)$, $-\left(\frac{dv}{db}\right)$, $-\left(\frac{dv}{dc}\right)$, will therefore be the same for all elliptical spheroids, similarly situated, which have the same excentricities $\sqrt{\theta}$ and $\sqrt{\varpi}$. Now by § 4,†
[1412'''] $-M.\left(\frac{dv}{da}\right)$, $-M.\left(\frac{dv}{db}\right)$, $-M.\left(\frac{dv}{dc}\right)$, express the attraction of the
[1412''''] spheroid, parallel to its three axes; *therefore the attractions of different ellipsoids, which have the same centre, the same position of the axes, and the same excentricities, upon an external point, are to each other as their masses.*

Important theorem on the attraction of an ellipsoid.

It is evident from formula [1410], that the dimensions of $U^{(0)}$, $U^{(1)}$,
[1412v] $U^{(2)}$, &c., in $\sqrt{\theta}$, $\sqrt{\varpi}$, increase by two units, so that $s = 2i$, $s' = 2i + 2$;‡

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* (961*a*) $U^{(0)}$ [1411] does not contain k, therefore $\left(\frac{dU^{(0)}}{dk}\right) = 0$. Hence the term depending on $\left(\frac{dU^{(0)}}{dk}\right)$ will vanish from the value of $U^{(1)}$, deduced from [1410], therefore $\left(\frac{dU^{(1)}}{dk}\right) = 0$, will also vanish from $U^{(2)}$, &c.

† (962) From $V = M.v$ [1399] we get

$$-\left(\frac{dV}{da}\right) = -M.\left(\frac{dv}{da}\right); \quad -\left(\frac{dV}{db}\right) = -M.\left(\frac{dv}{db}\right); \quad -\left(\frac{dV}{dc}\right) = -M.\left(\frac{dv}{dc}\right);$$

because M does not contain a, b, c, [1363''']. Hence from [1387, 1388], we find

[1412*a*] $$A = -M.\left(\frac{dv}{da}\right); \qquad B = -M.\left(\frac{dv}{db}\right); \qquad C = -M.\left(\frac{dv}{dc}\right);$$

A, B, C, being the attraction of the spheroid in directions parallel to its three axes [1347''].

‡ (963) We have found in [1410*h*], that if s and s' represent respectively the degrees of $U^{(i)}$, $U^{(i+1)}$, in k, $\sqrt{\theta}$, $\sqrt{\varpi}$, we shall have in general $s' = s + 2$. If $i = 0$, we have by [1411] $s = 0$, hence $s' = 2$ corresponds to $U^{(1)}$. Now putting $i = 1$, the

we have also, by the nature of homogeneous functions,*

$$\varpi \cdot \left(\frac{d\,U^{(i)}}{d\,\varpi}\right) = i \cdot U^{(i)} - \theta \cdot \left(\frac{d\,U^{(i)}}{d\,\theta}\right);$$ [1413]

therefore the formula [1410] will become†

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value s corresponding to $U^{(1)}$ will be $s=2$, and $s'=s+2=4$ will correspond to $U^{(2)}$. Proceeding in this way, we find in general that $s=2\,i$, and $s'=2\,i+2$, correspond to $U^{(i)}$ and $U^{(i+1)}$, as in [1412^{v}].

* (964) $U^{(i)}$ [1408′] is homogeneous relative to $\sqrt{\theta}$, $\sqrt{\varpi}$, k; and it is proved, in [1412′], that it does not contain k, it is therefore homogeneous in $\sqrt{\theta}$, $\sqrt{\varpi}$, and by [1412^{v}] it is of the degree $s=2\,i$, or in other words, it is homogeneous in θ, ϖ, and of the degree [1412b] $\frac{1}{2}s=i$. Therefore if we put $A=U^{(i)}$, $a=\theta$, $a'=\varpi$, $m=i$, in the equation [1403a], we shall get $\theta \cdot \left(\frac{d\,U^{(i)}}{d\,\theta}\right) + \varpi \cdot \left(\frac{d\,U^{(i)}}{d\,\varpi}\right) = i \cdot U^{(i)}$, as in [1413].

† (965) From [1412′], $U^{(i)}$ does not contain k, therefore $\left(\frac{d\,U^{(i)}}{dk}\right)=0$, also $s=2\,i$, [1412^{v}]. Substituting these, and [1413], in $W^{(i)}$ [1410e], it becomes, without reduction,

$$\begin{aligned} W^{(i)} = &-(2\,i+1)\cdot\theta^2\cdot\left(\frac{d\,U^{(i)}}{d\,\theta}\right) - (2\,i+1)\cdot\left\{ i\,\varpi \cdot U^{(i)} - \varpi\,\theta \cdot \left(\frac{d\,U^{(i)}}{d\,\theta}\right)\right\} \\ &-\left(\frac{2i+1}{2}\right)\cdot(\theta+\varpi)\cdot U^{(i)} - (2\,i+\tfrac{3}{2})\cdot b\,\theta\cdot\left(\frac{d\,U^{(i)}}{db}\right) - (2\,i+\tfrac{3}{2})\cdot c\,\varpi\cdot\left(\frac{d\,U^{(i)}}{dc}\right); \end{aligned}$$

or by connecting together the coefficients of $U^{(i)}$ and $\left(\frac{d\,U^{(i)}}{d\,\theta}\right)$ it becomes

$$\begin{aligned} W^{(i)} &= (2\,i+1)\cdot\left(\frac{d\,U^{(i)}}{d\,\theta}\right)\cdot(-\theta^2+\varpi\,\theta) - (2\,i+1)\cdot U^{(i)}\cdot\{i\,\varpi+\tfrac{1}{2}\cdot(\theta+\varpi)\} \\ &\quad - (2\,i+\tfrac{3}{2})\cdot b\,\theta\cdot\left(\frac{d\,U^{(i)}}{db}\right) - (2\,i+\tfrac{3}{2})\cdot c\,\varpi\cdot\left(\frac{d\,U^{(i)}}{dc}\right) \\ &= (2\,i+1)\cdot\theta\cdot(\varpi-\theta)\cdot\left(\frac{d\,U^{(i)}}{d\,\theta}\right) - (2\,i+\tfrac{3}{2})\cdot b\,\theta\cdot\left(\frac{d\,U^{(i)}}{db}\right) \\ &\quad - (2\,i+\tfrac{3}{2})\cdot c\,\varpi\cdot\left(\frac{d\,U^{(i)}}{dc}\right) - \tfrac{1}{2}\cdot(2\,i+1)\cdot\{\theta+(2\,i+1)\cdot\varpi\}\cdot U^{(i)}. \end{aligned}$$

Substituting this and $s'=2\,i+2$ [1412^{v}], or $\frac{1}{2}s'\cdot(s'+3)=(i+1)\cdot(2\,i+5)$, in $U^{(i+1)}$, [1410g], it becomes as in [1414].

[1414] $$U^{(i+1)}=\frac{\left\{\begin{array}{l}(2i+1).\theta.(\varpi-\theta).\left(\frac{dU^{(i)}}{d\theta}\right)-(2i+\tfrac{3}{2}).b\theta.\left(\frac{dU^{(i)}}{db}\right)\\ -(2i+\tfrac{3}{2}).c\varpi.\left(\frac{dU^{(i)}}{dc}\right)-\tfrac{1}{2}.(2i+1).\{\theta+(2i+1).\varpi\}.U^{(i)}\end{array}\right\}}{(i+1).(2i+5).(a^2+b^2+c^2)}. \qquad (4)$$

We shall have, by means of this equation, the value of v [1408], in a series,
[1414′] which will be very converging, when the excentricities $\sqrt{\theta}$ and $\sqrt{\varpi}$ are very small, or when the distance $\sqrt{a^2+b^2+c^2}$ from the attracted point to the centre of the spheroid,* is great in comparison with the dimensions of the ellipsoid.

[1414″] If the ellipsoid be a sphere, we shall have $\theta=0$, and $\varpi=0$, hence $U^{(1)}=0$, $U^{(2)}=0$, &c.; therefore†

[1415] $$v=U^{(0)}=\frac{1}{(a^2+b^2+c^2)^{\frac{1}{2}}};$$

and

[1416] $$V=\frac{M}{(a^2+b^2+c^2)^{\frac{1}{2}}}.$$

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* (966) The distance of the attracted point c, fig. 1, page 5, from the origin K, is

[1416a] $$Kc=\sqrt{(KH^2+Hf^2+fc^2)}=\sqrt{(a^2+b^2+c^2)}, \qquad [1355b];$$

the origin of the co-ordinates K being at the centre of the ellipsoid [1363b].

† (967) In the sphere the excentricities, $\sqrt{\theta}$, $\sqrt{\varpi}$, [1400b], are nothing; and by [1412b], $U^{(i)}$ is of the degree $2i$ in $\sqrt{\theta}$, $\sqrt{\varpi}$; therefore $U^{(1)}=0$, $U^{(2)}=0$, &c., and [1408] becomes $v=U^{(0)}=(a^2+b^2+c^2)^{-\frac{1}{2}}$, [1411], as in [1415]. Substituting this in $V=Mv$ [1399], we get V [1416]. If the whole mass of the ellipsoid M were collected in the centre of the ellipsoid, its distance from the attracted point would be $\sqrt{(a^2+b^2+c^2)}$ [1416a], and the corresponding value of V [1385‴] would be $\frac{M}{\sqrt{(a^2+b^2+c^2)}}$, which is equal to that in [1416] corresponding to the spherical mass; and as this value of V is the same function of a, b, c, the values of

$$A=-\left(\frac{dV}{da}\right), \qquad B=-\left(\frac{dV}{db}\right), \qquad C=-\left(\frac{dV}{dc}\right), \qquad [1387, 1388],$$

representing the attractions in directions of the axes, will also be the same in both cases, as is observed above.

hence it follows that the value of V is the same as if all the mass of the sphere were collected at its centre. Therefore *a sphere attracts a point, without its surface, as if all its mass were collected at its centre;* which result we have before obtained in the second book, § 12, [470″]. Theorem on the attraction of a sphere. [1416′]

7. The property of the function v, of being independent of k [1412′], furnishes a method of reducing its value to the most simple form, of which [1416″]
it is susceptible. For since we can vary k at pleasure, without changing the value of v, provided we preserve the same excentricities $\sqrt{\theta}$ and $\sqrt{\varpi}$; we [1416‴]
may suppose k to be such, that the ellipsoid may either be infinitely flattened, or reduced to such a form that its surface may pass through the attracted point. In both these cases, the investigation of the attractions of the ellipsoid [1416⁗]
becomes more simple; but as we have already determined, [1379, 1385], the attractions of an ellipsoid, upon a point placed at its surface; we shall suppose k to be such, that the surface of the spheroid passes through the attracted point.

If we put k', m', n', in this second ellipsoid, for what we have called k, m, [1416ᵛ]
n, in § 2, [1363′], relative to the first ellipsoid we have considered; the condition that the attracted point must be at the surface of this second [1416ᵛⁱ]
ellipsoid and therefore a, b, c, co-ordinates of a point of this surface, will give*

$$a^2 + m' \cdot b^2 + n' \cdot c^2 = k'^2 ; \qquad [1417]$$

and since it is supposed that the excentricities $\sqrt{\theta}$, $\sqrt{\varpi}$, remain the same, we shall have,†

$$\left(\frac{1-m'}{m'}\right) \cdot k'^2 = \theta ; \qquad \left(\frac{1-n'}{n'}\right) \cdot k'^2 = \varpi. \qquad [1418]$$

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* (968) This is deduced from [1363], accenting the letters m, n, k, to make it correspond to the second ellipsoid; and its general equation becomes $x^2 + m' y^2 + n' z^2 = k'^2$. Now [1417a]
by hypothesis, the attracted point, whose co-ordinates are a, b, c, is situated in this surface; therefore the preceding equation ought to be satisfied, by putting $x = a$, $y = b$, $z = c$; and this substitution being made, it becomes as in [1417].

† (969) The equations [1418] are deduced from [1400] by accenting the letters m, n, k, the excentricities θ, ϖ, remaining unchanged [1416‴]. From [1418], we get m', n', [1419], hence [1417] changes into [1420].

Hence we deduce

[1419] $$m' = \frac{k'^2}{k'^2 + \theta}; \qquad n' = \frac{k'^2}{k'^2 + \varpi};$$

we shall therefore have, to determine k', the equation

[1420] $$a^2 + \frac{k'^2}{k'^2 + \theta} \cdot b^2 + \frac{k'^2}{k'^2 + \varpi} \cdot c^2 = k'^2. \qquad (5)$$

It is evident from this, that there is but one ellipsoid, whose surface will pass through the attracted point, θ and ϖ remaining the same. For if we suppose
[1420′] θ and ϖ to be positive, which we can always do, it will be evident that when k'^2 is increased by any quantity, which may be considered as an aliquot part of k'^2, each of the terms of the first member* of this equation will increase in a less ratio than k'^2; therefore, if with the first value of k'^2, there was an
[1420″] equality between the two members of this equation, this equality would not take place with the second value of k'^2. Hence it follows, that k'^2 has but one real and positive value.

[1420‴] Now putting M' for the mass of this second ellipsoid; A', B', C', its attractions parallel to the axes of a, b, c; also†

[1421] $$\frac{1-m'}{m'} = \lambda^2; \qquad \frac{1-n'}{n'} = \lambda'^2;$$

$$F = \int_0^1 \frac{x^2\, dx}{\sqrt{(1+\lambda^2 . x^2) . (1+\lambda'^2 . x^2)}};$$

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* (970) Dividing [1420] by k'^2, it becomes

[1420a] $$\frac{a^2}{k'^2} + \frac{b^2}{k'^2+\theta} + \frac{c^2}{k'^2+\varpi} = 1.$$

If we now suppose θ, ϖ, to be positive, and increase k'^2, each term of the first member will decrease; and if we decrease k'^2, each term of the first member will increase; hence it is evident, that there is only one value of k'^2 which can render the first member equal to the second, and thus satisfy the equation [1420a] or [1420].

† (971) The equations [1421] are of the *same form* as in [1377], and the limits of the integral F are the same, namely from $x=0$ to $x=1$; but the *values* of λ^2, λ'^2, F, differ from those in [1377], except when $m'=m$, and $n'=n$. From these we obtain the equations [1422] in the same manner as the similar equations [1379] were found; these last differ from the others only by the accents on the letters.

we shall get, by § 3, [1379],

$$A' = \frac{3a \cdot M'F}{k'^3}; \qquad B' = \frac{3b \cdot M'}{k'^3} \cdot \left(\frac{d \cdot \lambda F}{d\lambda}\right); \qquad C' = \frac{3c \cdot M'}{k'^3} \cdot \left(\frac{d \cdot \lambda' F}{d\lambda'}\right). \quad [1422]$$

Changing in these values of A', B', C', M' into M, we shall get, from the preceding article, the values of A, B, C, corresponding to the first spheroid.* Now the equations [1418], [1422']

$$\left(\frac{1-m'}{m'}\right) \cdot k'^2 = \theta, \qquad \left(\frac{1-n'}{n'}\right) \cdot k'^2 = \varpi, \quad [1423]$$

give†

$$\lambda^2 = \frac{\theta}{k'^2}; \qquad \lambda'^2 = \frac{\varpi}{k'^2}; \quad [1424]$$

k'^2 being found from the equation [1420], which may be put under the following form,

$$0 = k'^6 - (a^2 + b^2 + c^2 - \theta - \varpi) \cdot k'^4 - \{a^2 + c^2) \cdot \theta + (a^2 + b^2) \cdot \varpi - \theta\varpi\} \cdot k'^2 - a^2 \cdot \theta\varpi; \quad [1425]$$

therefore we shall have

$$A = \frac{3a \cdot M}{k'^3} \cdot F; \qquad B = \frac{3b \cdot M}{k'^3} \cdot \left(\frac{d \cdot \lambda F}{d\lambda}\right); \qquad C = \frac{3c \cdot M}{k'^3} \cdot \left(\frac{d \cdot \lambda' F}{d\lambda'}\right). \quad [1426]$$

Attraction of an ellipsoid upon an external point.

These values exist for all points without the surface of the ellipsoid; and to apply them to points in the surface, and even to points within the surface, it will only be necessary to change k' into k.‡ [1426']

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* (972) It was proved in [1412'''''], that ellipsoids having the same centre, the same positions of the principal axes, and the same excentricities, $\sqrt{\theta}$, $\sqrt{\varpi}$, attract the proposed point in proportion to their masses M, M'; therefore by changing M' into M, in [1422], we shall obtain the values of A, B, C, [1426], corresponding to the attraction of the first ellipsoid upon an external point.

† (973) If in the equations [1423], or [1418], we substitute the values of $\frac{1-m'}{m'}$, $\frac{1-n'}{n'}$, [1421], they become $\lambda^2 \cdot k'^2 = \theta$, $\lambda'^2 \cdot k'^2 = \varpi$, as in [1424]. The value of k'^2 may be deduced from [1420], which being multiplied by $(k'^2 + \theta) \cdot (k'^2 + \varpi)$, and arranged according to the powers of k', becomes as in [1425].

‡ (974) If we change k' into k, in [1423], it will also be necessary to change m' into m, n' into n, in order that the equations [1423] may not become inconsistent with [1400]. By

[1426″] If the spheroid be of revolution about the axis k, so that $\theta = \varpi$, the equation [1420] will give*

[1427] $$2k'^2 = a^2 + b^2 + c^2 - \theta + \sqrt{(a^2 + b^2 + c^2 - \theta)^2 + 4a^2 \cdot \theta}\,;$$

and we shall have, by § 3,†

Attraction of an oblate ellipsoid of revolution upon an external point.

$$A = \frac{3a \cdot M}{k'^3 \cdot \lambda^3} \cdot \left\{ \lambda - \text{arc. tang.}\, \lambda \right\};$$

$$B = \frac{3b \cdot M}{2k'^3 \cdot \lambda^3} \cdot \left\{ \text{arc. tang.}\, \lambda - \frac{\lambda}{1 + \lambda^2} \right\};$$

[1428] $$C = \frac{3c \cdot M}{2k'^3 \cdot \lambda^3} \cdot \left\{ \text{arc. tang.}\, \lambda - \frac{\lambda}{1 + \lambda^2} \right\}.$$

We have thus obtained a complete theory of the attraction of ellipsoids. For the only thing that remains to be done is to integrate the differential

[1428′] expression F [1421], and this integration, in the general case, is impossible,

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these changes, the values of λ^2, λ'^2, [1421], will become the same as for the first spheroid, [1377], and the expression of F [1421] will become the same as that in [1377], or in [1379]. Therefore, by changing k' into k, the values [1426], corresponding to a point without the spheroid, will be reduced to the same form as those of [1379], corresponding to a point within or upon the surface of the same spheroid.

* (975) If the ellipsoid be of revolution, we shall have $\theta = \varpi$, and the equation [1420] will become $a^2 + \frac{k'^2}{k'^2+\theta} \cdot (b^2+c^2) = k'^2$. Multiplying this by $k'^2 + \theta$, and arranging it according to the powers of k', we get $k'^4 - (a^2 + b^2 + c^2 - \theta) \cdot k'^2 = a^2 \cdot \theta$, whence we deduce the value of $2k'^2$ [1427].

† (976) Putting $\theta = \varpi$ in [1424], we get $\lambda' = \lambda$; hence $F = \int_0^1 \frac{x^2 dx}{1 + \lambda^2 x^2}$ [1421]; and by [1380], this becomes $F = \frac{1}{\lambda^3} \cdot \{\lambda - \text{arc. tang.}\, \lambda\}$. Hence we find, as in [1384], $\left(\frac{d \cdot \lambda F}{d\lambda}\right) = \frac{1}{2\lambda^3} \cdot \left\{ \text{arc. tang.}\, \lambda - \frac{\lambda}{1+\lambda^2} \right\}$, which is also equal to $\left(\frac{d \cdot \lambda' F}{d\lambda'}\right)$; and by substitution in [1426], we obtain [1428], corresponding to an oblate ellipsoid of revolution.

Attraction of a prolate spheroid upon an external point

The values of A, B, C, corresponding to the attraction of a prolate spheroid of revolution, upon an external point, may be derived from [1385a], *by changing k into k'.*

by the usual methods ;* that is, the value of F cannot be expressed in finite terms, by algebraic quantities, logarithms, or circular arcs ;† or, in other

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* (976a) These integrals can easily be obtained by means of the elliptical functions, as was observed in [1379d].

† (976b) We shall now explain the method spoken of in note 943, page 27, by which Mr. Ivory avoids the complicated calculations of the fifth and sixth sections of this chapter, and reduces the attraction of an ellipsoid upon an external point, to the more simple computation [1379, 1385], for an internal point ; as in his memoir in the Philosophical Transactions of London for 1809 ; and in Le Gendre's Exercices de Calcul Intégral, T. 2, p. 512 ; or Traité des Fonctions Elliptiques, T. 1, p. 539.

For greater symmetry we shall put, as in [1369b], α, β, γ, for the three semi-axes of the ellipsoid, so that $k=\alpha$, $\frac{k}{\sqrt{m}}=\beta$, $\frac{k}{\sqrt{n}}=\gamma$, hence $m=\frac{\alpha^2}{\beta^2}$, $n=\frac{\alpha^2}{\gamma^2}$. Substituting [1428a] these values, in the equation of the ellipsoid [1363], divided by k^2, or α^2, it becomes,

Equation of the first ellipsoid.

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1. \qquad [1428b]$$

Moreover we have from [1400b], for the squares of the excentricities,

$$\theta=\beta^2-\alpha^2, \qquad \varpi=\gamma^2-\alpha^2. \qquad [1428c]$$

If the attracted point be on the surface of this ellipsoid, we may put $x=a$, $y=b$, $z=c$, in [1428b], and we shall get $\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}=1$; and if the attracted point fall [1428d] within the ellipsoid, upon the same radius, each term of the first member will be decreased, and we shall have $\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}<1$; if it fall without the ellipsoid, each term of the first [1428d′] member will be increased, and we shall have $\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}>1$.

The integral of the value of A [1348] may be taken relative to the variable quantity x, considering y, z, to be constant. In this case we shall have

$$\int\frac{dx.(a-x)}{[(a-x)^2+(b-y)^2+(c-z)^2]^{\frac{3}{2}}}=\frac{1}{[(a-x)^2+(b-y)^2+(c-z)^2]^{\frac{1}{2}}}+\text{constant}.$$

We must take the limits of this integral, from the greatest negative value of x, corresponding to the surface of the spheroid, and denoted by $-x_{,}$, to its greatest positive value, corresponding also to the same surface, and denoted by $x_{,}$; these values, deduced from [1428b], being $\mp\alpha.\left(1-\frac{y^2}{\beta^2}-\frac{z^2}{\gamma^2}\right)^{\frac{1}{2}}$, and then the preceding integral will become,

$$\int\frac{dx.(a-x)}{[(a-x)^2+(b-y)^2+(c-z)^2]^{\frac{3}{2}}}=\frac{1}{[(a-x_{,})^2+(b-y)^2+(c-z)^2]^{\frac{1}{2}}}-\frac{1}{[(a+x_{,})^2+(b-y)^2+(c-z)^2]^{\frac{1}{2}}}; \qquad [1428e]$$

[1428″] words, by an algebraical function of quantities, whose exponents are constant, nothing, or variable. The functions of this kind being the only ones that

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and the value of A [1348] will be

[1428f] $$A = \int\int dy \,.\, dz \,.\, \left\{ \frac{1}{[(a - x_{,})^2 + (b - y)^2 + (c - z)^2]^{\frac{1}{2}}} - \frac{1}{[(a + x_{,})^2 + (b - y)^2 + (c - z)^2]^{\frac{1}{2}}} \right\}.$$

In the same manner we may find the integral of B [1348] relative to y, or the integral of C relative to z, but it will not be necessary to take these into consideration.

For the sake of brevity, we shall put,

[1428g] $$D = \{(a + x_{,})^2 + (b - y)^2 + (c - z)^2\}^{\frac{1}{2}},$$
$$D_{,} = \{(a - x_{,})^2 + (b - y)^2 + (c - z)^2\}^{\frac{1}{2}}.$$

If we suppose the attracted point, whose co-ordinates are a, b, c, to be called P, then D will represent the distance of this attracted point P, from the point of the surface of the ellipsoid, whose co-ordinates are $-x_{,}$, y, z; and $D_{,}$ will be the distance of the same point P, from the point of the surface of the ellipsoid, whose co-ordinates are $x_{,}$, y, z; the line, connecting these two extreme points of the surface, being parallel to the axis of x. The expression [1428f] will then become

[1428h] $$A = \int\int dy \,.\, dz \,.\, \left\{ \frac{1}{D_{,}} - \frac{1}{D} \right\}.$$

Second ellipsoid. We shall now suppose a second ellipsoid to be formed, on the same centre, and with the principal axes in the same directions, and that the quantities, represented by α, β, γ, a, b, c,

[1428h′] $x_{,}$, y, z, A, B, C, D, $D_{,}$, may, in this second ellipsoid, be denoted by the same letters accented. We shall get, for the attraction of this second ellipsoid, upon a point P', resolved in a direction parallel to the axis of x, the following expression, similar to that in [1428h]; the co-ordinates of this point P' being a', b', c', parallel to the axes of x, y, z, respectively,

[1428i] $$A' = \int\int dy' \,.\, dz' \,.\, \left\{ \frac{1}{D'_{,}} - \frac{1}{D'} \right\}.$$

This second ellipsoid, and the attracted point P', are to be so taken as to make $D = D'$, $D_{,} = D'_{,}$, and it appears, from [1428g], that both these equations would be satisfied, if we had

[1428k] $$(a \pm x_{,})^2 + (b - y)^2 + (c - z)^2 = (a' \pm x'_{,})^2 + (b' - y')^2 + (c' - z')^2.$$

We shall suppose the co-ordinates a', b', c', x', y', z', of the second ellipsoid, to be so taken, with respect to the corresponding co-ordinates of the first ellipsoid, that we may have the relations between them as in the following equations [1428l], in which μ, ν, ω, are used, for brevity, to denote the ratios of the corresponding terms, $\frac{a}{a'}$, $\frac{b}{b'}$, $\frac{c}{c'}$,

can be expressed independent of the sign $\int$, all the integrals which cannot be reduced to similar functions, are impossible in finite terms.

$$a = \mu\, a', \qquad b = \nu\, b', \qquad c = \omega\, c',$$
$$\mu\, x_{\prime} = x_{\prime}', \qquad \nu\, y = y', \qquad \omega\, z = z'; \qquad [1428l]$$

then we shall get, by multiplying each of these equations by that immediately below it,

$$a\, x_{\prime} = a'\, x_{\prime}', \qquad b\, y = b'\, y', \qquad c\, z = c'\, z'. \qquad [1428m]$$

Substituting these in [1428k], we shall get, by rejecting the terms which mutually destroy each other,

$$a^2 + b^2 + c^2 + x_{\prime}^2 + y^2 + z^2 = a'^2 + b'^2 + c'^2 + x_{\prime}'^2 + y'^2 + z'^2$$
$$= \frac{a^2}{\mu^2} + \frac{b^2}{\nu^2} + \frac{c^2}{\omega^2} + \mu^2 x_{\prime}^2 + \nu^2 y^2 + \omega^2 z^2, \qquad [1428n]$$

or

$$(\mu^2 - 1) \,.\, x_{\prime}^2 + (\nu^2 - 1) \,.\, y^2 + (\omega^2 - 1) \,.\, z^2 = \frac{a^2}{\mu^2} \,.\, (\mu^2 - 1) + \frac{b^2}{\nu^2} \,.\, (\nu^2 - 1) + \frac{c^2}{\omega^2} \,.\, (\omega^2 - 1).$$

Putting for brevity $(\mu^2 - 1) \,.\, \alpha^2 = \varepsilon$, and then dividing by this quantity, we shall get [1428o]

$$\frac{x_{\prime}^2}{\alpha^2} + \frac{(\nu^2 - 1)}{\varepsilon} \,.\, y^2 + \frac{(\omega^2 - 1)}{\varepsilon} \,.\, z^2 = \frac{a^2}{\mu^2} \,.\, \frac{1}{\alpha^2} + \frac{b^2}{\nu^2} \,.\, \frac{(\nu^2 - 1)}{\varepsilon} + \frac{c^2}{\omega^2} \,.\, \frac{(\omega^2 - 1)}{\varepsilon}. \qquad [1428p]$$

This equation ought to agree with [1428b]. The coefficients of y^2, z^2, of the first members, being compared, give $\nu^2 - 1 = \frac{\varepsilon}{\beta^2}$, $\omega^2 - 1 = \frac{\varepsilon}{\gamma^2}$, by which means the second [1428q] member becomes $\frac{a^2}{\mu^2\,\alpha^2} + \frac{b^2}{\nu^2\,\beta^2} + \frac{c^2}{\omega^2\,\gamma^2}$, or $\frac{a^2}{\alpha^2 + \varepsilon} + \frac{b^2}{\beta^2 + \varepsilon} + \frac{c^2}{\gamma^2 + \varepsilon}$, as appears, by using the values $\mu^2\,\alpha^2 = \alpha^2 + \varepsilon$, &c., [1428o, q], and to make this equal to the second member of [1428b], we must put,

$$\frac{a^2}{\alpha^2 + \varepsilon} + \frac{b^2}{\beta^2 + \varepsilon} + \frac{c^2}{\gamma^2 + \varepsilon} = 1. \qquad [1428r]$$

The attracted point P being supposed without the first spheroid, we shall have,

$$\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2} > 1, \qquad [1428d'];$$

therefore, if we suppose $\varepsilon = 0$, the first member of [1428r] will become > 1; and as ε increases, this first member will always decrease, and will become nothing, when ε is infinite. Hence it follows, that the equation [1428r] has but one real positive root, or value of ε. Having found this quantity, we shall get [1428o, q],

$$\mu^2 = 1 + \frac{\varepsilon}{\alpha^2}, \qquad \nu^2 = 1 + \frac{\varepsilon}{\beta^2}, \qquad \omega^2 = 1 + \frac{\varepsilon}{\gamma^2}, \qquad [1428s]$$

Attraction of ellipsoids not homogeneous. If the ellipsoid be not homogeneous, but composed of elliptical strata, variable in position, excentricity and density, according to any law whatever,

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and the point P' will be determined by the co-ordinates [1428l]

[1428t] $$a'=\frac{a}{\mu}, \qquad b'=\frac{b}{\nu}, \qquad c'=\frac{c}{\omega}.$$

Moreover, if we substitute the values of $x_{\prime}$, y, z, [1428l], in [1428b], we shall find the following equation between $x_{\prime}'$, y', z',

[1428u] $$\frac{x_{\prime}'^2}{\mu^2\,\alpha^2}+\frac{y'^2}{\nu^2\,\beta^2}+\frac{z'^2}{\omega^2\,\gamma^2}=1.$$

[1428v] Comparing this with the equation of the second ellipsoid,

Equation of the second ellipsoid.

$$\frac{x_{\prime}'^2}{\alpha'^2}+\frac{y'^2}{\beta'^2}+\frac{z'^2}{\gamma'^2}=1\,;$$

which is of the same form as [1428b], accenting the letters as in [1428h']; we shall get, for the semi-axes of this second ellipsoid,

[1428w] $$\alpha'=\alpha\,\mu, \qquad \beta'=\beta\,\nu, \qquad \gamma'=\gamma\,\omega.$$

From the equations [1428o, q, w], we get

[1428x] $$\alpha'^2-\alpha^2=\varepsilon, \qquad \beta'^2-\beta^2=\varepsilon, \qquad \gamma'^2-\gamma^2=\varepsilon\,;$$

therefore, by taking the differences of these equations, we shall have,

[1428y] $$\beta'^2-\alpha'^2=\beta^2-\alpha^2, \qquad \gamma'^2-\beta'^2=\gamma^2-\beta^2, \qquad \gamma'^2-\alpha'^2=\gamma^2-\alpha^2.$$

Hence it follows, that the principal sections of these two ellipsoids, which are situated in the same plane, are described about the same foci [1428c].

Substituting the values of $\alpha^2+\varepsilon$, $\beta^2+\varepsilon$, $\gamma^2+\varepsilon$, [1428x], in [1428r], we shall find,

[1428z] $$\frac{a^2}{\alpha'^2}+\frac{b^2}{\beta'^2}+\frac{c^2}{\gamma'^2}=1\,;$$

therefore the point P, whose co-ordinates are a, b, c, is situated in the surface of the second ellipsoid M', [1428v]. If we substitute in [1428z] the values of a, b, c, α', β', γ', [1428l, w], we shall get,

[1428α] $$\frac{a'^2}{\alpha^2}+\frac{b'^2}{\beta^2}+\frac{c'^2}{\gamma^2}=1\,;$$

hence it follows, that the co-ordinates a', b', c', of the point P', are situated on the surface of the first ellipsoid M, [1428b].

Corresponding points. These corresponding points P, P', are so connected, that their co-ordinates have the relations mentioned in the equations [1428l, w], from which we get

[1428β] $$a:a'::\alpha':\alpha\,; \qquad b:b'::\beta':\beta\,; \qquad c:c'::\gamma':\gamma\,;$$

we may obtain the attraction of any one of its strata, by determining in the preceding manner, the difference of the attractions of two homogeneous [1428‴]

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therefore *the homologous co-ordinates are proportional to the corresponding axes, to which they are respectively parallel.*

Substituting in [1428*i*] the differentials of y', z', [1428*l*, *w*], namely,

$$dy'=\nu\,.\,dy=\frac{\beta'}{\beta}\,.\,dy,\quad dz'=\omega\,.\,dz=\frac{\gamma'}{\gamma}\,.\,dz,\quad \text{also}\quad D'=D,\quad D_{,}'=D_{,},$$ we [1428γ]

shall get,

$$A'=\nu\,\omega\,.\int\!\!\int dz\,.\,dy\,.\left\{\frac{1}{D_{,}}-\frac{1}{D}\right\}=\frac{\beta'\gamma'}{\beta\gamma}\,.\int\!\!\int dz\,.\,dy\,.\left\{\frac{1}{D_{,}}-\frac{1}{D}\right\}.$$ [1428δ]

Comparing this with [1428*h*], we find $A'=\frac{\beta'\gamma'}{\beta\gamma}\,.\,A.$ In like manner we may compute B, C; but these values may be much more easily derived from this value of A', by changing what relates to A into B, or C, and the contrary; hence we shall get the following system of equations,

$$A'=\frac{\beta'\gamma'}{\beta\gamma}\,.\,A,\qquad B'=\frac{\alpha'\gamma'}{\alpha\gamma}\,.\,B,\qquad C'=\frac{\alpha'\beta'}{\alpha\beta}\,.\,C.$$ [1428ε]

Therefore, if it be proposed to determine the attraction of an ellipsoid M, upon a point P, situated without this body, we must proceed in the same manner as the author has done in [1416ᵛ], and suppose a second ellipsoid M' to be formed, whose principal sections and foci are the same as those of the proposed ellipsoid, *and whose surface passes through the proposed point P.* These conditions will suffice for the determination of the magnitudes, and the positions of the semi-axes, α', β', γ', of this second ellipsoid. We must also take, in the surface of the first ellipsoid M, a *corresponding* point P', so that each co-ordinate of this [1428ζ] point P', may be to the *corresponding co-ordinate* of the point P, in the direct ratio of the semi-axes of the ellipsoids M and M', [1428β], situated in the directions of these co-ordinates. Then putting A', B', C', for the three forces, parallel to the axes of the co-ordinates, resulting from the attraction of the second ellipsoid M', upon the interior point P'; also A, B, C, for the similar forces of the first ellipsoid M, upon the exterior point P, we shall have, from what has been demonstrated, [1428ε],

$$A=\frac{\beta\gamma}{\beta'\gamma'}\,.\,A'\,;\qquad B=\frac{\alpha\gamma}{\alpha'\gamma'}\,.\,B'\,;\qquad C=\frac{\alpha\beta}{\alpha'\beta'}\,.\,C'\,;$$ [1428η]

by which means the second case treated of [1385′—1428] will be reduced to the same form as the first [1369″—1385]; and it is upon these principles the calculations are made in [1416″—1428]. The *corresponding* points P, P', being found according to the above directions [1428ζ], the results of the preceding investigation were enunciated by Mr. Ivory in the following theorem, observing that the area of the principal section of the ellipsis, whose

ellipsoids, of the same density as that stratum, of which the one has the

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semi-axes are β, γ, perpendicular to the axis α, is represented by $\pi.\beta\gamma$ [378v], and in like manner, $\pi.\alpha\gamma$, $\pi.\alpha\beta$, represent the areas of the principal sections, perpendicular to the axes β, γ, respectively.

"*If two ellipsoids of the same homogeneous matter have the same excentricities, and their*
[1428δ] *principal sections in the same planes; the attractions which one of the ellipsoids exerts upon a point in the surface of the other, perpendicularly to the planes of the principal sections, will be to the attractions which the second ellipsoid exerts upon the corresponding point in the surface of the first, perpendicularly to the same planes, in the direct proportion of the surfaces, or areas, of the principal sections to which the attractions are perpendicular.*"

Ivory's theorem on the attractions of ellipsoids;

generalized by Mr. Poisson.

This theorem is not restricted to the Newtonian law of gravity, but is true whatever be the law of its decrease, as has been observed by Mr. Poisson. For if we suppose gravity to vary
[1428ι] inversely as the power n of the distance, the expression [1428e] will become, by using the values of D, $D_{,}$, [1428g],

[1428κ]
$$\int\frac{dx\,.\,(a-x)}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{n+1}{2}}}=\frac{1}{n-1}\cdot\left\{\frac{1}{D_{,}^{n-1}}-\frac{1}{D^{n-1}}\right\}.$$

Hence A [1428h] will be of the following form,

[1428λ]
$$A=\frac{1}{n-1}\cdot\int\!\!\int dy\,.\,dz\,.\left\{\frac{1}{D_{,}^{n-1}}-\frac{1}{D^{n-1}}\right\},$$

and A' [1428i] will become, by using the values of D', $D_{,}'$, dy', dz', [1428γ],

[1428μ]
$$A'=\frac{1}{n-1}\cdot\int\!\!\int dy'\,.\,dz'\,.\left\{\frac{1}{D_{,}^{n-1}}-\frac{1}{D^{n-1}}\right\}=\frac{1}{n-1}\cdot\frac{\beta'\gamma'}{\beta\gamma}\cdot\int\!\!\int dy\,.\,dz\,.\left\{\frac{1}{D_{,}^{n-1}}-\frac{1}{D^{n-1}}\right\};$$

[1428ν] whence we shall get $A'=\frac{\beta'\gamma'}{\beta\gamma}\,.\,A$, as in [1428$\varepsilon$]. Similar results will be obtained relative to B', C'. Hence the truth of Mr. Poisson's remark is manifest.

[1428ν'] From [1369c] we have $M=\frac{4\pi}{3}\,.\,\alpha\beta\gamma$, $M'=\frac{4\pi}{3}\,.\,\alpha'\beta'\gamma'$; hence we get $\frac{M}{M'}=\frac{\alpha\beta\gamma}{\alpha'\beta'\gamma'}$, and $\frac{\beta\gamma}{\beta'\gamma'}=\frac{M}{M'}\cdot\frac{\alpha'}{\alpha}=\frac{M}{M'}\cdot\frac{a}{a'}$, [1428$\beta$]. In like manner we have
[1428ν''] $\frac{\alpha\gamma}{\alpha'\gamma'}=\frac{M}{M'}\cdot\frac{b}{b'}$; $\frac{\alpha\beta}{\alpha'\beta'}=\frac{M}{M'}\cdot\frac{c}{c'}$. Substituting these in [1428ε], we find

[1428ξ]
$$A=\frac{a\,.\,M}{a'\,.\,M'}\,.\,A',\qquad B=\frac{b\,.\,M}{b'\,.\,M'}\,.\,B',\qquad C=\frac{c\,.\,M}{c'\,.\,M'}\,.\,C';$$

in which A', B', C', represent the attractions of the second ellipsoid M' upon a point

same surface as the exterior surface of the stratum, and the other the same

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within its surface, whose co-ordinates are a', b', c', and these attractions may be found from [1379], by accenting A, B, C, a, b, c, k, using the values λ, λ', [1424], corresponding to this second ellipsoid M'. Hence we have [1428ξ']

$$A' = \frac{3a'.M'}{k'^3}.F, \qquad B' = \frac{3b'.M'}{k'^3}.\left(\frac{d.\lambda F}{d\lambda}\right), \qquad C' = \frac{3c'.M'}{k'^3}.\left(\frac{d.\lambda' F}{d\lambda'}\right). \qquad [1428\pi]$$

Substituting these in [1428ξ], we get the following values of A, B, C, being the same as found by La Place in [1426].

$$A = \frac{3a.M}{k^3}.F, \qquad B = \frac{3b.M}{k^3}.\left(\frac{d.\lambda F}{d\lambda}\right), \qquad C = \frac{3c.M}{k^3}.\left(\frac{d.\lambda' F}{d\lambda'}\right). \qquad [1428\rho]$$

If the co-ordinates of the attracted point a, b, c, and the excentricities θ, ϖ, be given, or constant, k', [1425], and λ, λ', [1424], will also be constant; then the values of A, B, C, [1428ρ], will be proportional to the mass M of the attracting spheroid, as was proved in another manner in [1412'''']. Hence it is evident, from [1387, 1388], that V must also be proportional to the mass M. [1428σ]

Le Gendre's theorem, corresponding to the action of the second ellipsoid, upon an internal point, whose co-ordinates are a', b', c', is $\frac{A'}{a'} + \frac{B'}{b'} + \frac{C'}{c'} = 4\pi$, being the same as in [1370g], accenting the letters, to correspond to the notation [1428ξ']. Multiplying this by $\frac{M}{M'} = \frac{\alpha\beta\gamma}{\alpha'\beta'\gamma'}$, [1428ν'], and substituting the values [1428σ']

$$\frac{M}{M'}.\frac{A'}{a'} = \frac{A}{a}, \qquad \frac{M}{M'}.\frac{B'}{b'} = \frac{B}{b}, \qquad \frac{M}{M'}.\frac{C'}{c'} = \frac{C}{c}, \qquad [1428\xi],$$

we get the following expression, for an external attracted point,

Theorem by Le Gendre for an external attracted point.

$$\frac{A}{a} + \frac{B}{b} + \frac{C}{c} = 4\pi.\frac{\alpha\beta\gamma}{\alpha'\beta'\gamma'}; \qquad [1428\tau]$$

which is similar to that in [1370g].

If we suppose $\alpha = \beta = \gamma$, $\alpha' = \beta' = \gamma'$, the two ellipsoids will become concentrical spheres; the radius of the least sphere being α, that of the greatest sphere α'. If we put s, s', for the surfaces of these spheres respectively, we shall have $\alpha^2 : \alpha'^2 :: s : s'$, [275b], and the equations [1428π] will become

$$A = \frac{s}{s'}.A', \qquad B = \frac{s}{s'}.B', \qquad C = \frac{s}{s'}.C'. \qquad [1428\upsilon]$$

surface as the interior surface of the stratum. Taking the integral of this differential of the attraction, we shall obtain the attraction of the whole ellipsoid.

Ivory's theorem applied to spheres.

[1428φ] Hence it evidently appears, that *the attraction of the least sphere upon a point of the surface of the greatest sphere, is to the attraction of the greatest sphere, upon a point of the surface of the least sphere, as the surface of the least sphere is to the surface of the greatest sphere.* From this it also follows, that the attraction of the least sphere upon *all* the points of the surface of the greatest sphere, is equal to the attraction of the greatest sphere upon *all* the points of the least sphere. The preceding theorem is proved by the author in another manner, in Book XII, [11245].

CHAPTER II.

ON THE DEVELOPMENT OF THE ATTRACTION OF ANY SPHEROID IN A SERIES.

8. We shall now consider the attraction of any spheroid whatever. We have seen in § 4, [1388′], that the quantity V, *which expresses the sum of the particles of the spheroid divided by their distances from the attracted point*, has the property of giving, by its differentiation, the attraction of this spheroid, parallel to any right line whatever. We shall also find, when we shall treat of the figure of the planets, that the attraction of their particles appears under this form, in the equation of equilibrium; we shall therefore enter into a particular investigation of the value of V. Function V. [1428′′′′]

We shall resume the equation [1386],

$$V = \int \frac{dM}{\sqrt{(a-x)^2+(b-y)^2+(c-z)^2}}. \qquad [1429]$$

a, b, and c, being the co-ordinates of the attracted point; x, y, z, the co-ordinates of the particle of the spheroid dM [1385′′′′]; the origin of the co-ordinates being within the spheroid. This integral is to be taken relative to the variable quantities x, y, z, and its limits are independent of a, b, c. This being premised, we shall have, by taking the differential of V,* [1429′] [1429′′]

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* (977) It was observed in [1386a], that the equation [1386], or [1429], might be deduced from [455], by putting $\rho = 1$, and changing x, y, z, x', y', z', into a, b, c, x, y, z, respectively. Now the equation [459] was deduced from [455], by merely taking the differentials; and if we make, in the equation [459], the changes in the letters just mentioned, we shall obtain the formula [1430]. [1430a]

It is supposed, in the equation [1430], as in the calculations [452—495], that no part of the attracting mass comes so near to the attracted point, as to render the denominator

$$\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}} \qquad [1430b]$$

La Place's fundamental theorem for the attraction of a spheroid. First form.

[1430] $$\left(\frac{ddV}{da^2}\right)+\left(\frac{ddV}{db^2}\right)+\left(\frac{ddV}{dc^2}\right)=0, \qquad [\text{or } -4\pi\rho'],$$

an equation which we have already obtained in § 11, Book II, [459].

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of the expression [455′] equal to nothing. For if this were the case, the particles of the spheroid, situated infinitely near to the attracted point, would produce sensible terms in the value of $\left(\frac{ddV}{dx^2}\right)+\left(\frac{ddV}{dy^2}\right)+\left(\frac{ddV}{dz^2}\right)$, which would alter the equation [459], or [1430], as has been observed by Mr. Poisson. This may be easily proved, by supposing the spheroid EIF, whose centre is C, to be divided into two parts. *First*, a very small sphere GHK, falling wholly

[1430c] within the attracting spheroid EIF, its centre being the attracted point D, and its radius $DG=u$, which, after all the calculations are completed, is to be made infinitely small; so as to include

[1430d] merely those parts of the spheroid EIF, which might produce sensible terms in [1430], from the vanishing of the denominator [1430b]. *Second*, the spheroidal shell comprised between the surfaces EIF, GHK. Then putting V', V'', for the parts of V corresponding to

[1430e] this small sphere, and to the shell, respectively, we shall have $V=V'+V''$. Substituting this in the second member of [459], we shall get for its value

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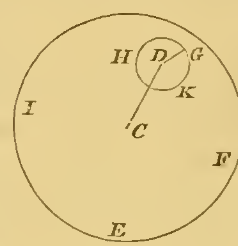

[1430f] $$\left(\frac{ddV}{dx^2}\right)+\left(\frac{ddV}{dy^2}\right)+\left(\frac{ddV}{dz^2}\right)$$
$$=\left\{\left(\frac{ddV'}{dx^2}\right)+\left(\frac{ddV'}{dy^2}\right)+\left(\frac{ddV'}{dz^2}\right)\right\}+\left\{\left(\frac{ddV''}{dx^2}\right)+\left(\frac{ddV''}{dy^2}\right)+\left(\frac{ddV''}{dz^2}\right)\right\}.$$

Now by hypothesis, all the particles of the shell are at a finite distance from the attracted point D; therefore the expression [459] will give, $\left(\frac{ddV''}{dx^2}\right)+\left(\frac{ddV''}{dy^2}\right)+\left(\frac{ddV''}{dz^2}\right)=0$, and the preceding equation will become

[1430g] $$\left(\frac{ddV}{dx^2}\right)+\left(\frac{ddV}{dy^2}\right)+\left(\frac{ddV}{dz^2}\right)=\left(\frac{ddV'}{dx^2}\right)+\left(\frac{ddV'}{dy^2}\right)+\left(\frac{ddV'}{dz^2}\right);$$

consequently the first member of this equation is the same, as it would be, if the whole spheroid EIF were reduced to the small sphere GHK, described about the attracted point D as a

[1430h] centre. The arbitrary radius of this sphere can be supposed so extremely small, that even in the case where the spheroid EIF is heterogeneous, we may neglect the variation of density in the sphere, supposing it to be homogeneous, and of the same density ρ' as at the central point D. For greater simplicity in the

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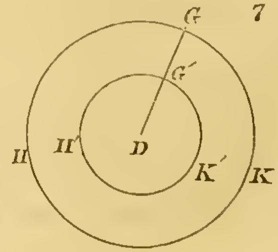

We shall transform the co-ordinates into others, which will be more convenient. For this purpose, we shall put r for the distance of the [1430′]

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calculation, we have assumed the mass GHK to be a sphere, but it is evident that we might have supposed this mass to be of any other form.

To determine the action of this sphere upon a point in its centre D, we shall, in the first place, compute the action of any finite homogeneous sphere GHK, fig. 7, whose radius is u, upon any internal point G', situated at the distance $DG'=r$ from its centre; taking [1430i] this centre as the origin of the co-ordinates. Then if we describe, about the centre D, a spherical surface $G'H'K'$, passing through the proposed point G'; it will be evident, from [469‴], that the spherical shell, included between the surfaces GHK, $G'H'K'$, produces no effect on the point G', and that the action of the sphere $G'H'K'$, upon the point G', is the same as if all its mass were collected in the centre D, [470″]. Now the radius DG' being r, and the density of the sphere ρ', its mass will be $\frac{4}{3}\pi\rho'.r^3$, [1363e]. [1430k] If this be divided by the square of the distance r^2, we shall get the whole action of the sphere upon the point G', equal to $\frac{4}{3}\pi\rho'.r$, in the direction $G'D$; and this, in [470, 470″], [1430l] represents the value of $-\left(\frac{dV'}{dr}\right)$, corresponding to the sphere $G'H'K'$, or to the whole sphere GHK. Hence we have

$$\left(\frac{dV'}{dr}\right)=-\tfrac{4}{3}\pi\rho'.r. \qquad [1430m]$$

This vanishes when $r=0$; but its partial differential relative to r, namely

$$\left(\frac{ddV'}{dr^2}\right)=-\tfrac{4}{3}\pi\rho', \qquad [1430n]$$

is finite and constant for all values of r.

The co-ordinates of the attracted point G', are, in [451$^{\text{viii}}$], denoted by x, y, z; and if we substitute their values, in terms of r, θ, ϖ, [460], the expression [459], corresponding to this sphere, will become, as in [465t],

$$\left(\frac{ddV'}{dx^2}\right)+\left(\frac{ddV'}{dy^2}\right)+\left(\frac{ddV'}{dz^2}\right)=\left(\frac{ddV'}{dr^2}\right)+\frac{2}{r}\cdot\left(\frac{dV'}{dr}\right), \qquad [1430o]$$

neglecting the terms depending on θ, ϖ, as in [466′]; it being evident that in this sphere the values of V' will not contain θ or ϖ; so that $\left(\frac{dV'}{d\theta}\right)=0$, $\left(\frac{ddV'}{d\theta^2}\right)=0$, &c. [1430p]

Substituting in [1430o], the values of $\left(\frac{dV'}{dr}\right)$, $\left(\frac{ddV'}{dr^2}\right)$, [1430m, n], the second member of that expression will become $-\frac{4}{3}\pi\rho'-\frac{8}{3}\pi\rho'=-4\pi\rho'$, and we shall finally get,

$$\left(\frac{ddV'}{dx^2}\right)+\left(\frac{ddV'}{dy^2}\right)+\left(\frac{ddV'}{dz^2}\right)=-4\pi\rho', \qquad [1430q]$$

attracted point from the origin of the co-ordinates, θ the angle which the

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for the value of this function, corresponding to any finite homogeneous sphere, whose radius is u, the attracted point being situated at any place whatever, within the surface of the sphere. The second member of this expression, being independent of r, will remain unchanged when
[1430r] r becomes infinitely small, as is supposed in [1430d, h]; therefore we may substitute this value in [1430g], and ρ' will represent the density of the spheroid at the attracted point. Hence *we shall get, for any spheroid whatever, when the attracted point is situated within its surface,*

Theorem when the attracted point is within the attracting mass.

[1430s]
$$\left(\frac{d\,d\,V}{d\,x^2}\right)+\left(\frac{d\,d\,V}{d\,y^2}\right)+\left(\frac{d\,d\,V}{d\,z^2}\right)=-4\,\pi\,\rho'.$$

To conform to the notation used in this article, we must, as in [1430a], change the co-ordinates of the attracted point x, y, z, into a, b, c, and we shall get,

[1430t]
$$\left(\frac{d\,d\,V}{d\,a^2}\right)+\left(\frac{d\,d\,V}{d\,b^2}\right)+\left(\frac{d\,d\,V}{d\,c^2}\right)=-4\,\pi\,\rho'.$$

This is to be used instead of [1430], when the whole of the attracted point forms a part of
[1430u] the attracting mass; or in other words, the *whole* of the infinitely small sphere must fall *within* the spheroid; otherwise we could not have neglected the consideration of the angle θ, ϖ, in [1430p], because the sphere would become a spherical segment, and the value of V' corresponding to this segment would differ from that we have here computed for the whole
[1430v] sphere. We have introduced the expression [or $-4\,\pi\,\rho'$], in the original formula, to correct for the defect in the attraction, upon an internal point. The necessity of this correction was first noticed by Mr. Poisson, who has also investigated it in a different manner, as we shall hereafter see, in [1447π]. Similar corrections have been made in the formulas [1434, 1435].

It may not be amiss to remark, that the equation [1430t] might have been deduced from the values of A, B, C, [1379], supposing the small body $H\,G\,K$, fig. 6, 7, to be an ellipsoid, whose semi-axes are as in [1369b], and $H'\,G'\,K'$ to be a similar and concentric ellipsoid. For in [1379] we have $A=\frac{3M.F}{k^3}\,.\,a$, in which M, F, k, are independent
[1430v'] of a; hence $\left(\frac{d\,A}{d\,a}\right)=\frac{3M.F}{k^3}$, or $\left(\frac{d\,A}{d\,a}\right)=\frac{A}{a}$; but from [1387] we have $\left(\frac{d\,A}{d\,a}\right)=-\left(\frac{d\,d\,V}{d\,a^2}\right)$; therefore $\left(\frac{d\,d\,V}{d\,a^2}\right)=-\frac{A}{a}$. In like manner, from the values of B, C, [1388, 1379], we get $\left(\frac{d\,d\,V}{d\,b^2}\right)=-\frac{B}{b}$, $\left(\frac{d\,d\,V}{d\,c^2}\right)=-\frac{C}{c}$. The sum of these three expressions, reduced by means of the formula [1370g], becomes

radius r makes with that of a, ϖ the angle which the plane formed by this [1430″]

$$\left(\frac{ddV}{da^2}\right)+\left(\frac{ddV}{db^2}\right)+\left(\frac{ddV}{dc^2}\right)=-\left\{\frac{A}{a}+\frac{B}{b}+\frac{C}{c}\right\}=-4\pi,$$ [1430v″]

which is the same as in [1430s], the density ρ' being put equal to unity, as in [1346]. In this calculation, the centre of the ellipsoid is taken for the origin of the co-ordinates; but it is evident that the same result would have been obtained, if the origin had been fixed at any other point. Instead of using [1370g] to obtain a second demonstration of the formula [1430t, v″], we might use [1430t], to demonstrate the formula [1370g], as is done by Mr. Poisson.

It is somewhat remarkable, that this defect in the formula [1430], as it was first published by La Place, should have remained unnoticed, nearly half a century; particularly as he had expressly called the attention of mathematicians to the necessity of having the limits of the integrals independent of the co-ordinates of the attracted point a, b, c, [1429″]; and had also conformed to this restriction, in the calculations of the first volume, [452, &c.]

In computing $\left(\frac{dV'}{dr}\right)$, $\left(\frac{ddV'}{dr^2}\right)$, for a sphere [1430i—n], we have, for greater simplicity, supposed it to be homogeneous. We shall now compute it in another manner, for a sphere, composed of concentrical spherical strata, of different densities. This calculation will serve as a method of showing how to find the differentials of formulas, depending on definite integrals, when the limits of these integrals are supposed to vary. Supposing ρ to represent the density of the concentrical spherical stratum, whose internal radius is R, and external radius $R+dR$, the mass of this stratum will be $4\pi\rho\,.\,R^2dR$, [275b]. Its integral, [1430w] taken from $R=0$ to $R=r$, will give the mass m of the internal sphere $G'H'K'$, whose radius is r. This mass will be represented by $4\pi\,.\int_0^r\rho\,.\,R^2dR$. If we divide it by r^2, we shall get, as in [1430l, m], $-\left(\frac{dV'}{dr}\right)=\frac{4\pi}{r^2}\,.\int_0^r\rho\,.\,R^2dR$. Putting for [1430x] brevity $\int_0^r\rho\,.\,R^2dR=r''$, we shall have $\left(\frac{dV'}{dr}\right)=-\frac{4\pi}{r^2}\,.\,r''$, r'' being a [1430y] function of r. The differential of this last value of $\left(\frac{dV'}{dr}\right)$ being taken relatively to r, we shall get $\left(\frac{ddV'}{dr^2}\right)=\frac{8\pi}{r^3}\,.\,r''-\frac{4\pi}{r^2}\,.\left(\frac{dr''}{dr}\right)$. In finding the differential of r'', [1430z] relative to r, we must observe that, when r is increased by the element dr, the last limit of the integral $r''=\int_0^r\rho\,.\,R^2dR$, is increased from r to $r+dr$, by which means r'' is increased by the element $\rho'\,.\,r^2dr$, corresponding to the point G'; hence

$$dr''=\rho'\,.\,r^2dr, \qquad \text{and} \qquad \left(\frac{dr''}{dr}\right)=\rho'\,.\,r^2.$$ [1430α]

radius and axis, makes with the plane of the axes of a, b, and we shall have,*

[1431] $$a = r\,.\cos.\theta\,; \qquad b = r\,.\sin.\theta\,.\cos.\varpi\,; \qquad c = r\,.\sin.\theta\,.\sin.\varpi.$$

[1431′] If we also put R, θ', ϖ', for what r, θ, ϖ, become at the point corresponding to the particle of the spheroid dM, we shall have†

[1432] $$x = R\,.\cos.\theta'\,; \qquad y = R\,.\sin.\theta'\,.\cos.\varpi'\,; \qquad z = R\,.\sin.\theta'\,.\sin.\varpi'.$$

Moreover, the particle of the spheroid dM is equal to a rectangular parallelopiped, whose sides are‡ dR, $Rd\theta'$, $Rd\varpi'\,.\sin.\theta'$; therefore,

[1432′] $$dM = \rho\,.\,R^2dR\,.\,d\theta'\,.\,d\varpi'\,.\sin.\theta,$$

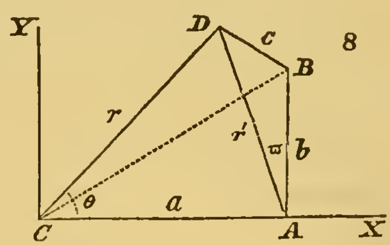
Substituting this in [1430z], we get $\left(\frac{ddV'}{dr^2}\right) = \frac{8\pi}{r^3}\,.\,r'' - \frac{4\pi}{r^2}\,.\,\rho' r^2 = \frac{8\pi}{r^3}\,.\,r'' - 4\pi\,.\,\rho'$.

Using this and the value of $\left(\frac{dV'}{dr}\right)$ [1430y], the second member of [1430o] becomes

[1430β] $\frac{8\pi}{r^3}\,.\,r'' - 4\pi\rho' - \frac{8\pi}{r^3}\,.\,r'' = -4\pi\rho'$, as in a homogeneous sphere, of the density ρ', [1430q].

* (978) The values [1431] are computed like those of [460], by changing, as in the last note, x, y, z, into a, b, c, respectively. In this case, C is the origin of the co-ordinates, D the attracted point, whose rectangular

[1431a] co-ordinates are $CA = a$, $AB = b$, $BD = c$, $CD = r$, angle $ACD = \theta$, $DAB = \varpi$. In the triangle CAD, we find



$$CA = CD\,.\cos.ACD = r\,.\cos.\theta = a, \qquad [1431]\,;$$
$$AD = CD\,.\sin.ACD = r\,.\sin.\theta\,;$$

and in the triangle ABD, $\quad AB = AD\,.\cos.DAB$, $\quad BD = AD\,.\sin.DAB$. Substituting the value of AD, we get

[1431a′] $$AB = CD\,.\sin.ACD\,.\cos.DAB, \qquad BD = CD\,.\sin.ACD\,.\sin.DAB,$$

which correspond to a, b, c, [1431].

† (979) The values [1432] are obtained in exactly the same way as [1431], putting x, y, z, R, θ', ϖ', for a, b, c, r, θ, ϖ, respectively.

‡ (980) These sides of the parallelopiped are similar to those computed in [1355h], fig. page 6, where they are called dr, rdp, $rdq\,.\sin.p$; in which $cA = r$,

ρ being its density ; hence we shall have,* [1432′′]

Function V.

$$V = \int_0^u \int_0^\pi \int_0^{2\pi} \frac{\rho \,.\, R^2 dR \,.\, d\theta' \,.\, d\varpi' \,.\, \sin.\theta'}{\sqrt{r^2 - 2Rr \,.\, \{\cos.\theta \,.\, \cos.\theta' + \sin.\theta \,.\, \sin.\theta' \,.\, \cos.(\varpi' - \varpi)\} + R^2}} ; \quad [1433]$$

the integral, relative to R, must be taken from $R = 0$, to the value of $R = u$ at the surface of the spheroid ; the integral relative to θ', must be [1433′]

$A c x = p$, $y c Q = q$; and if we put $r = R$, $p = \theta'$, $q = \varpi'$, in order to conform [1431b] to the notation of this article, the sides will become respectively dR, $R d\theta'$, $R\, d\varpi' \,.\, \sin.\theta'$, whose product multiplied by the density ρ, gives the mass of the particle

$$dM = \rho \,.\, R^2 dR \,.\, d\theta' \,.\, d\varpi' \,.\, \sin.\theta', \quad [1431c]$$

as above ; and the whole mass of the spheroid will be obtained by taking the integrals between the limits mentioned in [1433′, &c]. If we put $\cos.\theta' = \mu'$, we shall get $-d\theta' \,.\, \sin.\theta' = d\mu'$, and the limits of μ', corresponding to $\theta' = 0$, $\theta' = \pi$, [1433′′], will be $\mu' = 1$, $\mu' = -1$. If we change these limits, so that they may be from $\mu' = -1$ to $\mu' = 1$, we may put $d\theta' \,.\, \sin.\theta' = d\mu'$, and then the preceding value of dM will become $dM = \rho \,.\, R^2 dR \,.\, d\mu' \,.\, d\varpi'$; from which the whole mass M may be obtained, [1431d] by integrating from $R = 0$ to its value at the surface $R = u$, from $\mu' = -1$ to $\mu' = 1$, and from $\varpi' = 0$ to $\varpi' = 2\pi$.

* (981) If we substitute the values of a, b, c, x, y, z, [1431, 1432], in [1348a],

$$f^2 = (a - x)^2 + (b - y)^2 + (c - z)^2 \quad [1432a]$$

it becomes

$$\begin{aligned} f^2 &= \{r \,.\, \cos.\theta - R \,.\, \cos.\theta'\}^2 + \{r \,.\, \sin.\theta \,.\, \cos.\varpi - R \,.\, \sin.\theta' \,.\, \cos.\varpi'\}^2 \\ &\qquad + \{r \,.\, \sin.\theta \,.\, \sin.\varpi - R \,.\, \sin.\theta' \,.\, \sin.\varpi'\}^2 \\ &= r^2 \,.\, \{\cos.^2\theta + \sin.^2\theta \,.\, (\cos.^2\varpi + \sin.^2\varpi)\} \\ &\qquad - 2Rr \,.\, \{\cos.\theta \,.\, \cos.\theta' + \sin.\theta \,.\, \sin.\theta' \,.\, (\cos.\varpi \,.\, \cos.\varpi' + \sin.\varpi \,.\, \sin.\varpi')\} \\ &\qquad + R^2 \,.\, \{\cos.^2\theta' + \sin.^2\theta' \,.\, (\cos.^2\varpi' + \sin.^2\varpi')\} \\ &= r^2 \,.\, \{\cos.^2\theta + \sin.^2\theta\} - 2Rr \,.\, \{\cos.\theta \,.\, \cos.\theta' + \sin.\theta \,.\, \sin.\theta' \,.\, \cos.(\varpi' - \varpi)\} \\ &\qquad + R^2 \,.\, \{\cos.^2\theta' + \sin.^2\theta'\} \\ &= r^2 - 2Rr \,.\, \{\cos.\theta \,.\, \cos.\theta' + \sin.\theta \,.\, \sin.\theta' \,.\, \cos.(\varpi' - \varpi)\} + R^2. \end{aligned} \quad [1432b]$$

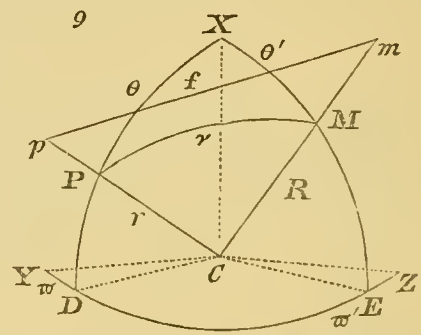
Substituting this, and the value of dM [1432′], in [1429], it becomes as in [1433]. The [1432c] limits of the integral correspond to those pointed out in note 338, page 294, Vol. I, to which this is similar.

taken from $\theta'=0$ to θ' equal to the semi-circumference; and the integral
[1433''] relative to ϖ' must be taken, from $\varpi'=0$ to ϖ' equal to the circumference.
Taking the differentials of this expression of V, we shall find,*

La Place's fundamental theorem. Second form.

[1434] $$\left(\frac{dd\,V}{d\,\theta^2}\right)+\frac{\cos.\theta}{\sin.\theta}\cdot\left(\frac{d\,V}{d\,\theta}\right)+\frac{\left(\frac{dd\,V}{d\,\varpi^2}\right)}{\sin.^2\theta}+r.\left(\frac{dd\,.\,r\,V}{d\,r^2}\right)=0,\ \ [\text{or } -4\pi\rho'\cdot r^2].\quad(2)$$

This equation is nothing more than a transformation of the equation [1430].



The preceding value of f, which represents the denominator of V [1433], or of T [1439], may also be computed geometrically, by means of the annexed figure; in which $f=m\,p$ represents the
[1432d] distance of the attracting particle m, from the attracted point p. C is the origin of the co-ordinates. The lines $C\,X=C\,Y=C\,Z=1$, are taken on the axes of x, y, z, respectively; and
[1432e] through the points X, Y, Z, a spherical surface is described about the centre C, cutting $C\,p$ in P, $C\,m$ in M. Lastly, the arcs $X\,P$, $X\,M$, are continued till they meet the circular arc $Y\,D\,E\,Z$, in D, and E. Then we shall have
[1432f] $X\,P=\theta$, $X\,M=\theta'$, $Y\,D=\varpi$, $Y\,E=\varpi'$, $D\,E=\varpi'-\varpi=$ spherical angle $P\,X\,M$; and if we put the arc $P\,M=\gamma$, we shall have, in the spherical triangle $P\,X\,M$, [1345[8]],

[1432g] $$\cos.\gamma=\cos.\theta\,.\,\cos.\theta'+\sin.\theta\,.\,\sin.\theta'\,.\,\cos.(\varpi'-\varpi).$$

Then in the plane triangle $p\,C\,m$, we have $C\,p=r$, $C\,m=R$, $p\,m=f$, and
[1432h] $\cos.p\,C\,m=\cos.\gamma$; hence we get $f^2=r^2-2\,R\,r\,.\cos.\gamma+R^2$, [62] Int. Substituting the preceding value of $\cos.\gamma$, we shall obtain for f, the same expression as above, which is used in [1433, 1439], namely,

[1432i] $$f=\{r^2-2\,R\,r\,.\cos.\gamma+R^2\}^{\frac{1}{2}}$$

[1432k] $$=\{r^2-2\,R\,r\,.\,[\cos.\theta\,.\,\cos.\theta'+\sin.\theta\,.\,\sin.\theta'\,.\,\cos.(\varpi'-\varpi)]+R^2\}^{\frac{1}{2}};$$

observing that the coefficient of $-2\,R\,r$, in both these expressions, is $\cos.\gamma$, or the
[1432l] cosine of the angle formed by the two lines, drawn from the origin of the co-ordinates, to the attracting and to the attracted points.

* (982) Changing x, y, z, into a, b, c, respectively, in [459, 460], they become identical with [1430, 1431]; if we make the same changes in [465, 466], which were deduced from the former, they will become identical with [1434, 1435], respectively. Moreover, in

If we make $\cos.\theta=\mu$, we may put this equation under the following form, [1434′]

La Place's fundamental theorem. Third form.

$$\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dV}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddV}{d\varpi^2}\right)}{1-\mu^2}+r.\left(\frac{dd.rV}{dr^2}\right)=0,\quad [\text{or } -4\pi\rho'.r^2]. \quad (3)$$ [1435]

We have before obtained these equations, in the second book, § 11, [459—466].

9. We shall suppose, in the first place, that the attracted point is without the spheroid. In this case we must develop V according to the descending powers of r, and it will therefore be of the form* [1435′]

Development of V for an external point.

$$V=\frac{U^{(0)}}{r}+\frac{U^{(1)}}{r^2}+\frac{U^{(2)}}{r^3}+\frac{U^{(3)}}{r^4}+\&c.$$ [1436]

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deducing [465] from [459], this last was multiplied by r^2, [465u], therefore the term $-4\pi\rho'$, introduced by Mr. Poisson, [1430t], must be multiplied by r^2, in [1434, 1435], [1432m] producing the term $-4\pi\rho'.r^2$, corresponding to the case omitted by La Place.

* (983) For the purpose of illustrating this method of developing V, we shall put,

$$\cos.\theta=\mu,\quad \cos.\theta'=\mu',\quad P_0=\mu,\quad P_1=\sqrt{(1-\mu^2)}.\cos.\varpi,\quad P_2=\sqrt{(1-\mu^2)}.\sin.\varpi,$$ [1433a]

$$P_3{}^2=P_0{}^2+P_1{}^2+P_2{}^2=\mu^2+(1-\mu^2).(\cos.^2\varpi+\sin.^2\varpi)=\mu^2+1-\mu^2=1,$$ [1433a']

also $\cos.(\varpi'-\varpi)=\cos.\varpi.\cos.\varpi'+\sin.\varpi.\sin.\varpi'$, [24] Int. Then the denominator of [1433], which is represented by the factor T, [1439], will give,

$$\begin{aligned}Tr&=\left\{1-\frac{2R}{r}.[\mu\mu'+\sqrt{(1-\mu^2)}.\sqrt{(1-\mu'^2)}.(\cos.\varpi.\cos.\varpi'+\sin.\varpi.\sin.\varpi')]+\frac{R^2}{r^2}\right\}^{-\frac{1}{2}}\\&=\left\{1-\frac{2R}{r}.[P_0.\mu'+P_1.\sqrt{(1-\mu'^2)}.\cos.\varpi'+P_2.\sqrt{(1-\mu'^2)}.\sin.\varpi']+\frac{R^2}{r^2}.P_3{}^2\right\}^{-\frac{1}{2}}.\end{aligned}$$ [1433b]

This expression of Tr is homogeneous in R, r, and of the degree 0: and being developed according to the powers of $\frac{R}{r}$, will become of the form,

$$Tr=Q^{(0)}+Q^{(1)}.\frac{R}{r}+Q^{(2)}.\frac{R^2}{r^2}+Q^{(3)}.\frac{R^3}{r^3}+\&c.;$$ [1433c]

in which $Q^{(0)}$, $Q^{(1)}$, $Q^{(2)}$, &c., are independent of R, r. Moreover, since the second [1433d]

Substituting this value of V in the equation [1435] of the preceding article, the comparison of the similar powers of r, will give, for any value of i,

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member of [1433b] is homogeneous in r, P_0, P_1, P_2, P_3, and of the degree 0; its development in the second member of [1433c], must also be homogeneous and of the degree 0, in r, P_0, P_1, P_2, P_3; and as $Q^{(i)}$ is divided by r^i in [1433c], therefore,
[1433d′] generally, when i is any positive integer, $Q^{(i)}$ will be homogeneous, and of the degree i, in P_0, P_1, P_2, P_3; and if $Q^{(i)}$ be arranged according to the powers of P_3^2, it will be of
[1433d″] the form $Q^{(i)} = A + B \,.\, P_3^2 + C \,.\, P_3^4 + D \,.\, P_3^6 + \&c.$; in which A, B, C, &c., are separately homogeneous functions of the three quantities P_0, P_1, P_2; it being evident that the uneven powers of P_3 do not occur in [1433b], and therefore will not be found in $Q^{(i)}$. Moreover, to render each of the terms of $Q^{(i)}$ homogeneous, we must have, A of the order i, B of the order $i-2$, C of the order $i-4$, &c.; negative values of $i-2$, $i-4$, &c., being neglected. Then putting, as in [1433a′], $P_3 = 1$, we shall get

[1433d‴]
$$Q^{(i)} = A + B + C + D + \&c.$$

The comparison of the formulas [1433b, c] will also show, that $Q^{(0)} = 1$, and that $Q^{(1)}$
[1433d⁗] depends on the first power of the quantities P_0, P_1, P_2. Dividing [1433c] by r, we shall find, as in [1441″],

[1433e]
$$T = Q^{(0)} \,.\, \frac{1}{r} + Q^{(1)} \,.\, \frac{R}{r^2} + Q^{(2)} \,.\, \frac{R^2}{r^3} + \&c.$$

If we put $P_4 = \surd(1-\mu^2) \,.\, \cos.(\varpi' - \varpi)$, the expression of T [1439] will give, by
[1433e′] using as above $P_3 = 1$,

[1433f]
$$\begin{aligned} T r &= \left\{ 1 - \frac{2R}{r} \,.\, [\mu\mu' + \surd(1-\mu^2) \,.\, \surd(1-\mu'^2) \,.\, \cos.(\varpi' - \varpi)] + \frac{R^2}{r^2} \right\}^{-\frac{1}{2}} \\ &= \left\{ 1 - \frac{2R}{r} \,.\, [P_0 \,.\, \mu' + P_4 \,.\, \surd(1-\mu'^2)] + \frac{R^2}{r^2} \,.\, P_3^2 \right\}^{-\frac{1}{2}}. \end{aligned}$$

This expression of Tr can be developed, in the same form as in [1433c]. Hence we shall get T in the same form as in [1433e], or [1441″]; in which $Q^{(i)}$ is a homogeneous, rational and integral function of P_0, P_3, P_4, of the degree i, and of the same form as [1433d″];
[1433g] which, by putting $P_3 = 1$, becomes $Q^{(i)} = A + B + C + D + \&c.$; A, B, C, &c., being separately homogeneous functions of P_0, P_4. Moreover, A is of the order i in P_0, P_4; B of the order of the positive number $i-2$, C of the order of the positive number $i-4$, &c., as in [1433d″, &c.]

Differential equation in $U^{(i)}$.

$$0=\left\{\frac{d.\left\{(1-\mu\mu).\left(\frac{dU^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddU^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i.(i+1).U^{(i)}.\ * \qquad [1437]$$

Substituting $d\mu'=-d\theta'.\sin.\theta'$, and T [1433e] in [1433], we shall get,

$$V=-\textstyle\iiint\rho.R^2dR.d\mu'.d\varpi'.\left\{Q^{(0)}.\frac{1}{r}+Q^{(1)}.\frac{R}{r^2}+Q^{(2)}.\frac{R^2}{r^3}+\&c.\right\}. \qquad [1433h]$$

In the integrations of this function, μ, ϖ, and therefore also P_0, P_1, P_2, which occur in $Q^{(1)}$, $Q^{(2)}$, &c., are considered as constant. If we perform the integrations relative to each of the quantities $Q^{(0)}$, $Q^{(1)}$, &c., separately, we may put, generally,

$$-\textstyle\iiint\rho.R^2dR.d\mu'.d\varpi'.Q^{(i)}.\frac{R^{(i)}}{r^{i+1}}=\frac{U^{(i)}}{r^{i+1}}; \qquad [1433i]$$

taking the integrals between the given limits of R, μ', ϖ'; and we may incidentally remark, that $Q^{(i)}$ does not contain r, therefore $U^{(i)}$ will not contain r. In this manner of [1433k] development and integration, we perceive that $U^{(i)}$, like $Q^{(i)}$, [1433d'''], will not contain terms of the powers and products of P_0, P_1, P_2, which exceed the degree i; but it might, as in [1433d'''], contain powers and products of these quantities, of a less order, represented by the integral positive numbers $i-2$, $i-4$, &c. It appears also, from the preceding method of finding $Q^{(i)}$, $U^{(i)}$, that they must be rational and integral functions of P_0, P_1, P_2. [1433l]
Finally, by comparing the expressions [1433h, i], we evidently obtain V in the form [1436]. [1433m]

* (984) Any term of V [1436], as $\frac{U^{(i)}}{r^{i+1}}$, being substituted in the second member of [1435], produces an expression, which we shall call $\frac{W^{(i)}}{r^{i+1}}$, of the following form,

$$\frac{W^{(i)}}{r^{i+1}}=\left\{\frac{d.(1-\mu^2).\left(\frac{d.\frac{U^{(i)}}{r^{i+1}}}{d\mu}\right)}{d\mu}\right\}+\frac{\left(\frac{dd.\frac{U^{(i)}}{r^{i+1}}}{d\varpi^2}\right)}{1-\mu^2}+r.\left(\frac{dd.\frac{U^{(i)}}{r^i}}{dr^2}\right). \qquad [1436a]$$

Now as $U^{(i)}$ does not contain r [1433k], we shall have,

$$r.\left(\frac{dd.\left(\frac{U^{(i)}}{r^i}\right)}{dr^2}\right)=U^{(i)}.r.\left(\frac{dd.r^{-i}}{dr^2}\right)=U^{(i)}.i.(i+1).r^{-i-1}.$$

Substituting this in [1436a], we may bring the denominator r^{i+1} from under the signs of

[1437] It is evident, from the expression of V [1433, 1436], that $U^{(i)}$ is a rational and integral function of μ, $\sqrt{1-\mu^2}\,.\,\sin.\varpi$, $\sqrt{1-\mu^2}\,.\,\cos.\varpi$, depending
[1437′′] on the nature of the spheroid.* When $i=0$, this function becomes a constant quantity, and in case $i=1$, it is of the following form,

[1438] $$U^{(1)}=H\,.\,\mu+H'\,.\sqrt{1-\mu^2}\,.\,\sin.\varpi+H''\,.\sqrt{1-\mu^2}\,.\,\cos.\varpi\,;$$

H, H', H'', being constant quantities.

To determine the general value of $U^{(i)}$, we shall put

[1439] $$T=\frac{1}{\sqrt{r^2-2Rr\,.\{\cos.\theta\,.\cos.\theta'+\sin.\theta\,.\sin.\theta'\,.\cos.(\varpi'-\varpi)\}+R^2}}\,;$$

and we shall have†

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differentiation, relative to μ, ϖ, in the two first terms; and then the whole expression will contain the factor r^{-i-1}; and if we divide by this factor, we shall get,

[1436b] $$W^{(i)}=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,U^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{d\,d\,U^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i\,.(i+1)\,.\,U^{(i)}.$$

Hence the expression [1435] will become,

[1437a] $$0=\frac{W^{(0)}}{r}+\frac{W^{(1)}}{r^2}+\frac{W^{(2)}}{r^3}+\frac{W^{(3)}}{r^4}+\&c.\,,$$

in which $W^{(0)}$, $W^{(1)}$, $W^{(2)}$, &c., [1436b], are independent of r. This equation cannot be
[1437b] satisfied, *for all values of* r, except we have generally $W^{(i)}=0$. This is evident; for if we multiply [1437a] by r, and then put $r=\infty$, we shall get $W^{(0)}=0$. Substituting this in [1437a], multiplied by r^2, and then putting $r=\infty$, we shall get $W^{(1)}=0$, and so on; making generally $W^{(i)}=0$, which is the same as the formula [1437].

* (985) $U^{(i)}$ is a rational and integral function of P_0, P_1, P_2, of an order not exceeding
[1438a] i [1433k—m]. Therefore, when $i=0$, the terms P_0, P_1, P_2, must be of the degree 0, so that $U^{(0)}$ is equal to a constant quantity; and when $i=1$, the terms P_0, P_1, P_2, must be of the degree 1, so that $U^{(1)}=H\,.\,P_0+H'\,.\,P_1+H''\,.\,P_2$, as in [1438].

† (986) The equations [1434, 1435] were deduced from [1433], by differentiation, in a manner that is not affected by the triple sign of integration $\int\!\int\!\int$ in [1433], nor by the
[1439a] value of the numerator $\rho\,.\,R^2\,dR\,.\,d\theta'\,.\,d\varpi'\,.\sin.\theta'$; and the same resulting equation

$$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,T}{d\,\mu}\right)\right\}}{d\,\mu}\right\}+\frac{\left(\frac{d\,d\,T}{d\,\varpi^2}\right)}{1-\mu^2}+r\,.\left(\frac{dd\,.\,rT}{d\,r^2}\right). \qquad [1440]$$

This equation will also be satisfied if θ be changed into θ', ϖ into ϖ', and the contrary; because T [1439] is composed of θ, ϖ, in like manner as it is of θ', ϖ'. [1440′]

If we develop T [1439] in a descending series relative to r, we shall get, [1433*f*, *e*], Function T.

$$T=\frac{1}{\sqrt{r^2-2Rr\,.\{\cos.\theta\,.\,\cos.\theta'+\sin.\theta\,.\,\sin.\theta'\,.\,\cos.(\varpi'-\varpi)\}+R^2}}; \qquad [1441]$$

$$=\frac{1}{\sqrt{r^2-2Rr\,.\{\mu\,\mu'+\sqrt{1-\mu^2}\,.\,\sqrt{1-\mu'^2}\,.\,\cos.(\varpi'-\varpi)\}+R^2}}; \qquad [1441']$$

$$=\frac{Q^{(0)}}{r}+Q^{(1)}\,.\,\frac{R}{r^2}+Q^{(2)}\,.\,\frac{R^2}{r^3}+Q^{(3)}\,.\,\frac{R^3}{r^4}+\&c.\,; \qquad [1441'']$$

$Q^{(i)}$ being for all values of i subjected to this equation,*

[1434] would have been obtained, if we had neglected fff, and put

$$\rho\,.\,R^2\,d\,R\,.\,d\,\theta'\,.\,d\,\varpi'\,.\,\sin.\theta'=1.$$

These changes being made in V [1433], it becomes like T [1439]; and the same change, of T for V, may be made in [1434], or in [1435], and the last will become as in [1440]. Moreover, since T [1439] is not altered by changing θ, ϖ, into θ', ϖ', and the contrary; the same changes may be made in [1440], as is observed in [1440′].

* (987) If in the equations [1436, 1435], we change V into T, and $U^{(i)}$ into $Q^{(i)}\,.\,R^i$, they will respectively become, as in [1441″, 1440]; and the same changes being made in [1437], which was derived from [1436, 1435], it will become equal to the product of the function [1442] by the quantity R; because this factor R^i is not affected by the partial differentials relative to μ, ϖ, and may therefore be brought from under the sign of differentiation relative to μ, ϖ; and the whole expression, being divided by R^i, will become as in [1442]. If we put $\mu=\cos.\theta$, this will become by reduction as in the two following formulas, [1442*a*, *b*], which bear the same relation to [1442], that [1434] does to [1435], and they may be computed as in [466*b*]. Differential equations in $Q^{(i)}$.

$$0=\left(\frac{d\,d\,Q^{(i)}}{d\,\theta^2}\right)+\frac{\cos.\theta}{\sin.\theta}\,.\left(\frac{d\,Q^{(i)}}{d\,\theta}\right)+\frac{\left(\frac{d\,d\,Q^{(i)}}{d\,\varpi^2}\right)}{\sin.^2\theta}+i\,.\,(i+1)\,.\,Q^{(i)}\,; \qquad [1442a]$$

$$0=\frac{1}{\sin.\theta}\,.\left\{\frac{d\,.\left\{\sin.\theta\,.\left(\frac{d\,Q^{(i)}}{d\,\theta}\right)\right\}}{d\,\theta}\right\}+\frac{\left(\frac{dd\,Q^{(i)}}{d\,\varpi^2}\right)}{\sin.^2\theta}+i\,.\,(i+1)\,.\,Q^{(i)}. \qquad [1442b]$$

Differential equation in $Q^{(i)}$.

[1442] $$0 = \left\{ \frac{d \cdot \left\{ (1-\mu^2) \cdot \left(\frac{d Q^{(i)}}{d \mu} \right) \right\}}{d \mu} \right\} + \frac{\left(\frac{d d Q^{(i)}}{d \varpi^2} \right)}{1-\mu^2} + i \cdot (i+1) \cdot Q^{(i)};$$

and it is also evident that $Q^{(i)}$ is a rational and integral function of μ,

[1442'] $\sqrt{1-\mu^2} \cdot \cos. (\varpi' - \varpi)$, [1433g]. $Q^{(i)}$ being known, we shall obtain $U^{(i)}$, by means of the equation,*

External point, $U^{(i)}$.

[1443] $$U^{(i)} = \int_0^{2\pi} \int_0^{\pi} (\int \rho \cdot R^{i+2} \cdot dR) \cdot d\varpi' \cdot d\theta' \cdot \sin. \theta' \cdot Q^{(i)}.$$

If we now suppose the attracted point to be within the spheroid, we must develop the value of V [1433], in a series ascending relative to r, which gives for V an expression of this form,†

Development of V for an internal point.

[1444] $$V = v^{(0)} + r \cdot v^{(1)} + r^2 \cdot v^{(2)} + r^3 \cdot v^{(3)} + \&c. ;$$

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* (988) This is the same as [1433i], multiplied by r^{i+1}, which is not affected by the

[1442c] signs of integration $\int\int\int$; observing also that $-d\mu' = d\theta' \cdot \sin. \theta'$, [1433a].

† (989) The value of T [1439] is not altered by changing r into R, and R into r. The same changes being made in [1441''], it becomes as in [1446], the quantities $Q^{(0)}$,

[1443a] $Q^{(1)}$, $Q^{(2)}$, &c., being independent of R, r, [1433d], and are the same in both formulas. Substituting this value of T [1446] in [1433], we get,

[1444a] $$V = \int\int\int \rho \cdot R^2 dR \cdot d\theta' \cdot d\varpi' \cdot \sin. \theta' \cdot \left\{ \frac{Q^{(0)}}{R} + Q^{(1)} \cdot \frac{r}{R^2} + Q^{(2)} \cdot \frac{r^2}{R^3} + Q^{(3)} \cdot \frac{r^3}{R^4} + \&c. \right\}.$$

Now if we put the part of this integral depending on r^i equal to $r^i \cdot v^{(i)}$, i being any integral positive number, we shall have,

[1444b] $$r^i \cdot v^{(i)} = \int\int\int \rho \cdot R^2 dR \cdot d\theta' \cdot d\varpi' \cdot \sin. \theta' \cdot Q^{(i)} \cdot \frac{r^i}{R^{i+1}},$$

and the expression of V [1444a] will become as in [1444]. The signs of integration refer to R, θ', ϖ', and we may bring r from under those signs; then dividing by r^i, we shall get

[1444c] $v^{(i)}$, [1447]; $v^{(i)}$ being, like $Q^{(i)}$ [1433d'''], a rational and integral function of P_0, P_1, P_2. If we substitute V [1444] in [1435], the general term $r^i \cdot v^{(i)}$ will produce the quantity,

[1444d] $$\left\{ \frac{d \cdot \left\{ (1-\mu^2) \cdot \left(\frac{d \cdot (r^i \cdot v^{(i)})}{d \mu} \right) \right\}}{d \mu} \right\} + \frac{\left(\frac{d^2 \cdot (r^i \cdot v^{(i)})}{d \varpi^2} \right)}{1-\mu^2} + r \cdot \left(\frac{d^2 \cdot (r^{i+1} \cdot v^{(i)})}{d r^2} \right);$$

and if we neglect the variations of the limits of $v^{(i)}$, the last term, $r \cdot \left(\frac{d^2 \cdot (r^{i+1} \cdot v^{(i)})}{d r^2} \right)$, will

$v^{(i)}$ being a rational and integral function of μ, $\sqrt{1-\mu^2}\,.\,\sin.\varpi$, $\sqrt{1-\mu^2}\,.\,\cos.\varpi$, [1444'] [1444c], which satisfies the same equation of partial differentials as $U^{(i)}$, [1437]; so that we shall have [1444e, h],

Internal point, $v^{(i)}$.

$$0=\left\{\frac{d\,.\left\{(1-\mu\mu)\,.\left(\frac{d\,v^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}+\frac{\left(\frac{d\,d\,v^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,v^{(i)}. \qquad [1445]$$

To determine $v^{(i)}$, we shall reduce the radical T to an ascending series relative to r, and we shall get, [1443a],

$$T=\frac{Q^{(0)}}{R}+Q^{(1)}\,.\,\frac{r}{R^2}+Q^{(2)}\,.\,\frac{r^2}{R^3}+Q^{(3)}\,.\,\frac{r^3}{R^4}+\&c.; \qquad [1446]$$

the quantities $Q^{(0)}$, $Q^{(1)}$, $Q^{(2)}$, &c., being the same as above, [1441'', 1443a]; hence we shall have, [1444b],

Internal point, $v^{(i)}$.

$$v^{(i)}=\int_0^{2\pi}\int_0^{\pi}\left(\int\frac{\rho\,.\,dR}{R^{i-1}}\right)d\,\varpi'\,.\,d\,\theta'\,.\,\sin.\theta'\,.\,Q^{(i)}. \qquad [1447]$$

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become $i\,.\,(i+1)\,.\,r^i\,.\,v^{(i)}$. In the other two terms, the differentials relative to μ and ϖ, do not affect r, and it may be brought from under the signs d^2, d. Then each term will have the factor r^i, and the whole expression will be equal to r^i multiplied by the function in the second member of [1445]. If we put this function equal to $w^{(i)}$, the whole expression of the [1444e] first member of [1435], will become $w^{(0)}+r\,.\,w^{(1)}+r^2\,.\,w^{(2)}+r^3\,.\,w^{(3)}+\&c.$, exclusive of the terms depending on the variations of the limits of r, in $v^{(i)}$, or $U^{(i)}$. It will be seen hereafter, [1447π], that this last part of the expression produces the term $-4\pi\rho'\,.\,r^2$, [1447π], which destroys the like term in the second member of [1435], so that we shall [1444f] finally have,

$$w^{(0)}+r\,.\,w^{(1)}+r^2\,.\,w^{(2)}+r^3\,.\,w^{(3)}+\&c.=0. \qquad [1444g]$$

In this equation, $w^{(0)}$, $w^{(1)}$, &c., are independent of r [1444e, 1445], and *it exists for all the values of* $r<R$. If we put $r=0$, it becomes $w^{(0)}=0$; subtracting this from [1444g], and dividing by r, we get $w^{(1)}+r\,.\,w^{(2)}+\&c.=0$. Again, putting $r=0$, we find $w^{(1)}=0$; and by proceeding in this way, we get generally $w^{(i)}=0$, [1444h] as in [1445].

But as the expression of T [1446] converges, only when R is equal
[1447′] to or exceeds r; the preceding value of V, [1444], must be used only for the strata of the spheroid without the attracted point. This point* being above the other strata, we must determine the part of V

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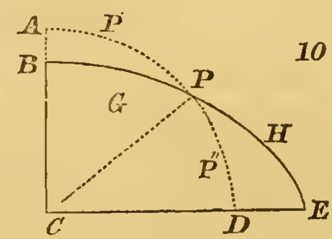
* (990) If the attracted point be near the surface of the spheroid, it may happen that
[1447a] for some values of θ', ϖ', we shall have $r > R$, and for other values, $r < R$. This is a much more complex form than either of the other cases, because neither of the formulas [1436, 1444] can be separately applied to the whole surface; and it is necessary to notice accurately the limits of θ', ϖ', corresponding to each formula; restricting ourselves to the use of converging series only; taking in [1436] those values of θ', ϖ', corresponding to $r > R$,
[1447b] and in [1444] those in which $r < R$. A very simple instance of this nature, mentioned by Mr. Poisson, is that of an ellipsoid CBE, in which the origin of the co-ordinates is at the centre C, and the attracted point anywhere on the spherical surface $AP'PP''D$, described about the centre C, with the radius $CP = r$, this radius being greater than the shortest semi-axis CB, and less than the longest semi-axis CE. We may however observe, that this complex case may generally be avoided; since the attraction upon an internal point can be obtained, as in [1447′], by merely changing the position, or the origin of the co-ordinates, so that the distance from this origin to the
[1447c] attracted point, may be less than any of the radii, drawn from the origin to the surface of the spheroid.



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Mr. Poisson, who first called the attention of mathematicians to this excepted case, has altered a little the formulas of La Place, and given in an abridged and somewhat improved form, the values of V, corresponding to the different situations of the attracted point. We
[1447d] shall here give his formulas, varying however a little the notation, so as to conform to that
Signs used in this work. u is the value of the radius R corresponding to the surface of the
$\int'$, spheroid. The sign $\int'$, with the accent *above* the letter, is used to denote the integration relative to those elements in which $r > u$, or where the spherical surface AP falls
$\int_{,}$. *above* the corresponding parts of the spheroid BGP; and the sign $\int_{,}$, with the
[1447e] accent *below* the letter, is used to denote the integration of those elements where

which corresponds to these lower strata, by means of the first expression of V [1436].

Attraction of a spheroid differing but little from a sphere.

10. We shall in the first place consider a spheroid which differs but little from a sphere, and shall determine the functions $U^{(0)}$, $U^{(1)}$, $U^{(2)}$, &c., [1447″]
$v^{(0)}$, $v^{(1)}$, $v^{(2)}$, &c., relative to this spheroid. There exists a differential

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$r<u$, or the spherical surface $PP''D$ falls *below* the corresponding parts of the spheroid PHE.

The area of the spherical surface [1432′] whose sides are $Rd\theta'$, $Rd\varpi'.\sin.\theta'$, becomes, when R is unity, equal to $\sin.\theta'.d\theta'.d\varpi'$; and if for brevity we put

$$dw=\sin.\theta'.d\theta'.d\varpi', \qquad [1447f]$$

it is evident that the limits of θ', ϖ', [1433″], will include all the values of w corresponding to the whole surface of the sphere whose radius is unity. Therefore, if we use the sign Σ_0^∞ of finite integrals to include all integral values of i, from $i=0$ to $i=\infty$, the [1447f′]
equation [1436, 1443], may be put under the form [1447k], corresponding to an external point. The equation [1444, 1447] may be put under the form [1447l], corresponding to an internal point. If we wish it to be approximative, we must use it only in those points where $R>r$; it is therefore particularly adapted to the case of a hollow spheroid, in which the attracted point is in the internal void space, the density ρ being nothing [1447g]
in that space. The combination of these two formulas gives [1447m], corresponding to the attraction of a solid spheroid, upon a point placed below its surface; the first term includes the part of the spheroid below the attracted point, and the second term the parts above the [1447h]
attracted point. Lastly, the formulas [1447n, o] represent the action of a spheroid, upon a point, which may be either internal or external, but placed so near the surface that the value of r exceeds some of the values of u, and is less than other values of u; so that it becomes [1447i]
necessary to separate the values of θ', ϖ', corresponding to these two cases; and if for the sake of illustration, we refer to the above figure, the first term of [1447n] will correspond to the part of the spheroid $BGPC$, where $u<r$; the second term to the part of the sphere $PP''DC$, and the third to the external part $PP''DEH$.

[1447‴] equation in V, which takes place at the surface of a spheroid of this form,

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Attracted point, External.

[1447k] $$V=\Sigma_0^{\infty}\cdot\left\{\frac{1}{r^{i+1}}\cdot\int\left(\int_0^u\rho\,.\,R^{i+2}dR\right).\,Q^{(i)}.\,dw\right\}$$

Internal void space.

[1447l] $$V=\Sigma_0^{\infty}\cdot\left\{r^i\cdot\int\left(\int_0^u\frac{\rho\,.\,dR}{R^{i-1}}\right).\,Q^{(i)}.\,dw\right\}$$

Internal solid mass.

[1447m] $$V=\Sigma_0^{\infty}\cdot\left\{\frac{1}{r^{i+1}}\cdot\int\left(\int_0^r\rho\,.\,R^{i+2}dR\right).Q^{(i)}.dw\right\}+\Sigma_0^{\infty}\cdot\left\{r^i.\int\left(\int_r^u\frac{\rho.dR}{R^{i-1}}\right).\,Q^{(i)}.\,dw\right\}$$

Point near the surface.

[1447n] $$V=\Sigma_0^{\infty}\cdot\left\{\frac{1}{r^{i+1}}.\int'\left(\int_0^u\rho.R^{i+2}dR\right).Q^{(i)}.dw\right\}+\Sigma_0^{\infty}\cdot\left\{\frac{1}{r^{i+1}}.\int_{,}\left(\int_0^r\rho\,.\,R^{i+2}dR\right).Q^{(i)}.dw\right\}$$
$$+\Sigma_0^{\infty}\cdot\left\{r^i\cdot\int_{,}\left(\int_r^u\frac{\rho.dR}{R^{i-1}}\right).\,Q^{(i)}.\,dw\right\}$$

[1447o] $$=\Sigma_0^{\infty}\cdot\left\{\frac{1}{r^{i+1}}.\int\left(\int_0^r\rho.R^{i+2}dR\right).Q^{(i)}.dw\right\}+\Sigma_0^{\infty}\cdot\left\{\frac{1}{r^{i+1}}.\int'\left(\int_r^u\rho.R^{i+2}dR\right).Q^{(i)}.dw\right\}$$
$$+\Sigma_0^{\infty}\cdot\left\{r^i\cdot\int_{,}\left(\int_r^u\frac{\rho.dR}{R^{i-1}}\right).\,Q^{(i)}\,dw\right\}.$$

[1447p] The expression [1447o] is easily deduced from [1447n], by changing, in the second term, the characteristic $\int_{,}$ into its equivalent $\int-\int'$, $\int$ indicating the integral for all values of R. For by this means that second term will produce the two following,

[1447q] $$\Sigma_0^{\infty}.\left\{\frac{1}{r^{i+1}}.\int\left(\int_0^r\rho.R^{i+2}dR\right).Q^{(i)}.dw\right\}-\Sigma_0^{\infty}.\left\{\frac{1}{r^{i+1}}.\int'\left(\int_0^r\rho.R^{i+2}dR\right).Q^{(i)}.dw\right\};$$

of which the first term is the same as the first of [1447o], and the second can be connected

[1447r] with the first of [1447n], observing that $\int_0^u-\int_0^r=\int_r^u$, by a mere change of the limits; and the result will be the second term of [1447o]. The third terms of [1447n, o] are the same.

The limits of the integrals $\int'$, $\int_{,}$, relative to θ', ϖ', depend implicitly on the situation of the

[1447s] attracted point, or on the co-ordinates r, θ, ϖ. This circumstance ought to be noticed when we take the differential of V relative to these variable quantities; but upon examination, it will be found that this has no effect on the partial differentials of the first order; and it will

[1447t] not therefore affect the values of the forces, resolved in the direction of the radius R, and perpendicular to it in the direction of the meridian and parallel of latitude, which are computed in [1811l], and represented by R', R'', R'''. This will be evident from the consideration that the differentials of V [1447o], arising from the limits of $\int'$, and $\int_{,}$, are merely the elements of these integrals corresponding to the same limits, or to the points where

[1447u] $u-r=0$; which also makes these elements vanish, because the integrals relative to r, contained under the signs $\int'$ and $\int_{,}$ vanish when $u=r$. Therefore in taking the first

and is remarkable, on account of its furnishing a method of computing these functions without any integration. [1447‴]

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differential of V it will not be necessary to notice these limits, but it must be done in taking the differentials of the second and higher orders. [1447v]

The general term of the values of V [1447k, l, m], relative to the angles θ, ψ, are of the same nature as the function $Q^{(i)}$; that is, they must satisfy the equations [1437, 1445], which are similar to [1442]. This follows from [1437b, 1444h], observing that the limits of the integrals in these values of V, are independent of the two angles θ, ψ; but this is not the case with the other two values [1447n, o]. [1447w]

We have seen that *when the attracted point does not make a part of the spheroid*, the corresponding value of V [1447k, l] will satisfy the equation [465], which may be put under the following form, as is evident by developing the first term, [1447x]

External attracted point.

$$\frac{1}{\sin.\theta}\cdot\left\{\frac{d\cdot\left\{\sin.\theta\cdot\left(\frac{dV}{d\theta}\right)\right\}}{d\theta}\right\}+\frac{\left(\frac{ddV}{d\varpi^2}\right)}{\sin.^2\theta}+r\cdot\left(\frac{dd.rV}{dr^2}\right)=0. \qquad [1447y]$$

This corresponds to the values of V [1447k, l].

When the attracted point makes a part of the spheroid, we may reduce the value of V to the form [1447m]. Now the two parts of which this value of V is composed, being of the same forms as those in [1447k, l], would like them satisfy this equation [1447y], and make its second member become nothing, if we were to neglect the terms arising from the variations of the limits of r, connected with the signs of integration $\int_0^r$, $\int_r^u$. Therefore in finding the value of the function [1447y], corresponding to this value of V [1447m], we need only notice the terms depending on this change of the limits of r; so that we may neglect the consideration of the two first terms of [1447y], depending on $d\theta$, $d\varpi$, and restrict ourselves to the last term $r\cdot\left(\frac{d^2.rV}{dr^2}\right)$, or its equivalent developed value [465u], [1447z]

$$2r\cdot\left(\frac{dV}{dr}\right)+r^2\cdot\left(\frac{ddV}{dr^2}\right); \qquad [1447\alpha]$$

by this means we may obtain another demonstration of the correction of the formula [1430s], by Mr. Poisson. For this purpose we shall put for brevity,

$$r'=\int_0^r \rho\,.\,R^{i+2}\,dR\,; \qquad r''=\int_r^u \rho\,.\,R^{-i+1}\,dR\,; \qquad [1447\beta]$$

r' and r'' will be functions of r, and the expression of V [1447m] may be put under the form

$$V=\Sigma_0^\infty.\{\int(r^{-i-1}\,r'+r^i\,r'').Q^{(i)}.dw\}; \qquad [1447\gamma]$$

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[1447''''] We shall suppose generally, that gravity is proportional to the power n of the distance, that dM is a particle of the spheroid, and f its distance from

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the sign $\int$ referring to the differential dw, or to the differentials $d\theta'$, $d\varpi'$, [1447f], $Q^{(i)}$ being independent of r [1433d]. Therefore if we take the partial differential of this expression relative to r, we shall get,

[1447δ] $$\left(\frac{dV}{dr}\right)=\Sigma_0^{\infty}.\int\left\{-(i+1).r^{-i-2}r'+i.r^{i-1}r''+r^{-i-1}.\left(\frac{dr'}{dr}\right)+r^{i}.\left(\frac{dr''}{dr}\right)\right\}.Q^{(i)}.dw.$$

In finding $\left(\frac{dr'}{dr}\right)$, $\left(\frac{dr''}{dr}\right)$, from [1447β], it changes the limit r into $r+dr$, which

[1447ε] increases r' by the element $\rho.r^{i+2}dr$, and decreases r'' by the element $\rho.r^{-i+1}dr$, because at that limit $R=r$; hence we have,

[1447ζ] $$\left(\frac{dr'}{dr}\right)=\rho.r^{i+2}; \qquad \left(\frac{dr''}{dr}\right)=-\rho.r^{-i+1};$$

consequently

[1447η] $$r^{-i-1}.\left(\frac{dr'}{dr}\right)+r^{i}.\left(\frac{dr''}{dr}\right)=\rho.r-\rho.r=0;$$

therefore the terms depending on the variations of the limits of the integrals, vanish from the expression of $\left(\frac{dV}{dr}\right)$ [1447δ]; and we shall only have to compute the remaining

[1447θ] term, $r^2.\left(\frac{ddV}{dr^2}\right)$ [1447α], depending on the variations of the same limits.

Substituting [1447η] in [1447δ], we get for the complete value of $\left(\frac{dV}{dr}\right)$,

[1447ι] $$\left(\frac{dV}{dr}\right)=\Sigma_0^{\infty}.\{\int[-(i+1).r^{-i-2}r'+i.r^{i-1}r''].Q^{(i)}.dw\}.$$

As we only have to compute the part of $\left(\frac{ddV}{dr^2}\right)$ depending on the variation of the limits of r', r'', [1447z, β], we need only to notice the differentials of these two quantities in this last equation, and we shall have,

[1447κ] $$\left(\frac{ddV}{dr^2}\right)=\Sigma_0^{\infty}.\left\{\int\left[-(i+1).r^{-i-2}.\left(\frac{dr'}{dr}\right)+i.r^{i-1}.\left(\frac{dr''}{dr}\right)\right].Q^{(i)}.dw\right\}.$$

Substituting the values [1447ζ], we obtain,

[1447λ] $$\left(\frac{ddV}{dr^2}\right)=\Sigma_0^{\infty}.\{\int[-(i+1).\rho-i\rho].Q^{(i)}.dw\}=-\Sigma_0^{\infty}.\{\int(2i+1).\rho.Q^{(i)}.dw\}.$$

General form of V.

the attracted point. We shall also put

$$V = \int f^{n+1} \,.\, dM\,; \qquad [1448]$$

Hence the term $r^2 . \left(\frac{ddV}{dr^2}\right)$ [1447α, θ], which represents the value of the function [1447y], for the case under consideration, will become

$$-r^2 . \Sigma_0^{\infty} . \{(2i+1) . \int \rho \,.\, Q^{(i)} \,.\, dw\},$$

and we shall get,

$$\frac{1}{\sin.\theta} . \left\{ \frac{d . \left\{ \sin.\theta . \left(\frac{dV}{d\theta}\right) \right\}}{d\theta} \right\} + \frac{\left(\frac{ddV}{d\varpi^2}\right)}{\sin.^2\theta} + r . \left(\frac{dd . rV}{dr^2}\right)$$

$$= -r^2 . \Sigma_0^{\infty} . \{(2i+1) . \int \rho \,.\, Q^{(i)} \,.\, dw\}. \qquad [1447\mu]$$

In making farther reductions of the second member of this equation, we shall have to use the formulas [1476a, 1532a, 1533m], which will hereafter be demonstrated. This was the most convenient arrangement to avoid a great increase of the present note, and to conform as much as possible to the general plan of the work. This will cause no embarrassment to the learner, because the object of the present calculation is merely to prove that this method will produce, in [1447π], the same result as had been found by a different process in [1430s]; and it was important to show that both methods would agree.

The density ρ, being a function of θ', ϖ', may be developed in a series of the form [1532a],

$$\rho = Y'^{(0)} + Y'^{(1)} + \&c. = \Sigma\, Y'^{(i)}\,; \qquad [1447\nu]$$

and if we denote by $Y^{(i)}$, the value of $Y'^{(i)}$, when θ' is changed into θ, ϖ' into ϖ, ρ' into ρ; ρ' being the density of the body at the attracted part, the integral $\int \rho \,.\, Q^{(i)} \,.\, dw$ will be composed of terms of the form $\int Y'^{(i')} . Q^{(i)} . dw$, or $\int Y'^{(i')} . Q^{(i)} . d\theta' . \sin.\theta' . d\varpi'$; which vanish when i' differs from i [1476a]; and when $i' = i$, it becomes

$$\int Y'^{(i)} \,.\, Q^{(i)} \,.\, d\theta' \,.\, \sin.\theta' \,.\, d\varpi'. \qquad [1447\xi]$$

This, in [1533m], will be found equal to $\frac{4\pi}{2i+1} . Y^{(i)}$; therefore the second member of [1447μ] will become

$$-r^2 . \Sigma_0^{\infty} . \left\{(2i+1) . \frac{4\pi . Y^{(i)}}{2i+1}\right\} = -4\pi . r^2 . \Sigma_0^{\infty} . Y^{(i)} = -4\pi . r^2 . \rho'\,; \qquad [1447\nu];$$

and we shall finally get, when the attracted point forms a part of the mass of the spheroid,

Internal attracted point.

$$\frac{1}{\sin.\theta} . \left\{ \frac{d . \left\{ \sin.\theta . \left(\frac{dV}{d\theta}\right) \right\}}{d\theta} \right\} + \frac{\left(\frac{ddV}{d\varpi^2}\right)}{\sin.^2\theta} + r . \left(\frac{dd . rV}{dr^2}\right) = -4\pi . r^2 . \rho'\,; \qquad [1447\pi]$$

being the same as was found by another method in [1430s].

this integral being taken so as to include the whole mass of the spheroid. In the case of nature, in which $n=-2$, it becomes,*

[1449] $$V=\int\frac{dM}{f};$$

which we have also denoted by V in the preceding articles, [1428'''', &c., 1386]. The function V has the advantage of giving, by means of its
[1449'] differential, the attraction of the spheroid, parallel to any right line whatever. For if we consider f as a function of the three rectangular co-ordinates of the attracted point, one of which is parallel to this right line, and we put this co-ordinate equal to r, then the attraction of the spheroid in the
[1449''] direction of the line r, towards its origin, will be† $\int f^n.\left(\frac{df}{dr}\right).dM$; consequently it will be equal to $\frac{1}{n+1}.\left(\frac{dV}{dr}\right)$, which, in the case of
[1449'''] nature, becomes $-\left(\frac{dV}{dr}\right)$, conformably to what we have before found, [1388'].

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* (991) This tenth section treats of the attraction of a spheroid differing but little from a sphere; the case of the attraction of a spheroid which differs considerably from a sphere, is treated of [1507—1563].

The value of V [1449], deduced from [1448], by putting $n=-2$, is of the same form as that assumed in [1386].

† (992) Gravity being supposed proportional to the power n of the distance f; the attraction of the particle dM, in the direction f, will be $f^n.dM$; but

[1448a] $$f=\sqrt{\{(a-x)^2+(b-y)^2+(c-z)^2\}},$$

[1348a]; hence, by differentiation,

$$\left(\frac{df}{da}\right)=\frac{a-x}{\sqrt{\{(a-x)^2+(b-y)^2+(c-z)^2\}}}=\frac{a-x}{f}.$$

The quantity, here called f, is what is named r, in [1355b']; and is represented by the line Ac, fig. 1, page 5. Now it is evident, that if the preceding force $f^n.dM$, in the direction cA, be resolved, in the direction cb, parallel to the axis x, it will be

$$f^n.dM.\frac{cb}{Ac}=f^n.dM.\frac{a-x}{f}, \qquad [1355b];$$

We shall now suppose the spheroid to differ but very little from a sphere whose radius is a ; the centre of this sphere being on the line r, drawn [1449′′′′] *perpendicular to the surface of the spheroid ; the origin of this line being arbitrary, but very near to the centre of gravity of the spheroid. We shall also suppose that the sphere touches the spheroid, and that the attracted point is on the point of contact of the two surfaces.** The spheroid is equal to the sphere [1449v] increased by the *excess* of the mass of the spheroid above that of the sphere ; now we may suppose this *excess* to be composed of an infinite number of particles spread over the surface of the sphere, these particles being supposed [1449vi] to be negative, wherever the sphere exceeds, or falls without the spheroid. We may therefore obtain the value of V, by determining, *first*, its value relative to a sphere, *secondly*, relative to the particles just mentioned. [1449vii]

With respect to the sphere, V is a function of a, which we shall denote by [1449viii] A. If we put dm for one of the particles of the difference between the

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and by substituting the preceding value of $\frac{a-x}{f}$, it becomes $f^n \cdot \left(\frac{df}{da}\right) \cdot dM$; therefore the attraction of the whole spheroid on that point, in a direction parallel to the axis of x, will be represented by $\int f^n \cdot \left(\frac{df}{da}\right) \cdot dM$.

The particle $dM = dx \cdot dy \cdot dz$, [1347], is independent of a ; therefore the differential of V, [1448], taken relative to a, will be $\left(\frac{dV}{da}\right) = (n+1) \cdot \int f^n \cdot \left(\frac{df}{da}\right) \cdot dM.$ Hence $\frac{1}{n+1} \cdot \left(\frac{dV}{da}\right) = \int f^n \cdot \left(\frac{df}{da}\right) \cdot dM$; and as the second member is equal to the preceding expression of the attraction of the spheroid, in the direction parallel to the axis of x, that attraction must be $\frac{1}{n+1} \cdot \left(\frac{dV}{da}\right)$. The axes x, y, z, are arbitrary, and we may take the axis of x so as to be parallel to the proposed line r, and then it will become $\frac{1}{n+1} \cdot \left(\frac{dV}{dr}\right)$, as in [1449′′′]. In the case of nature, where $n = -2$, this becomes simply $-\left(\frac{dV}{dr}\right)$, as above, and in [1387, 1388].

* (992a) In figure 11, let $cHGD$ be a section of the proposed sphere, whose centre is E ; $chgd$ the corresponding section of the spheroid, whose centre of gravity is e ; this centre is very near to E, but may be above or below the plane of the present figure. Now to compute the value of V, representing the sum of the particles of the spheroid, divided each by its distance from the point c, we may first compute the value of V,

spheroid and sphere, and f for its distance from the attracted point, the value
[1449ix] of V, relative to this excess, will be $\int f^{n+1}\,.\,dm$. Hence we shall have, for the whole value of V, corresponding to the spheroid,

[1450] $$V = A + \int f^{n+1}\,.\,dm.$$

Supposing now the attracted point to be elevated an infinitely small quantity dr above the surface of the spheroid and the sphere, upon the
[1450′] continuation of the line r or a; the value of V, corresponding to this last
[1451] position of the attracted point, will become,* $V + \left(\frac{dV}{dr}\right).dr$; A will
[1452] increase by a quantity proportional to dr, which we shall represent by $A'.dr$.

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corresponding to the whole sphere $cHGD$; to which we must apply a correction, arising from the difference of the masses of the sphere and spheroid; and it is evident, that this correction must be added, in the parts of the spheroid, which fall without the sphere, at g; but subtracted, in the points near h, d, where it falls within the sphere. To compute this last part, we may draw about c, as a centre, with the radius $ch = f$, an arc hH, meeting the sphere in H, and the spheroid in h. Then, in computing the part of the value of V, corresponding to the mass situated upon the arc hH, we may suppose the whole of that mass to be concentrated at H; because the value of the mass, divided by its distance from the point c, is the same in both hypotheses. The part corresponding to hH, falls within the sphere, and is therefore subtractive; the corresponding mass, collected at H, may therefore be supposed *negative* at H; but it would be *positive*, in the points situated near G. Putting dm for the mass collected at any point H, whose distance from c is f, the part of V, arising from this mass, will be $f^{n+1}\,.\,dm$, [1448], whose integral, added to A, gives the whole value of V, as in [1450].

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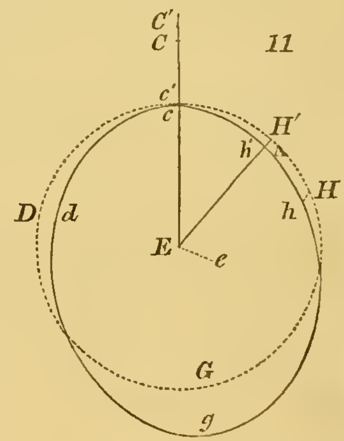


* (993) Instead of supposing the attracted point c to be situated on the *surface* of the
[1449a] sphere $cHGD$, fig. 11, 12, we shall suppose it to be placed at C, on the continuation of the line Ec, so that $EC = r$. Now V, [1436, 1437′], is a function of

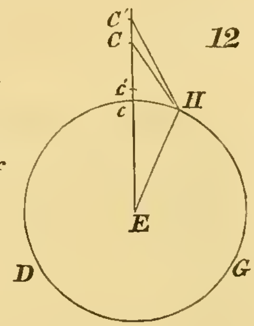
$$r, \qquad \mu, \qquad \sqrt{(1-\mu^2)}\,.\cos.\varpi, \qquad \sqrt{(1-\mu^2)}\,.\sin.\varpi,$$

[1449b] depending on the nature of the spheroid; and if we suppose r to be increased by $dr = CC'$, the attracted point will be at C'; moreover, as μ, ϖ, do not vary by this increase of r, the

Moreover, if we put γ for the angle formed by the two radii, drawn from the centre of the sphere to the attracted point, and to the particle dm; the distance f from this particle to the attracted point, will be, in the first position of this point, equal to $\sqrt{2a^2.(1-\cos.\gamma)}$; in the second position, it will be, [1452'] [1452'']

$$\sqrt{(a+dr)^2-2a.(a+dr).\cos.\gamma+a^2}; \qquad [1453]$$

or $f.\left(1+\frac{dr}{2a}\right)$; the integral $\int f^{n+1}.dm$, will by this means become, [1453']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value of V will evidently increase by the differential $\left(\frac{dV}{dr}\right).dr$; V being the value corresponding to the point C. Hence the first member of [1450] will become $V+\left(\frac{dV}{dr}\right).dr$, as in [1451]. Moreover, if A increase by $A'.dr$, as in [1452], and $HC=f$ change into $HC'=f'$, the equation [1450] will become [1449c]

$$V+\left(\frac{dV}{dr}\right).dr=A+A'.dr+\int f'^{n+1}.dm.$$

Subtracting from this the expression [1450], we shall obtain

$$\left(\frac{dV}{dr}\right).dr=A'.dr+\int(f'^{n+1}-f^{n+1}).dm; \qquad [1450a]$$

in which we must substitute the values of f', f. Now if the angle $HEC=\gamma$, $EC=r$, $EH=Ec=a$, we shall have, as in [1432i],

$$f^2=r^2-2ra.\cos.\gamma+a^2. \qquad [1450b]$$

When the point C falls in c, r will be equal to a, and this value of f^2 will become, as in [1452''], $f^2=2a^2.(1-\cos.\gamma)$; and then the point C' will fall in c', making $cc'=dr$. [1450c] We shall have the value of f'^2, corresponding to the point c', by writing $a+dr$ for r, in [1450b], by which means it will become, as in [1453],

$$\begin{aligned} f'^2 &= (a+dr)^2-2a.(a+dr).\cos.\gamma+a^2 \\ &= 2a^2.(1-\cos.\gamma)+2a.dr.(1-\cos.\gamma)+dr^2 \\ &= f^2+f^2.\frac{dr}{a}+dr^2. \end{aligned} \qquad [1450d]$$

If we neglect terms of the order dr^2, and extract the square root, we shall get, as in [1453'], $f'=f.\left(1+\frac{dr}{2a}\right)$. [1453a]

[1454] $$\left\{1+\left(\frac{n+1}{2}\right).\frac{dr}{a}\right\}.\int f^{n+1}.dm;\ *$$

therefore we shall have

[1455] $$\left(\frac{dV}{dr}\right).dr=A'.dr+\left(\frac{n+1}{2}\right).\frac{dr}{a}.\int f^{n+1}.dm.$$

[1455′] Substituting for $\int f^{n+1}.dm$, its value $V-A$, we shall get, [*supposing, as will be shown in* [1459*s*], *n to be any positive number from* 0 *to* ∞, *or negative from* 0 *to* -2, *inclusive*],

Limits of n in this general formula of La Place.

[1456] $$\left(\frac{dV}{dr}\right)=A'-\frac{(n+1).A}{2a}+\frac{n+1}{2a}.V. \qquad (1)$$

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* (993*a*) The value of f', [1453*a*], gives $f'^{n+1}=f^{n+1}.\left\{1+\frac{(n+1).dr}{2a}\right\}$, neglecting the square and the higher powers of dr. Substituting this in [1450*a*], we get,

[1453*b*] $\left(\frac{dV}{dr}\right).dr=A'.dr+\int\frac{(n+1)}{2}.\frac{dr}{a}.f^{n+1}.dm$; and as the sign of integration does not

[1453*c*] affect $\frac{n+1}{2}.\frac{dr}{a}$, it becomes $\left(\frac{dV}{dr}\right).dr=A'.dr+\frac{(n+1)}{2}.\frac{dr}{a}.\int f^{n+1}dm$, as in [1455]. Substituting in this the expression $\int f^{n+1}.dm=V-A$, [1450], we get $\left(\frac{dV}{dr}\right).dr=A'.dr+\frac{(n+1)}{2}.\frac{dr}{a}.(V-A)$; dividing this by dr, we obtain [1456], which is free from the integral $\int f^{n+1}.dm$. The method of eliminating this integral appears at first to be unexceptionable; and it was so considered by mathematicians, from the time it was first made known, in the Memoirs of the Academy of Arts and Sciences of Paris, for 1782, till about the year 1809, when Mr. La Grange published some strictures upon it, in *Cahier* XV *du Journal de l'École Polytechnique*; and the same was done in the

[1453*d*] Philosophical Transactions of the Royal Society of London for 1812, by Mr. Ivory. But it will be shown hereafter, [1459*a*—*t*], that the formula [1456] must be restricted to the values which I have mentioned in [1455′]; these restrictions are not in the original work, though it was very apparent that for certain negative values of n, the element $f^{n+1}.dm$ would be infinite, and then this method of elimination might be defective and erroneous. In the case

Correction of La Place's formula, by La Grange and Ivory.

[1453*e*] of nature, where $n=-2$, the solution is correct, neglecting terms of the order of the square of the excentricity of the spheroid, and this value of n is on the limit between the true and erronnous solutions, as will be seen hereafter, [1459*s*].

In the case of nature, the equation [1456] becomes,*

$$-a\cdot\left(\frac{dV}{dr}\right)=-aA'-\tfrac{1}{2}A+\tfrac{1}{2}V. \qquad [1457]$$

The value of V relative to a sphere whose radius is a, is by § 6, equal to $\frac{4\pi\cdot a^3}{3r}$, which gives $A=\frac{4}{3}\pi\cdot a^2$, $A'=-\frac{4}{3}\pi\cdot a$;† therefore we shall have, [1457′] [1457″]

La Place's formula for spheroids differing but little from a sphere, in the law of nature.

$$-a\cdot\left(\frac{dV}{dr}\right)=\tfrac{2}{3}\pi\cdot a^2+\tfrac{1}{2}V. \qquad (2) \qquad [1458]$$

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* (994) In the case of nature, $n=-2$. Substituting this in [1456], and multiplying by $-a$, we get [1457].

† (995) If the spheroid dhg, fig. 11, page 86, coincided with the sphere DHG, the particles of matter dm by which the spheroid differs from the sphere, would become nothing, consequently $\int f^{n+1}\cdot dm=0$, and from [1450] we should get $A=V$, therefore by [1416] $A=\frac{M}{\sqrt{(a^2+b^2+c^2)}}=\frac{M}{r}$, [1416$a$, 1430′]; but $M=\frac{4}{3}\pi\cdot a^3$, [1363$e$], [1457$a$] the density ρ being equal to unity; hence for a sphere, fig. 12, p. 87, whose attracted point is C, radius $Ec=a$, and $EC=r$, we have $A=\frac{4\pi\cdot a^3}{3r}$. If in this we change r into [1457a'] $r+dr$, it will be increased by the quantity $A'\cdot dr$, [1452]; hence

$$A+A'\cdot dr=\frac{4\pi\cdot a^3}{3\cdot(r+dr)}=\frac{4\pi\cdot a^3}{3r}\cdot\left(1-\frac{dr}{r}\right),$$

neglecting dr^2. Subtracting the preceding value of A, we get $A'\cdot dr=-\frac{4\pi\cdot a^3}{3r}\cdot\frac{dr}{r}$; [1457$b$] hence $A'=-\frac{4\pi\cdot a^3}{3r^2}$, being the same as the partial differential $\left(\frac{dA}{dr}\right)$, which would have given A' immediately, without going through the preceding calculations, by the method of the author.

When the point C falls at c, infinitely near to the surface of the sphere, the distance r becomes equal to a, and then the preceding values of A, A', become

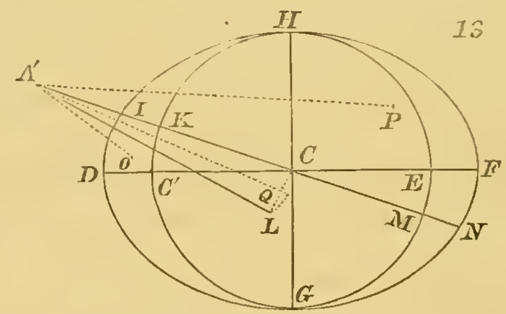
$$A=\frac{4\pi\cdot a^3}{3a}=\frac{4\pi\cdot a^2}{3}, \qquad \text{and} \qquad A'=\frac{-4\pi\cdot a^3}{3a^2}=\frac{-4\pi\cdot a}{3}, \qquad [1457c]$$

as in [1457″]. Substituting these in [1457], we get, $-a\cdot\left(\frac{dV}{dr}\right)=\frac{2\pi\cdot a^2}{3}+\frac{1}{2}V$, as in [1458]. We shall show, in [1459v], that this equation is correct, in the hypothesis here

We may here observe, that this equation exists, whatever be the situation of
[1458′] the right line r, even in the case where it is not perpendicular to the surface of the spheroid, provided it pass very near its centre of gravity. For it is evident, that the attraction of the spheroid, resolved in the direction of the
[1458″] right line r, and expressed by $-\left(\frac{dV}{dr}\right)$, as we have seen, [1449‴], is always the same, whatever be the position of that line, neglecting quantities
[1458‴] of the order of the square of the excentricity of the spheroid.*

[1457d] used, notwithstanding the defects mentioned in [1453d, e]; and on account of the importance of this equation, we shall also give another demonstration of it, by Mr. Poisson, in [1561y], by a different method from that used by the author, either in this work, or in the Mémoires de l'Académie Royale des Sciences, Paris, 1782, where this theorem was first published.

* (996) To illustrate this, we shall suppose the body to be a homogeneous spheroid of revolution $HDGF$, the axis of revolution being HG, and the inscribed sphere
[1458a] $HKC'GE$. Then this sphere attracts the point placed at A' in the direction of the radius $A'C$, with a force proportional to the mass of the sphere. The parts of the the spheroidal excess corresponding to



$IKGMNDI$, and $IKHEMNFHI$,

are to the mass of the sphere, of the same order as the excentricity, or excess, $C'D$, is to the radius of the sphere; so that these
[1458b] parts may be supposed to attract the point A' with forces of the same order as their masses, in directions which we shall suppose to be $A'O$, $A'P$, respectively; but the former mass, being nearer to the point A', will attract it more powerfully than the latter; so that if we resolve these two forces into two others, in the directions $A'C$ and CL, perpendicular to each other, the whole force of the sphere, and the forces in the directions $A'P$, $A'O$, may be reduced to a force $A'C$, and a force CL perpendicular to $A'C$, and falling on the same side of $A'C$ as the point O; the resultant of all these forces will be the single force F of the spheroid in the direction $A'L$; the quantity CL being to $A'C$, of
[1458c] the same order as $C'D$ is to the radius of the spheroid. This force, resolved by the usual method, in any other direction, as $A'Q$, inclined to $A'L$, by the small angle $LA'Q=2z$, will be $F.\cos.2z$, which may be put under the form $F-2F.\sin.^2 z$, [1] Int.; therefore the force F, in the direction $A'L$, differs from that in the direction
[1458d] $A'Q$, by the quantity $2F.\sin.^2 z$; and as $\sin. z$ is of the same order as $\frac{LQ}{LA'}$, the
[1458e] term $2\sin.^2 z$ must be of the same order as the square of the excentricity of the spheroid. If we neglect quantities of this order, the force, in the direction $A'Q$, will be

11. We shall now resume the general expression of V, [1436], corresponding to an attracted point without the spheroid,

$$V=\frac{U^{(0)}}{r}+\frac{U^{(1)}}{r^2}+\frac{U^{(2)}}{r^3}+\&c.\,; \qquad [1459]$$

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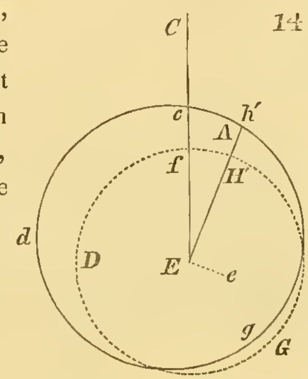
the same as that in the direction $A'L$, each of them being equal to the force in the direction $A'C$, neglecting quantities of the order just mentioned, as in [1458″]. [1458f]

(997) We shall here investigate the limits of the values of n, mentioned in [1455′], between which the formula [1456] is correct. To avoid the embarrassment arising from the attracted point being at the surface of the spheroid at c, fig. 14, we shall suppose, in the first place, as in [1449a], that it is elevated above the surface of the spheroid at C, and that the attracting particle dm, included between the spheroid and sphere, is situated at the point A, on the right line $EH'Ah'$. To include the most general case, we shall suppose the sphere $DH'G$ not to meet the spheroid at the point c, but to cut the line EC in another point f. Then if we put $Ec=\rho$, $Eh'=\rho'$, $EA=\rho''$, $EC=r$, $Ef=EH'=a$, angle $CEH'=\gamma$, $CA=f$, we shall have, as in [1450b], by changing a into ρ'',



$$f^2=r^2-2r\rho''.\cos.\gamma+\rho''^2\,; \qquad [1459a]$$

also, as in [1450],

$$V=A+\int f^{n+1}.dm, \qquad [1459b]$$

V being the whole value of V, corresponding to the spheroid $ch'gd$, according to the definition in [1448]; A its value for the sphere $H'GDf$, and $\int f^{n+1}.dm$, its value for the mass m, which represents the difference between the mass M of the sphere, and the mass M' of the spheroid, [1449viiii]. It is evident, that V, A, and $\int f^{n+1}.dm$, are functions of r; so that if we put, as in [1457b], $\left(\frac{dA}{dr}\right)=A'$, and take the partial differential of [1459b], relative to r, we shall get, [1459c]

$$\left(\frac{dV}{dr}\right)=A'+(n+1).\int f^n.\left(\frac{df}{dr}\right).dm. \qquad [1459d]$$

Multiplying [1459b] by $-\frac{(n+1)}{2r}$, and adding it to [1459d], we get, by observing that the sign $\int$ does not affect r,

$$\left(\frac{dV}{dr}\right)=A'-\frac{(n+1)}{2r}.A+\frac{(n+1)}{2r}.V+(n+1).\left\{\int f^n.\left(\frac{df}{dr}\right).dm-\frac{1}{2r}.\int f^{n+1}.dm\right\}$$

$$=A'-\frac{(n+1)}{2r}.A+\frac{(n+1)}{2r}.V+\frac{(n+1)}{2}.\int f^{n-1}.dm.\left\{2f.\left(\frac{df}{dr}\right)-\frac{f^2}{r}\right\}. \qquad [1459e]$$

the function $U^{(i)}$ being subjected to the following equation of partial differentials, whatever be the value of i [1437],

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Now the differential of [1459a], relative to r, gives

[1459f] $$2f.\left(\frac{df}{dr}\right)=2r-2\rho''.\cos.\gamma=\frac{2r^2-2r.\rho''.\cos.\gamma}{r}=\frac{f^2+r^2-\rho''^2}{r};$$

hence the preceding expression becomes,

[1459g] $$\left(\frac{dV}{dr}\right)=A'-\frac{(n+1)}{2r}.A+\frac{(n+1)}{2r}.V+\frac{(n+1)}{2r}.\int f^{n-1}.dm.(r^2-\rho''^2),$$

in which nothing is neglected. If for brevity we put

[1459h] $$W=\int f^{n-1}.dm.(r^2-\rho''^2),$$

this will become

[1459i] $$\left(\frac{dV}{dr}\right)=A'-\frac{(n+1)}{2r}.A+\frac{(n+1)}{2r}.V+\frac{(n+1)}{2r}.W.$$

This formula, divided by $n+1$, represents the attraction of the whole spheroid, upon a point C, situated without its surface, and resolved in the direction of the radius r, no quantities being neglected, [1449'''], except those of a less order than the terms of W. If we suppose, with the author, the attracted point C to be at the surface of the spheroid at c, and the point f to fall in c; so that the surface of the inscribed sphere may pass through
[1459k] the point c; we shall have $r=\rho=a$, and the expression [1459i] will become

[1459l] $$\left(\frac{dV}{dr}\right)=A'-\frac{(n+1)}{2a}.A+\frac{(n+1)}{2a}.V+\frac{(n+1)}{2a}.W;$$

which will not agree with the fundamental equation [1456], except we have W equal to nothing, or of the order of the neglected terms α^2; α being used, as in [1461'], to denote
[1459l'] terms of the same order as the excentricity of the spheroid. Now in [1449^{vi}], the particles dm are supposed by the author to be placed on the surface of the sphere whose radius is a, so that $\rho''=a$. Substituting this and $r=a$ [1459k], in W [1459h], we get,

$$W=\int f^{n-1}.dm.(a^2-a^2)=(a^2-a^2).\int f^{n-1}.dm;$$

and this is neglected by the author, because the factor $a^2-a^2=0$; without adverting to the circumstance, that $\int f^{n-1}.dm$, has, in some cases, a divisor of a much less order than this factor, which prevents W from vanishing. This defect was discovered, about the same time, both by Mr. La Grange and Mr. Ivory.

The quantity m [1459c], representing the difference between the mass of the spheroid and
[1459m] that of the sphere, is of the order α [1459l']. Moreover, when $r=\rho$, the factor $r^2-\rho''^2$, which occurs in W, becomes $\rho^2-\rho''^2=(\rho+\rho'').(\rho-\rho'')$, or $2\rho.(\rho-\rho'')$ nearly; and as $\rho-\rho''$ is of the order α, the elements of the integral

[1159m'] $$W=\int f^{n-1}.dm.(\rho^2-\rho''^2)$$

Differential equation in $U^{(i)}$.

$$0=\left\{\frac{d\cdot\left\{(1-\mu^2)\cdot\left(\frac{dU^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddU^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i\cdot(i+1)\cdot U^{(i)}. \qquad [1460]$$

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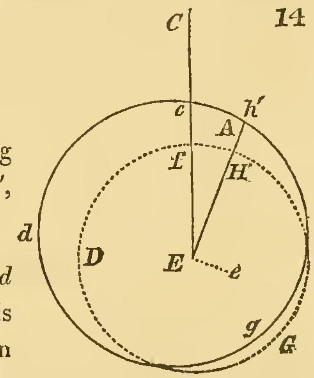
must be of the order α^2, when f has a finite value; and these elements can become sensible only when the exponent of f is negative, and f excessively small or nothing. Therefore in the investigation of the value of W, we may limit the integration from $f=0$, to $f=f'$, [1459n]
f' being a very small quantity, which for the purpose of illustration we may suppose to be of the order α, and the question will be to determine, with what values of n, the expression
[1459h]

$$W=\int_0^{f'} f^{n-1}\cdot dm\cdot(\rho^2-\rho''^2), \qquad [1459o]$$

becomes of a finite magnitude, or of the order α, always neglecting quantities of the order α^2, as the author has done in [1458''', 1461', 1614^{v}, &c.] [1459o']



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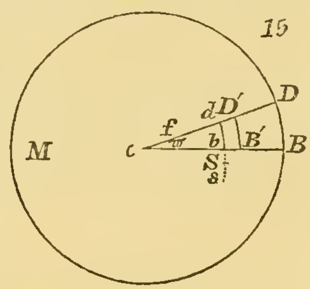
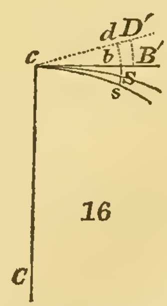
The sphere is supposed to touch the spheroid at the attracted point c [1449^{v}], and we shall also suppose, that through this point a plane is drawn, *tangent* to the sphere and spheroid. Upon this plane, we shall describe about the centre c, with the radius f', a circle MBD. This is drawn separately for distinctness in fig. 15, and is divided into elementary triangles like BcD, so that if we put the angle $BcD=d\varpi$, $cb=cd=f$, the arc $bd=fd\varpi$, $bB'=dD'=df$, the elementary area $bB'D'd$, included by the circular arcs bd, $B'D'$, will be $fd\varpi.df$. [1459p]
We shall now suppose a line bSs to be drawn through the point b perpendicular to this tangent plane, cutting the sphere in s, and the spheroid in S, as in fig. 16, which is drawn separately for distinctness. C is the centre of the sphere, and also of the circular arc cs, whose tangent is cb. Then we shall have the versed sine



15

$bs=\frac{cb^2}{2Cc}=\frac{f^2}{2a}$, nearly. Multiplying this by the area of the elementary base $bB'D'd=fd\varpi.df$, we shall obtain $\frac{f^3df.d\varpi}{2a}$, for the magnitude of the rectangular prism formed upon that base, and included between the sphere and the tangent plane. In like manner, if we take a' for the radius of curvature of the spheroidal arc cS, which radius we shall suppose to differ from a only by quantities of the order $a\alpha$, we shall obtain the magnitude of the rectangular prism, limited between the same base $bB'D'd$, and the surface of the spheroid passing through the point S, by changing a into a', in the [1459q] [1459q']



16

We shall have, by taking the differential of V [1459], relative to r,

[1461] $$-\left(\frac{dV}{dr}\right)=\frac{U^{(0)}}{r^2}+\frac{2\,U^{(1)}}{r^3}+\frac{3\,U^{(2)}}{r^4}+\&c. \;*$$

We shall represent the radius drawn from the origin of r, to the surface of

Coefficient α.

the spheroid, by $a.(1+\alpha y)$; α *being a very small constant coefficient,*

[1461′] *whose square and higher powers may be neglected,* and y a function of μ and ϖ, depending on the figure of the spheroid. We shall have, by neglecting

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preceding expression, and it will therefore become $\frac{f^3 df.d\varpi}{2a'}$. The difference of these two expressions $\frac{f^3 df.d\varpi}{2a'}-\frac{f^3 df.d\varpi}{2a}=\frac{(a-a')}{2aa'}.f^3 df.d\varpi$, represents the value of dm to be substituted in [1459o], which, by this means, becomes

[1459r] $$W=\int_0^{f'}\frac{(a-a')}{2aa'}.f^{n+2}.df.d\varpi.(\rho^2-\rho''^2).$$

Now it is evident that while n has any positive value from 0 to ∞, or any negative value between 0 and -2, the quantity f^{n+2} will vanish when f becomes nothing; and in no case, within the assigned limits of f, will it become infinite. In the case of nature, where

[1459s] $n=-2$, we shall have $f^{n+2}=f^0=1$; but as every element of the expression [1459r] is multiplied by the factors $a-a'$, $\rho^2-\rho''^2$, each of which is of the order α,

[1459t] [1459m, q′], the whole will then be of the order α^2, and may therefore be neglected. This result would not generally take place, if the negative values of n exceeded -2, for if we put $n=-2-n'$, n' being positive, the expression $f^{n+2}=f^{-n'}=\frac{1}{f^{n'}}$, becomes infinite when $f=0$.

Hence it appears that the formula [1456] was not sufficiently restricted by the author, and

[1459u] that all negative values of n *between* -2 and $-\infty$ must in general be excluded. In the

[1459v] case of nature we have $n=-2$, and as the method is correct for this value of n, the formulas [1457, 1458] must be accurate, neglecting quantities of the order α^2 [1459o′, t];

[1459w] always supposing, as in [1459q′], that the difference $a-a'$, between the radii of curvature of the spheroid, and tangent sphere, at the attracted point, is of the order $a\alpha$. With these

[1459x] limitations, the method of the author appears to be satisfactory; and the objections made to it, apply only to the cases we have here excepted.

* (997a) As $U^{(i)}$ does not contain r [1433k], the differential of [1459] relative to r, divided by $-dr$, will give [1461].

quantities of the order α, $V = \frac{4\pi . a^3}{3r}$.* Hence it follows, that in the [1461″]
preceding expression of V, we shall have, *first*, the quantity $U^{(0)}$ equal
to $\frac{4}{3}\pi . a^3$, increased by a very small quantity of the order α, which we [1461‴]
shall denote by $U'^{(0)}$; *second*, the quantities $U^{(1)}$, $U^{(2)}$, &c., are very
small quantities of the order α. Substituting $a . (1 + \alpha y)$ for r, in the
preceding expressions of V, and $-\left(\frac{dV}{dr}\right)$, and neglecting quantities of [1461⁗]
the order α^2; we shall have, for any attracted point situated upon the surface,

$$\tfrac{1}{2} V = \tfrac{2}{3}\pi . a^2 . (1 - \alpha y) + \frac{U'^{(0)}}{2a} + \frac{U^{(1)}}{2a^2} + \frac{U^{(2)}}{2a^3} + \&c.$$

[1462]

$$-a . \left(\frac{dV}{dr}\right) = \tfrac{4}{3}\pi . a^2 . (1 - 2\alpha y) + \frac{U'^{(0)}}{a} + \frac{2U^{(1)}}{a^2} + \frac{3U^{(2)}}{a^3} + \&c.$$

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* (998) As the spheroid differs from the sphere, only by quantities of the order α, the value of V [1428⁗] will differ from that of the sphere $\frac{4\pi . a^3}{3r}$ [1457a, d] by a quantity of the order α; so that if we subtract $\frac{4\pi . a^3}{3r}$ from both members of the equation [1459], the first member $V - \frac{4\pi . a^3}{3r}$ will be of the order α; therefore the second member must be of the same order; and if we, for brevity, put $U^{(0)} - \frac{4}{3}\pi . a^3 = U'^{(0)}$, the quantity [1461$a$]
$U'^{(0)}$, as well as $U^{(1)}$, $U^{(2)}$, &c., will also be of the order α, and we shall get,

$$V - \frac{4\pi . a^3}{3r} = \frac{U'^{(0)}}{r} + \frac{U^{(1)}}{r^2} + \frac{U^{(2)}}{r^3} + \&c., \quad \text{or} \quad V = \frac{4\pi . a^3}{3r} + \frac{U'^{(0)}}{r} + \frac{U^{(1)}}{r^2} + \frac{U^{(2)}}{r^3} + \&c.; \qquad [1461b]$$

whose differential, relative to r, will give

$$-\left(\frac{dV}{dr}\right) = \frac{4\pi . a^3}{3r^2} + \frac{U'^{(0)}}{r^2} + \frac{2U^{(1)}}{r^3} + \frac{3U^{(2)}}{r^4} + \&c. \qquad [1461c]$$

Putting $r = a . (1 + \alpha y)$, and neglecting α^2, we get

$$\frac{4\pi . a^3}{3r} = \frac{4\pi . a^2}{3} . (1 - \alpha y); \qquad \frac{4\pi . a^3}{3r^2} = \frac{4\pi . a}{3} . (1 - 2\alpha y). \qquad [1461d]$$

Substituting these in the preceding values of V, $-\left(\frac{dV}{dr}\right)$, we shall obtain the expressions [1462], observing that in all the terms depending on $U'^{(0)}$, $U^{(1)}$, $U^{(2)}$, &c., which are of the order α, we may put $r = a$.

If we substitute these values in the equation [1458] of the preceding article, we shall obtain,*

[1463] $$4\,\alpha\pi\,.\,a^2\,y=\frac{U'^{(0)}}{a}+\frac{3U^{(1)}}{a^2}+\frac{5U^{(2)}}{a^3}+\frac{7U^{(3)}}{a^4}+\&c.$$

Important develop-ment of the function y.

Hence it follows, that the function y is of this form,†

[1464] $$y=Y^{(0)}+Y^{(1)}+Y^{(2)}+Y^{(3)}+\&c.\,;$$

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* (999) Substituting the values [1462] in [1458], put under the form

$$0=-\tfrac{1}{2}\,V-a\,.\left(\frac{dV}{dr}\right)-\tfrac{2}{3}\,\pi\,.\,a^2,$$

and connecting the similar terms, we get,

[1463a] $$0=-2\,\alpha\pi\,.\,a^2y+\frac{U'^{(0)}}{2\,a}+\frac{3\,U^{(1)}}{2\,a^2}+\frac{5\,U^{(2)}}{2\,a^3}+\&c.$$

Multiplying this by 2, we get [1463].

† (1000) Dividing [1463] by $4\,\alpha\pi\,.\,a^2$, we shall get the value of y, in the form given in [1464]; the term depending on $U^{(i)}$ is evidently equal to $\frac{(2i+1)\,.\,U^{(i)}}{4\,\alpha\pi\,.\,a^{i+3}}$; putting this equal to $Y^{(i)}$ [1464], we shall get $U^{(i)}$ [1466]. Substituting this value of $U^{(i)}$ in [1437],

[1464a] the factor $\frac{4\,\alpha\pi}{2i+1}\,.\,a^{i+3}$ can be brought from under the signs of the differentials, and the whole expression, being divided by that factor, will become as in [1465]. This exists for all integral positive values of i, including also $i=0$. For $U^{(0)}$ [1433k] does not contain

[1464a′] P, P', P'', and is therefore independent of μ, ϖ; hence $Y^{(0)}$ [1466] must be independent of μ, ϖ, so that $\left(\frac{d\,Y^{(0)}}{d\mu}\right)=0$, $\left(\frac{d\,Y^{(0)}}{d\varpi}\right)=0$; and as the factor $i\,.\,(i+1)$ also vanishes when $i=0$, each term of [1465] will separately vanish when $i=0$.

If we put $\mu=\cos.\theta$ [1434′], and proceed with $Y^{(i)}$ as was done with V in [465w, &c.], we shall get,

$$\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}=\frac{\cos.\theta}{\sin.\theta}\,.\left(\frac{d\,Y^{(i)}}{d\theta}\right)+\left(\frac{dd\,Y^{(i)}}{d\theta^2}\right)$$

[1464b] $$=\frac{1}{\sin.\theta}\,.\left\{\frac{d\,.\left\{\sin.\theta\,.\left(\frac{d\,Y^{(i)}}{d\theta}\right)\right\}}{d\theta}\right\}.$$

the quantities $Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}$, $Y^{(3)}$, &c., as well as $U^{(0)}$, $U^{(1)}$, $U^{(2)}$, &c., [1464']
being subjected to the following equation of partial differentials,

Important differential equation in $Y^{(i)}$. First form.

$$0=\left\{\frac{d\,.\left\{(1-\mu\mu)\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}+\frac{\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,Y^{(i)}\,; \qquad [1465]$$

therefore this expression of y is not arbitrary, but is derived from the development of the attractions of spheroids in a series.* We shall see, in [1465']
the following article, [1479'], that y cannot be developed in this manner but in one form ; therefore we shall have, by comparing the similar functions, this general value of $U^{(i)}$ [1464a],

$U^{(i)}$.

$$U^{(i)}=\frac{4\,\alpha\,\pi}{2\,i+1}\,.\,a^{i+3}\,.\,Y^{(i)}. \qquad [1466]$$

Hence it follows, that whatever be r, we shall have,†

General value of V for a homogeneous spheroid attracting an external point.

$$V=\frac{4\,\pi\,.\,a^3}{3\,r}+\frac{4\,\alpha\,\pi\,.\,a^3}{r}\,.\left\{Y^{(0)}+\frac{a}{3\,r}\,.\,Y^{(1)}+\frac{a^2}{5r^2}\,.\,Y^{(2)}+\frac{a^3}{7r^3}\,.\,Y^{(3)}+\text{\&c.}\right\}; \qquad (3) \qquad [1467]$$

Substituting this in [1465], it becomes of the following form, which is frequently used,

Differential equations in $Y^{(i)}$. Second form.

$$0=\frac{1}{\sin.\,\theta}\,.\left\{\frac{d\,.\left\{\sin.\,\theta\,.\left(\frac{d\,Y^{(i)}}{d\,\theta}\right)\right\}}{d\,\theta}\right\}+\frac{1}{\sin.^2\theta}\,.\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)+i\,.\,(i+1)\,.\,Y^{(i)}. \qquad [1464c]$$

* (1001) Considerable discussion has arisen upon this development of the function y, between several of the first mathematicians of Europe ; some contending, that this form can be used only when y is actually *a rational and integral function of the quantities* μ, $\sqrt{(1-\mu^2)}\,.\,\cos.\,\varpi$, $\sqrt{(1-\mu^2)}\,.\,\sin.\,\varpi$; whilst others, with more satisfactory reasons, assert that it is not restricted to this particular class of functions, but embraces the general value of y, even when it contains fractional, or surd expressions, of the same quantities, as we shall hereafter in [1530l—1535k] more fully explain.

† (1002) Supposing V to be reduced to series, according to the powers of $\frac{1}{r}$, as in [1459] ; then if this series were given, we should know the values of $U^{(0)}$, $U^{(1)}$, $U^{(2)}$, &c., also $U'^{(0)}=U^{(0)}-\frac{4}{3}\pi a^3$, [1461$a$] ; all of which are independent of r [1433k]. Substituting these in [1466], we shall get successively $Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}$, &c., and then, as in [1464], $y=Y^{(0)}+Y^{(1)}+\text{\&c.}$ So that the value of V being given, we may thence deduce the value of y. On the contrary, if the value of y were given, in a series of the form

To obtain the value of V, it is now only necessary to reduce y to the form we have supposed; we shall hereafter [1530'''', 1533i] give a very simple method of doing it.

[1467'] If we have have $y = Y^{(i)}$, the part of V, corresponding to the excess of the spheroid above the sphere whose radius is a, or, in other words, the part corresponding to the spherical stratum whose radius is a, and thickness

[1467''] $\alpha \,.\, a y$, will be $\frac{4\alpha\pi \,.\, a^{i+3} \,.\, Y^{(i)}}{(2i+1) \,.\, r^{i+1}}$; therefore this value will be proportional to y; and it is evident that *this is the only case in which this proportionality exists.*

[1467'''] 12. We may simplify the expression of $y = Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$, and make the two first terms of the series vanish, by taking, for a, the radius of a sphere, whose mass is equal to that of the spheroid; and fixing

[1467''''] the arbitrary origin of r, at the centre of gravity of the spheroid. To prove this, we shall observe that the mass of the spheroid M, supposed to be homogeneous, and of a density equal to unity, is, by § 8, [1431d] equal to

[1467^{v}] $$\textstyle\int R^2 dR \,.\, d\mu \,.\, d\varpi, \qquad \text{or} \qquad \frac{1}{3}\int R'^3 \,.\, d\mu \,.\, d\varpi,$$

$Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$, we might, by a similar operation, deduce the value of $U^{(0)}$, $U^{(1)}$, &c., and thence the value of V [1459]. For greater simplicity, we may express this value of V, in terms of $Y^{(0)}$, $Y^{(1)}$, &c. For in [1461b] we have

$$V = \frac{4\pi \,.\, a^3}{3r} + \frac{U'^{(0)}}{r} + \frac{U^{(1)}}{r^2} + \&c.,$$

and from [1466], putting successively, $i = 0$, $i = 1$, &c., we have,

[1466a] $$U'^{(0)} = \frac{4\alpha\pi}{1} \,.\, a^3 \,.\, Y^{(0)}, \qquad U^{(1)} = \frac{4\alpha\pi}{3} \,.\, a^4 \,.\, Y^{(1)}, \qquad U^{(2)} = \frac{4\alpha\pi}{5} \,.\, a^5 \,.\, Y^{(2)}, \qquad \&c.$$

These values of $U^{(0)}$, $U^{(1)}$, &c., are computed, in [1466], upon the supposition that the attracted point is upon the surface of the spheroid; but being independent of r [1433k], they will be the same for all values of r corresponding to an external attracted point, and may therefore be substituted in [1461b], to obtain the general value of V [1467], including, however, only the terms of the order α.

We shall hereafter [1560α] give Mr. Poisson's value of V, in which all the powers of α are retained. La Place has treated of the same subject in [1820'', &c.]

R' being the radius R continued till it meets the surface of the spheroid.* [1467vi]
Substituting for R' its value $a.(1+\alpha y)$, [1461′], we shall have,

Mass of a spheroid.

$$M=\frac{4\pi.a^3}{3}+\alpha a^3.\int y.d\mu.d\varpi;$$ [1468]

in which we must substitute the value of $y=Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.$, [1464], and perform the integrations. For this purpose, we shall give the following general theorem, which is very useful in this analysis.

"If $Y^{(i)}$, and $Z^{(i')}$, be *any rational and integral functions whatever* of μ, $\sqrt{1-\mu^2}.\sin.\varpi$, and $\sqrt{1-\mu^2}.\cos.\varpi$, which satisfy the following equations, [1468′]

$$0=\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dY^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddY^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i.(i+1).Y^{(i)};$$

$$0=\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dZ^{(i')}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddZ^{(i')}}{d\varpi^2}\right)}{1-\mu^2}+i'.(i'+1).Z^{(i')};$$ [1469]

we shall have generally,

Important theorem in definite integrals.

$$\int_{-1}^{1}\int_{0}^{2\pi}Y^{(i)}.Z^{(i')}.d\mu.d\varpi=0,$$ [1470]

when i, i', *are integral but not equal positive numbers*; the integrals being taken from $\mu=-1$ to $\mu=1$, and from $\varpi=0$ to $\varpi=2\pi$, 2π [1470′]
being the circumference whose radius is unity."

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* (1003) Putting the density $\rho=1$ in [1431d] we get, by neglecting the accents on μ', ϖ',

$$M=\int R^2dR.d\mu.d\varpi=\tfrac{1}{3}\int R'^3.d\mu.d\varpi=\tfrac{1}{3}\int a^3.(1+\alpha y)^3.d\mu.d\varpi$$
$$=\tfrac{1}{3}a^3.\int d\mu.d\varpi+\alpha a^3.\int y.d\mu.d\varpi.$$ [1467a]

the limits of the integrals being as in [1431d]. But $\int_0^{2\pi}d\varpi=2\pi$, and $\int_{-1}^{1}d\mu=2$. [1467b]
Hence $\int_0^{2\pi}\int_{-1}^{1}d\varpi.d\mu=2\pi.2=4\pi$, and the preceding value of M becomes, by substitution, as in [1468]. [1468a]

To demonstrate this theorem, we shall observe, that by means of the first of the preceding equations of partial differentials [1469], we have*

[1471]
$$\int Y^{(i)} \,.\, Z^{(i')} \,.\, d\mu \,.\, d\varpi = -\frac{1}{i.(i+1)} \,.\int Z^{(i')} \,.\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}.\,d\mu\,.\,d\varpi$$
$$-\frac{1}{i.(i+1)}\,.\int\frac{Z^{(i')}\,.\left(\frac{d\,d\,Y^{(i)}}{d\varpi^2}\right)}{1-\mu^2}\,.\,d\mu\,.\,d\varpi\,;$$

and if we integrate by parts, relative to μ, we shall get,†

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* (1004) Multiplying the first of the equations [1469] by $\frac{d\mu\,.\,d\varpi\,.\,Z^{(i')}}{i.(i+1)}$, transposing the two first terms, depending on $\left(\frac{d\,Y^{(i)}}{d\mu}\right)$, $\left(\frac{d\,d\,Y^{(i)}}{d\varpi^2}\right)$, and annexing the sign of integration $\int$, we get the equation [1471].

† (1004a) Putting, for brevity,

[1471a]
$$W=(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\mu}\right), \qquad W'=(1-\mu^2)\,.\left(\frac{d\,Z^{(i')}}{d\mu}\right),$$

and integrating successively by parts, relative to μ, we shall get,

$$\int Z^{(i')}\,.\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}.\,d\mu=$$
$$\int Z^{(i')}\,.\left(\frac{dW}{d\mu}\right).\,d\mu \qquad = Z^{(i')}\,.\,W-\int W\,.\left(\frac{d\,Z^{(i')}}{d\mu}\right).\,d\mu$$
$$= Z^{(i')}\,.\,W-\int(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\mu}\right).\left(\frac{d\,Z^{(i')}}{d\mu}\right).\,d\mu$$
$$= Z^{(i')}\,.\,W-\int W'\,.\left(\frac{d\,Y^{(i)}}{d\mu}\right).\,d\mu$$

[1471b]
$$= Z^{(i')}\,.\,W-W'\,.\,Y^{(i)}+\int Y^{(i)}\,.\left(\frac{d\,W'}{d\mu}\right).\,d\mu\,;$$

which, by replacing the values of W, W', becomes as in [1472]. Now at the limits of the integral relative to μ [1470′] we have $1-\mu^2=0$; therefore W, W', [1471a], will vanish at these limits, and the expression [1471b] will become,

[1471c]
$$\int Z^{(i')}\,.\left(\frac{dW}{d\mu}\right).\,d\mu=\int Y^{(i)}\,.\left(\frac{d\,W'}{d\mu}\right).\,d\mu.$$

$$\int Z^{(i')} \cdot \left\{ \frac{d \cdot \left\{ (1-\mu^2) \cdot \left(\frac{d\,Y^{(i)}}{d\mu} \right) \right\}}{d\mu} \right\} \cdot d\mu = (1-\mu^2) \cdot \left(\frac{d\,Y^{(i)}}{d\mu} \right) \cdot Z^{(i')}$$

$$- (1-\mu^2) \cdot Y^{(i)} \cdot \left(\frac{d\,Z^{(i')}}{d\mu} \right) \qquad [1472]$$

$$+ \int Y^{(i)} \cdot \left\{ \frac{d \cdot \left\{ (1-\mu^2) \cdot \left(\frac{d\,Z^{(i')}}{d\mu} \right) \right\}}{d\mu} \right\} \cdot d\mu.$$

Now it is evident, that if we take the integral from $\mu = -1$ to $\mu = 1$, [1472′] the second member of this equation will be reduced to its last term. We shall have, in like manner, by integrating by parts, relative to ϖ,*

$$\int Z^{(i')} \cdot \left(\frac{ddY^{(i)}}{d\varpi^2} \right) \cdot d\varpi = \text{const.} + Z^{(i')} \cdot \left(\frac{dY^{(i)}}{d\varpi} \right) - Y^{(i)} \cdot \left(\frac{d\,Z^{(i')}}{d\varpi} \right) + \int Y^{(i)} \cdot \left(\frac{ddZ^{(i')}}{d\varpi^2} \right) \cdot d\varpi \,; \qquad [1473]$$

and this second member is also reduced to its last term, when the integral is taken from $\varpi = 0$ to $\varpi = 2\varpi$, because the values of $Y^{(i)}$, $\left(\frac{d\,Y^{(i)}}{d\varpi} \right)$, $Z^{(i')}$, $\left(\frac{d\,Z^{(i')}}{d\varpi} \right)$, are the same at these two limits ;† therefore we shall have, [1473′]

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* (1005) Integrating successively by parts, relative to ϖ, we get,

$$\int Z^{(i')} \cdot \left(\frac{ddY^{(i)}}{d\varpi^2} \right) \cdot d\varpi = Z^{(i')} \cdot \left(\frac{d\,Y^{(i)}}{d\varpi} \right) - \int \left(\frac{d\,Y^{(i)}}{d\varpi} \right) \cdot \left(\frac{d\,Z^{(i')}}{d\varpi} \right) \cdot d\varpi$$

$$= Z^{(i')} \cdot \left(\frac{d\,Y^{(i)}}{d\varpi} \right) - Y^{(i)} \cdot \left(\frac{d\,Z^{(i')}}{d\varpi} \right) + \int Y^{(i)} \cdot \left(\frac{d\,d\,Z^{(i')}}{d\varpi^2} \right) \cdot d\varpi + \text{const.} \,; \qquad [1471d]$$

as in [1473], Now $Y^{(i)}$, $Z^{(i')}$, are functions of $\sin. \varpi$, $\cos. \varpi$, [1468′], which have the same values at the limits of the integral $\varpi = 0$, $\varpi = 2\pi$; therefore, by taking the integrals between these limits, the two terms of [1471d] without the sign $\int$ will vanish, and we shall have, $\int Z^{(i')} \cdot \left(\frac{d\,d\,Y^{(i)}}{d\varpi^2} \right) \cdot d\varpi = \int Y^{(i)} \cdot \left(\frac{d\,d\,Z^{(i')}}{d\varpi^2} \right) \cdot d\varpi$, as in [1473′]. [1471$e$]

† (1006) Substituting in [1471] the values of $\int Z^{(i')} \cdot \left(\frac{dW}{d\mu} \right) \cdot d\mu$, $\int Z^{(i')} \cdot \left(\frac{ddY^{(i)}}{d\varpi^2} \right) \cdot d\varpi$, [1471$c$, e], we get,

[1474] $$\int Y^{(i)}.Z^{(i')}.d\mu.d\varpi = -\frac{1}{i.(i+1)}.\int Y^{(i)}.d\mu.d\varpi.\left\{\left(\frac{d.\left\{(1-\mu^2).\left(\frac{dZ^{(i')}}{d\mu}\right)\right\}}{d\mu}\right)+\frac{\left(\frac{ddZ^{(i')}}{d\varpi^2}\right)}{1-\mu^2}\right\}.$$

Hence we deduce, by means of the second of the two preceding equations of partial differentials,*

[1475] $$\int Y^{(i)}.Z^{(i')}.d\mu.d\varpi = \frac{i'.(i'+1)}{i.(i+1)}.\int Y^{(i)}.Z^{(i')}.d\mu.d\varpi;$$

$$\int Y^{(i)}.Z^{(i')}.d\mu.d\varpi = -\frac{1}{i.(i+1)}.\int Y^{(i)}.\left(\frac{dW'}{d\mu}\right).d\mu.d\varpi - \frac{1}{i.(i+1)}.\int \frac{Y^{(i)}.\left(\frac{ddZ^{(i')}}{d\varpi^2}\right)}{1-\mu^2}.d\mu.d\varpi$$

$$= -\frac{1}{i.(i+1)}.\int Y^{(i)}.d\mu.d\varpi.\left\{\left(\frac{dW'}{d\mu}\right)+\frac{\left(\frac{ddZ^{(i')}}{d\varpi^2}\right)}{1-\mu^2}\right\};$$

which, by resubstituting the value of W' [1471a], becomes as in [1474].

* (1007) Transposing the last term of the second equation [1469], we obtain,

$$\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dZ^{(i')}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddZ^{(i')}}{d\varpi^2}\right)}{1-\mu^2} = -i'.(i'+1).Z^{(i')}.$$

Substituting this in the second member of [1474], we get [1475]; and if in this last we transpose the terms of the first member, we obtain,

$$0 = \left\{\frac{i'.(i'+1)}{i.(i+1)} - 1\right\}.\int Y^{(i)}.Z^{(i')}.d\mu.d\varpi.$$

Now the second member can vanish, like the first, only in two ways; *first*, by putting the factor $\frac{i'.(i'+1)}{i.(i+1)} - 1 = 0$; *second*, by putting $\int Y^{(i)}.Z^{(i')}.d\mu.d\varpi = 0$. But by hypothesis, i, i', are integral positive numbers; and if we suppose $i' > i$, we shall have $\frac{i'.(i'+1)}{i.(i+1)} > 1$, but if we suppose $i' < i$, we shall have $\frac{i'.(i'+1)}{i.(i+1)} < 1$; neither of which values would therefore satisfy the equation $\frac{i'.(i'+1)}{i.(i+1)} - 1 = 0$; but $i' = i$ would satisfy it, therefore the equation $0 = \left\{\frac{i'.(i'+1)}{i.(i+1)} - 1\right\}.\int Y^{(i)}.Z^{(i')}.d\mu.d\varpi$, when i' differs from i, cannot be satisfied, except by putting the second factor

$$\int Y^{(i)}.Z^{(i')}.d\mu.d\varpi$$

[1475a] equal to nothing, as in [1476]. If we substitute for $d\mu$ its value $-d\theta.\sin.\theta$ and put

Important theorem in definite Integrals. First form.

therefore we shall have, when i differs from i',

$$0=\int_{-1}^{1}\int_{0}^{2\varpi} Y^{(i)} \,.\, Z^{(i')} \,.\, d\mu \,.\, d\varpi. \qquad [1476]$$

[1476'] Hence it is easy to prove, that y cannot be developed in more than in one expression of the form $Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.$ [1464]; for we have generally,*

$$\int y \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi = \int Y^{(i)} \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi. \qquad [1477]$$

If we could develop y in another series of the same form

$$Y_{\prime}^{(0)}+Y_{\prime}^{(1)}+Y_{\prime}^{(2)}+\&c.,$$

we should have,

$$\int y \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi = \int Y_{\prime}^{(i)} \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi\,; \qquad [1478]$$

therefore

$$\int Y_{\prime}^{(i)} \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi = \int Y^{(i)} \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi. \qquad [1479]$$

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for brevity, as in [1447f], $d\theta \,.\, \sin.\theta \,.\, d\varpi = dw$, this equation will become, for all values of i which differ from i',

Theorem in definite integrals. Second form.

$$0=\int_{0}^{\pi}\int_{0}^{2\pi} Y^{(i)} \,.\, Z^{(i')} \,.\, d\theta \,.\, \sin.\theta \,.\, d\varpi = \int Y^{(i)} \,.\, Z^{(i')} \,.\, dw\,; \qquad [1476a]$$

the integrations relative to w extending to the whole surface of a sphere whose radius is unity.

If $Y^{(i)}$ be a constant quantity, it will be of the form $Y^{(0)}$, and from [1476a] we shall have generally, when i' differs from 0

$$0=\int Y^{(0)} \,.\, Z^{(i')} \,.\, dw = Y^{(0)} \,.\, \int Z^{(i')} \,.\, dw, \qquad \text{or} \qquad \int Z^{(i')} \,.\, dw = 0, \qquad [1476b]$$

$Z^{(i')}$ being any integral function whatever of μ, $\sqrt{(1-\mu^2)} \,.\, \cos.\varpi$, $\sqrt{(1-\mu^2)} \,.\, \sin.\varpi$, satisfying an equation, similar to that in [1469].

* (1008) Substituting $y=Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.$ [1464] in the first member of the equation [1477], it becomes

$$\int y \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi = \int Y^{(0)} \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi + \int Y^{(i)} \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi + \&c.$$

Now from [1476], all the terms of the second member, except that depending on $Y^{(i)}$, vanish; therefore $\int y \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi = \int Y^{(i)} \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi$, as in [1477]. The expression [1478] is obtained in a similar manner. Putting these two integrals equal to each other, we get [1479].

[1479′] Now it is evident that if we take for $Z^{(i)}$ the most general expression of that kind, the preceding equation cannot take place, except in the case where $Y_{\prime}^{(i)}=Y^{(i)}$; *therefore the function y can be developed in this manner only in one form.**

There can be only one development of y in the form $\Sigma Y^{(i)}$.

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* (1009) If we put $Y_{\prime}^{(i)}-Y^{(i)}=Y_{\prime\prime}^{(i)}$, and suppose, as above, that $Y_{\prime}^{(i)}$, $Y^{(i)}$, satisfy the equation [1465], it will be evident that $Y_{\prime\prime}^{(i)}$ will also satisfy the same equation; and if we transpose the term in the second member of [1479], we shall get

[1479a] $$\int(Y_{\prime}^{(i)}-Y^{(i)}).Z^{(i)}.d\mu.d\varpi=0, \qquad \text{or} \qquad \int Y_{\prime\prime}^{(i)}.Z^{(i)}.d\mu.d\varpi=0;$$

now this equation cannot exist, if we use the most general value of $Z^{(i)}$ [1479′], unless we have $Y_{\prime\prime}^{(i)}=0$; i being, as above, any positive integer. To prove this, we shall find it convenient to use the theorem [1548′]; and on this account the reader may, if he thinks it necessary, pass over this note, until he shall have examined the demonstration of that formula, which is made independently of the theorem treated of in this place.

Now if we suppose, as in [1541‴], that the part of $Y_{\prime\prime}^{(i)}$, depending on the angle $n\varpi$, is represented by $\lambda.(A^{(n)}.\sin.n\varpi+B^{(n)}.\cos.n\varpi)$; and the part of $Z^{(i)}$, depending on the same angle is $\lambda.(A'^{(n)}.\sin.n\varpi+B'^{(n)}.\cos.n\varpi)$, [1547′]; we shall have the

[1479b] corresponding terms of $\int Y_{\prime\prime}^{(i)}.Z^{(i)}.d\mu.d\varpi=\frac{4\pi}{(2i+1).\gamma}.(A^{(n)}.A'^{(n)}+B^{(n)}.B'^{(n)})$,

as in [1548′]. The equation [1479a] ought to exist for all values of $Z^{(i)}$; and if we suppose it to be reduced to one term, $\lambda.A'^{(n)}.\sin.n\varpi$, all the other terms, $A'^{(1)}$, $A'^{(2)}$, &c., $B'^{(0)}$, $B'^{(1)}$, &c., vanishing, the integral [1479b] will become

$$\frac{4\pi}{(2i+1).\gamma}.A^{(n)}.A'^{(n)};$$

and this will not vanish, while $A'^{(n)}$ retains a finite value, unless we have $A^{(n)}=0$. In like manner, if all the terms of $Z^{(i)}$ are supposed to vanish, except $\lambda.B^{(n)}.\cos.n\varpi$, the integral [1479b] will become $\frac{4\pi}{(2i+1).\gamma}.B^{(n)}.B'^{(n)}$; and this will not vanish, as is required by the formula [1479a], unless we have $B^{(n)}=0$. In this way we find that all the terms of $Y_{\prime\prime}^{(i)}$, depending on any angle $n\varpi$, vanish; and we shall have, generally, $Y_{\prime\prime}^{(i)}=0$, or $Y_{\prime}^{(i)}=Y^{(i)}$; hence we may conclude, as above, that any function y can be developed in only one manner, in a series of the form [1464]. We shall give another demonstration of this important proposition in [1533i′].

It has been observed, that a similar result would not necessarily follow, if the function y were represented in a series of sines and cosines of the *multiples* of θ and ϖ. Since the same function y, for a given extent of the values of each variable quantity θ, ϖ, may be expressed, in this last manner, in many different ways; and if we have obtained two expressions of the *whole* value of y in such series, we cannot generally conclude, that the similar terms of each are severally equal in both expressions; whereas, in functions of the form $Y^{(i)}$, this equality must, of necessity, exist.

If in the integral $\int y\,.\,d\mu\,.\,d\varpi$, we substitute for y, its value [1464], $Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.$, we shall have generally $0=\int Y^{(i)}\,.\,d\mu\,.\,d\varpi$, i being equal to, or greater than, unity; for unity, by which $d\mu\,.\,d\varpi$ is [1479″] multiplied, is comprised in the form $Z^{(0)}$, which corresponds to any constant quantity, independent of μ and ϖ. The integral $\int y\,.\,d\mu\,.\,d\varpi$, is therefore reduced to $\int Y^{(0)}\,.\,d\mu\,.\,d\varpi$, equal to $4\pi\,.\,Y^{(0)}$;* hence we shall have, [1479‴]

$$M=\tfrac{4}{3}\pi\,.\,a^3+4\alpha\pi\,.\,a^3\,.\,Y^{(0)}\,; \qquad [1480]$$

Case in which $Y^{(0)}$ vanishes.

therefore, by taking for a *the radius of the sphere equal in mass to that of the spheroid* [1457a], *we shall have* $Y^{(0)}=0$, *and the term* $Y^{(0)}$ *will disappear from the expression of* y. [1480′]

The distance of the particle $dM=R^2dR\,.\,d\mu\,.\,d\varpi$ [1467a], from the [1480″] plane of the meridian, from which the angle ϖ is counted, is equal to

$$R\,.\sqrt{1-\mu^2}\,.\,\sin.\,\varpi\,;\dagger \qquad [1480‴]$$

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* (1010) Substituting y [1464] in [1468], we get

$$M=\tfrac{4}{3}\pi\,.\,a^3+\alpha\,.\,a^3\,.\int(Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.)\,.\,d\mu\,.\,d\varpi\,.\,Z^{(0)},$$

supposing $Z^{(0)}=1$. This, by means of [1476], is reduced to

$$M=\tfrac{4}{3}\pi\,.\,a^3+\alpha\,.\,a^3\,.\int Y^{(0)}\,.\,Z^{(0)}\,.\,d\mu\,.\,d\varpi=\tfrac{4}{3}\pi\,.\,a^3\,.+\alpha\,.\,a^3\,.\,Y^{(0)}\,.\int d\mu\,.\,d\varpi$$
$$=\tfrac{4}{3}\pi\,.\,a^3+\alpha\,.\,a^3\,.\,Y^{(0)}\,.\,4\pi, \qquad [1468a]\,;$$

observing that $Y^{(0)}$, being independent of μ, ϖ, is brought from under the sign of integration. This value of M is the same as in [1480].

† (1011) To conform to the present notation, we must reject the accents on θ', ϖ', in the expression of the co-ordinates of the attracted point dM [1432], and we shall have,

$$x=R\,.\,\cos.\,\theta. \qquad y=R\,.\,\sin.\,\theta\,.\,\cos.\,\varpi, \qquad z=R\,.\,\sin.\,\theta\,.\,\sin.\,\varpi\,; \qquad [1480a]$$

and if we put $\cos.\,\theta=\mu$, they will become, as above,

$$x=R\,.\,\mu, \qquad y=R\,.\,\surd(1-\mu^2)\,.\,\cos.\,\varpi, \qquad z=R\,.\,\surd(1-\mu^2)\,.\,\sin.\,\varpi\,; \qquad [1480b]$$

in which the plane of xy, is the plane from which the angle ϖ is counted, ϖ being 0, when $z=0$; and the plane of yz, is the plane of the equator. In this case, $x=R\,.\,\mu$, represents the distance of the particle dM from the plane of the equator, yz, as in [1480$^{\rm vi}$]; $y=R\,.\surd(1-\mu^2)\,.\,\cos.\,\varpi$ represents its distance from the plane xz, as in [1480c] [1480$^{\rm v}$]; and $z=R.\surd(1-\mu^2).\,\sin.\,\varpi$ its distance from the plane xy, as in [1480‴]. If we multiply these distances respectively, by the value [1480″] of the particle

$$dM=R^2dR\,.\,d\mu\,.\,d\varpi, \qquad [1480d]$$

the distance of the centre of gravity of the spheroid from this plane, multiplied by the mass M, will therefore be $\int R^3 dR\,.\,d\mu\,.\,d\varpi\,.\sqrt{1-\mu^2}\,.\,\sin.\varpi$: and by integrating with respect to R, it will be

[1480''''] $$\tfrac{1}{4}\int R'^4\,.\,d\mu\,.\,d\varpi\,.\sqrt{1-\mu^2}\,.\,\sin.\varpi,$$

R' being the radius continued to the surface of the spheroid. In like manner, the distance of the particle dM from the plane of the meridian perpendicular

[1480v] to the preceding, being $R\,.\sqrt{1-\mu^2}\,.\,\cos.\varpi$; the distance of the centre of gravity of the spheroid from this plane, multiplied by the mass M, will be $\tfrac{1}{4}\int R'^4\,.\,d\mu\,.\,d\varpi\,.\sqrt{1-\mu^2}\,.\,\cos.\varpi$. Lastly, the distance of the particle

[1480vi] $d\,M$ from the plane of the equator, being $R\mu$; the distance of the centre of gravity of the spheroid from this plane, multiplied by the mass of the spheroid, will be $\tfrac{1}{4}\int R'^4\,.\,\mu\,d\mu\,.\,d\varpi$. The functions μ, $\sqrt{1-\mu^2}\,.\,\sin.\varpi$, and

[1480vii] $\sqrt{1-\mu^2}\,.\,\cos.\varpi$, are of the form* $Z^{(1)}$, $Z^{(1)}$ being subjected to the equation of partial differentials,

[1481] $$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{dZ^{(1)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddZ^{(1)}}{d\varpi^2}\right)}{1-\mu^2}+2\,.\,Z^{(1)}.$$

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and integrate the products, so as to include the whole mass M of the spheroid, these integrals will represent, by the nature of the centre of gravity [126, 127], the products of the mass M, by the distances X, Y, Z, of that centre from the planes yz, xz, xy, respectively. Hence we shall have, as in [1480''''—1480vii],

[1480e]
$$M\,.\,X=\int R^3 dR\,.\,\mu\,d\mu\,.\,d\varpi = \tfrac{1}{4}\int R'^4\,.\,\mu\,d\mu\,.\,d\varpi;$$
$$M\,.\,Y=\int R^3 dR\,.\,d\mu\,.\,d\varpi\,.\sqrt{(1-\mu^2)}\,.\,\cos.\varpi=\tfrac{1}{4}\int R'^4\,.\,d\mu\,.\,d\varpi\,.\sqrt{(1-\mu^2)}\,.\,\cos.\varpi;$$
$$M\,.\,Z=\int R^3 dR\,.\,d\mu\,.\,d\varpi\,.\sqrt{(1-\mu^2)}\,.\,\sin.\varpi=\tfrac{1}{4}\int R'^4\,.\,d\mu\,.\,d\varpi\,.\sqrt{(1-\mu^2)}\,.\,\sin.\varpi.$$

* (1012) The expression [1437], by putting $i=1$, $U^{(i)}=Z^{(1)}$, becomes as in [1481]; the same changes being made in the general value of $U^{(1)}$ [1438], it becomes

[1480f] $Z^{(1)}=H\,.\,\mu+H'\,.\sqrt{(1-\mu^2)}\,.\,\sin.\varpi+H''\,.\sqrt{(1-\mu^2)}\,.\,\cos.\varpi$; H, H', H'', being arbitrary constant quantities; and by assigning to them particular values, we shall obtain corresponding values of $Z^{(1)}$. Thus, *first*, putting $H=1$, $H'=0$, $H''=0$, we

[1480g] obtain $Z^{(1)}=\mu$; *second*, putting $H=0$, $H'=1$, $H''=0$, we obtain

[1480h] $Z^{(1)}=\sqrt{(1-\mu^2)}\,.\,\sin.\varpi$, *third*, putting $H=0$, $H'=0$, $H''=1$, we obtain

[1480i] $Z^{(1)}=\sqrt{(1-\mu^2)}\cos.\varpi$; therefore the values μ, $\sqrt{(1-\mu^2)}.\sin.\varpi$, and $\sqrt{(1-\mu^2)}.\cos.\varpi$, being substituted for $Z^{(1)}$, in [1481], will satisfy that equation.

If we suppose R'^4 to be developed in a series $N^{(0)}+N^{(1)}+N^{(2)}+\&c.$, [1481′]
$N^{(i)}$ being an integral and rational function of μ, $\sqrt{1-\mu^2}\,.\,\sin.\varpi$, $\sqrt{1-\mu^2}\,.\,\cos.\varpi$, subjected to the following equation of partial differentials,

$$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,N^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}+\frac{\left(\frac{d\,d\,N^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,N^{(i)}\,; \qquad [1482]$$

the distances of the centre of gravity of the spheroid from the three preceding [1482′]
planes, multiplied by the mass of the spheroid, will be, by means of the general theorem we have just demonstrated, [1476],*

$$\tfrac{1}{4}\int N^{(1)}\,.\,d\,\mu\,.\,d\,\varpi\,.\,\sqrt{1-\mu^2}\,.\,\sin.\varpi\,;$$
$$\tfrac{1}{4}\int N^{(1)}\,.\,d\,\mu\,.\,d\,\varpi\,.\,\sqrt{1-\mu^2}\,.\,\cos.\varpi\,; \qquad [1483]$$
$$\tfrac{1}{4}\int N^{(1)}\,.\,d\,\mu\,.\,d\,\varpi\,.\,\mu.$$

$N^{(1)}$ is, by § 9, [1438], of the form,†

$$N^{(1)}=A\,.\,\mu+B\,.\,\sqrt{1-\mu^2}\,.\,\sin.\varpi+C\,.\,\sqrt{1-\mu^2}\,.\,\cos.\varpi\,; \qquad [1483']$$

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* (1013) If we substitute, in the equations [1480e], the values of $Z^{(1)}$ [1480f], corresponding to each of them, they will all become of the form $\tfrac{1}{4}\int R'^4\,.\,d\,\mu\,.\,d\,\varpi\,.\,Z^{(1)}$. This, by means of the value of R'^4 [1481′], is equal to

$$\tfrac{1}{4}\int\{N^{(0)}+N^{(1)}+N^{(2)}+\&c.\}\,.\,d\,\mu\,.\,d\,\varpi\,.\,Z^{(1)},$$

and by using the theorem [1470], it is reduced to $\tfrac{1}{4}\int N^{(1)}\,.\,Z^{(1)}\,.\,d\,\mu\,.\,d\,\varpi$. Now [1482a]
resubstituting the values of $Z^{(1)}$ [1480i], we shall obtain the three expressions [1483].

† (1014) This form of $N^{(1)}$ is the same as that in [1438], changing the constant coefficients H, H', H'', into A, B, C. If we substitute this value of $N^{(1)}$ in the first of the expressions [1483], it will become,

$$\begin{aligned}\tfrac{1}{4}\,A\,.\int\mu\,.\,\sqrt{(1-\mu^2)}\,.\,d\,\mu\,.\int d\,\varpi\,.\,\sin.\varpi&+\tfrac{1}{4}\,B\,.\int(1-\mu^2)\,.\,d\,\mu\,.\int.\,d\,\varpi\,.\,\sin.^2\varpi\\&+\tfrac{1}{4}\,C\,.\int(1-\mu^2)\,.\,d\,\mu\,.\int d\,\varpi\,.\,\sin.\varpi\,.\,\cos.\varpi.\end{aligned} \qquad [1483a]$$

Now if n be any integral number, we shall have, by taking the integrals between $\varpi=0$, and $\varpi=2\,\pi$,

$$0=\int_0^{2\pi}\sin.\,n\,\varpi\,.\,d\,\varpi=0\,; \qquad \int_0^{2\pi}\cos.\,n\,\varpi\,.\,d\,\varpi=0\,; \qquad [1483b]$$

for $\int\sin.\,n\varpi\,.\,d\,\varpi=-\frac{1}{n}\,.\,\cos.\,n\varpi+\frac{1}{n}$, which vanishes if $\varpi=0$, and when $\varpi=2\pi$ [1483c]

A, B, C, being constant quantities; the preceding distances, multiplied by
[1483′] the mass, will therefore become $\frac{\pi}{3}\,.\,B$, $\frac{\pi}{3}\,.\,C$, $\frac{\pi}{3}\,.\,A$. The position of the centre of gravity of the spheroid, depends, therefore, only upon the
[1483‴] function $N^{(1)}$, which furnishes a very simple method of determining its value. If the origin of the radius R', which is the point of intersection of the three
[1483⁗] preceding planes, be the centre of gravity of the spheroid, the distances of this centre from these planes will be nothing; therefore $A=0$, $B=0$, $C=0$, [1483″]; hence $N^{(1)}=0$, [1483′].

$N^{(1)}$ vanishes if the origin of the co-ordinates be at the centre of gravity.

These results exist for a homogeneous spheroid of any form whatever. When it differs but very little from a sphere, we shall have, [1461′, 1480⁗],

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it becomes also $-\frac{1}{n}+\frac{1}{n}=0$. The same result is obtained from

$$\int \cos.\, n\varpi\,.\,d\varpi=\frac{1}{n}\,.\,\sin.\, n\varpi.$$

Hence the terms of [1483a], multiplied by A, C, vanish; and in the term multiplied by B, we may put $\sin.^2\varpi=\frac{1}{2}-\frac{1}{2}\,.\cos.\,2\varpi$, and then, for the same reason, we may neglect
[1483d] $\cos.\,2\varpi$, and we shall have $\int_0^{2\pi}\sin.^2\varpi\,.\,d\varpi=\frac{1}{2}\int_0^{2\pi}d\varpi=\frac{1}{2}\,.\,2\pi=\pi$. Hence the expression [1483a] will become $\frac{1}{4}\pi\,.\,B\,.\int(1-\mu^2)\,.\,d\mu=\frac{1}{4}\pi\,.\,B\,.\,(\mu-\frac{1}{3}\mu^3+\frac{2}{3})$, the constant quantity being taken so as to make it vanish when $\mu=-1$, [1470′], and when
[1483e] $\mu=1$, it becomes $\frac{1}{3}\pi\,.\,B$, as in [1483″]. In like manner we may compute the result of the second of the equations [1483]; or we may derive it from the preceding calculation, in a more simple manner. For if we change, in $N^{(1)}$, [1483′], B, C, ϖ, into C, $-B$, $\varpi+\frac{1}{2}\pi$, respectively, its value will not be altered; and the same alteration being made in the first equation [1483], it will change into the second of these equations; and the result
[1483f] $\frac{1}{3}\pi\,.\,B$, [1483e], will become $\frac{1}{3}\pi\,.\,C$, as in [1483″]. Again, the same value of $N^{(1)}$ [1483′], being substituted in the last of the equations [1483], it becomes

$$\tfrac{1}{4}A\,.\int\mu^2\,d\mu\,.\int d\varpi+\tfrac{1}{4}B\,.\int\mu\,d\mu\,.\sqrt{(1-\mu^2)}\,.\int d\varpi\,.\,\sin.\,\varpi$$
$$+\tfrac{1}{4}C\,.\int\mu\,d\mu\,.\sqrt{(1-\mu^2)}\,.\int d\varpi\,.\,\cos.\,\varpi.$$

The terms multiplied by B, C, vanish, because, by [1483b],

[1483g] $$\int_0^{2\pi}d\varpi\,.\,\sin.\,\varpi=0,\qquad \int_0^{2\pi}d\varpi\,.\,\cos.\,\varpi=0;$$

and as $\int_0^{2\pi}d\varpi=2\pi$, the whole expression will become

[1483h] $$\tfrac{1}{4}A\,.\int\mu^2\,d\mu\,.\,2\pi=\tfrac{1}{2}\pi\,.\,A\,.\int\mu^2\,d\mu=\tfrac{1}{2}\pi\,.\,A\,.\,(\tfrac{1}{3}\mu^3+\tfrac{1}{3}),$$

which vanishes when $\mu=-1$, and when $\mu=1$, it becomes $\frac{1}{2}\pi\,.\,A\,.\,\frac{2}{3}=\frac{1}{3}\pi\,.\,A$, as in [1483″].

$R' = a \cdot (1 + \alpha y)$, and $R'^4 = a^4 \cdot (1 + 4 \alpha y)$;* moreover, as y is equal to $Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$, [1464] ; we shall have $N^{(1)} = 4\alpha \cdot a^4 \cdot Y^{(1)}$, [1481'] ; the function $Y^{(1)}$ *will therefore vanish from the expression of* y, *when we fix the origin of* R' *at the centre of gravity of the spheroid.* [1483v] also $Y^{(1)}$ vanishes in the same case. [1483vi]

13. If the attracted point be placed within the spheroid, we shall have, by § 9, [1444, 1447],†

$$V = v^{(0)} + r \cdot v^{(1)} + r^2 \cdot v^{(2)} + r^3 \cdot v^{(3)} + \&c. ;$$
$$v^{(i)} = \int \frac{dR \cdot d\varpi' \cdot d\theta' \cdot \sin.\theta' \cdot Q^{(i)}}{R^{i-1}} . \qquad [1484]$$

If we suppose this value of V to correspond to a stratum, whose internal surface is spherical, and radius a ; the radius of the external surface being [1484']

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* (1014a) If we retain all the powers of α, we shall have

$$R'^4 = a^4 \cdot (1 + 4\alpha \cdot y + 6\alpha^2 \cdot y^2 + 4\alpha^3 \cdot y^3 + \alpha^4 \cdot y^4).$$

Developing y^2, y^3, y^4, in functions similar to that of [1464], of the forms

$$y^2 = Y_2^{(0)} + Y_2^{(1)} + Y_2^{(2)} + \&c. = \Sigma Y_2^{(i)} ; \qquad y^3 = Y_3^{(0)} + Y_3^{(1)} + \&c. = \Sigma Y_3^{(i)} ;$$
$$y^4 = Y_4^{(0)} + Y_4^{(1)} + \&c. = \Sigma Y_4^{(i)} ; \qquad [1483i]$$

we shall get,

$$R'^4 = a^4 \cdot (1 + 4\alpha \cdot \Sigma Y^{(i)} + 6\alpha^2 \cdot \Sigma Y_2^{(i)} + 4\alpha^3 \cdot \Sigma Y_3^{(i)} + \alpha^4 \cdot \Sigma Y_4^{(i)}) ; \qquad [1483k]$$

all the terms $Y_2^{(i)}$, $Y_3^{(i)}$, $Y_4^{(i)}$, satisfying a differential equation similar to [1465]. Comparing this value of R'^4, with the similar development [1481'], we get, for $N^{(1)}$, the following expression, $N^{(1)} = a^4 \cdot (4\alpha \cdot Y^{(1)} + 6\alpha^2 \cdot Y_2^{(1)} + 4\alpha^3 \cdot Y_3^{(1)} + \alpha^4 \cdot Y_4^{(1)})$; but by [1483''''], we have $N^{(1)} = 0$; substituting this, and dividing by $4\alpha \cdot a^4$, we obtain, Equation if the origin of the coordinates be at the centre of gravity.

$$0 = Y^{(1)} + \tfrac{3}{2}\alpha \cdot Y_2^{(1)} + \alpha^2 \cdot Y_3^{(1)} + \tfrac{1}{4}\alpha^3 \cdot Y_4^{(1)} ; \qquad [1483l]$$

which is the equation resulting from the condition, that the origin of R' is at the centre of gravity. If we neglect terms of the order α in [1483l], which is the same as to neglect α^2 in $N^{(1)}$, we shall get $Y^{(1)} = 0$, as in [1483vi]. Similar results would be obtained, in the computation [1480], by retaining all the powers of α, in the expression [1467v, &c.] $\frac{1}{3}\int R'^3 \cdot d\mu \cdot d\varpi$. [1483m]

† (1015) The equations [1484] are the same as [1444, 1447], putting the density $\rho = 1$ [1467v]. The process used in § 13, for computing the value of V [1496], is

$a \,.\, (1 + \alpha y)$; the thickness of this stratum will be $\alpha \,.\, a\, y$. If we denote
[1484''] by y', what y becomes, when θ, ϖ, are changed into θ', ϖ' ; we may, by neglecting quantities of the order α^2, change R into a, and dR into $\alpha \,.\, a y'$.
[1484'''] in the integral expression of $v^{(i)}$ [1484] ; we shall by this means have,*

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divided into *three* parts, which we shall explain by means of the annexed figure ; in which $a\,b\,g\,d$ is any section of the proposed spheroid, passing through the line $C\,E\,f$ connecting the origin of the co-ordinates C, and the attracted point E; $E\,H\,I$ is the section, formed by the same plane, and the *first* spherical surface, described about the centre C, with the radius $C\,E = r$; $A\,B\,O\,F\,G\,D$ is the section, formed by the plane and the *second* spherical surface, described about the same centre C, with the radius $CF = a$. CP is the axis from which the angles θ, θ', are counted ; so that the angles $P\,C\,F = \theta$, $P\,C\,G = \theta'$; the point P being on the surface of the second sphere ; and on this surface are drawn also the arcs PO, PF, PG, making the angles $O\,P\,F = \varpi$, $OPG = \varpi'$; PO being the meridian from which the angles ϖ, ϖ', are counted, and the attracting particle being situated on the radius $C\,g$. Then the *first* part of V is that depending on the attraction of the shell, included between the spheroid $a\,b\,g\,d$, and the sphere $A\,B\,G\,D$, the thickness of this shell, in the direction of the line $C\,E\,F\!f$, drawn through the attracted point E, being represented by $F\!f = a \,.\, \alpha y$; the thickness in the direction of the line $C\,g$, drawn through the attracting particles, being $G\,g = a \,.\, \alpha y'$. The *second* part of V depends upon the spherical shell included between the surfaces $A\,B\,G\,D$, and EHI. The *third* part depends on the sphere EHI. The
[1484a] *first* of these parts is computed in [1491], the *second* in [1494], the *third* in [1495] ; the sum of these three parts gives the complete value of V [1496].

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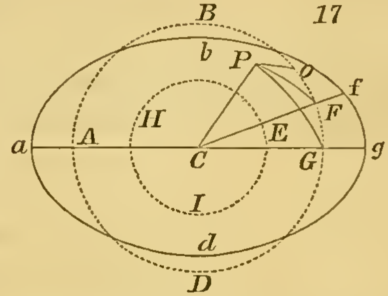

* (1016) If we take the integral of [1484], relative to R, that is, the integral of $\frac{dR}{R^{i-1}} = R^{1-i} \,.\, dR$, it will become $\frac{1}{2-i} \,.\, (R^{2-i} - a^{2-i})$, the constant quantity being taken so as to make it vanish at the point G of the preceding figure, where $R = a$. At the point g, which is the other limit of this integral, we have $R = a \,.\, (1 + \alpha y')$, and by neglecting α^2, we have $R^{2-i} = a^{2-i} \,.\, (1 + \alpha y')^{2-i} = a^{2-i} \,.\, \{1 + (2-i) \,.\, \alpha y'\}$. Hence the preceding integral becomes

$$\int \frac{dR}{R^{i-1}} = \frac{1}{2-i} \,.\, \{a^{2-i} \,.\, [1 + (2-i) \,.\, \alpha y'] - a^{2-i}\} = \frac{a^{2-i}}{2-i} \,.\, (2-i) \,.\, \alpha y' = \frac{\alpha y'}{a^{i-2}}.$$

Substituting this in $v^{(i)}$ [1484], we obtain [1485], the constant quantities α, a, being brought from under the sign of integration.

$$v^{(i)}=\frac{\alpha}{a^{i-2}}\cdot\int y'\cdot d\varpi'\cdot d\theta'\cdot\sin.\theta'\cdot Q^{(i)}.$$ [1485]

For a point placed without the spheroid, we have, by § 9, [1436, 1443],*

$$V=\frac{U^{(0)}}{r}+\frac{U^{(1)}}{r^2}+\frac{U^{(2)}}{r^3}+\&c.\,;$$ [1486]

$$U^{(i)}=\int R^{i+2}\cdot dR\cdot d\varpi'\cdot d\theta'\cdot\sin.\theta'\cdot Q^{(i)}.$$

If we suppose that this value of V corresponds to a stratum, whose internal radius is a, and external radius $a\cdot(1+\alpha y)$, we shall have, [1486']

$$U^{(i)}=\alpha\cdot a^{i+3}\cdot\int y'\cdot d\varpi'\cdot d\theta'\cdot\sin.\theta'\cdot Q^{(i)}\,;$$ [1487]

therefore,†

$$v^{(i)}=\frac{U^{(i)}}{a^{2i+1}}.$$ [1488]

We have, by [1466],

$$U^{(i)}=\frac{4\alpha\pi\cdot a^{i+3}\cdot Y^{(i)}}{2i+1}\,;$$ [1489]

therefore,

$$v^{(i)}=\frac{4\alpha\pi\cdot Y^{(i)}}{(2i+1)\cdot a^{i-2}}\,;$$ [1490]

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* (1017) From [1436, 1443], we get, by putting $\rho=1$, the two equations [1486]. Integrating the last of these relative to R, we get $\int R^{i+2}\cdot dR=\frac{1}{i+3}\cdot(R^{i+3}-a^{i+3})$, the limits being as in the last note, Putting $R=a\cdot(1+\alpha y')$, we find

$$R'^{i+3}=a^{i+3}\cdot\{1+(i+3)\cdot\alpha y'\},\qquad\text{hence}\qquad\int R^{i+2}\cdot dR=a^{i+3}\cdot\alpha y'.$$

Substituting this in $U^{(i)}$ [1486], we get [1487].

† (1018) Dividing [1487] by a^{2i+1}, we get, by using [1485],

$$\frac{U^{(i)}}{a^{2i+1}}=\frac{\alpha}{a^{i-2}}\cdot\int y'\cdot d\varpi'\cdot d\theta'\cdot\sin.\theta'\cdot Q^{(i)}=v^{(i)},$$

as in [1488]; substituting the value of $U^{(i)}$ [1489], it becomes as in [1490]. Putting successively $i=0,\ 1,\ 2,$ &c., we obtain, $v^{(0)}=4\alpha\pi\cdot a^2\cdot Y^{(0)}$, $v^{(1)}=\frac{4}{3}\alpha\pi\cdot a\cdot Y^{(1)}$, $v^{(2)}=\frac{4}{5}\alpha\pi\cdot Y^{(2)}$, &c.; substituting these in V [1484], we get [1491], which is the *first* part of the complete value of V, mentioned in [1484a].

which gives,

[1491] $$V = 4\alpha\pi \,.\, a^2 \,.\, \left\{ Y^{(0)} + \frac{r}{3a} \,.\, Y^{(1)} + \frac{r^2}{5a^2} \,.\, Y^{(2)} + \&c. \right\}.$$

We must add, to this value of V, the part corresponding to a spherical
[1491′] stratum, of the thickness $a - r$, above the attracted point; also the part corresponding to a sphere described with the radius r, and situated below the
[1491″] attracted point. If we put $\cos.\theta' = \mu'$, we shall have, for the first of these parts of V,*

[1492] $$v^{(i)} = \int \frac{dR \,.\, d\varpi' \,.\, d\mu' \,.\, Q^{(i)}}{R^{i-1}};$$

[1492′] the integral relative to μ', must be taken from $\mu' = -1$ to $\mu' = 1$. Integrating relative to R, from $R = r$ to $R = a$, we shall have,

[1493] $$v^{(i)} = \frac{1}{2-i} \,.\, \left\{ \frac{1}{a^{i-2}} - \frac{1}{r^{i-2}} \right\} \,.\, \int d\varpi' \,.\, d\mu' \,.\, Q^{(i)};$$

now by the preceding theorem, we have in general $\int d\varpi' \,.\, d\mu' \,.\, Q^{(i)} = 0$,
[1493′] when i is equal to, or greater than, unity.† When $i = 0$, we have $Q^{(0)} = 1$,

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[1492a] * (1019) If we put $\rho = 1$, $\mu' = \cos.\theta'$, $d\mu' = -d\theta' . \sin.\theta'$, in [1447], we shall get, $v^{(i)} = -\int \frac{dR \,.\, d\varpi' \,.\, d\mu' \,.\, Q^{(i)}}{R^{i-1}}$; the limits relative to θ' being as in [1433″], from $\theta' = 0$ to $\theta' = \pi$, or from $\mu' = 1$ to $\mu' = -1$. If we invert the order of the limits, as
[1492b] in [1431d], and take them from $\mu' = -1$ to $\mu = 1$, we may change the sign of this value of $v^{(i)}$, and it will become as in [1492]. The integral, relative to R, is

$$\int \frac{dR}{R^{i-1}} = \int R^{1-i} \,.\, dR = \frac{1}{2-i} \,.\, (R^{2-i} - r^{2-i});$$

the constant quantity being taken so as to make the integral vanish at the first limit, where $R = r$; and at the second limit, where $R = a$, the whole integral becomes

$$\frac{1}{2-i} \,.\, \{a^{2-i} - r^{2-i}\} = \frac{1}{2-i} \,.\, \left\{ \frac{1}{a^{i-2}} - \frac{1}{r^{i-2}} \right\}.$$

Substituting this in [1492], and bringing a, r, from under the sign $\int$, because they are independent of ϖ', μ', it becomes as in [1493].

† (1020) The quantity 1 may be considered as a factor of $d\varpi'$, $d\mu'$, and put equal to $Z^{(0)}$, as in [1479″], and then the expression $\int d\varpi' \,.\, d\mu' \,.\, Q^{(i)} = \int d\varpi' \,.\, d\mu' \,.\, Z^{(0)} \,.\, Q^{(i)}$; and by the theorem [1470], all the terms, except that depending on $Q^{(0)}$, will vanish from

[1433d''''] ; moreover, the integration relative to ϖ' ought to be taken from $\varpi'=0$ to $\varpi'=2\pi$; hence we shall have,

$$v^{(0)}=2\pi.(a^2-r^2).$$ [1494]

This value of $v^{(0)}$ is the part of V, corresponding to the spherical stratum whose thickness is $a-r$. [1494']

The part of V corresponding to the sphere whose radius is r, is equal to the mass of that sphere, divided by the distance of the attracted point from its centre; therefore it is equal to*

$$\tfrac{4}{3}\pi.r^2.$$ [1495]

General value of V for a homogeneous spheroid attracting an internal point, neglecting terms of the order α^2.

Adding together these several parts of V [1491, 1494, 1495], we shall have, for its complete value,

$$V=2\pi.a^2-\tfrac{2}{3}\pi.r^2+4\alpha\pi.a^2.\left\{Y^{(0)}+\frac{r}{3a}.Y^{(1)}+\frac{r^2}{5a^2}.Y^{(2)}+\frac{r^3}{7a^3}.Y^{(3)}+\&c.\right\}.\quad(4)$$ [1496]

If we suppose the attracted point to be placed within a stratum of nearly a spherical form, the internal radius being

$$a+\alpha a.\{Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.\},$$ [1497]

and the external radius

$$a'+\alpha a'.\{Y'^{(0)}+Y'^{(1)}+Y'^{(2)}+\&c.\},$$ [1498]

we may include the terms $\alpha a.Y^{(0)}$, and $\alpha a'.Y'^{(0)}$, in the quantities a, a'; moreover, by fixing the origin of the co-ordinates at the centre of [1498']

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this integral; consequently all the values of $v^{(i)}$ [1493], except $v^{(0)}$, must vanish, and when $i=0$, we shall get $v^{(0)}=\frac{1}{2}.(a^2-r^2).\int d\varpi'.d\mu'.Q^{(0)}$. Now by [1433$d''''$], $Q^{(0)}=1$, hence $\int d\varpi'.d\mu'.Q^{(0)}=\int d\varpi'.d\mu'=4\pi$ [1468a]. Substituting this in the preceding value of $v^{(0)}$, it becomes as in [1494]. This is the *second* part of V [1484a]. [1494a]

* (1021) The quantity V [1457a], corresponding to a sphere whose mass is M, is $A=\frac{M}{r}$, and if the radius of this sphere be r, we shall have $M=\frac{4}{3}\pi.r^3$, [1457a], hence $A=\frac{4}{3}.\pi r^2$, as in [1495]. This is the *third* part of V [1484a]. The sum of these three parts, [1491, 1494, 1495], is the complete value of V [1496], *neglecting terms depending on the second and higher powers of* α. These last terms are noticed in [1560α, 1820'].

[1498″] gravity of the spheroid, whose radius is $a+\alpha a.\{Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.\}$, we may make $Y^{(1)}$ vanish [1483vi] from this expression of the radius; and then the radius of the internal stratum will be

Internal radius.

[1499]
$$a+\alpha a.\{Y^{(2)}+Y^{(3)}+Y^{(4)}+\&c.\};$$

and the external radius will be

External radius.

[1500]
$$a'+\alpha a'.\{Y'^{(1)}+Y'^{(2)}+Y'^{(3)}+\&c.\}.$$

We may obtain the value of V corresponding to this stratum, by taking the
[1500′] difference of the values of V, corresponding to two spheroids, of which the least has the first quantity [1499] for the radius of its surface, and the greatest has the second quantity [1500] for the radius of its surface.
[1500″] Putting therefore $\Delta.V$, for what the quantity V becomes, relative to this stratum, we shall have,*

Value of V for any homogeneous spheroidal stratum, attracting an internal point, neglecting terms of the order α^2.

[1501]
$$\Delta.V=2\pi.(a'^2-a^2)+4\alpha\pi.\left\{\frac{ra'}{3}.Y'^{(1)}+\frac{r^2}{5}.(Y'^{(2)}-Y^{(2)})+\frac{r^3}{7}.\left(\frac{Y'^{(3)}}{a'}-\frac{Y^{(3)}}{a}\right)+\&c.\right\}.$$

If we wish to find the form of the stratum, so that a point placed within it shall be equally attracted in every direction, it is necessary that $\Delta.V$ should
[1501′] be reduced to a constant quantity, independent of r, θ, ϖ; for we have seen that the partial differentials of $\Delta.V$,† taken relatively to these variable

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* (1022) Putting $Y^{(0)}=0$, $Y^{(1)}=0$, in [1496], it becomes

$$V=2\pi.a^2-\tfrac{2}{3}\pi.r^2+4\alpha\pi.a^2.\left\{\frac{r^2}{5a^2}.Y^{(2)}+\frac{r^3}{7a^3}.Y^{(3)}+\&c.\right\},$$

corresponding to the least spheroid; and if we accent the letters V, a, $Y^{(i)}$, [1496], and then put $Y'^{(0)}=0$, [1498′], we shall have the value V', corresponding to the greatest spheroid,

$$V'=2\pi.a'^2-\tfrac{2}{3}\pi.r^2+4\alpha\pi.a'^2.\left\{\frac{r}{3a'}.Y'^{(1)}+\frac{r^2}{5a'^2}.Y'^{(2)}+\frac{r^3}{7a'^3}.Y'^{(3)}+\&c.\right\}.$$

Hence we get $\Delta.V=V'-V$, as in [1501]. This is given, in another form, in [1561n].

† (1023) $-\left(\frac{dV}{dr}\right)$, $-\left(\frac{dV'}{dr}\right)$, [1458″], represent the attractions of the spheroids, in the direction r; consequently their difference $-\left\{\left(\frac{dV'}{dr}\right)-\left(\frac{dV}{dr}\right)\right\}$, or $-\left(\frac{d.\Delta V}{dr}\right)$, will express the attraction of the spheroidal stratum, treated of above.

Form of the surfaces of a hollow stratum, which attracts an internal point equally in every direction;

quantities, express the partial attractions of the stratum upon the attracted point; hence we have $Y'^{(1)}=0$, and in general,

$$Y'^{(i)}=\left(\frac{a'}{a}\right)^{i-2}.\,Y^{(i)}; \qquad [1502]$$

so that the radius of the internal surface being given, that of the external surface will be obtained.

When the internal surface is elliptical, we shall have $Y^{(3)}=0$, [1502']
$Y^{(4)}=0$, &c.;* consequently $Y'^{(3)}=0$, $Y'^{(4)}=0$, &c.; the radii of

these surfaces may be elliptical.

Now as $Y^{(2)}$, $Y^{(3)}$, &c., $Y'^{(1)}$, $Y'^{(2)}$, &c., are independent of r, we shall obtain, from [1501],

$$-\left(\frac{d.\Delta V}{dr}\right)=-4\alpha\pi.\left\{\frac{a'}{3}.Y'^{(1)}+\frac{2r}{5}.(Y'^{(2)}-Y^{(2)})+\frac{3r^2}{7}.\left(\frac{Y'^{(3)}}{a'}-\frac{Y^{(3)}}{a}\right)+\&c.\right\};$$

and since by hypothesis the attraction in this direction is nothing, the second member of this expression must be nothing. Dividing it by the common factor $-4\alpha\pi$, we obtain

$$0=\frac{a'}{3}.Y'^{(1)}+\frac{2r}{5}.\{Y'^{(2)}-Y^{(2)}\}+\frac{3r^2}{7}.\left\{\frac{Y'^{(3)}}{a'}-\frac{Y^{(3)}}{a}\right\}+\frac{4r^3}{9}.\left\{\frac{Y'^{(4)}}{a'^2}-\frac{Y^{(4)}}{a^2}\right\}+\&c.; \qquad [1501a]$$

and this ought to exist for all values of r. If we now put $r=0$, it becomes $0=\frac{a'}{3}.Y'^{(1)}$; hence $Y'^{(1)}=0$, as above, for all values of r, because it is independent of r. Substituting this in [1501a], and dividing by r, we get

$$0=\tfrac{2}{5}.(Y'^{(2)}-Y^{(2)})+\frac{3r}{7}.\left\{\frac{Y'^{(3)}}{a'}-\frac{Y^{(3)}}{a}\right\}+\&c.; \qquad [1501b]$$

which also exists for all values of r. If we put $r=0$, it becomes $0=\tfrac{2}{5}.(Y'^{(2)}-Y^{(2)})$; hence $Y'^{(2)}=Y^{(2)}$. Substituting this in [1501b], dividing by r, and again putting $r=0$, we obtain $0=\frac{Y'^{(3)}}{a'}-\frac{Y^{(3)}}{a}$. Proceeding in this manner, we find generally,

$$\frac{Y'^{(i)}}{a'^{i-2}}-\frac{Y^{(i)}}{a^{i-2}}=0,$$

which is easily reduced to the form [1502].

* (1024) If we put $m=1-\alpha m'$, $n=1-\alpha n'$, in the equation of the ellipsoid [1363], it will become $x^2+y^2+z^2=k^2+\alpha.(m'y^2+n'z^2)$, in which $\alpha m'$, $\alpha n'$, are of the same order as the excentricity of the ellipsoid [1400]. Substituting in this equation the values of x, y, z, [1432], omitting the accents on θ', ϖ', in order to conform to the present notation, the first member of the preceding equation will be R^2, and we shall get $R^2=k^2+\alpha.R^2.\sin.^2\theta.(m'.\cos.^2\varpi+n'.\sin.^2\varpi)$; so that R, k, differ from each

the internal and external surfaces will therefore be of the following forms respectively,

[1503] $$a.\{1+\alpha.Y^{(2)}\}; \qquad a'.\{1+\alpha.Y'^{(2)}\}.$$

Hence it is evident that these two surfaces are similar and similarly situated, which is conformable to what we have found in [1369v].

14. The formulas [1467, 1496], include the whole theory of the
[1503'] attraction of a homogeneous spheroid, which varies but little from a sphere. Hence it is easy to find the attraction of a heterogeneous spheroid, whatever be the law of the variation of the figure, and of the density of the strata. For this purpose we shall suppose that $a.(1+\alpha y)$ is the radius of one of the strata of a heterogeneous spheroid, and that y is reduced to the form

[1503''] $$y = Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$$

The coefficients, included in the quantities $Y^{(0)}$, $Y^{(1)}$, &c., are functions of
[1503'''] a, and therefore variable from one stratum to another. If we then take the differential of V [1467], relative to a; and put ρ *for the density of the*
[1503''''] *stratum, whose radius is* $a.(1+\alpha y)$, ρ *being a function of* a *only;* the

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other only by quantities of the order α. If we take the square root, neglecting α^2 as in [1461', &c.], we shall get $R=k+\frac{1}{2}\alpha k.\sin.^2\theta.(m'.\cos.^2\varpi+n'.\sin.^2\varpi)$. Substituting in this $\cos.^2\varpi=\frac{1}{2}+\frac{1}{2}\cos.2\varpi$, $\sin.^2\varpi=\frac{1}{2}-\frac{1}{2}\cos.2\varpi$, $\sin.^2\theta=1-\mu^2$, it becomes,

$$R=k+\tfrac{1}{4}\alpha k.(1-\mu^2).\{m'+n'+(m'-n').\cos.2\varpi\}$$
$$=\{k+\tfrac{1}{4}\alpha k.\tfrac{2}{3}.(m'+n')\}-\tfrac{1}{4}\alpha k.(m'+n').(\mu^2-\tfrac{1}{3})+\tfrac{1}{4}\alpha k.(m'-n').(1-\mu^2).\cos.2\varpi.$$

Radius of an ellipsoid, neglecting α^2.

If we put $k+\frac{1}{4}\alpha k.\frac{2}{3}.(m'+n')=a$, $-\frac{1}{4}k.(m'+n')=a.B^{(0)}$, $\frac{1}{4}k.(m'-n')=a.B^{(2)}$, it becomes

[1503a] $$R=a.\{1+\alpha.B^{(0)}.(\mu^2-\tfrac{1}{3})+\alpha.B^{(2)}.(1-\mu^2).\cos.2\varpi\}=a.(1+\alpha.Y^{(2)});$$

in which the terms depending on $B^{(0)}$, $B^{(2)}$, satisfy the equation [1465] in the case of $i=2$. This also is very easily perceived by referring to the general value of $Y^{(2)}$ [1528c], supposing $A_2^{(1)}=0$, $B_2^{(1)}=0$, $A_2^{(2)}=0$. This form of R agrees with that in [1503], in which $Y^{(3)}=0$, $Y^{(4)}=0$, &c.; hence, by means of [1502], we shall have $Y'^{(3)}=0$, $Y'^{(4)}=0$, &c., as in [1503]. This result, in which terms of the order α^2 are neglected, agrees with the more general demonstration [1369v], in which all the powers of α are retained.

value of V, corresponding to this stratum, will be, for an external attracted point,*

$$\frac{4\pi}{3r}.\rho.d.a^3+\frac{4\alpha\pi.\rho}{r}.d.\left\{a^3.Y^{(0)}+\frac{a^4}{3r}.Y^{(1)}+\frac{a^5}{5r^2}.Y^{(2)}+\&c.\right\}. \qquad [1504]$$

Value of V for a heterogeneous spheroid attracting an external point, neglecting terms of the order α^2.

The value of V for the whole spheroid will therefore be,

$$V=\frac{4\pi}{3r}.\int_0^{\mathrm{a}}\rho.d.a^3+\frac{4\alpha\pi}{r}.\int_0^{\mathrm{a}}\rho.d.\left\{a^3.Y^{(0)}+\frac{a^4}{3r}.Y^{(1)}+\frac{a^5}{5r^2}.Y^{(2)}+\frac{a^6}{7r^3}.Y^{(3)}+\&c.\right\}; \qquad (5) \qquad [1505]$$

the integral being taken from $a=0$, *to the value of a, corresponding to the surface of the spheroid, and denoted by* a. [1505′]

To obtain the value of V, corresponding to an attracted point, placed within the spheroid, we shall first determine the part of this value, corresponding [1505″] to all the strata within that point. This first part is given by the formula [1505], taking the integral from $a=0$ to $a=a$; *a corresponding to the stratum upon which the attracted point is situated.* We shall determine the [1505‴] second part of V corresponding to all the strata which include the attracted point, by taking the differential of the formula [1496], of the preceding article, relative to a; multiplying this differential by ρ, and then taking its [1505⁗] integral from $a=a$ to $a=\mathrm{a}$; the sum of these two parts of V will be its whole value, corresponding to a point within the spheroid, and we shall have for this sum,†

Value of V for a heterogeneous spheroid attracting an internal point, neglecting terms of the order α^2.

$$V=\frac{4\pi}{3r}.\int_0^{a}\rho.d.a^3+\frac{4\alpha\pi}{r}.\int_0^{a}\rho.d.\left\{a^3.Y^{(0)}+\frac{a^4}{3r}.Y^{(1)}+\frac{a^5}{5r^2}.Y^{(2)}+\frac{a^6}{7r^3}.Y^{(3)}+\&c.\right\}$$
$$+2\pi.\int_a^{\mathrm{a}}\rho.d.a^2+4\alpha\pi.\int_a^{\mathrm{a}}\rho.d.\left\{a^2.Y^{(0)}+\frac{ar}{3}.Y^{(1)}+\frac{r^2}{5}.Y^{(2)}+\frac{r^3}{5a}.Y^{(3)}+\&c.\right\}. \qquad (6) \qquad [1506]$$

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* (1025) The expression [1504] is the differential of [1467], relative to a, and represents the attraction of a stratum of a spheroid whose thickness corresponds to da, included between the values a and $a+da$; the expression being multiplied by the density ρ. The integral of this, relative to a, gives [1505].

† (1026) The two first integrals [1506] are the same as in [1505], taking the limits from $a=0$ to $a=a$. The two last integrals are derived from [1496], as the two first were from [1467]; namely, by taking the differential relative to a, multiplying by ρ, and then integrating relative to a, between the limits $a=a$ and $a=\mathrm{a}$. Lastly, as $r=a.(1+\alpha y)$, we have $\frac{1}{r}=\frac{1-\alpha y}{a}$, neglecting α^2, and in terms multiplied by α, we may write a for r, after integration, as in [1506″].

[1506′] The two first integrals being taken from $a=0$ to $a=a$; and the two last, from $a=a$ to $a=\mathrm{a}$; we must also, after taking the integrals, substitute a for r, in the terms multiplied by α, and $\frac{1-\alpha y}{a}$ for

[1506″] $\frac{1}{r}$, in the term $\frac{4\pi}{3r}\cdot\int_0^a \rho\,.\,d\,.\,a^3$.*

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* (1027) If we put $\alpha=0$, in [1505], we shall get the value of

[1505a]
$$V=\frac{4\pi}{3r}\cdot\int_0^{\mathrm{a}}\rho\,.\,d\,.\,a^3,$$

corresponding to the attraction of a sphere, composed of concentrical strata, in which the density ρ of any stratum is a function of its radius a; the attracted point being without the body. Now, from [1363e], the solidity of a homogeneous sphere, whose radius is a, is $\frac{4}{3}\pi\,.\,a^3$; the differential of this being multiplied by ρ, and then integrated, from $a=0$ to $a=\mathrm{a}$, will give the whole mass M of the sphere whose radius is a, namely,

[1506a]
$$M=\tfrac{4}{3}\pi\,.\int_0^{\mathrm{a}}\rho\,.\,d\,.\,a^3.$$

[1506b] Attraction of a sphere of a variable density on an external point; Substituting this in the preceding value of V, we get $V=\frac{M}{r}$; hence $-\left(\frac{dV}{dr}\right)=\frac{M}{r^2}$ = the attraction of the sphere in the direction of the radius, as in [470, &c.]

In like manner, by putting $\alpha=0$ in [1506], we shall obtain the value of V, corresponding to the attraction of the same sphere, upon an internal point of the body of the sphere, situated at the distance r from its centre, namely,

[1506c]
$$V=\frac{4\pi}{3r}\cdot\int_0^r\rho\,.\,d\,.\,a^3+2\pi\,.\int_r^{\mathrm{a}}\rho\,.\,d\,.\,a^2=\frac{4\pi}{r}\cdot\int_0^r\rho\,.\,a^2da+4\pi\,.\int_r^{\mathrm{a}}\rho\,.\,a\,d\,a.$$

In taking the differential of this relative to r, the terms depending on the variations of the limits destroy each other, as in [1447n]. This will be evident, by observing that when the limit r is changed into $r+dr$, the integral $\int_0^r\rho\,.\,a^2\,d\,a$ is increased, by the element $\rho\,.\,r^2\,d\,r$; and the integral $\int_r^{\mathrm{a}}\rho\,.\,a\,d\,a$ is decreased, by the element $-\rho\,.\,r\,d\,r$. Neglecting therefore these terms, depending on the limits, we get,

on an internal point. [1506d]
$$-\left(\frac{dV}{dr}\right)=\frac{4\pi}{r^2}\cdot\int_0^r\rho\,.\,a^2\,d\,a.$$

If the sphere be hollow, and the attracted point be situated in the internal void space, we must put $\rho=0$, from $a=0$, to a value of a which exceeds r; and then the expression [1506d] will vanish, as in [469‴, &c.]

15. We shall now consider spheroids of any form. The investigation of their attractions is reduced, in § 9, to the formation of the quantities $U^{(i)}$ [1443], and $v^{(i)}$ [1447]; we shall have, by the same article,*

$$U^{(i)}=\int\rho\,.\,R^{i+2}dR\,.\,d\mu'\,.\,d\varpi'\,.\,Q^{(i)}. \qquad [1507]$$

The integrals corresponding to R, must be taken from $R=0$ to its value at the surface; from $\mu'=-1$ to $\mu'=1$, and from $\varpi'=0$ to $\varpi'=2\pi$, [1433″]. To determine this integral, we must find the value of $Q^{(i)}$. This quantity may be developed in a *finite* function of cosines of the angle $\varpi-\varpi'$, and its multiples.† Let $\beta\,.\cos.n\,.(\varpi-\varpi')$ be the term of $Q^{(i)}$, depending on $\cos.n\,.(\varpi-\varpi')$; β being a function of μ and μ'. If we substitute the value of $Q^{(i)}$, in the equation of partial differentials [1442], we shall get, by comparing the terms multiplied by $\cos.n\,.(\varpi-\varpi')$, this equation of common differentials, [1507′] [1507″]

Differential equation in β.

$$0=\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\beta}{d\mu}\right)\right\}}{d\mu}-\frac{n^2\,.\,\beta}{1-\mu^2}+i\,.\,(i+1)\,.\,\beta\,; \qquad [1508]$$

If the sphere be homogeneous, we shall have $\int_0^r\rho\,.\,d\,.\,a^3=\rho\,.\int_0^r d\,.\,a^3=\rho\,.\,r^3$, and $\int_r^{\mathrm{a}}\rho\,.\,d\,.\,a^2=\rho\,.\int_r^{\mathrm{a}}d\,.\,a^2=\rho\,.\,(\mathrm{a}^2-r^2)$. Substituting these in [1506c], we get, [1506e]

$$V=2\pi\rho\,.\,\mathrm{a}^2-\tfrac{2}{3}\pi\rho\,.\,r^2, \qquad -\left(\frac{dV}{dr}\right)=\tfrac{4}{3}\pi\rho\,.\,r. \qquad [1506f]$$

* (1028) Substituting in [1443] the value $\mu'=\cos.\theta'$, and making the limits of the integral from $\mu'=-1$ to $\mu'=1$, as in [1492a, b], it becomes as in [1507].

† (1029) It is evident from [1441″, 1433g], that if T be developed, in a series proceeding according to the powers of $\frac{1}{r}$, the term $Q^{(i)}\,.\,\frac{R^i}{r^{i+1}}$, will contain terms [1508a] depending on the powers of $\cos.(\varpi-\varpi')$, as high as the power i; and these terms may be reduced, as in [6—10] Int., to the form $\beta\,.\cos.n\,.(\varpi-\varpi')$; n being equal to, or less than, i, and β a function of μ, μ'; so that we may put $Q^{(i)}=\Sigma\,.\,\beta\,.\cos.n\,.(\varpi-\varpi')$, using [1508b] the characteristic of finite integrals Σ. Substituting this in [1442], it becomes of the form $0=\Sigma\,.\,B\,.\cos.n\,.(\varpi-\varpi')$; B being a function of μ, μ', represented by the second [1508c] member of the equation [1508], and n any integral number, not exceeding i. The angle $\varpi-\varpi'$ may be varied at pleasure, without changing μ, μ', or B; hence it is evident that we cannot satisfy generally the equation [1508c], except by putting $B=0$, which is the same as the equation [1508].

[1508′] $Q^{(i)}$ being the coefficient of $\frac{R^{(i)}}{r^{i+1}}$ in the development of the radical [1441],

Function T.

[1509] $$T=\frac{1}{\sqrt{r^2-2Rr.\{\mu\mu'+\sqrt{1-\mu^2}.\sqrt{1-\mu'^2}.\cos.(\varpi-\varpi')\}+R^2}}.$$

The term, depending on $\cos.n.(\varpi-\varpi')$, in the development of this radical, can be produced only by the powers of $\cos.(\varpi-\varpi')$, represented
[1509′] by n, $n+2$, $n+4$, &c.;* hence, as $\cos.(\varpi-\varpi')$ has for a factor the
[1509″] quantity $\sqrt{1-\mu^2}$, β ought to have the factor $(1-\mu^2)^{\frac{n}{2}}$. It is evident, from the consideration of the development of the radical [1509], that β is of the following form,†

[1510] $$\beta=(1-\mu^2)^{\frac{n}{2}}.\{A.\mu^{i-n}+A^{(1)}.\mu^{i-n-2}+A^{(2)}.\mu^{i-n-4}+\text{\&c.}\}$$

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* (1030) This is evident from the inspection of the formulas [6—10] Int., putting $\varpi-\varpi'=z$. For if $n=1$, the term $\cos.n.(\varpi-\varpi')=\cos.z$, is found in the formulas [7, 9, &c.] Int., corresponding to $\cos.^3z$, $\cos.^5z$, &c.; whose exponents are
[1509a] $n+2$, $n+4$, &c. If $n=2$, the term $\cos.2.(\varpi-\varpi')=\cos.2z$, is found in [6, 8, 10, &c.] Int., corresponding to $\cos.^2z$, $\cos.^4z$, $\cos.^6z$, &c., whose exponents
[1509b] are n, $n+2$, $n+4$, &c.; and it is evident that the same law prevails for other values of n. In like manner we may infer, from formulas [1—5] Int., that the term $\sin.nz$, or $\cos.nz$, can be produced by the powers of $\sin.z$ whose exponents are n, $n+2$, $n+4$, &c.; and generally we may conclude, from [1—10, 17—20] Int., that *terms of the form*
Theorem. $\sin.nz$, *or* $\cos.nz$, *n being an integer, can be produced only by products of the form*
[1509c] $\cos.^bz.\sin.^cz$, *in which the sum of the exponents* $b+c$ *is equal to* n, $n+2$, $n+4$, &c. This theorem will be frequently used.

† (1031) The coefficient of β, in the development of the radical T [1509], arises from the terms connected with powers of $\cos.(\varpi-\varpi')$, of the degree n, $n+2$, $n+4$, &c., [1509′]; and since $\cos.(\varpi-\varpi')$ is multiplied by the factor $(1-\mu^2)^{\frac{1}{2}}$, in that value of T, these powers of $\cos.(\varpi-\varpi')$ will have the factors $(1-\mu^2)^{\frac{n}{2}}$, $(1-\mu^2)^{\frac{n+2}{2}}$, &c., or, as they may be written, $(1-\mu^2)^{\frac{n}{2}}$, $(1-\mu^2)^{\frac{n}{2}}.(1-\mu^2)$, $(1-\mu^2)^{\frac{n}{2}}.(1-\mu^2)^2$, &c., so that all the terms of β will have the common factor
[1510a] $(1-\mu^2)^{\frac{n}{2}}$, and we may therefore put $\beta=(1-\mu^2)^{\frac{n}{2}}.\beta'$, β' being an integral function of $(1-\mu^2)$ and μ, this last quantity μ being contained in the first term of

If we substitute this value in the differential equation in β [1508], the comparison of the similar powers of μ, will give,

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$$p = \mu\,\mu' + \surd(1-\mu^2)\,.\,\surd(1-\mu'^2)\,.\,\cos.(\varpi - \varpi').$$

This symbol p is used for brevity to denote the coefficient of $-2Rr$, in the denominator of T [1509]; the powers of p being produced in the development of T in a series, and also in the general value of β, or β'. Again, from [1508a], the quantity $Q^{(i)}$ contains no power of p exceeding p^i; it will therefore contain no power or product of μ, $\surd(1-\mu^2)$, exceeding the power i; therefore the term $(1-\mu^2)^{\frac{n}{2}}\,.\,\mu^{i-n}$, will express the form of that term of β, which contains the highest power of μ; and as $\beta' = \beta\,.\,(1-\mu^2)^{-\frac{n}{2}}$, the term μ^{i-n}, will express the highest power of μ which occurs in β'. If we therefore assume for β' a function of the form

$$A\,.\,\mu^{i-n} + A_{,}\,.\,\mu^{i-n-1} + A^{(1)}\,.\,\mu^{i-n-2} + \&c. = \mu^{i-n}\,.\,(A + A_{,}\,.\,\mu^{-1} + A^{(1)}\,.\,\mu^{-2} + \&c.);$$

we shall have $\beta = (1-\mu^2)^{\frac{n}{2}}\,.\,\mu^{i-n}\,.\,(A + A_{,}\,.\,\mu^{-1} + A^{(1)}\,.\,\mu^{-2} + \&c.)$; and it is easy [1510$b$] to prove that the coefficients of the odd powers of μ, namely μ^{-1}, μ^{-3}, &c., must vanish from this expression. For if we change the sign of the three quantities, R, μ, $(1-\mu^2)^{\frac{1}{2}}$, the value of T will not be altered, as is evident from the inspection of the formula [1509]; and the same changes ought not to alter its development [1441″]; that is, it ought not to produce any change in the general terms $Q^{(i)}\,.\,\frac{R^i}{r^{i+1}}$, $Q^{(i)}\,.\,R^i$, or $R^i\,.\,\beta$, [1507″]. This last expression is equal to $R^i\,.\,(1-\mu^2)^{\frac{n}{2}}\,.\,\mu^{i-n}\,.\,(A + A_{,}\,.\,\mu^{-1} + A^{(1)}\,.\,\mu^{-2} + \&c.)$, [1510$b$]; and the alteration of the signs changes R^i into $R^i\,.\,(-1)^i$; $(1-\mu^2)^{\frac{n}{2}}$ into $(1-\mu^2)^{\frac{n}{2}}\,.\,(-1)^n$; μ^{i-n} into $\mu^{i-n}\,.\,(-1)^{i-n}$; hence the factor $R^i\,.\,(1-\mu^2)^{\frac{n}{2}}\,.\,\mu^{i-n}$ becomes $R^i\,.\,(1-\mu^2)^{\frac{n}{2}}\,.\,\mu^{i-n}\,.\,(-1)^{2i}$; but i being an integer, $(-1)^{2i}=1$, therefore this factor will not change its sign; consequently the other factor,

$$(A + A_{,}\,.\,\mu^{-1} + A^{(1)}\,.\,\mu^{-2} + \&c.),$$

must not change its value, by writing $-\mu$ for μ. Now this will not take place, except the coefficients of the odd powers of μ, namely μ^{-1}, μ^{-3}, &c., vanish; in which case this factor will be $(A + A^{(1)}\,.\,\mu^{-2} + A^{(2)}\,.\,\mu^{-4} + \&c.)$, and the expression of β [1510b] will

[1511] $$A^{(s)} = -\frac{(i-n-2s+2).(i-n-2s+1)}{2s.(2i-2s+1)}.A^{(s-1)}.\,*$$

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become as in [1510]. If we use the characteristic of finite integrals Σ, this value of β may be put under the form,

[1510c] $$\beta = \Sigma.(1-\mu^2)^{\frac{n}{2}}.A^{(s)}.\mu^{i-n-2s}.$$

* (1032) The differential of [1510c] gives,

$$\left(\frac{d\beta}{d\mu}\right) = \Sigma.A^{(s)}.\left\{-n\mu.(1-\mu^2)^{\frac{n}{2}-1}.\mu^{i-n-2s} + (i-n-2s).(1-\mu^2)^{\frac{n}{2}}.\mu^{i-n-2s-1}\right\}$$

$$= \Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}-1}.\left\{-n\mu^{i-n-2s+1} + (i-n-2s).(1-\mu^2).\mu^{i-n-2s-1}\right\}$$

[1510d] $$= \Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}-1}.\left\{-(i-2s).\mu^{i-n-2s+1} + (i-n-2s).\mu^{i-n-2s-1}\right\}.$$

[1510e] Multiplying this last expression by $(1-\mu^2)$, and putting, for brevity, $i-n-2s=m$, we get,

[1510f] $$(1-\mu^2).\left(\frac{d\beta}{d\mu}\right) = \Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}}.\left\{-(m+n).\mu^{m+1} + m.\mu^{m-1}\right\};$$

its differential, divided by $d\mu$, is

$$\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{d\beta}{d\mu}\right)\right\}}{d\mu}\right\}$$

$$= \Sigma.A^{(s)}.\left\{\begin{array}{l} (1-\mu^2)^{\frac{n}{2}-1}.\{(m+n).n.\mu^{m+2} - mn.\mu^m\} \\ +(1-\mu^2)^{\frac{n}{2}}.\{-(m+n).(m+1).\mu^m + m.(m-1).\mu^{m-2}\} \end{array}\right\}$$

$$= \Sigma.A^{(s)}.\left\{\begin{array}{l} (1-\mu^2)^{\frac{n}{2}-1}.\{n^2.\mu^{m+2} - mn.(1-\mu^2).\mu^m\} \\ +(1-\mu^2)^{\frac{n}{2}}.\{-(m+n).(m+1).\mu^m + m.(m-1).\mu^{m-2}\} \end{array}\right\}$$

$$= \Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}-1}.n^2.\mu^{m+2}$$
$$+ \Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}}.\{-mn.\mu^m - (m+n).(m+1).\mu^m + m.(m-1).\mu^{m-2}\}$$

[1510g] $$= \Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}-1}.n^2.\mu^{m+2}$$
$$+ \Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}}.\{(-m^2-2mn-m-n).\mu^m + m.(m-1).\mu^{m-2}\}.$$

Hence we may obtain the values of $A^{(1)}$, $A^{(2)}$, &c., by making successively $s=1$, $s=2$, &c.; consequently,

$$\beta = A.(1-\mu^2)^{\frac{n}{2}}.\left\{\begin{array}{l}\mu^{i-n} - \frac{(i-n).(i-n-1)}{2.(2i-1)}.\mu^{i-n-2} + \frac{(i-n).(i-n-1).(i-n-2).(i-n-3)}{2.4.(2i-1).(2i-3)}.\mu^{i-n-4} \\ - \frac{(i-n).(i-n-1).(i-n-2).(i-n-3).(i-n-4).(i-n-5)}{2.4.6.(2i-1).(2i-3).(2i-5)}.\mu^{i-n-6} + \&c.\end{array}\right\}. \quad [1512]$$

The same value of β [1510c] gives

$$-\frac{n^2.\beta}{1-\mu^2} = -\Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}-1}.n^2.\mu^m; \quad [1510h]$$

$$i.(i+1).\beta = \Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}}.i.(i+1).\mu^m. \quad [1510i]$$

Adding together the three equations [1510g, h, i], the first member becomes like the second of [1508], and is therefore equal to nothing. In taking the sum of the second members, we may connect together, the first term of [1510g], with that in [1510h], and the sum of these two terms becomes,

$$\Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}-1}.n^2.\{\mu^{m+2}-\mu^m\} = -\Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}}.n^2.\mu^m;$$

therefore the sum of the three equations has the common factor $(1-\mu^2)^{\frac{n}{2}}$, and is expressed by

$$0 = \Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}}.[\{-m^2-2mn-m-n+i.(i+1)-n^2\}.\mu^m + m.(m-1).\mu^{m-2}]. \quad [1510k]$$

The coefficient of μ^m may be reduced, by means of $m+n=i-2s$, [1510e], since

$$-m^2-2mn-m-n+i.(i+1)-n^2 = -(m+n)^2-(m+n)+i.(i+1)$$
$$= -(i-2s)^2-(i-2s)+i.(i+1) \qquad = -(i^2-4is+4s^2)-(i-2s)+i.(i+1)$$
$$= 4is-4s^2+2s \qquad = 2s.(2i-2s+1).$$

Hence [1510k] becomes, by the resubstitution of the value of m [1510e],

$$0 = \Sigma.A^{(s)}.(1-\mu^2)^{\frac{n}{2}}.\{2s.(2i-2s+1).\mu^{i-n-2s} + (i-n-2s).(i-n-2s-1).\mu^{i-n-2s-2}\}. \quad [1510l]$$

This may be put under another form, so that both terms may contain the same power of μ; by changing, in the second term, s into $s-1$, and therefore $A^{(s)}$ into $A^{(s-1)}$, and we shall get,

$$0 = \Sigma.(1-\mu^2)^{\frac{n}{2}}.\mu^{i-n-2s}.\{2s.(2i-2s+1).A^{(s)} + (i-n-2s+2).(i-n-2s+1).A^{(s-1)}\}.$$

This equation exists for all values of μ; hence it follows, that the coefficient of each power of μ must be separately equal to nothing; therefore we shall have,

$$0 = 2s.(2i-2s+1).A^{(s)} + (i-n-2s+2).(i-n-2s+1).A^{(s-1)}. \quad [1510m]$$

A is a function of μ' independent of μ.* Now as μ, μ', are contained in the
[1512'] same manner in the preceding radical [1509], they ought to appear under the same form in the expression of β; therefore we shall have,†

Dividing by $2s.(2i-2s+1)$, we get $A^{(s)}$ [1511]; and if in this we put successively $s=1$, $s=2$, $s=3$, &c., we shall obtain the following values of $A^{(1)}$, $A^{(2)}$, $A^{(3)}$, expressed in terms of $A^{(0)}$, or A.

$$A^{(1)}=-\frac{(i-n).(i-n-1)}{2.(2i-1)}.A;$$

$$A^{(2)}=-\frac{(i-n-2).(i-n-3)}{4.(2i-3)}.A^{(1)}=\frac{(i-n).(i-n-1).(i-n-2).(i-n-3)}{2.4.(2i-1).(2i-3)}.A;$$

$$A^{(3)}=-\frac{(i-n-4).(i-n-5)}{6.(2i-5)}.A^{(2)}$$

[1510n] $$=-\frac{(i-n).(i-n-1).(i-n-2).(i-n-3).(i-n-4).(i-n-5)}{2.4.6.(2i-1).(2i-3).(2i-5)}.A.$$

Substituting these in β [1510], it becomes as in [1512]; the series being continued till the
[1510o] exponent of μ becomes 1 or 0, all negative powers of μ being excluded.

* (1033) The radical T [1509], is a function of R, r, μ, μ', $\varpi-\varpi'$, and when it is
[1512a] developed according to the powers of $\frac{R}{r}$, [1441''], its general term is $Q^{(i)}.\frac{R^i}{r^{i+1}}$; in which $Q^{(i)}$ is independent of R, r, and must therefore be a function of μ, μ', $\varpi-\varpi'$, and constant quantities. Now by hypothesis [1507''], $\beta.\cos.n.(\varpi-\varpi')$ represents the part of $Q^{(i)}$ depending on the angle $n.(\varpi-\varpi')$; therefore β is independent of $\varpi-\varpi'$, and must be a function of μ, μ', only. Lastly, when β is arranged according to the powers of μ, as in [1510], the quantities A, $A^{(1)}$, &c., must be independent of μ, and they must
[1512b] therefore be functions of the remaining quantity μ', and are wholly independent of μ, R, r, $\varpi-\varpi'$.

† (1034) We shall put, for brevity,

[1513a] $$(1-\mu^2)^{\frac{n}{2}}.\left\{\mu^{i-n}-\frac{(i-n).(i-n-1)}{2.(2i-1)}.\mu^{i-n-2}+\&c.\right\}=\varphi(\mu);$$

[1513b] $\varphi(\mu)$ representing a function of μ, and the formula [1512] will become $\beta=A.\varphi(\mu)$; A being a function of μ', independent of μ [1512b]. Now in the radical, which represents the value of T [1441'], we may change μ into μ', and μ' into μ, without altering its value; we may therefore do the same in the value of β, so that the value of β must contain the same function of μ' as it does of μ; hence the factor A must be of the form $A=\gamma.\varphi(\mu')$, as in [1513], γ being independent of μ', for the same reason that A [1512'] is independent of μ. Hence γ must be a constant quantity, independent of μ, μ', and we shall have, as in [1514],

[1513c] $$\beta=\gamma.\varphi(\mu).\varphi(\mu').$$

$$A=\gamma\,.\,(1-\mu'^2)^{\frac{n}{2}}.\left\{\mu'^{\,i-n}-\frac{(i-n)\,.\,(i-n-1)}{2\,.\,(2i-1)}\,.\,\mu'^{\,i-n-2}+\&c.\right\},\qquad [1513]$$

γ being a coefficient independent of μ and μ', therefore

General value of β.

$$\beta=\gamma\,.\,(1-\mu'^2)^{\frac{n}{2}}.\left\{\mu'^{\,i-n}-\frac{(i-n)\,.\,(i-n-1)}{2\,.\,(2i-1)}\,.\,\mu'^{\,i-n-2}+\&c.\right\}$$
$$\times\,(1-\mu^2)^{\frac{n}{2}}.\left\{\mu^{i-n}-\frac{(i-n)\,.\,(i-n-1)}{2\,.\,(2i-1)}\,.\,\mu^{i-n-2}+\&c.\right\}.\qquad [1514]$$

Hence we find that β is separated into three factors. The first is independent of μ and μ'; the second is a function of μ' only; and the third is a similar function of μ. It now remains to determine γ. [1514′]

For this purpose we shall observe, that if $i-n$ be even, we shall have, by supposing $\mu=0$, and $\mu'=0$.* [1514″]

$$\beta=\frac{\gamma.\{1.2.3\,.....\,(i-n)\}^2}{\{2.4\,.....(i-n).(2i-1).(2i-3).....(i+n+1)\}^2}=\frac{\gamma.\{1.3.5\,.....(i-n-1).1.3.5\,.....(i+n-1)\}^2}{\{1.3.5\,.....\,(2i-1)\}^2}.\qquad [1515]$$

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* (1035) If we examine the part between the braces, in the expression [1512], we shall find that the term depending on μ^{i-n-2s} is

$$\pm\frac{(i-n).(i-n-1).(i-n-2)\,.....(i-n-2s+1)}{2.4.6\,.....2s\,.\,(2i-1).(2i-3).(2i-5).(2i-2s+1)}\,.\,\mu^{i-n-2s};\qquad [1515a]$$

and if $\mu=0$, the factor $\mu^{i-n-2s}=0$, except in the case where $i-n-2s=0$, there being no negative exponents of μ [1510*o*]. This excepted case corresponds to $i-n=2s$, an even number, and $\mu^{i-n-2s}=\mu^0=1$. The preceding expression then becomes $\pm\dfrac{(i-n).(i-n-1).(i-n-2)\,.....\,3.2.1}{2.4.6\,.....(i-n).(2i-1).(2i-3).(2i-5)\,.....(i+n+1)}$. Inverting the order of the terms of the numerator, and observing that when $\mu=0$, the quantity $(1-\mu^2)^{\frac{n}{2}}=1$, the whole factor of A, in the value of β [1513*b*, *a*], will become

$$\varphi(\mu)=\pm\frac{1.2.3\,.....(i-n-2).(i-n-1).(i-n)}{2.4.6\,.....(i-n).(2i-1).(2i-3)\,.....(i+n+1)};\qquad [1515b]$$

and in like manner, when $\mu'=0$, we shall have,

$$\varphi(\mu')=\pm\frac{1.2.3\,.....(i-n-2).(i-n-1).(i-n)}{2.4.6\,.....(i-n).(2i-1).(2i-3)\,.....(i+n+1)}.\qquad [1515b']$$

The product of these two quantities is

$$\varphi(\mu).\varphi(\mu')=\frac{\{1.2.3\,.....(i-n)\}^2}{\{2.4.6\,.....(i-n).(2i-1).(2i-3)\,.....(i+n+1)\}^2};\qquad [1515c]$$

[1515'] If $i-n$ be odd, we shall have, by retaining only the first power of μ and μ',*

[1516] $$\beta=\frac{\gamma.\mu\mu'.\{1.2.3.....(i-n)\}^2}{\{2.4.....(i-n-1).(2i-1).(2i-3).....(i+n+2)\}^2}=\frac{\gamma.\mu\mu'.\{1.3.5.....(i-n).1.3.5.....(i+n)\}^2}{\{1.3.5.....(2i-1)\}^2}.$$

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and $\beta=\gamma.\varphi(\mu).\varphi(\mu')$ [1513c] becomes, in this case,

[1515d] $$\beta=\frac{\gamma.\{1.2.3.....(i-n)\}^2}{\{2.4.6.....(i-n).(2i-1).(2i-3).....(i+n+1)\}^2}.$$

This is the first of the values [1515]; and as the terms $2.4.6.....(i-n)$ are included both in the numerator and denominator, we may reject them, and we shall get

[1515e] $$\beta=\frac{\gamma.\{1.3.5.....(i-n-1)\}^2}{\{(2i-1).(2i-3).....(i+n+1)\}^2}.$$

If we multiply the numerator and denominator by the factor $\{1.3.5.(i+n-1\}^2$, the new numerator will be $\gamma.\{1.3.5.....(i-n-1).1.3.5.....(i+n-1)\}^2$, and the new denominator, $\{(2i-1).(2i-3).....(i+n+1).(i+n-1).....5.3.1\}^2$; or, by inverting the order of the terms, $\{1.3.5.....(2i-1)\}^2$; hence the expression [1515e], will become, as in the second form [1515].

* (1036) If $i-n$ be odd, and we put $i-n-2s=1$, the expression [1515a] will become,

[1516a] $$\pm\frac{(i-n).(i-n-1).(i-n-2).....3.2}{2.4.6.....(i-n-1).(2i-1).(2i-3).....(i+n+2)}.\mu;$$

and this will be the term depending on the first power of μ, in $\varphi(\mu)$ [1513a], because $(1-\mu^2)^{\frac{n}{2}}$, contains no powers of μ less than μ^2.

Changing μ into μ', we obtain the similar term of $\varphi(\mu')$. The product of these two quantities gives the part of $\varphi(\mu).\varphi(\mu')$ depending on the first power of $\mu\mu'$, which will be $\frac{\mu\mu'.\{(i-n).(i-n-1).....3.2\}^2}{\{2.4.6.....(i-n-1).(2i-1).(2i-3).....(i+n+2)\}^2}$. Substituting this in [1513c], and inverting the order of the terms of the numerator, we shall get,

[1516b] $$\beta=\frac{\gamma.\mu\mu'.\{1.2.3.....(i-n)\}^2}{\{2.4.6.....(i-n-1).(2i-1).(2i-3).....(i+n+2)\}^2},$$

as in the first expression [1516]. Now it is evident that the terms $2.4.6.....(i-n-1)$ of the denominator, occur also in the numerator, and by rejecting them from both, we have $\beta=\frac{\gamma.\mu\mu'.\{1.3.5.....(i-n)\}^2}{\{(2i-1).(2i-3).....(i+n+2)\}^2}$. If we multiply the numerator by

The preceding radical [1509] becomes, by neglecting the squares of μ and μ',*

$$\{r^2-2Rr.\cos.(\varpi-\varpi')+R^2\}^{-\frac{1}{2}}+Rr.\mu\mu'.\{r^2-2rR.\cos.(\varpi-\varpi')+R^2\}^{-\frac{3}{2}}. \qquad (f)$$ [1517]

If we substitute, for $\cos.(\varpi-\varpi')$, its value in imaginary exponential quantities, and put c for the number whose hyperbolic logarithm is unity; the part independent of $\mu\mu'$ becomes,† [1517']

$$\{r^2-2Rr.\cos.(\varpi-\varpi')+R^2\}^{-\frac{1}{2}}$$
$$=\{r-R.c^{(\varpi-\varpi').\sqrt{-1}}\}^{-\frac{1}{2}}.\{r-R.c^{-(\varpi-\varpi').\sqrt{-1}}\}^{-\frac{1}{2}}.$$ [1518]

The coefficient of $\frac{R^i}{r^{i+1}}.\left\{\frac{c^{n.(\varpi-\varpi').\sqrt{-1}}+c^{-n.(\varpi-\varpi').\sqrt{-1}}}{2}\right\}$, or of [1518']

$\frac{R^i}{r^{i+1}}.\cos.n.(\varpi-\varpi')$, in the development of this function, is,‡

$\{1.3.5\,....\,(i+n)\}^2$, and the denominator by the same quantity, in an inverted order, we shall get, $\beta=\frac{\gamma.\mu\mu'.\{1.3.5\,....\,(i-n).1.3.5\,....\,(i+n)\}^2}{\{(2i-1).(2i-3)\,.....\,5.3.1\}^2}$; which, by inverting [1516c] the order of the terms of the denominator, becomes like the second expression [1516].

* (1037) If we neglect μ^2, μ'^2, and put for brevity $S=r^2-2rR.\cos.(\varpi-\varpi')+R^2$, the expression [1509] will become, as in [1517],

$$T=(S-2Rr.\mu\mu')^{-\frac{1}{2}}=S^{-\frac{1}{2}}+Rr.\mu\mu'.S^{-\frac{3}{2}}.$$ [1517a]

† (1038) Substituting $\cos.(\varpi-\varpi')=\frac{1}{2}.\{c^{(\varpi-\varpi').\sqrt{-1}}+c^{-(\varpi-\varpi').\sqrt{-1}}\}$, [1518a] [12] Int., in the first term of [1517], it becomes,

$$\{r^2-2Rr.\cos.(\varpi-\varpi')+R^2\}^{-\frac{1}{2}}=\{r^2-Rr.c^{(\varpi-\varpi).\sqrt{-1}}-Rr.c^{-(\varpi-\varpi').\sqrt{-1}}+R^2\}^{-\frac{1}{2}}$$
$$=\{(r-R.c^{(\varpi-\varpi').\sqrt{-1}}).(r-R.c^{-(\varpi-\varpi').\sqrt{-1}})\}^{-\frac{1}{2}}$$
$$=(r-R.c^{(\varpi-\varpi').\sqrt{-1}})^{-\frac{1}{2}}.(r-R^{-(\varpi-\varpi').\sqrt{-1}})^{-\frac{1}{2}},$$ as in [1518]. [1518b]

‡ (1039) Putting for brevity $(\varpi-\varpi').\sqrt{-1}=z$, the expression [1518] becomes [1518c] $(r-R.c^z)^{-\frac{1}{2}}.(r-R.c^{-z})^{-\frac{1}{2}}$. Developing each of these factors, and noting the general terms corresponding to c^{gz}, $c^{-g'z}$, we get,

[1519] $$\frac{2\,.\,1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i+n-1)\,.\,1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i-n-1)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i+n)\,.\,2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i-n)};$$

[1518d]
$$(r-R\,.\,c^{z})^{-\frac{1}{2}}=\frac{1}{r^{\frac{1}{2}}}+\tfrac{1}{2}\,.\,\frac{R}{r^{\frac{3}{2}}}\,.\,c^{z}+\tfrac{1}{2}\,.\tfrac{3}{4}\,.\,\frac{R^2}{r^{\frac{5}{2}}}\,.\,c^{2z}+\,.\,.\,.\,.\,.\,\frac{1.3.5\,.\,.\,.\,.\,.\,(2g-1)}{2.4.6\,.\,.\,.\,.\,.\,2g}\,.\,\frac{R^{g}}{r^{g+\frac{1}{2}}}\,.\,c^{gz},$$
$$(r-R\,.\,c^{-z})^{-\frac{1}{2}}=\frac{1}{r^{\frac{1}{2}}}+\tfrac{1}{2}\,.\,\frac{R}{r^{\frac{3}{2}}}\,.\,c^{-z}+\tfrac{1}{2}\,.\tfrac{3}{4}\,.\,\frac{R^2}{r^{\frac{5}{2}}}\,.\,c^{-2z}+\,.\,.\,.\,.\,.\,\frac{1.3.5\,.\,.\,.\,.\,.\,(2g'-1)}{2.4.6\,.\,.\,.\,.\,.\,2g'}\,.\,\frac{R^{g'}}{r^{g'+\frac{1}{2}}}\,.\,c^{-g'z}.$$

Multiplying these quantities together, the general term of the product will be

[1518e] $$\frac{1.3.5\,.\,.\,.\,.\,.\,(2g-1)}{2.4.6\,.\,.\,.\,.\,.\,2g}\,.\,\frac{1.3.5\,.\,.\,.\,.\,.\,(2g'-1)}{2.4.6\,.\,.\,.\,.\,.\,2g'}\,.\,\frac{R^{g+g'}}{r^{g+g'+1}}\,.\,c^{(g-g').z}.$$

Therefore, to obtain the terms corresponding to $\frac{R^i}{r^{i+1}}\,.\,c^{nz}$, we must put $g+g'=i$, $g-g'=n$, whence $g=\frac{1}{2}\,.\,(i+n)$, $g'=\frac{1}{2}\,.\,(i-n)$. Substituting these in [1518e],

[1520a] it becomes $\frac{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i+n-1)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i+n)}\,.\,\frac{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i-n-1)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i-n)}\,.\,\frac{R^i}{r^{i+1}}\,.\,c^{nz}$. Changing n into $-n$, we get the factors of the term depending on c^{-nz}; which will be found to be the product of the same factors as those of c^{nz}; the order of the two factors being changed. Connecting these two expressions together, the sum will be

[1520b] $$\frac{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i+n-1)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i+n)}\,.\,\frac{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i-n-1)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i-n)}\,.\,\frac{R^i}{r^{i+1}}\,.\,\{c^{nz}+c^{-nz}\}.$$

[1520c] Substituting $c^{nz}+c^{-nz}=c^{n\,.\,(\varpi-\varpi')\,.\,\sqrt{-1}}+c^{-n\,.\,(\varpi-\varpi')\,.\,\sqrt{-1}}=2\,.\,\cos.\,n\,.\,(\varpi-\varpi')$, [1518a, c], it becomes,

[1520d] $$\frac{2\,.\,1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i+n-1)\,.\,1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i-n-1)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i+n)\,.\,2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i-n)}\,.\,\frac{R^i}{r^{i+1}}\,.\,\cos.\,n\,.\,(\varpi-\varpi');$$

which is the part of $\beta\,.\,\frac{R^i}{r^{i+1}}\,.\,\cos.\,n\,.\,(\varpi-\varpi')$, [1507″, 1517], independent of μ, μ'. Comparing this part of β with the second value [1515], which also corresponds to the terms independent of μ, μ', [1514″], we shall get,

[1520e] $$\frac{2.1.3.5\,.\,.\,.\,.\,.\,(i+n-1).1.3.5\,.\,.\,.\,.\,.\,(i-n-1)}{2.4.6\,.\,.\,.\,.\,.\,(i+n)\,.2.4.6\,.\,.\,.\,.\,.\,(i-n)}=\frac{\gamma.\{1.3.5\,.\,.\,.\,.\,.\,(i-n-1).1.3.5\,.\,.\,.\,.\,.\,(i+n-1)\}^2}{\{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(2i-1)\}^2};$$

[1520f] rejecting the first power of the factors $1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i+n-1)\,.\,1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i-n-1)$, which occur in both numerators, and then dividing by the coefficient of γ, we get,

[1520g] $$\gamma=\frac{2\,.\,\{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(2i-1)\}^2}{\{1.3.5\,.\,.\,.\,.\,.\,(i-n-1)\,.2\,.4\,.6\,.\,.\,.\,.\,.\,(i-n)\}.\{1.3.5\,.\,.\,.\,.\,.\,(i+n-1)\,.2\,.4\,.6\,.\,.\,.\,.\,.\,(i+n)\}}.$$

which is the value of β when $i-n$ is even. Comparing it with that we have found in the same case, we shall have, [1519′]

$$\gamma=2.\left(\frac{1.3.5\,.....\,(2i-1)}{1.2.3\,.....\,i}\right)^2.\frac{i.(i-1)\,.....\,(i-n+1)}{(i+1).(i+2)\,.....\,(i+n)}. \qquad [1520]$$

Values of γ.

*When $n=0$, we must take but the half of this coefficient, and then we shall have,**

$$\gamma=\left(\frac{1.3.5\,.....\,(2i-1)}{1.2.3\,.....\,i}\right)^2. \qquad [1521]$$

Likewise the coefficient of $\frac{R^i}{r^{i+1}}.\mu\mu'.\cos.n.(\varpi-\varpi')$, in the function [1517], is

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Now we have, evidently,

$$1.3.5.....(i-n-1).2.4.6.....(i-n)=1.2.3.4.....(i-n)=\frac{1.2.3\,.....\,i}{i.(i-1).....(i-n+1)}; \qquad [1520h]$$

$$1.3.5.....(i+n-1).2.4.6.....(i+n)=1.2.3.4.....(i+n)$$
$$=\{1.2.3.....i\}.\{(i+1).(i+2)\,.....\,(i+n)\}. \qquad [1520h']$$

Substituting this in [1520*g*], it becomes as in [1520].

* (1040) In general the product of the factors [1518*d*] produces, as in [1520*a*—*b*], *one* factor depending on c^{nz}, and *another* on c^{-nz}, each being multiplied by [1521*a*]

$$\frac{1.3.5\,.....\,(i+n-1)}{2.4.6\,.....\,(i+n)}.\frac{1.3.5\,.....\,(i-n-1)}{2.4.6\,.....\,(i-n)}.\frac{R^i}{r^{i+1}};$$

and if we put, for $c^{nz}+c^{-nz}$, its value $2.\cos.n.(\varpi-\varpi')$ [1520*c*], it multiplies this factor by 2; but we must except the case of $n=0$, for then only *one* term, depending on the factor $\frac{R^i}{r^{i+1}}$, will be produced, namely,

$$\frac{1.3.5\,.....\,(i-1)}{2.4.6\,.....\,i}.\frac{1.3.5\,.....\,(i-1)}{2.4.6\,.....\,i}.\frac{R^{(i)}}{r^{i+1}},$$

which is but half the value resulting from making $n=0$, in the general expression [1520*d*]. Therefore, when $n=0$, we must take half the value of γ [1520]; observing that the factor $\frac{i.(i-1)\,.....\,(i-n+1)}{(i+1).(i+2)\,.....\,(i+n)}=1$, as will evidently appear by the inspection of the formulas [1520*h*, *h*′], in which these terms of this factor are produced. Hence the value of γ will become, for this case, as in [1521]. [1521*b*]

[1522] $$\frac{2\,.\,1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i+n)\,.\,1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(i-n)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i+n-1)\,.\,2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i-n-1)};*$$

which is the coefficient of $\mu\mu'$ in the value of β, when we neglect the
[1522′] squares of μ and μ', and when $i-n$ is odd. Comparing it with the

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* (1041) The term depending on $\mu\mu'$, in [1517], is, by using [1518, 1518c],

$$R\,r\,.\,\mu\mu'.\{r^2-2\,R\,r\,.\cos.(\varpi-\varpi')+R^2\}^{-\frac{3}{2}}=Rr.\,\mu\mu'.(r-R.c^{z})^{-\frac{3}{2}}.(r-R.c^{-z})^{-\frac{3}{2}}$$

[1523a] $$=\frac{R}{r^2}.\mu\mu'.\left(1-\frac{R}{r}.c^{z}\right)^{-\frac{3}{2}}.\left(1-\frac{R}{r}.c^{-z}\right)^{-\frac{3}{2}}.$$

Developing these factors, we get,

$$\left\{1-\frac{R}{r}.c^{z}\right\}^{-\frac{3}{2}}=1+\tfrac{3}{2}.\frac{R}{r}.c^{z}+\tfrac{3}{2}.\tfrac{5}{4}.\frac{R^2}{r^2}.c^{2z}+\ldots..\frac{3.5.7\ldots..(2g+1)}{2.4.6\ldots..2g}.\frac{R^g}{r^g}.c^{gz};$$

[1523b]

$$\left\{1-\frac{R}{r}.c^{-z}\right\}^{-\frac{3}{2}}=1+\tfrac{3}{2}.\frac{R}{r}.c^{-z}+\tfrac{3}{2}.\tfrac{5}{4}.\frac{R^2}{r^2}.c^{-2z}+\ldots..\frac{3.5.7\ldots..(2g'+1)}{2.4.6\ldots..2g'}.\frac{R^{g'}}{r^{g'}}.c^{-g'z}.$$

The product of the general terms of these expressions depending on c^{gz}, $c^{-g'z}$, produces, in [1523a], the expression

[1523c] $$\frac{3\,.\,5\,.\,7\,.\,.\,.\,.\,.\,(2g+1)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,2g}\,.\,\frac{3\,.\,5\,.\,7\,.\,.\,.\,.\,.\,(2g'+1)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,2g'}\,.\,\frac{R^{g+g'+1}}{r^{g+g'+2}}\,.\,c^{(g-g').z}\,.\,\mu\mu'.$$

[1523d] To obtain the term depending on $\frac{R^i}{r^{i+1}}.c^{nz}$, we must put $i=g+g'+1$, $n=g-g'$, or $g=\frac{1}{2}.(i+n-1)$, $g'=\frac{1}{2}.(i-n-1)$, and if for brevity we put

[1523e] $$\mathcal{N}=\frac{3\,.\,5\,.\,7\,.\,.\,.\,.\,.\,(i+n)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i+n-1)}\,.\,\frac{3\,.\,5\,.\,7\,.\,.\,.\,.\,.\,(i-n)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(i-n-1)},$$

the preceding expression [1523c] will become $\mathcal{N}.\frac{R^i}{r^{i+1}}.c^{nz}.\mu\mu'$. If we change n into $-n$, which does not alter the value of $\mathcal{N}$, we shall obtain the term corresponding to c^{-nz}, namely, $\mathcal{N}.\frac{R^i}{r^{i+1}}.c^{-nz}.\mu\mu'$. The sum of these two expressions, reduced as in [1520c], is

[1523f] $$\mathcal{N}.\frac{R^i}{r^{i+1}}.\mu\mu'.(c^{nz}+c^{-nz})=2\,\mathcal{N}.\frac{R^i}{r^{i+1}}.\mu\mu'.\cos.\frac{nz}{\sqrt{-1}}=2\,\mathcal{N}.\frac{R^i}{r^{i+1}}.\mu\mu'.\cos.n.(\varpi-\varpi');$$

and the coefficient $2\,\mathcal{N}$ is the same as in [1522]. The values of i, n, [1523d], make $i-n=2g'+1$, which is an odd number, as in [1522′].

expression which we have found for this coefficient in the same case, we shall have*

$$\gamma = 2 \cdot \left(\frac{1 \cdot 3 \cdot 5 \, \ldots \ldots (2i-1)}{1 \cdot 2 \cdot 3 \, \ldots \ldots \, i}\right)^2 \cdot \frac{i \cdot (i-1) \, \ldots \ldots (i-n+1)}{(i+1) \cdot (i+2) \, \ldots \ldots (i+n)}; \qquad [1523]$$

an expression which is the same as in the case where $i-n$ is even [1520]. [1523′]
If $n=0$, we shall also have,† as in the former case [1521],

$$\gamma = \left(\frac{1 \cdot 3 \cdot 5 \, \ldots \ldots (2i-1)}{1 \cdot 2 \cdot 3 \, \ldots \ldots \, i}\right)^2. \qquad [1524]$$

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* (1042) Putting the coefficient of $\frac{R^i}{r^{i+1}} \cdot \cos. \; n(\varpi - \varpi')$, [1523*f*], equal to the second expression of the same quantity [1516], we get

$$2\mathcal{N} \cdot \mu\mu' = \frac{\gamma \cdot \mu\mu' \cdot \{1.3.5 \ldots\ldots (i-n).1.3.5 \ldots\ldots (i+n)\}^2}{\{1.3.5 \ldots\ldots (2i-1)\}^2}. \qquad [1523g]$$

Substituting $\mathcal{N}$ [1523*e*], and rejecting the factor $1.3.5 \ldots\ldots (i-n) \cdot 1.3.5 \ldots\ldots (i+n) \cdot \mu\mu'$, which is common to both members, we shall get,

$$\gamma = \frac{2 \cdot \{1 \cdot 3 \cdot 5 \, \ldots \ldots (2i-1)\}^2}{\{2.4.6 \ldots\ldots (i+n-1).2.4.6 \ldots\ldots (i-n-1)\} \cdot \{1.3.5 \ldots\ldots (i+n).1.3.5 \ldots\ldots (i-n)\}}$$

$$= \frac{2 \cdot \{1 \cdot 3 \cdot 5 \, \ldots \ldots (2i-1)\}^2}{\{1.3.5 \ldots\ldots (i+n).2.4.6 \ldots\ldots (i+n-1)\} \cdot \{1.3.5 \ldots\ldots (i-n).2.4.6 \ldots\ldots (i-n-1)\}}$$

$$= \frac{2 \cdot \{1 \cdot 3 \cdot 5 \, \ldots \ldots (2i-1)\}^2}{\{1 \cdot 2 \cdot 3 \cdot 4 \, \ldots \ldots (i+n)\} \cdot \{1 \cdot 2 \cdot 3 \cdot 4 \, \ldots \ldots (i-n)\}}.$$

Substituting in the denominator of this last expression $1.2.3 \ldots\ldots (i-n) = \frac{(1.2.3 \ldots\ldots i)}{i.(i-1) \ldots\ldots (i-n+1)}$, and $1 \cdot 2 \cdot 3 \, \ldots \ldots (i+n) = \{1 \cdot 2 \cdot 3 \, \ldots \ldots \, i\} \cdot (i+1) \cdot (i+2) \, \ldots \ldots (i+n)$, it becomes as in [1523].

† (1043) Proceeding in this case as in [1521*a*, *b*], we easily perceive that the product of the two factors [1523*b*] does not produce *two* terms multiplied by $\frac{R^{i-1}}{r^{i-1}}$, but simply *one* term, $\mathcal{N} \cdot \frac{R^{i-1}}{r^{i-1}} \cdot \mu\mu'$, found by putting $g = g' = \frac{1}{2} \cdot (i-1)$ in [1523*d*, *e*]; the corresponding term of [1523*a*] being $\mathcal{N} \cdot \frac{R^i}{r^{i+1}} \cdot \mu\mu'$. The factor of $\frac{R^i}{r^{i+1}}$, in this last expression, namely, $\mathcal{N} \cdot \mu\mu'$, being put equal to the second value of β, [1516],

16. From what has been said, we may determine the general form of
[1524'] the function $Y^{(i)}$, composed of μ, $\sqrt{1-\mu^2}\,.\,\sin.\varpi$, $\sqrt{1-\mu^2}\,.\,\cos.\varpi$, which satisfies the equation of partial differentials [1465],*

Differential equation in $Y^{(i)}$.

[1525]
$$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}+\frac{\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,Y^{(i)}.$$

[1525'] Denoting the coefficient of $\sin.n\,\varpi$, or $\cos.n\,\varpi$, in this function $Y^{(i)}$, by β, we shall have,

Differential equation in β.

[1526]
$$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,\beta}{d\,\mu}\right)\right\}}{d\,\mu}\right\}-\frac{n^2\,.\,\beta}{1-\mu^2}+i\,.\,(i+1)\,.\,\beta.$$

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corresponding to the case of $n=0$, we get $\mathcal{N}\,.\,\mu\mu'=\frac{\gamma\,.\,\mu\mu'.\{1.3.5\,.....\,i\}^4}{\{1.3.5\,.....\,(2i-1)\}^2}$. Putting $n=0$, in [1523e], we get the value of $\mathcal{N}$, corresponding to this case,

$$\mathcal{N}=\frac{\{1\,.\,3\,.\,5\,.....\,i\}^2}{\{2\,.\,4\,.\,6\,.....\,(i-1)\}^2}.$$

This being substituted in the preceding equation, divided by the coefficient of γ, we get

$$\gamma=\frac{\{1\,.\,3\,.\,5\,.....\,i\}^2}{\{2\,.\,4\,.\,6\,.....\,(i-1)\}^2}\,.\,\frac{\{1\,.\,3\,.\,5\,.....\,(2i-1)\}^2}{\{1\,.\,3\,.\,5\,.....\,i\}^4}$$

$$=\frac{\{1\,.\,3\,.\,5\,.....\,(2i-1)\}^2}{\{2\,.\,4\,.\,6\,......\,(i-1)\}^2\,.\,\{1\,.\,3\,.\,5\,.....\,i\}^2}=\frac{\{1\,.\,3\,.\,5\,.....\,(2i-1)\}^2}{\{1\,.\,2\,.\,3\,.....\,i\}^2}.$$

as in [1524].

* (1044) If we suppose $Y^{(i)}$ to contain a term, depending on $\sin.n\,\varpi$, of the form $\beta\,.\,\sin.n\,\varpi$, n being a finite integer; this will produce in $\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)$, the term $\left(\frac{d\,\beta}{d\,\mu}\right)\,.\,\sin.n\,\varpi$; and in $\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)$, the term $-n^2\,.\,\sin.n\,\varpi$. Substituting these in [1525], it will produce a term of the form $B\,.\,\sin.n\,\varpi$, B being used for brevity to represent the second member of [1526], which is a function of μ and constant quantities,
[1526a] independent of ϖ. This term will not vanish from the second member of [1525] for all values of ϖ, unless we have generally $B=0$, as in [1526]. Similar remarks may be made relative to the term $\beta\,.\,\cos.n\,\varpi$.

β is equal to $(1-\mu^2)^{\frac{n}{2}}$, multiplied by a rational and integral function of [1526′]
μ;* and in this case we shall have, by the preceding article,†

$$\beta = A^{(n)} \cdot (1-\mu^2)^{\frac{n}{2}} \cdot \left\{ \mu^{i-n} - \frac{(i-n)\cdot(i-n-1)}{2\cdot(2i-1)} \cdot \mu^{i-n-2} + \&c. \right\}; \qquad [1527]$$

$A^{(n)}$ being an arbitrary quantity. Hence the part of $Y^{(i)}$, depending on the angle $n\varpi$, is

$$(1-\mu^2)^{\frac{n}{2}} \cdot \left\{ \mu^{i-n} - \frac{(i-n)\cdot(i-n-1)}{2\cdot(2i-1)} \cdot \mu^{i-n-2} + \&c. \right\} \cdot \{A^{(n)}\cdot \sin. n\varpi + B^{(n)} \cdot \cos. n\varpi\}, \qquad [1528]$$

$A^{(n)}$, $B^{(n)}$, being two arbitrary quantities. If in this function we put
successively $n=0$, $n=1$, $n=2 \ldots\ldots n=i$, the sum of all the [1528′]
functions thus obtained, will be the general expression of $Y^{(i)}$, and this

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* (1045) In [1524′], $Y^{(i)}$ is supposed to be a rational and integral function of the
quantities μ, $\sqrt{(1-\mu^2)}\cdot \sin.\varpi$, $\sqrt{(1-\mu^2)}\cdot\cos.\varpi$; and the term of $Y^{(i)}$ represented [1526b]
by $\beta\cdot\sin. n\varpi$, or $\beta\cdot\cos. n\varpi$, can be produced only by powers and products of $\sin.\varpi$,
$\cos.\varpi$, of the order n, $n+2$, $n+4$, &c., [1509c]; the least of these powers being n. [1526c]
Now $\sin.\varpi$, $\cos.\varpi$, are connected with the factor $(1-\mu^2)^{\frac{1}{2}}$, in [1526b]; therefore
the powers or products of $\sin.\varpi$, $\cos.\varpi$, of the order n, must be connected with the
factor $(1-\mu^2)^{\frac{n}{2}}$, as in [1526′]; the higher powers and products of the order $n+2$,
$n+4$, &c., will contain the same factor, multiplied by integral powers of $1-\mu^2$, or μ;
as is evident from a slight examination.

† (1046) The expression β [1512], satisfies the equation [1508], which is the same as [1526]; we may therefore assume for β the value [1527], which is like that in [1512]; and we shall see, in [1530′′′′], that this form is sufficiently general for the determination of the complete value of any function S, like that treated of in [1530′′′′, &c.]. Hence the value of β, corresponding to $\sin. n\varpi$, may be put equal to

$$A^{(n)} \cdot (1-\mu^2)^{\frac{n}{2}} \cdot \left\{ \mu^{i-n} - \frac{(i-n)\cdot(i-n-1)}{2\cdot(2i-1)} \cdot \mu^{i-n-2} + \&c. \right\} \cdot \sin. n\varpi.$$

In like manner, by changing the constant term $A^{(n)}$ into $B^{(n)}$, and $\sin. n\varpi$ into $\cos. n\varpi$, we shall get the term corresponding to $\cos\, n\varpi$. The sum of these two expressions is the part of $Y^{(i)}$ corresponding to the angle $n\varpi$, [1528].

expression will contain $2i+1$ arbitrary quantities $B^{(0)}$, $A^{(1)}$, $B^{(1)}$, $A^{(2)}$, $B^{(2)}$, &c.*

General expression of $Y^{(0)}$, $Y^{(1)}$, &c.

* (1047) To illustrate the formula [1528], we shall give the values of $Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}$, $Y^{(3)}$, $Y^{(4)}$, deduced from it by putting successively $i=0$, $i=1$, $i=2$, $i=3$, $i=4$, and including, in each expression, all the values of n from $n=0$, to $n=i$.

[1528a] $Y^{(0)}=B_0^{(0)}$;

[1528b] $Y^{(1)}=B_1^{(0)}.\mu+(1-\mu^2)^{\frac{1}{2}}.\{A_1^{(1)}.\sin.\varpi+B_1^{(1)}.\cos.\varpi\}$;

[1528c] $$Y^{(2)}=B_2^{(0)}.\{\mu^2-\tfrac{1}{3}\}+(1-\mu^2)^{\frac{1}{2}}.\mu.\{A_2^{(1)}.\sin.\varpi+B_2^{(1)}.\cos.\varpi\}$$
$$+(1-\mu^2).\{A_2^{(2)}.\sin.2\varpi+B_2^{(2)}.\cos.2\varpi\};$$

[1528d] $$Y^{(3)}=B_3^{(0)}.\{\mu^3-\tfrac{3}{5}.\mu\}+(1-\mu^2)^{\frac{1}{2}}.\{\mu^2-\tfrac{1}{5}\}.\{A_3^{(1)}.\sin.\varpi+B_3^{(1)}.\cos.\varpi\}$$
$$+(1-\mu^2).\mu.\{A_3^{(2)}.\sin.2\varpi+B_3^{(2)}.\cos.2\varpi\}+(1-\mu^2)^{\frac{3}{2}}.\{A_3^{(3)}.\sin.3\varpi+B_3^{(3)}.\cos.3\varpi\};$$

[1528e] $$Y^{(4)}=B_4^{(0)}.\{\mu^4-\tfrac{6}{7}\mu^2+\tfrac{3}{35}\}+(1-\mu^2)^{\frac{1}{2}}.\{\mu^3-\tfrac{3}{7}\mu\}.\{A_4^{(1)}.\sin.\varpi+B_4^{(1)}.\cos.\varpi\}$$
$$+(1-\mu^2).\{\mu^2-\tfrac{1}{7}\}.\{A_4^{(2)}.\sin.2\varpi+B_4^{(2)}.\cos.2\varpi\}$$
$$+(1-\mu^2)^{\frac{3}{2}}.\mu.\{A_4^{(3)}.\sin.3\varpi+B_4^{(3)}.\cos.3\varpi\}$$
$$+(1-\mu^2)^2.\{A_4^{(4)}.\sin.4\varpi+B_4^{(4)}.\cos.4\varpi\}.$$

Thus we find that $Y^{(0)}$ contains *one* arbitrary constant quantity, $B^{(0)}$; $Y^{(1)}$ contains *three*,
[1528f] $B_1^{(0)}$, $A_1^{(1)}$, $B_1^{(1)}$; $Y^{(2)}$ contains *five*, $B_2^{(0)}$, $A_2^{(1)}$, $B_2^{(1)}$, $A_2^{(2)}$, $B_2^{(2)}$; $Y^{(3)}$ contains *seven*, $B_3^{(0)}$, $A_3^{(1)}$, $B_3^{(1)}$, $A_3^{(2)}$, $B_3^{(2)}$, $A_3^{(3)}$, $B_3^{(3)}$; and in like manner the general expression of $Y^{(i)}$ contains $2i+1$ arbitrary constant quantities. Hence the number of arbitrary quantities contained in the $i+1$ functions $Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}\ldots\ldots Y^{(i)}$, is equal to the sum of the arithmetical progression $1, 3, 5, 7\ldots\ldots 2i+1$, which is evidently equal
[1528g] to $(i+1)^2$.

From the preceding formulas [1528a—e], we easily perceive the method of putting μ^2, μ^3, μ^4, &c., under the form $Y^{(0)}+Y^{(1)}+Y^{(2)}+$ &c., as in the following expressions, in which the terms represented by $Y^{(0)}$, $Y^{(1)}$, &c., are separated by braces, and arranged according to the order of these terms respectively.

[1528h] $$\mu^2=\tfrac{1}{3}+\{\mu^2-\tfrac{1}{3}\},$$

[1528i] $$\mu^3=\tfrac{3}{5}\mu+\{\mu^3-\tfrac{3}{5}\mu\},$$

[1528k] $$\mu^4=\tfrac{1}{5}+\tfrac{6}{7}.\{\mu^2-\tfrac{1}{3}\}+\{\mu^4-\tfrac{6}{7}.\mu^2+\tfrac{3}{35}\}.$$

We shall now consider a rational integral function S of the order s, [1528″]
relative to the three rectangular co-ordinates x, y, z. If we represent the distance of the point, determined by these co-ordinates, from their origin, by R; the angle formed by R and the axis of x, by θ; the angle formed by the plane of xy, and the plane passing through R and the axis of x, by ϖ; [1528‴]
we shall have [1480b],

$$x = R\,.\,\mu\,; \qquad y = R\,.\sqrt{1-\mu^2}\,.\,\cos.\varpi\,; \qquad z = R\,.\sqrt{1-\mu^2}\,.\,\sin.\varpi. \qquad [1529]$$

We shall substitute these values in the function S, and then develop it in sines and cosines of the angle ϖ and its multiples. If S be the most general [1529′]
function of the order s, the terms $\sin. n\varpi$, and $\cos. n\varpi$, will be multiplied by functions of the form,*

$$(1-\mu^2)^{\frac{n}{2}}\,.\,\{A\,.\,\mu^{s-n} + B\,.\,\mu^{s-n-1} + C\,.\,\mu^{s-n-2} + \&c.\}. \qquad [1530]$$

Hence the part of S depending on the angle $n\varpi$, will contain † [1530′]

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* (1048) The function S is supposed to be composed of terms formed by the powers and products of x, y, z, as $G\,.\,x^a\,.\,y^b\,.\,z^c$, in which the sum of the integral exponents a, b, c, *is equal to, or less than*, s, and G a constant quantity. If we substitute in this term, [1530a]
$G\,.\,x^a\,.\,y^b\,.\,z^c$, the values [1529], it will become,

$$G\,.\,R^{a+b+c}\,.\,\mu^a\,.\,(1-\mu^2)^{\frac{b+c}{2}}\,.\,\cos.^b\varpi\,.\,\sin.^c\varpi\,;$$

and if we put $b+c$ successively equal to n, $n+2$, $n+4$, &c., it will produce, as in [1509c] terms depending on $\sin. n\varpi$, or $\cos. n\varpi$, multiplied by $(1-\mu^2)^{\frac{n}{2}}$, and by a function of μ, of the form $\{A\,.\,\mu^{s-n} + B\,.\,\mu^{s-n-1} + C\,.\,\mu^{s-n-2} + \&c.\}$, which may [1530b]
contain all the integral powers of μ, of the degree $s-n$, $s-n-1$, $s-n-2$, &c., to 0. This function will not contain any power of μ exceeding $s-n$, because x, y, z, [1529], are of the first degree in μ, $\sqrt{(1-\mu^2)}$; therefore $G\,.\,x^a\,.\,y^b\,.\,z^c$ must be of the [1530c]
degree $a+b+c$, which is equal to or less than s, [1530a], relative to the same quantities μ, $\sqrt{(1-\mu^2)}$; and as the factor $(1-\mu^2)^{\frac{n}{2}}$ is of the degree n relative to $\sqrt{(1-\mu^2)}$, the greatest exponent of μ must be $s-n$, corresponding to the term $A\,.\,\mu^{s-n}$ [1530b], so as to render the whole expression [1530] of the degree s, as in [1529′[.

† (1049) The number of arbitrary constant quantities A, B, C, &c., in [1530], is evidently equal to the number of terms μ^{s-n}, μ^{s-n-1}, $\mu^{s-n-2}\,.\,.\,.\,.\,.\,\mu^3$, μ^2, μ^1, μ^0; which

$2.(s-n+1)$ arbitrary indeterminate quantities. The part of S^* depending

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is equal to $s-n+1$, and as there is the same number of terms for $\sin. n\varpi$ and for $\cos. n\varpi$, the number corresponding to the angle $n\varpi$ must be double this quantity, or
[1530d] $2.(s-n+1)$, as in [1530′]; observing that there are no *negative* powers of x, y, z, or μ, contained in the integral function S [1528″, 1529], but it may contain the *positive* powers and products of $\sin.\varpi$, $\cos.\varpi$, as high as the order s, which will produce, as in [1530a—c], terms of the form $\sin. n\varpi$, $\cos. n\varpi$, in which n may be any integral number from s to 1; and if we substitute in $2.(s-n+1)$ [1530d], the values of n corresponding to the series s, $s-1 \ldots\ldots 1$, we shall obtain the corresponding number of arbitrary constant quantities depending on each of these values of n, which will therefore be represented by the
[1530e] arithmetical series $2, 4, 6 \ldots\ldots 2s$, whose sum is evidently $s.(s+1)$, [1530″]. Again, when $n=0$, the term $\sin. n\varpi$ vanishes, but $\cos. n\varpi$ becomes unity, and this value of n produces in [1530] the expression $A.\mu^s + B.\mu^{s-1} + C.\mu^{s-2} + \&c.$; which contains $s+1$ constant quantities. Adding this to the preceding sum [1530e], we get $s.(s+1)+(s+1)=(s+1)^2$, for the number of arbitrary constant quantities in the function S, as in [1530‴].

Now it appears from [1528g], that the general function $Y^{(0)}+Y^{(1)}+\ldots\ldots Y^{(s)}$, contains
[1530e′] $(s+1)^2$ arbitrary constant quantities, which is the same number as in the most general expression of S [1530‴]; therefore, by assigning proper values to the constant quantities, contained in $Y^{(0)}+Y^{(1)}+\ldots\ldots Y^{(s)}$, it may generally be made equal to any given function S, of the order s.

* (1050) A few examples will illustrate this method of development of the function S.
[1530f] *First*, let it be required to put $S=1-\mu^2$ under the form $Y^{(0)}+Y^{(1)}+Y^{(2)}$; all terms above $Y^{(2)}$ being neglected, because $1-\mu^2$ is of the second degree in μ; moreover, as S does not contain ϖ, we may neglect ϖ in [1528c], and we shall have

$$Y^{(2)}=B_2^{(0)}.(\mu^2-\tfrac{1}{3});$$

hence $S'=S-Y^{(2)}=1-\mu^2-B_2^{(0)}.(\mu^2-\tfrac{1}{3})=1+\tfrac{1}{3}B_2^{(0)}-\mu^2.(1+B_2^{(0)})$; and by putting the coefficient of μ^2 equal to nothing, we get $B_2^{(0)}=-1$, and then

$$B_0^{(0)}=1+\tfrac{1}{3}B_2^{(0)}=\tfrac{2}{3}=Y^{(0)};$$

hence S' becomes $S-Y^{(2)}=Y^{(0)}$, or $S=Y^{(0)}+Y^{(2)}=\tfrac{2}{3}-(\mu^2-\tfrac{1}{3})$, which is of the required form [1530f], supposing $Y^{(1)}=0$. *Second*, let it be required to reduce the function $S=(\mu^2+2\mu).(1-\mu^2)^{\frac{1}{2}}.\sin.\varpi$, of the third degree, to the form $S=Y^{(0)}+Y^{(1)}+Y^{(2)}+Y^{(3)}$. In this case, all the terms are multiplied by $\sin.\varpi$; therefore we must retain only such terms in [1528a—e], and put

$$Y^{(0)}=0, \qquad Y^{(1)}=A_1^{(1)}.(1-\mu^2)^{\frac{1}{2}}.\sin.\varpi.$$

[1530g] $$Y^{(2)}=A_2^{(1)}.(1-\mu^2)^{\frac{1}{2}}.\mu.\sin.\varpi, \qquad Y^{(3)}=A_3^{(1)}.(1-\mu^2)^{\frac{1}{2}}.(\mu^2-\tfrac{1}{5}).\sin.\varpi.$$

on the angle ϖ and its multiples, will therefore contain $s.(s+1)$ [1530'']

Hence $S^{(1)}=S-Y^{(3)}=(\mu^2+2\mu-A_3^{(1)}.\mu^2+\tfrac{1}{5}A_3^{(1)}).(1-\mu^2)^{\frac{1}{2}}.\sin.\varpi$; which, by putting $A_3^{(1)}=1$, becomes of the second degree, $S^{(1)}=(2\mu+\tfrac{1}{5}).(1-\mu^2)^{\frac{1}{2}}.\sin.\varpi$. Hence $S^{(2)}=S^{(1)}-Y^{(2)}=(2\mu+\tfrac{1}{5}-A_2^{(1)}.\mu).(1-\mu^2)^{\frac{1}{2}}.\sin.\varpi$; and if we put $A_2^{(1)}=2$, it will be reduced to the first degree $S^{(2)}=\tfrac{1}{5}.(1-\mu^2)^{\frac{1}{2}}.\sin.\varpi=Y^{(1)}$, supposing $A_1^{(1)}=\tfrac{1}{5}$ [1530g]. Therefore the values [1530g] will become $Y^{(0)}=0$, $Y^{(1)}=\tfrac{1}{5}.(1-\mu^2)^{\frac{1}{2}}.\sin.\varpi$, $Y^{(2)}=2.(1-\mu^2)^{\frac{1}{2}}.\mu.\sin.\varpi$, $Y^{(3)}=(1-\mu^2)^{\frac{1}{2}}.(\mu^2-\tfrac{1}{5}).\sin.\varpi$, [1530h] and we shall have $S=(\mu^2+2\mu).(1-\mu^2)^{\frac{1}{2}}.\sin.\varpi=Y^{(0)}+Y^{(1)}+Y^{(2)}+Y^{(3)}$. We have gone through the calculation in the general manner, proposed by the author; finding successively $S^{(0)}$, $S^{(1)}$, $S^{(2)}$, &c.; but in the present instance, we might obtain the same result, in a shorter way, by adding the expressions $Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}$, $Y^{(3)}$, [1530g], and putting the sum equal to the proposed value of S, then rejecting the common factor $(1-\mu^2)^{\frac{1}{2}}.\sin.\varpi$; by which means we shall get,

$$\mu^2+2\mu=A_1^{(1)}+A_2^{(1)}.\mu+A_3^{(1)}.(\mu^2-\tfrac{1}{5})=A_3^{(1)}.\mu^2+A_2^{(1)}.\mu+\{A_1^{(1)}-\tfrac{1}{5}A_3^{(1)}\}. \quad [1530i]$$

This becomes identical, by putting $A_3^{(1)}=1$, $A_2^{(1)}=2$, $A_1^{(1)}=\tfrac{1}{5}.A_3^{(1)}=\tfrac{1}{5}$; hence we obtain the same values of $Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}$, $Y^{(3)}$, as in [1530h].

In this way we may obtain, in the required form, the complete value of the quantity S, when it is a rational and integral function of μ, $\surd(1-\mu^2).\cos.\varpi$, and $\surd(1-\mu^2).\sin.\varpi$, [1530k] of the order s, and if we represent this proposed function by $f(\mu,\varpi)$, we shall have,

$$S=f(\mu,\varpi)=Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.=\Sigma.Y^{(i)}, \quad [1530l]$$

in which $Y^{(i)}$ satisfies the equation [1525]. If we put $\mu=\cos.\theta$ [1434'], we may suppose S to be a rational and integral function of $\cos.\theta$, $\sin.\theta.\cos.\varpi$, $\sin.\theta.\sin.\varpi$, represented by $f(\theta,\varpi)$, and we shall have,

General expressions of the function S.

$$S=f(\theta,\varpi)=Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.=\Sigma.Y^{(i)}, \quad [1530l']$$

in which $Y^{(i)}$ will satisfy the equation [1464c]. This development of S is the same as in [1530l], changing μ into $\cos.\theta$.

The method of finding the terms $Y^{(0)}$, $Y^{(1)}$, &c., [1530^v, &c.], is attended with some inconvenience when s is large, because we are obliged to compute *all* the terms of the series, beginning with the highest $Y^{(s)}$, even when only two or three of the first terms, or perhaps only one, is wanted; and this method becomes impracticable when s is expressed by an infinite series, because $Y^{(s)}$ has an infinite exponent. In this case we may use the method of definite integrals, by which any term may be computed, independently of the rest, [1530m] in the manner we shall explain in this note. We have introduced this method of definite integrals in this place, as the most convenient, for a note of so great a length; but it will be best for the reader to examine the original work as far as [1542], before commencing upon this subject, as we shall have to refer to some of the formulas contained in that part of the work.

indeterminate quantities; the part independent of ϖ will contain $(s+1)$.

Supposing, as in [1441''], $Q^{(i)}$ to be the coefficient of $\frac{R^i}{r^{i+1}}$, in the development of the

[1530n] radical $\{r^2-2Rr.[\mu\mu'+\sqrt{(1-\mu^2)}.\sqrt{(1-\mu'^2)}.\cos.(\varpi'-\varpi)]+R^2\}^{-\frac{1}{2}}$, and

Values of $Q_n^{(i)}$, that $Q_n^{(i)}$ is the term of $Q^{(i)}$ depending on the angle $n.(\varpi'-\varpi)$, we shall have, as in [1543'', 1523, 1524, 1542],

[1530o]
$$Q_n^{(i)}=\gamma.\lambda\lambda'.\cos.n.(\varpi'-\varpi);$$

γ, [1530p]
$$\gamma=2.\left(\frac{1.3.5.....(2i-1)}{1.2.3.....i}\right)^2.\frac{i.(i-1).....(i-n+1)}{(i+1).(i+2).....(i+n)}.$$

[1530p']
$$\gamma=\left(\frac{1.3.5.....(2i-1)}{1.2.3.....i}\right)^2, \quad \text{when} \quad n=0,$$

λ, [1530q]
$$\lambda=(1-\mu^2)^{\frac{n}{2}}.\left\{\mu^{i-n}-\frac{(i-n).(i-n-1)}{2.(2i-1)}.\mu^{i-n-2}+\&c.\right\};$$

λ'. [1530r]
$$\lambda'=(1-\mu'^2)^{\frac{n}{2}}.\left\{\mu'^{i-n}-\frac{(i-n).(i-n-1)}{2.(2i-1)}.\mu'^{i-n-2}+\&c.\right\}.$$

[1530s] Now if we multiply by $Q^{(i)}.d\mu'.d\varpi'$ the function $f(\mu',\varpi')=Y'^{(0)}+Y'^{(1)}+Y'^{(2)}+\&c$, which was deduced from [1530l], by changing μ, ϖ, $Y^{(i)}$, into μ', ϖ', $Y'^{(i)}$, respectively, and then take the integral of the product relatively to μ', from $\mu'=-1$ to $\mu'=1$; also

[1530s'] relatively to ϖ', from $\varpi'=0$ to $\varpi'=2\pi$, we shall get, by using successively the formulas [1476, 1540'''],

[1530t]
$$\int_{-1}^{1}\int_0^{2\pi}f(\mu',\varpi').Q^{(i)}.d\mu'.d\varpi'=\int_{-1}^{1}\int_0^{2\pi}\{Y'^{(0)}+Y'^{(1)}+Y'^{(2)}+\&c.\}.Q^{(i)}.d\mu'.d\varpi'$$

[1530u]
$$=\int_{-1}^{1}\int_0^{2\pi}Y'^{(i)}.Q^{(i)}.d\mu'.d\varpi'=\frac{4\pi}{2i+1}.Y^{(i)};$$

hence

$Y^{(i)}$. [1530v]
$$Y^{(i)}=\frac{2i+1}{4\pi}.\int_{-1}^{1}\int_0^{2\pi}f(\mu',\varpi').Q^{(i)}.d\mu'.d\varpi'.$$

If we change $Q^{(i)}$ into $Q_n^{(i)}$, we shall get $Y_n^{(i)}$, or the part of $Y^{(i)}$ depending on the angle $n.(\varpi'-\varpi)$, namely,

$Y_n^{(i)}$. [1530w]
$$Y_n^{(i)}=\frac{2i+1}{4\pi}.\int_{-1}^{1}\int_0^{2\pi}f(\mu',\varpi').Q_n^{(i)}.d\mu'.d\varpi'.$$

Substituting, in this, the value of $Q_n^{(i)}$ [1530o], we may bring γ, λ, from under the signs of integration, and we shall get, by using the value of λ' [1530r],

$Y_n^{(i)}$. [1530x]
$$Y_n^{(i)}=\frac{2i+1}{4\pi}.\gamma.\lambda.\int_{-1}^{1}\int_0^{2\pi}(1-\mu'^2)^{\frac{n}{2}}.\left\{\mu'^{i-n}-\frac{(i-n).(i-n-1)}{2.(2i-1)}.\mu'^{i-n-2}+\&c.\right\}$$
$$\times f(\mu',\varpi').\cos.n.(\varpi'-\varpi).d\mu'.d\varpi'.$$

S will therefore contain $(s+1)^2$ indeterminate constant quantities. [1530‴]

We may apply this formula to the case where $f(\mu',\varpi')$ becomes a function of μ' independent of ϖ', represented by $f(\mu')$, by putting $n=0$, and then we shall have

$$(1-\mu^2)^{\frac{n}{2}}=1 \qquad (1-\mu'^2)^{\frac{n}{2}}=1, \qquad \cos. n\,.\,(\varpi'-\varpi)=1.$$

The integration relative to ϖ' can then be obtained, since

$$\frac{2i+1}{4\pi}\cdot\int_0^{2\pi} d\varpi'=\frac{2i+1}{4\pi}\,.\,2\pi=\frac{2i+1}{2}\,; \qquad [1530y]$$

hence, by using [1530p', q], we get, for this case,

$$Y_0^{(i)}=\frac{2i+1}{2}\cdot\left(\frac{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(2i-1)}{1\,.\,2\,.\,3\,.\,.\,.\,.\,.\,i}\right)^2\cdot\left\{\mu^i-\frac{i\,.\,(i-1)}{2\,.\,(2i-1)}\cdot\mu^{i-2}+\&c.\right\} \qquad Y_0^{(i)}.$$
$$\times\int_{-1}^{1}\left\{\mu'^i-\frac{i\,.\,(i-1)}{2\,.\,(2i-1)}\cdot\mu'^{i-2}+\&c.\right\}.f(\mu')\,.\,d\mu'\,; \qquad [1530z]$$

$f(\mu')$ being the part of $f(\mu',\varpi')$ [1530s] independent of the angle ϖ'. If $i=0$, we shall have $Q^{(0)}=1$ [1433d''''], and we shall get, from [1530v],

$$Y_0^{(0)}=\frac{1}{4\pi}\cdot\int_{-1}^{1}\int_0^{2\pi}f(\mu')\,.\,d\mu'\,.\,d\varpi'\,;$$

and since $\int_0^{2\pi} d\varpi'=2\pi$, this becomes $Y_0^{(0)}=\frac{1}{2}\,.\int_{-1}^{1}f(\mu')\,.\,d\mu'.$ [1531a]

As an example of the formula [1530z], we shall suppose that it is required to find, separately from the other terms, the value of $Y^{(2)}$, in the development of $S=\mu^4$, in the form [1530l]. In this case, $i=2$, and $f(\mu')=\mu'^4$; hence [1530z] becomes

$$Y^{(2)}=\tfrac{5}{2}\cdot\left(\frac{1.3}{1.2}\right)^2\cdot(\mu^2-\tfrac{1}{3})\cdot\int_{-1}^{1}(\mu'^2-\tfrac{1}{3})\cdot\mu'^4\,.\,d\mu'\,; \qquad [1531b]$$

but $\int(\mu'^2-\frac{1}{3})\,.\,\mu'^4\,.\,d\mu'=\frac{1}{7}\mu'^7-\frac{1}{15}\mu'^5+\frac{8}{105}$, which vanishes when $\mu'=-1$; and at the other limit $\mu'=1$, it becomes $\int_{-1}^{1}(\mu'^2-\frac{1}{3})\,.\,\mu'^4\,.\,d\mu'=\frac{16}{105}$; hence $Y^{(2)}=\frac{5}{2}\cdot\left(\frac{1.3}{1.2}\right)^2\cdot(\mu^2-\frac{1}{3})\cdot\frac{16}{105}=\frac{6}{7}\cdot(\mu^2-\frac{1}{3})$ being the same as was found in [1528k].

There is no difficulty in developing $f(\mu,\varpi)$, in the form [1530l], when it is actually a finite rational and integral function of μ, $\sqrt{(1-\mu^2)}\,.\cos.\varpi$, $\sqrt{(1-\mu^2)}\,.\sin.\varpi$; but if it be irrational, fractional, or discontinuous in any arbitrary manner, the series, when reduced to a rational and integral form, will most commonly be composed of an infinite number of terms, which may be computed by means of the formulas [1530v, x, z]; *the function $f(\mu',\varpi')$ being supposed not to become infinite within the proposed limits of the integrals* [1530s']. To illustrate this by a simple example, we shall take the irrational quantity $S=(1-\mu^2)^{\frac{1}{2}}$, which does not contain ϖ, so that $f(\mu,\varpi)$ becomes $f(\mu)=(1-\mu^2)^{\frac{1}{2}}$, and $f(\mu')=(1-\mu'^2)^{\frac{1}{2}}$. If we develop $f(\mu')$ in a series, proceeding according to the powers of μ', we shall get, [1531c] [1531d]

$$f(\mu')=1-\tfrac{1}{2}\mu'^2-\tfrac{1}{8}\mu'^4-\tfrac{1}{16}\mu'^6-\&c.\,; \qquad [1531e]$$

The function $Y^{(0)}+Y^{(1)}+Y^{(2)}+\ldots\ldots Y^{(s)}$, will likewise contain

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in which s the highest exponent of μ' is *infinite;* and as this is the first term wanted, in the method of [1530v], the development could not, in this way, be obtained; therefore it would be necessary to recur to the process of definite integrals. Substituting in [1530z] the preceding value of $f(\mu')=(1-\mu'^2)^{\frac{1}{2}}$, we shall get,

[1531f]
$$Y_0^{(i)}=\frac{2i+1}{2}\cdot\left(\frac{1.3.5\ldots\ldots(2i-1)}{1.2.3\ldots\ldots i}\right)^2\cdot\left\{\mu^i-\frac{i.(i-2)}{2.(2i-1)}\cdot\mu^{i-2}+\&c.\right\}$$
$$\times\int_{-1}^{1}\left\{\mu'^i-\frac{i.(i-1)}{2.(2i-1)}\cdot\mu'^{i-2}+\&c.\right\}\cdot(1-\mu'^2)^{\frac{1}{2}}\cdot d\mu'.$$

If we take the differential of $\mu'^{n-1}\cdot(1-\mu'^2)^{\frac{3}{2}}$, we shall find,

$$d.\{\mu'^{n-1}\cdot(1-\mu'^2)^{\frac{3}{2}}\}=(n-1)\cdot\mu'^{n-2}d\mu'\cdot(1-\mu'^2)^{\frac{3}{2}}-3\mu'^n d\mu'\cdot(1-\mu'^2)^{\frac{1}{2}}$$
$$=(1-\mu'^2)^{\frac{1}{2}}\cdot\{(n-1)\cdot\mu'^{n-2}\cdot(1-\mu'^2)-3\mu'^n\}\cdot d\mu'$$
$$=(1-\mu'^2)^{\frac{1}{2}}\cdot\{(n-1)\cdot\mu'^{n-2}-(n+2)\cdot\mu'^n\}\cdot d\mu'.$$

Integrating this, and supposing the arbitrary constant quantity to be included in the terms under the sign $\int$, we have,

[1531g]
$$\mu'^{n-1}\cdot(1-\mu'^2)^{\frac{3}{2}}=(n-1)\cdot\int\mu'^{n-2}\cdot d\mu'\cdot(1-\mu'^2)^{\frac{1}{2}}-(n+2)\cdot\int\mu'^n d\mu'\cdot(1-\mu'^2)^{\frac{1}{2}}.$$

If we take these integrals from $\mu'=-1$ to $\mu'=1$, the first member of the equation will vanish at both limits, and then dividing by $n+2$, we shall get the following theorem.

Theorem. [1531h]
$$\int_{-1}^{1}\mu'^n d\mu'\cdot(1-\mu'^2)^{\frac{1}{2}}.=\frac{n-1}{n+2}\cdot\int_{-1}^{1}\mu'^{n-2}d\mu'\cdot(1-\mu'^2)^{\frac{1}{2}}.$$

If we now put successively $n=2,\ 4,\ 6$, &c., we shall get the general value of

[1531i]
$$\int_{-1}^{1}\mu'^n d\mu'\cdot(1-\mu'^2)^{\frac{1}{2}},$$

observing that $\int_{-1}^{1}d\mu'\cdot(1-\mu'^2)^{\frac{1}{2}}=\frac{1}{2}\pi$, which represents the area of a semicircle

[1531k] whose radius is 1, absciss $\mu'=x$, ordinate $y=\sqrt{(1-\mu'^2)}$, and area, by the usual formula $\int y\,dx$. Hence we get successively

$$\int_{-1}^{1}d\mu'\cdot(1-\mu'^2)^{\frac{1}{2}}=\frac{\pi}{2};$$

[1531l]
$$\int_{-1}^{1}\mu'^2 d\mu'.(1-\mu'^2)^{\frac{1}{2}}=\tfrac{1}{4}\cdot\int_{-1}^{1}d\mu'\cdot(1-\mu'^2)^{\frac{1}{2}}=\tfrac{1}{4}\cdot\frac{\pi}{2};$$

$$\int_{-1}^{1}\mu'^4 d\mu'.(1-\mu'^2)^{\frac{1}{2}}=\tfrac{3}{6}\cdot\int_{-1}^{1}\mu'^2 d\mu'\cdot(1-\mu'^2)^{\frac{1}{2}}=\tfrac{1}{4}\cdot\tfrac{3}{6}\cdot\frac{\pi}{2};$$

Definite integrals.

and generally, when n is an even number,

[1531m]
$$\int_{-1}^{1}\mu'^n d\mu'.(1-\mu'^2)^{\frac{1}{2}}=\frac{1.3.5\ldots\ldots(n-1)}{2.4.6\ldots\ldots(n+2)}\cdot\pi.$$

All the terms of this last integral will vanish if n be an odd number, because they will all

$(s+1)^2$ indeterminate constant quantities [1528g], since the function [1530''''']

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depend on $\int_{-1}^{1}\mu' d\mu' . (1-\mu'^2)^{\frac{1}{2}} = -\frac{1}{3} . (1-\mu'^2)^{\frac{3}{2}} + \text{const.}$, which vanishes at both limits of the integral. Substituting these values in the integrals of [1531f], corresponding successively to $i=2$, $i=4$, &c., we get, [1531n]

$$\int_{-1}^{1}\{\mu'^2-\tfrac{1}{3}\} . (1-\mu'^2)^{\frac{1}{2}} . d\mu' = \tfrac{1}{4} . \tfrac{\pi}{2} - \tfrac{1}{3} . \tfrac{\pi}{2} = -\tfrac{\pi}{24};$$

$$\int_{-1}^{1}\{\mu'^4-\tfrac{6}{7}\mu'^2+\tfrac{3}{35}\} . (1-\mu'^2)^{\frac{1}{2}} . d\mu' = \tfrac{1}{4} . \tfrac{3}{6} . \tfrac{\pi}{2} - \tfrac{6}{7} . \tfrac{1}{4} . \tfrac{\pi}{2} + \tfrac{3}{35} . \tfrac{\pi}{2} = -\tfrac{\pi}{560}; \quad \&c. \qquad [1531o]$$

Now from [1531a, l], we find $Y^{(0)} = \frac{1}{2} . \int_{-1}^{1}(1-\mu'^2)^{\frac{1}{2}} . d\mu' = \frac{1}{2} . \frac{\pi}{2} = \frac{\pi}{4}$; and the values of $Y^{(2)}$, $Y^{(4)}$, &c., deduced from the formulas [1531f, o], are

$$Y^{(2)} = -\frac{15\pi}{64} . (\mu^2-\tfrac{1}{3}); \qquad Y^{(4)} = -\frac{315\pi}{2048} . (\mu^4-\tfrac{6}{7}\mu^2+\tfrac{3}{35}); \qquad \&c.$$

Hence we finally obtain,

$$(1-\mu^2)^{\frac{1}{2}} = \frac{\pi}{4} - \frac{15\pi}{64} . \{\mu^2-\tfrac{1}{3}\} - \frac{315\pi}{2048} . \{\mu^4-\tfrac{6}{7}\mu^2+\tfrac{3}{35}\} - \&c.; \qquad [1531p]$$

which is of the required form. We shall hereafter, [1535k], show that series of this kind are, in their nature, converging, in successive intervals.

The results we have obtained, in [1530l—z, 1540, &c.], have been found in another manner by Mr. Poisson, who has also given a different demonstration of the formula [1540]; and as the subject has excited considerable discussion, we have thought it might be useful to give the substance of his method. He has shown that any function whatever of θ, ϖ, represented by $f(\theta,\varpi)$, can be developed in the form [1530l'], namely,

Development of S, or $f(\theta,\varpi)$.

$$S = f(\theta,\varpi) = Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c. = \Sigma_0^s Y^{(i)}; \qquad [1532a]$$

$Y^{(i)}$ being a rational and integral function of $\cos.\theta$, $\sin.\theta . \cos.\varpi$, $\sin.\theta . \sin.\varpi$, satisfying the differential equation [1464c]. The limits of the values of θ, ϖ, between which the development [1532a] is to be used, are from $\theta=0$ to $\theta=\pi$, and from $\varpi=0$ to $\varpi=2\pi$; *and the function $f(\theta,\varpi)$ is supposed not to become infinite anywhere between these limits.* We shall take, for α, a positive number, which, at the end of the operation, may be supposed to differ from unity, by an infinitely small quantity g. This value of α is entirely different from that used in [1461'], and is analogous to that in [963iv]. We shall also put [1532b]

Formulas.

$$p = \cos.\theta . \cos.\theta' + \sin.\theta . \sin.\theta' . \cos.(\varpi'-\varpi) = \cos.\gamma; \qquad [1532c]$$

$$g = 1-\alpha; \qquad \theta' = \theta + h; \qquad \varpi' = \varpi + k; \qquad [1532d]$$

$$Q^{(i)} = \frac{1.3.5.....(2i-1)}{1.2.3.....i} . \left\{p^i - \frac{i.(i-1)}{2.(2i-1)} . p^{i-2} + \frac{i.(i-1).(i-2).(i-3)}{2.4.(2i-1).(2i-3)} . p^{i-4} - \&c.\right\}; \qquad [1532e]$$

$$\rho = (1-2\alpha.p+\alpha^2)^{-\frac{1}{2}} \qquad [1532f]$$

$$= 1 + \alpha . Q^{(1)} + \alpha^2 . Q^{(2)} + \alpha^3 . Q^{(3)} + \&c. = \Sigma_0^{\infty} . \alpha^i . Q^{(i)} \qquad [1532g]$$

$$= \frac{1}{\alpha} + \frac{1}{\alpha^2} . Q^{(1)} + \frac{1}{\alpha^3} . Q^{(2)} + \frac{1}{\alpha^4} . Q^{(4)} + \&c. = \Sigma_0^{\infty} . \frac{1}{\alpha^{i+1}} . Q^{(i)}; \qquad [1532g']$$

$$X = \frac{1}{4\pi} . \int_0^{\pi}\int_0^{2\pi} \frac{(1-\alpha^2).f(\theta',\varpi').\sin.\theta'.d\theta'.d\varpi'}{(1-2\alpha.p+\alpha^2)^{\frac{3}{2}}}; \qquad [1532h]$$

Development of a function in terms of $Y^{(i)}$. $Y^{(i)}$ contains $2i+1$; we may therefore transform S into a function of that

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[1532h'] the integration of this value of X being taken, relatively to θ', from $\theta'=0$ to $\theta'=\pi$; and relatively to ϖ', from $\varpi'=0$ to $\varpi'=2\pi$. The expression $p=\cos.\gamma$, [1532e],
[1532h''] is the same quantity as that called δ in [1629, 1432g], changing v into θ', and ψ into ϖ'; and by these changes, the symbol $P^{(i)}$, used by the author in [1628, 1532c], becomes the same as the quantity $Q^{(i)}$ [1441'', 1532e]. We have used the letter p in preference to δ, because δ is usually employed as the sign of a differential or variation. If we also change, in [1626], f into $\frac{1}{\rho}$, s into 1, r into α, and divide by S, it will become like the preceding value ρ, [1532f]; its development is easily obtained as in [1441'', 1627]. Then when g or $1-\alpha$ becomes infinitely small, we shall have, as we shall soon see,

Theorem by Mr. Poisson.

[1532i] $$f(\theta,\varpi)=\pm X;$$

for all values of θ, ϖ, *included between the limits* [1532b]; *the upper sign being used when*
[1532k] $\alpha<1$, *the lower sign when* $\alpha>1$.

To demonstrate this theorem, we must observe, that the factor $\frac{1-\alpha^2}{(1-2\alpha.p+\alpha^2)^{\frac{3}{2}}}$ of the
[1532l] expression [1532h], becomes infinitely small with $1-\alpha$, except p be infinitely near to unity; which happens only when θ', ϖ', differ from θ, ϖ, respectively, by infinitely small
[1532l'] quantities h, k, [1532d]; observing that when $\theta'=\theta$, and $\varpi'=\varpi$, the expression [1532e] becomes $p=\cos.^2\theta+\sin.^2\theta=1$. Therefore we may, in the integral X [1532h], neglect
[1532m] all the elements, except those in which h and k are excessively small; and for these values, we have very nearly, by means of [44] Int., neglecting always h^3, k^3, h^2g, k^2g,

[1532n] $$\cos.(\varpi'-\varpi)=\cos.k=1-\tfrac{1}{2}k^2, \qquad \cos.(\theta'-\theta)=\cos.h=1-\tfrac{1}{2}h^2.$$

Substituting these in [1532c], we get,

[1532o] $$p=\cos.\theta.\cos.\theta'+\sin.\theta.\sin.\theta'-\tfrac{1}{2}k^2.\sin.\theta.\sin.\theta'=\cos.(\theta'-\theta)-\tfrac{1}{2}k^2.\sin.\theta.\sin.\theta'=1-\tfrac{1}{2}h^2-\tfrac{1}{2}k^2.\sin.^2\theta;$$

hence, by putting $\alpha=1$, in the terms $\alpha.h^2$, $\alpha.k^2$, we have,

$$1-2\alpha.p+\alpha^2=1-2\alpha.(1-\tfrac{1}{2}h^2-\tfrac{1}{2}k^2.\sin.^2\theta)+\alpha^2=(1-2\alpha+\alpha^2)+\alpha.(h^2+k^2.\sin.^2\theta)$$

[1532p] $$=(1-\alpha)^2+h^2+k^2.\sin.^2\theta \qquad =g^2+h^2+k^2.\sin.^2\theta;$$

$$(1-\alpha^2).f(\theta',\varpi').\sin.\theta'=(2g-g^2).f(\theta',\varpi').\sin.\theta'.$$

Moreover, the differentials of θ', ϖ', [1532d], are $d\theta'=dh$, $d\varpi'=dk$; hence the expression [1532h] becomes,

[1532q] $$X=\frac{1}{4\pi}.\iint\frac{(2g-g^2).f(\theta',\varpi').\sin.\theta'.dh.dk}{(g^2+h^2+k^2.\sin.^2\theta)^{\frac{3}{2}}}.$$

The numerator and denominator of this expression are of the third order, in g, h, k; if we neglect terms of a higher order, we may change θ' into θ, ϖ' into ϖ, and it will become of the following form; observing that $f(\theta,\varpi)$ is constant, relatively to these integrations, and may be brought from under the signs of integration; hence we shall get,

[1532r] $$X=\frac{f(\theta,\varpi)}{2\pi}.\iint\frac{g.\sin.\theta.dh.dk}{(g^2+h^2+k^2.\sin.^2\theta)^{\frac{3}{2}}};$$

which may be easily integrated, relatively to h, k, observing that g, θ, are considered as

form; the following is the most simple method of making this transformation.

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constant. The limits of this integral relative to $\theta' = \theta + h$, are $\theta' = \theta + h = 0$, and $\theta' = \theta + h = \pi$, [1532h'], corresponding to $h = -\theta$, $h = \pi - \theta$; the first being [1532s]
negative, the second positive. Now the elements of this integral vanish [1532m], unless h be excessively small; we may therefore extend the limits of h from $-\infty$ to $+\infty$, without altering the value of the integral. In like manner, the limits of ϖ' being $\varpi' = \varpi + k = 0$, [1532s']
and $\varpi' = \varpi + k = 2\pi$, [1532h'], the corresponding limits of k are $k = -\varpi$, $k = 2\pi - \varpi$; which may also be extended from $-\infty$ to $+\infty$. In finding the integral of [1532r], relatively to k, it will be convenient to use a new variable quantity k', formed by putting $k \,.\, \sin.\,\theta = k' . \sqrt{(g^2+h^2)}$. The differential of this equation is $dk . \sin.\theta = dk' . \sqrt{(g^2+h^2)}$, [1532t]
h being considered constant in finding the integral relative to k, because the two quantities h, k, are independent of each other. Substituting these in [1532r], and reducing, we find,

$$X = \frac{f(\theta,\varpi)}{2\pi} . \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g\,dh\,.\,dk'\,.\,(g^2+h^2)^{\frac{1}{2}}}{\{g^2+h^2+(g^2+h^2).k'^2\}^{\frac{3}{2}}}$$

$$= \frac{f(\theta,\varpi)}{2\pi} . \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g\,dh\,.\,dk'}{(g^2+h^2).(1+k'^2)^{\frac{3}{2}}} = \frac{f(\theta,\varpi)}{2\pi} . \int_{-\infty}^{\infty} \frac{g\,dh}{g^2+h^2} . \int_{-\infty}^{\infty} \frac{dk'}{(1+k'^2)^{\frac{3}{2}}} . \qquad [1532u]$$

But $\int \frac{dk'}{(1+k'^2)^{\frac{3}{2}}} = 1 + \frac{k'}{(1+k'^2)^{\frac{1}{2}}}$, as in Vol. I, page 324, and it may be easily proved [1532v]
by differentiation; the constant quantity 1 being inserted, in the second member, in order that the integral may vanish at the first limit $k' = -\infty$; then at the second limit $k' = \infty$, it becomes 2, and [1532u] changes into

$$X = \frac{f(\theta,\varpi)}{\pi} . \int_{-\infty}^{\infty} \frac{g\,dh}{g^2+h^2} . \qquad [1532w]$$

If we put $h = g\,t$, we shall get $\int \frac{g\,dh}{g^2+h^2} = \int \frac{dt}{1+t^2} = \text{arc tang.}\, t$, [51] Int. The [1532x]
constant quantity $\frac{1}{2}\pi$ is to be added, to make it vanish when $h = -\infty$, or $t = -\infty$, and when $h = \infty$, $t = \infty$, it becomes $\frac{1}{2}\pi + \frac{1}{2}\pi = \pi$; substituting this in [1532w], we get,

$$X = f(\theta,\varpi); \qquad [1532y]$$

which is the limit of the value of X when $1 - \alpha = g$, is supposed to be positive, but infinitely small. If $\alpha > 1$, g will become negative, and the sign of the second member of X [1532r, &c.], will be changed, and we shall get for the limit, in this case,

$$X = -f(\theta,\varpi). \qquad [1532z]$$

The results obtained in [1532, y, z,], agree with the theorem proposed to be demonstrated in [1532i].

We may remark, that in the preceding demonstration, the function $f(\theta',\varpi')$ is supposed to be constant, for all the values of θ', ϖ', in which the corresponding elements of the value of X do not vanish, when $1-\alpha$ becomes infinitely small; which is a necessary result, from the supposition, that $f(\theta',\varpi')$ never becomes infinite between the proposed limits [1532h'].
For if we put $f(\theta',\varpi') = f(\theta,\varpi) + \zeta$, ζ will be an infinitely small quantity between the [1533a]

Development of a function in terms of the form $Y^{(i)}$.

We must take, in the preceding manner, the most general expression of

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[1533a'] proposed limits of h, k; and if we also, for brevity, put $d\sigma = \frac{(1-\alpha^2)\,.\,\sin.\,\theta'\,.\,d\theta'\,.\,d\varpi'}{(1-2\alpha\,.\,p+\alpha^2)^{\frac{3}{2}}}$; the expression [1532h] will become,

[1533b] $$X = \frac{1}{4\pi}\,.\,\int\int\{f(\theta,\varpi)+\zeta\}\,.\,d\sigma = \frac{f(\theta,\varpi)}{4\pi}\,.\int\int d\sigma + \frac{1}{4\pi}\,.\int\int\zeta\,.\,d\sigma.$$

When α differs from unity by an infinitely small quantity, it will suffice to take the integral for infinitely small values of $\theta'-\theta$, $\varpi'-\varpi$; and if we suppose ζ' to be the greatest value of ζ, independent of its sign, between these limits we shall evidently have,

$$\frac{1}{4\pi}\,.\int\int\zeta\,.\,d\sigma < \frac{1}{4\pi}\,.\,\zeta'\,.\int\int d\sigma\,;$$

and as the factor $\frac{1}{4\pi}\,.\,\zeta'$ is infinitely small in comparison with $\frac{f(\theta,\varpi)}{4\pi}$ [1533a], this

[1533c] second integral may be neglected, and we shall have, as before, $X = \frac{f(\theta,\varpi)}{4\pi}\,.\int\int d\sigma$.

Some objections were made to this demonstration, in the London Philosophical Magazine for May 1827, which were satisfactorily answered, in the same Journal for July 1827. In the *Connoisance des Tems* for 1831, Mr. Poisson varied the form of the integral [1532h], by taking for c, any constant positive quantity, he put

[1533c'] $$X = \frac{c}{\pi\,.\,2^{1+\frac{1}{2}c}}\,.\int_0^{\pi}\int_0^{2\pi}\frac{(1-\alpha^2)^c\,.\,f(\theta',\varpi')\,.\,\sin.\,\theta'\,.\,d\theta'\,.\,d\varpi'}{(1-2\alpha\,.\,p+\alpha^2)^{1+\frac{1}{2}c}}\,;$$

and with this more general value, he obtained the same results, as with the former expression, by proceeding in nearly the same manner as in [1532l—z]; but we shall not enter into any particular explanation of this method, because the former is sufficient for all the purposes required in this work.

The value of $\rho = (1-2\alpha\,.\,p+\alpha^2)^{-\frac{1}{2}}$ [1532f], gives

[1533d] $$\left(\frac{d\rho}{d\alpha}\right) = (p-\alpha)\,.\,(1-2\alpha\,.\,p+\alpha^2)^{-\frac{3}{2}}\,; \qquad \text{hence}$$

$$\rho+2\alpha.\left(\frac{d\rho}{d\alpha}\right) = \{(1-2\alpha.p+\alpha^2)+2\alpha.(p-\alpha)\}.(1-2\alpha.p+\alpha^2)^{-\frac{3}{2}} = (1-\alpha^2).(1-2\alpha.p+\alpha^2)^{-\frac{3}{2}}\,;$$

substituting this in X [1532h], we get,

[1533e] $$X = \frac{1}{4\pi}\,.\int_0^{\pi}\int_0^{2\pi}\left\{\rho+2\alpha\,.\left(\frac{d\rho}{d\alpha}\right)\right\}.\,f(\theta',\varpi')\,.\,\sin.\,\theta'\,.\,d\theta'\,.\,d\varpi'.$$

Now from [1532g] we have,

[1533f]
$$\rho = 1+\alpha\,.\,Q^{(1)}+\alpha^2\,.\,Q^{(2)}+\alpha^3\,.\,Q^{(3)}+\ldots\ldots\ldots\ldots\,\alpha^i\,.\,Q^{(i)}+\&c.\,;$$
$$2\alpha\,.\left(\frac{d\rho}{d\alpha}\right) = \quad 2\alpha\,.\,Q^{(1)}+4\alpha^2\,.\,Q^{(2)}+6\alpha^3\,.\,Q^{(3)}+\ldots\ldots\ldots\,2i\,.\,\alpha^i\,.\,Q^{(i)}+\&c.\,;$$
$$\rho+2\alpha.\left(\frac{d\rho}{d\alpha}\right) = 1+3\alpha\,.\,Q^{(1)}+5\alpha^2\,.\,Q^{(2)}+7\alpha^3\,.\,Q^{(3)}+\ldots\ldots(2i+1)\,.\,\alpha^i\,.\,Q^{(i)}+\&c.$$

Substituting this in [1533e], we find,

Theorem.

[1533g] $$X = \frac{1}{4\pi}\,.\int_0^{\pi}\int_0^{2\pi}\{1+3\alpha.Q^{(1)}+5\alpha^2.Q^{(2)}+\ldots..(2i+1).\alpha^i.Q^{(i)}+\&c.\}.f(\theta',\varpi').\sin.\theta'.d\theta'.d\varpi'\,;$$

and the quantity X will be developed according to the powers of α.

$Y^{(s)}$; we must subtract it from S, and determine the arbitrary terms of — Development of a function in terms of $Y^{(i)}$.

The quantity $Q^{(i)}$ satisfies the differential equation [1442], in which the variable quantities are μ, ϖ, and the coefficient of $Q^{(i)}$, in the element of the integral [1533g], does not contain μ, ϖ, being $\frac{1}{4\pi} \cdot (2i+1) \cdot \alpha^i \cdot f(\theta',\varpi') \cdot \sin.\theta' \cdot d\theta' \cdot d\varpi'$; therefore this element, as well as its double integral, will also satisfy the same equation [1442]; and if we put this double integral equal to $\alpha^i \cdot Y^{(i)}$, we shall have,

$$Y^{(i)} = \frac{2i+1}{4\pi} \cdot \int_0^{\pi} \int_0^{2\pi} Q^{(i)} \cdot f(\theta',\varpi') \cdot \sin.\theta' \cdot d\theta' \cdot d\varpi'. \qquad [1533h]$$

Supposing now α *to be less than unity, by an infinitely small quantity*, the expression of X [1533g], or its equal [1532y], will become, — Development of $f(\theta,\varpi)$.

$$X = f(\theta,\varpi) = Y^{(0)} + Y^{(1)} + Y^{(2)} + \ldots\ldots Y^{(i)} + \&c.; \qquad [1533i]$$

which is the theorem proposed to be demonstrated in [1532a].

From [1533h, i], we obtain another demonstration of the theorem [1479′], namely, that the function y can be developed only in one series of the form $Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$ For $Q^{(i)}$ is given in [1441″], and if all the values of $f(\theta',\varpi')$, between the proposed limits of the integral [1533h], be given, the form and value of $Y^{(i)}$ will be fully determined by means of this integral, and the resulting *series* $Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$ [1533i], *will also have, in general, a single form and value, every term being given by a definite integral, all the elements of which are known, within the proposed limits of the integral.* — There can be only one development of a function [1533i'] in terms of the form $\Sigma\, Y^{(i)}$. Second demonstration.

Changing θ, ϖ, $Y^{(i)}$, into θ', ϖ', $Y'^{(i)}$, respectively, in [1533i], we get,

$$f(\theta',\varpi') = Y'^{(0)} + Y'^{(1)} + Y'^{(2)} + \ldots\ldots Y'^{(i)} + \&c. \qquad [1533k]$$

Substituting this in [1533h], and reducing, by means of the theorem [1476a], it becomes

$$Y^{(i)} = \frac{2i+1}{4\pi} \cdot \int_0^{\pi} \int_0^{2\pi} Q^{(i)} \cdot Y'^{(i)} \cdot \sin.\theta' \cdot d\theta' \cdot d\varpi'. \qquad [1533l]$$

Hence

$$\int_0^{\pi} \int_0^{2\pi} Y'^{(i)} \cdot Q^{(i)} \cdot \sin.\theta' \cdot d\theta' \cdot d\varpi' = \frac{4\pi}{2i+1} \cdot Y^{(i)}, \qquad [1533m]$$

Very important theorem in definite integrals.

which is the same as the important theorem given by La Place in [1540], with a different demonstration; and if we use the value $dw = \sin.\theta' \cdot d\theta' \cdot d\varpi'$ [1447f], it will become

$$\int Y'^{(i)} \cdot Q^{(i)} \cdot dw = \frac{4\pi}{2i+1} \cdot Y^{(i)}; \qquad [1533n]$$

the integral relative to w being taken, so as to include the whole surface of the sphere, whose radius is unity.

A formula so general as that here given [1532h, i] for the development of X, or $f(\theta,\varpi)$, is not to be expected to be free from cases of failure, such as frequently happen in much less complicated formulas; as for example, the very simple formula $\frac{x^2-1}{x-1} = x+1$, [1533o] which is generally correct, for all values of x, becomes illusory, and of the form $\frac{0}{0} = 2$, when $x = 1$. In like manner, if the quantity $\sqrt{(1-x)}$ be developed, by the common process of the binomial theorem, or by Maclaurin's theorem [607a], it will become of the

[1530v] $Y^{(s)}$, so that the powers and products of μ and $\sqrt{1-\mu^2}$, of the order s,

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[1533o′] form $1-\frac{1}{2}x-\frac{1}{8}x^2-$ &c., which is correct and approximative, when $x<1$; but if x exceed 1, the expression $\sqrt{(1-x)}$ becomes imaginary, and the development is wholly defective. The experienced analyst will therefore, in using a general formula, for the development of a function of any kind, examine carefully into the nature of the series, and in those particular cases where it might fail, will make the necessary modifications, or apply another method. For instance, if it be found that the series become diverging, when applied to the solution of any particular problem, it will be necessary to proceed with great caution; since the results obtained, from the use of diverging series, are frequently illusory; as will be seen in [1548λ]. A similar difficulty occurs if any of the terms of the proposed solution become infinite. In such instances, the usual development might be found inconvenient or impracticable; and we may find cases in which the formulas [1532*h*, *i*], will require some modification, or adjustment, to adapt them to particular values of θ, ϖ, in the same
[1533p] manner as in the examples [1533*o*, *o′*]. For in the development of $f(\theta,\varpi)$, θ is supposed to be limited between the values 0, π; also ϖ between 0 and 2π; and it will be found, upon examination, that for the particular values $\theta=0$, $\theta=\pi$, $\varpi=0$, $\varpi=2\pi$, there are certain conditions necessary to render the general development of $f(\theta,\varpi)$ applicable; as will appear by the following investigation.

[1533q] If we have $\varpi=0$, there will be *two* values, $\varpi'=0$, $\varpi'=2\pi$, which being taken with $\theta'=\theta$, will render p [1532*c*] equal to unity. In this case, we must not, in finding the value of X, when $1-\alpha$ is infinitely small, restrict ourselves to *one* form of $\varpi'=\varpi+k$, as in [1532*d*]; but must take successively $\varpi'=k$, $\varpi'=2\pi+k$; but as the variable
[1533r] quantity ϖ' ought always to be positive, and not *exceed* 2π [1532*h′*], we must, in the first case, use only the *positive* values of k; and in the second case only the *negative* values; therefore, in the calculation of X [1532*r*], we must, in the first case, integrate only from $k=0$ to $k=\infty$; and in the second case from $k=-\infty$ to $k=0$; which will reduce,
[1533s] in the two cases, the integral relative to k, to half its preceding value [1532*i*]. Hence it follows, that for the particular value $\varpi=0$, the limit of X will be,

[1533t] $$X=\pm\tfrac{1}{2}\cdot\{f(\theta,0)+f(\theta,2\pi)\};$$

the upper sign being used, as in [1532*k*], when $\alpha<1$, the lower sign when $\alpha>1$. In
[1533u] like manner we shall obtain a similar result for the other limit $\varpi=2\pi$.

Therefore at the two extremities of the values of ϖ, for which the equation [1532*i*] exists,
[1533v] its second member expresses the half sum of the corresponding values of $f(\theta,\varpi)$, instead of the values themselves; consequently the equation [1532*i*] will not agree with these extreme
[1533w] values, unless we suppose $f(\theta,0)=f(\theta,2\pi)$. We may observe, that this equation would generally be satisfied, if $f(\theta,\varpi)$ were given in a finite integral expression of the sines and cosines of θ, ϖ, and their multiples; but it might not be satisfied for other functions of θ, ϖ; as for example, if it contained the arcs θ, ϖ, independently of their sines or cosines.

[1533x] In the case of $\theta=0$, we have $p=\cos.\theta'$ [1532*c*], $f(\theta,\varpi)=f(0,\varpi)$, $f(\theta',\varpi')=f(0,\varpi')$; and then [1532*h*] will become,

may disappear from the difference $S-Y^{(s)}$; this difference will then

Development of a function in terms of the form $Y^{(i)}$.

$$X=\frac{1}{4\pi}\cdot\int_0^{2\pi}f(0,\varpi')\,.\,d\varpi'\,.\int_0^{\pi}\frac{(1-\alpha^2)\,.\,\sin.\theta'\,.\,d\theta'}{(1-2\alpha\,.\cos.\theta'+\alpha^2)^{\frac{3}{2}}};\qquad[1533y]$$

but whatever be the value of the constant quantity α, we shall have,

$$\int\frac{(1-\alpha^2)\,.\,\sin.\theta'\,.\,d\theta'}{(1-2\alpha\,.\cos.\theta'+\alpha^2)^{\frac{3}{2}}}=-\frac{(1-\alpha^2)}{\alpha\,.\,(1-2\alpha\,.\cos.\theta'+\alpha^2)^{\frac{1}{2}}}+\text{constant};\qquad[1533z]$$

as is easily proved by taking the differential of the second member, relative to θ'. At the two limits of this integral $\theta'=0$, and $\theta'=\pi$, the radical in the denominator becomes $\pm(1-\alpha)$, $\pm(1+\alpha)$; and as this ought to be positive throughout the whole limits of the integration, because the radical does not change sign, while θ' varies from 0 to π, we must take $1+\alpha$, at the limit $\theta'=\pi$; and $\pm(1-\alpha)$, at the limit $\theta'=0$; the upper sign being used when $\alpha<1$, the lower when $\alpha>1$; in the first case, the integral [1533z] is expressed by [1533α]

$$-\frac{(1-\alpha^2)}{\alpha\,.\,(1+\alpha)}+\frac{(1-\alpha^2)}{\alpha\,.\,(1-\alpha)}=-\frac{1}{\alpha}\,.\,(1-\alpha)+\frac{1}{\alpha}\,.\,(1+\alpha)=\frac{\alpha}{\alpha}+\frac{\alpha}{\alpha}=2;\qquad[1533\beta]$$

in the second case it becomes,

$$-\frac{(1-\alpha^2)}{\alpha\,.\,(1+\alpha)}-\frac{(1-\alpha^2)}{\alpha\,.\,(1-\alpha)}=-\frac{1}{\alpha}\,.\,(1-\alpha)-\frac{1}{\alpha}\,.\,(1+\alpha)=-\frac{1}{\alpha}-\frac{1}{\alpha}=-\frac{2}{\alpha}.\qquad[1533\gamma]$$

Substituting these in X [1533y], it becomes,

$$X=\frac{1}{2\pi}\cdot\int_0^{2\pi}f(0,\varpi')\,.\,d\varpi',\qquad\text{or}\qquad X=-\frac{1}{2\pi\,.\,\alpha}\cdot\int_0^{2\pi}f(0,\varpi')\,.\,d\varpi'.\qquad[1533\delta]$$

In like manner we shall find, in the case of $\theta=\pi$,

$$X=\frac{1}{2\pi}\cdot\int_0^{2\pi}f(\pi,\varpi')\,.\,d\varpi',\qquad\text{or}\qquad X=-\frac{1}{2\pi\,.\,\alpha}\cdot\int_0^{2\pi}f(\pi,\varpi')\,.\,d\varpi'.\qquad[1533\varepsilon]$$

Hence we see, that for the extreme values of θ, the expression $\pm X$ [1532i], will not give the corresponding values $f(0,\varpi)$, and $f(\pi,\varpi)$, of the proposed function; but will [1533ζ] express the mean of all the values it has, between $\varpi=0$ and $\varpi=2\pi$; that is to say, $\frac{1}{2\pi}\cdot\int_0^{2\pi}f(0,\varpi)\,.\,d\varpi$, and $\frac{1}{2\pi}\cdot\int_0^{2\pi}f(\pi,\varpi)\,.\,d\varpi$. Therefore the equation [1532a] will not exist for the cases of $\theta=0$ and $\theta=\pi$, unless ϖ disappears from $f(\theta,\varpi)$, for [1533η] each of these values of θ; which requires that $f(0,\varpi)$ and $f(\pi,\varpi)$ should be constant quantities. [1533θ]

We may observe, that when the proposed quantity $f(\theta,\varpi)$ is actually a finite and integral function of the three co-ordinates $\cos.\theta$, $\sin.\theta\,.\,\sin.\varpi$, $\sin.\theta\,.\,\cos.\varpi$, both the conditions [1533ι] [1533w, η] will be satisfied; because these co-ordinates remain unaltered, when ϖ is changed from 0 to 2π, as is required in [1533w]; and the two last co-ordinates vanish, when $\theta=0$ [1533κ] or $\theta=\pi$; by which means ϖ disappears from the proposed function, as in [1533θ]; this is also evident from the inspection of the formulas [1528a—e].

The function $f(\theta,\varpi)$ is always supposed not to become infinite [1532b] between the extreme limits of θ, ϖ; but it may otherwise be of the most general form. It may even be discontinuous, or such that its values are not comprised in the same analytical expression. [1533λ] For example, $f(\theta,\varpi)$ may be composed of two parts, the one expressed by $\varphi(\theta,\varpi)$, from

[1530vi] become a function of the order $s-1$, which we shall denote by $S^{(1)}$. We

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$\theta=0$ to $\theta=\theta_{,}$; the other $\varphi_{,}(\theta,\varpi)$, from $\theta=\theta_{,}$ to $\theta=\pi$. The series [1533i] will then represent the values of φ, or those of $\varphi_{,}$, according as we shall substitute, for θ, a quantity
[1533μ] less, or greater, than $\theta_{,}$, but always comprised between the extreme limits $\theta=0$ and $\theta=\pi$. The general term of this series may be obtained by separating the integral [1533h], relative to θ', into two parts, so that we may have,

[1533ν]
$$Y^{(i)}=\frac{2i+1}{4\pi}\cdot\int_0^{2\pi}\left\{\int_0^{\theta_{,}}\varphi(\theta',\varpi').Q^{(i)}.\sin.\theta'.d\theta'+\int_{\theta_{,}}^{\pi}\varphi_{,}(\theta',\varpi').Q^{(i)}.\sin.\theta'.d\theta'\right\}.d\varpi'.$$

We may proceed in like manner, if the given function be discontinuous, relative to the other variable quantity ϖ. We must however observe, that when the function $f(\theta,\varpi)$ is
[1533ξ] discontinuous, it is necessary that the two values, corresponding to each point of junction, of the two different expressions, should be the same; that is, $\varphi(\theta_{,},\varpi)=\varphi_{,}(\theta_{,},\varpi)$; and if differentials of $f(\theta,\varpi)$ were required, it might also be necessary to apply similar conditions to the differentials of the functions φ, $\varphi_{,}$, to prevent any of the partial differentials $\left(\frac{d.f(\theta,\varpi)}{d\theta}\right)$, $\left(\frac{d.f(\theta,\varpi)}{d\varpi}\right)$, &c., from becoming infinite at the point of junction.

In the case mentioned in [1530l—l'], where $f(\theta,\varpi)$ is actually a finite integral function of $\cos.\theta$, $\sin.\theta.\cos.\varpi$, $\sin.\theta.\sin.\varpi$; we have seen that the development [1533i], can
[1533ξ'] be completely obtained, in finite terms, as in [1530''', &c.] This case includes an immense variety of spheroids, differing but little from a sphere; to these we might join numerous classes of irrational quantities, which could be reduced to rational converging series, and arranged in the form [1533i]; *so that for all practical purposes, the development of* $f(\theta,\varpi)$ *may be considered as embracing all cases, which would be wanted for the investigation of the*
[1533π] *general properties of the attraction of a spheroid, differing but little from a sphere.* Therefore we shall not enter into a more minute discussion of the cases of discontinuous or fractional functions; any one who wishes to pursue this subject, may refer to the paper published by Mr. Poisson, in the *Connoisance des Tems* for the year 1829. Finally we may remark, for the sake of illustrating and confirming what has been said, that the case of a spheroid, differing but little from a sphere, and of an irregular form, may generally, by the method of interpolation explained in [812, &c.], be reduced approximatively, to a curve surface, depending on a finite and integral function of the co-ordinates $\cos.\theta$, $\sin.\theta.\cos.\varpi$, $\sin.\theta.\sin.\varpi$, treated of in [1530'''']; and by increasing the number n [811viii] of the points of the surface, taken into consideration, we may generally obtain a surface, which will differ but very little from
[1533π'] the actual surface of the spheroid. In this way, the attraction of any irregular surface may be reduced, very nearly, to the finite form used in [1530''''], including a great variety of figures, about which there has been no doubt or discussion.

The greatest positive value of p [1532c], corresponds to $\gamma=0$, and is $p=1$. Substituting this in ρ [1532f, g], it becomes

[1533ρ]
$$\rho=(1-2\alpha+\alpha^2)^{-\frac{1}{2}}=(1-\alpha)^{-1}=1+\alpha+\alpha^2+\alpha^3\,\&c.;$$

must then take the most general expression of $Y^{(s-1)}$, and subtract it

Develop-ment of a function in terms of the form $Y^{(i)}$.

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therefore we have in this case $Q^{(i)}=1$. The greatest negative value of p corresponds to $\gamma=180^d$, and then $p=\cos.\gamma=-1$. Substituting this in [1532f, g], it becomes $(1+2\alpha+\alpha^2)^{-\frac{1}{2}}=(1+\alpha)^{-1}=1-\alpha+\alpha^2-\&c.$; hence $Q^{(0)}=1$, $Q^{(1)}=-1$, $Q^{(2)}=1$, and generally $Q^{(i)}=\pm 1$, the upper sign being used if i be even, the lower if odd. [1533σ]

Putting for brevity $x=c^{\gamma.\sqrt{-1}}$, $x^{-1}=c^{-\gamma.\sqrt{-1}}$, we shall get, from [12] Int.,

$$2\cos.n\gamma=c^{n\gamma.\sqrt{-1}}+c^{-n\gamma.\sqrt{-1}}=x^n+x^{-n}, \qquad \text{and}$$

[1533τ]

$$p=\cos.\gamma=\tfrac{1}{2}c^{\gamma.\sqrt{-1}}+\tfrac{1}{2}c^{-\gamma.\sqrt{-1}}=\tfrac{1}{2}.(x+x^{-1}), \qquad [1532c].$$

Substituting this in [1532f], we get,

$$\rho=\left\{1-\alpha.c^{\gamma.\sqrt{-1}}-\alpha.c^{-\gamma.\sqrt{-1}}+\alpha^2\right\}^{-\frac{1}{2}}=\{1-\alpha x-\alpha x^{-1}+\alpha^2\}^{-\frac{1}{2}}=\{1-\alpha x\}^{-\frac{1}{2}}.\{1-\alpha x^{-1}\}^{-\frac{1}{2}}$$

$$=\{1+\tfrac{1}{2}.\alpha x+\tfrac{1}{2}.\tfrac{3}{4}.\alpha^2x^2+\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}.\alpha^3x^3+\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}.\tfrac{7}{8}.\alpha^4x^4+\&c.\}.\{1+\tfrac{1}{2}.\alpha x^{-1}+\tfrac{1}{2}.\tfrac{3}{4}.\alpha^2x^{-2}+\&c.\}$$

$$\begin{aligned}
=1&+(\tfrac{1}{2}).\alpha x+(\tfrac{1}{2}.\tfrac{3}{4}).\alpha^2x^2+(\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}).\alpha^3x^3+(\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}.\tfrac{7}{8}).\alpha^4x^4+\&c.\\
&+(\tfrac{1}{2}).\alpha x^{-1}+(\tfrac{1}{2}).(\tfrac{1}{2}).\alpha^2+(\tfrac{1}{2}).(\tfrac{1}{2}.\tfrac{3}{4}).\alpha^3x+(\tfrac{1}{2}).(\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}).\alpha^4x^2+\&c.\\
&+(\tfrac{1}{2}.\tfrac{3}{4}).\alpha^2x^{-2}+(\tfrac{1}{2}.\tfrac{3}{4}).(\tfrac{1}{2}).\alpha^3x^{-1}+(\tfrac{1}{2}.\tfrac{3}{4}).(\tfrac{1}{2}.\tfrac{3}{4}).\alpha^4+\&c.\\
&+(\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}).\alpha^3x^{-3}+(\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}).(\tfrac{1}{2}).\alpha^4x^{-2}+\&c.\\
&+(\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}.\tfrac{7}{8}).\alpha^4x^{-4}+\&c.
\end{aligned}$$ [1533υ]

$$=1+\alpha.\left\{\frac{x+x^{-1}}{2}\right\}+\alpha^2.\{\tfrac{1}{2}.\tfrac{3}{4}.(x^2+x^{-2})+\tfrac{1}{2}.\tfrac{1}{2}\}+\&c.$$

$$\begin{aligned}
=1+\alpha.\cos.\gamma\ &+\alpha^2.\{\tfrac{1}{2}.\tfrac{3}{4}.2\cos.2\gamma+\tfrac{1}{2}.\tfrac{1}{2}\}\\
&+\alpha^3.\{\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}.2\cos.3\gamma+\tfrac{1}{2}.(\tfrac{1}{2}.\tfrac{3}{4}).2\cos.\gamma\}\\
&+\alpha^4.\{\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}.\tfrac{7}{8}.2\cos.4\gamma+\tfrac{1}{2}.(\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}).2\cos.2\gamma+(\tfrac{1}{2}.\tfrac{3}{4}).(\tfrac{1}{2}.\tfrac{3}{4})\}+\&c.
\end{aligned}$$ [1533φ]

The limits of $Q^{(i)}$ are ± 1.

The numerical coefficients of all these terms being positive, the greatest positive coefficient of α^i must evidently correspond to $\gamma=0$; and for the same reason, the greatest negative coefficient of any odd power of α must correspond to $\gamma=180^d$. Now we have shown, in [1533σ], that these greatest and least values are represented by ± 1. Hence we may conclude that the greatest positive value of $Q^{(i)}$ is $Q^{(i)}=1$, and the greatest negative value is $Q^{(i)}=-1$. [1533χ]

In the fifth volume of this work, the author has shown the convergency of the terms of the series $\rho=\{1-2\alpha.\cos.\gamma+\alpha^2\}^{-\frac{1}{2}}=Q^{(0)}+\alpha.Q^{(1)}+\alpha^2.Q^{(2)}+\&c.$; and in the *Connoisance des Tems* for 1831, Mr. Poisson has also proved the convergency of the series $f(\theta,\varpi)=Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.$, [1532a]. The necessity of an examination into the convergency of such series will more fully appear, from some examples we shall hereafter give [1548λ]; and on account of the importance of the subject, we have thought it necessary to insert the demonstrations in this place, in order to prevent any doubt in relation to it. In doing this, we shall have to investigate the values of some definite integrals, which will be wanted in this computation, and in other parts of the work. [1533ψ]

[1530vii] from $S^{(1)}$; then determine the arbitrary quantities of $Y^{(s-1)}$, so that the

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[1534a] Supposing s *to be any integral positive number*, and hyp. log. $c=1$, we shall have generally $\int x^s \cdot c^{-x} \cdot dx = -x^s \cdot c^{-x} + s \cdot \int x^{s-1} \cdot c^{-x} \cdot dx$, as is easily proved by differentiation and reduction. The term $-x^s \cdot c^{-x}$, without the sign of integration, evidently vanishes when $x=0$. It also vanishes when $x=\infty$, for $x^s \cdot c^{-x} = \left(\frac{x}{c^{\frac{x}{s}}}\right)^s$, and if we suppose x to increase from 1 to ∞, in an arithmetical progression, the numerator

[1534a′] of the fraction $\frac{x}{c^{\frac{x}{s}}}$, will increase in an *arithmetical* progression from 1 to ∞, and the denominator $c^{\frac{x}{s}}$, will increase in a *geometrical* progression; so that when $x=\infty$, the denominator will be infinitely greater than the numerator, and the fraction itself must become nothing. Hence if we take the preceding integral between the limits $x=0$ and $x=\infty$, we shall have generally,

Theorem.

[1534b]
$$\int_0^\infty x^s \cdot c^{-x} \cdot dx = s \cdot \int_0^\infty x^{s-1} \cdot c^{-x} \cdot dx.$$

Changing successively s into $s-1$, $s-2$, &c. ..... 1, we get

$$\int_0^\infty x^{s-1} \cdot c^{-x} \cdot dx = (s-1) \cdot \int_0^\infty x^{s-2} \cdot c^{-x} \cdot dx,$$

$$\int_0^\infty x^{s-2} \cdot c^{-x} \cdot dx = (s-2) \cdot \int_0^\infty x^{s-3} \cdot c^{-x} \cdot dx, \qquad \text{\&c.};$$

[1534c] and by successive substitutions, $\int_0^\infty x^s \cdot c^{-x} \cdot dx = s \cdot (s-1) \cdot (s-2) \ldots\ldots 1 \cdot \int_0^\infty dx \cdot c^{-x}$. Now $\int dx \cdot c^{-x} = 1 - c^{-x}$, vanishes at the first limit $x=0$, and at the second limit

[1534c′] $x=\infty$, it becomes $\int_0^\infty dx \cdot c^{-x} = 1$; hence we finally get,

Theorem.

[1534d]
$$\int_0^\infty x^s \cdot c^{-x} \cdot dx = 1 \cdot 2 \cdot 3 \ldots\ldots s.$$

The integral of the expression $dv \cdot c^{-v\cdot(1+xx)}$, supposing x to be constant, and commencing

[1534e] the integral with $v=0$, is $\frac{1}{1+x^2} \cdot \{1 - c^{-v\cdot(1+xx)}\}$, as is easily proved by differentiation.

[1534e′] This, when $v=\infty$, becomes $\int_0^\infty dv \cdot c^{-v\cdot(1+xx)} = \frac{1}{1+x^2}$; which may be used in finding the values of the double integral $\int_0^\infty \int_0^\infty dv \cdot dx \cdot c^{-v\cdot(1+xx)}$. For the integration being taken relatively to v, in the manner just mentioned, between the proposed limits, it becomes $\int_0^\infty \frac{dx}{1+x^2}$. Now $\int \frac{dx}{1+x^2} = \text{arc} \cdot (\text{tang.}\, x)$ [51] Int. vanishes when $x=0$, and when $x=\infty$, it becomes $\frac{1}{2}\pi$; hence we finally get,

Theorem.

[1534f]
$$\int_0^\infty \int_0^\infty dv \cdot dx \cdot c^{-v\cdot(1+xx)} = \tfrac{1}{2}\pi;$$

powers and products of μ and $\sqrt{1-\mu^2}$, of the order $s-1$, may

Development of a function in terms of $Y^{(i)}$.

We may find this integral in a different manner, using another variable quantity t instead of x, putting $t=x\,.\,v^{\frac{1}{2}}$. This gives $x=t\,.\,v^{-\frac{1}{2}}$; and its differential, taken upon the supposition that the independent variable quantity v is constant, is $dx=dt\,.\,v^{-\frac{1}{2}}$. Substituting this in the first member of [1534f], it becomes [1534g]

$$\int_0^\infty\int_0^\infty dv\,.\,dt\,.\,v^{-\frac{1}{2}}\,.\,c^{-v-tt}=\int_0^\infty dv\,.\,v^{-\frac{1}{2}}\,.\,c^{-v}\,.\int_0^\infty dt\,.\,c^{-tt}; \qquad [1534h]$$

and if we put $K=\int_0^\infty dt\,.\,c^{-tt}$, the preceding integral will become

$$\int_0^\infty dv\,.\,v^{-\frac{1}{2}}\,.\,c^{-v}\,.\,K=K\,.\int_0^\infty dv\,.\,v^{-\frac{1}{2}}\,.\,c^{-v}.$$

Supposing $v=\tau^2$, we get $\int_0^\infty dv\,.\,v^{-\frac{1}{2}}\,.\,c^{-v}=2\,.\int_0^\infty d\tau\,.\,c^{-\tau^2}=2K$; therefore the preceding integral becomes, [1534i]

Theorem.

$$\int_0^\infty\int_0^\infty dv\,.\,dx\,.\,c^{-v\,.\,(1+x^2)}=2K^2. \qquad [1534k]$$

Putting the two expressions [1534f, k] equal to each other, we get $2K^2=\frac{1}{2}\pi$; whence $K=\frac{1}{2}\,.\sqrt{\pi}$, or

Theorem.

$$\int_0^\infty dt\,.\,c^{-t^2}=\tfrac{1}{2}\,.\sqrt{\pi}. \qquad [1534l]$$

This well known theorem is demonstrated by the author, in nearly this manner, in the tenth book [8331].

As c^{-tt} does not alter its value by changing the sign of t, it is evident that the preceding integral would be doubled, by including all the negative values of t, from $t=-\infty$ to $t=0$. Hence we shall have

Theorem.

$$\int_{-\infty}^\infty dt\,.\,c^{-t^2}=\sqrt{\pi}. \qquad [1534m]$$

The integral $\int t\,dt\,.\,c^{-tt}=-\frac{1}{2}c^{-tt}$, vanishes when $t=\mp\infty$, hence we have,

Theorem.

$$\int_{-\infty}^\infty t\,dt\,.\,c^{-t^2}=0. \qquad [1534n]$$

We may easily prove, by differentiation, that

$$\int t^m\,dt\,.\,c^{-tt}=-\tfrac{1}{2}t^{m-1}\,.\,c^{-tt}+\tfrac{1}{2}\,.\,(m-1)\,.\int t^{m-2}\,dt\,.\,c^{-tt}; \qquad [1534o]$$

and as the term $-\frac{1}{2}t^{m-1}\,.\,c^{-tt}$ vanishes when $t=\mp\infty$ [1534a′], we shall have, by taking $\pm\infty$ for the limits of the integral [1534o],

Theorem.

$$\int_{-\infty}^\infty t^m\,dt\,.\,c^{-t^2}=\tfrac{1}{2}\,.\,(m-1)\,.\int_{-\infty}^\infty t^{m-2}\,dt\,.\,c^{-t^2}. \qquad [1534p]$$

Putting now successively $m=2$, 4, 6, &c., and using the formula [1534m], we get,

Definite integrals.

$$\int_{-\infty}^\infty t^2\,dt\,.\,c^{-t^2}=\tfrac{1}{2}\,.\int_{-\infty}^\infty dt\,.\,c^{-t^2}=\tfrac{1}{2}\,.\sqrt{\pi};$$

$$\int_{-\infty}^\infty t^4\,dt\,.\,c^{-t^2}=\tfrac{3}{2}\,.\int_{-\infty}^\infty t^2\,dt\,.\,c^{-t^2}=\tfrac{1}{2}\,.\,\tfrac{3}{2}\,.\sqrt{\pi}; \qquad [1534q]$$

$$\int_{-\infty}^\infty t^6\,dt\,.\,c^{-t^2}=\tfrac{5}{2}\,.\int_{-\infty}^\infty t^4\,dt\,.\,c^{-t^2}=\tfrac{1}{2}\,.\,\tfrac{3}{2}\,.\,\tfrac{5}{2}\,.\sqrt{\pi}; \qquad \&c.$$

[1530viii] disappear from the difference $S^{(1)}-Y^{(s-1)}$. Continuing in this manner, we may

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By a similar reduction with the odd values of m, 3, 5, 7, &c., they will all be reduced to the form [1534n], and will therefore vanish within the proposed limits. Hence, if s be any positive integral number, we shall have,

Theorem. [1534r]
$$\int_{-\infty}^{\infty} t^{2s+1}\, dt \,.\, c^{-t^2} = 0.$$

If we put $x = s + z$ in [1534d], so that z may be the variable quantity whose differential is $dz = dx$, we shall have, for the limits of z, $z = -s$, and $z = \infty$, corresponding to $x = 0$, and $x = \infty$; and this expression will become

[1534s]
$$1\,.\,2\,.\,3\,\ldots\ldots s = \int_{-s}^{\infty} (s+z)^s \,.\, c^{-s-z} \,.\, dz = s^s \,.\, c^{-s} \,.\, \int_{-s}^{\infty} \left(1+\frac{z}{s}\right)^s \,.\, c^{-z} \,.\, dz.$$

Now putting $\left(1+\frac{z}{s}\right)^s \,.\, c^{-z} = c^{-t^2}$, or $c^{t^2} = c^z \,.\, \left(1+\frac{z}{s}\right)^{-s}$; we have $\mp\infty$ for the limits of t, corresponding to the preceding limits of z; and the logarithm of this expression of c^{t^2} being found, as in [58] Int., and multiplied by $\frac{2}{s}$, gives,

[1534t]
$$\frac{2t^2}{s} = \frac{2z}{s} - 2 \,.\, \log.\left(1+\frac{z}{s}\right) = \frac{2z}{s} - 2 \,.\, \left\{\frac{z}{s} - \tfrac{1}{2} \,.\, \frac{z^2}{s^2} + \tfrac{1}{3} \,.\, \frac{z^3}{s^3} - \&c.\right\}$$
$$= \frac{z^2}{s^2} - \frac{2z^3}{3s^3} + \frac{2z^4}{4s^4} - \&c.$$

Taking the square root of this expression, we get $t \,.\, s^{-\frac{1}{2}} \,.\, \sqrt{2} = \frac{z}{s} - \tfrac{1}{3} \,.\, \frac{z^2}{s^2} + \tfrac{7}{36} \,.\, \frac{z^3}{s^3} + \&c.$; in which the second member proceeds according to the powers of $\frac{z}{s}$; and by inverting the series in the usual manner, we shall get, for $\frac{z}{s}$, an expression ascending according to the powers of $t.s^{-\frac{1}{2}}.\sqrt{2}$, namely $\frac{z}{s} = t.s^{-\frac{1}{2}}.\sqrt{2} + \tfrac{2}{3}t^2.s^{-1} + \frac{1}{9.\sqrt{2}} \,.\, t^3 \,.\, s^{-\frac{3}{2}} + \&c.$; as is very easily proved by substituting this value of $\frac{z}{s}$ in the preceding equation; thence we have

[1534u] $z = t \,.\, s^{\frac{1}{2}} \,.\, \sqrt{2} + \tfrac{2}{3}t^2 + \frac{1}{9.\sqrt{2}} \,.\, t^3 \,.\, s^{-\frac{1}{2}} + \&c.$ The differential of this is

[1534v]
$$dz = dt \,.\, s^{\frac{1}{2}} \,.\, \sqrt{2} + \tfrac{4}{3}\, t\, dt + \frac{1}{3.\sqrt{2}} \,.\, t^2\, dt \,.\, s^{-\frac{1}{2}} + \&c.\,;$$

therefore the expression [1534s] becomes, by using the formulas [1534m, n, q],

$$1\,.\,2\,.\,3\,\ldots\ldots s = s^s \,.\, c^{-s} \,.\, \int_{-\infty}^{\infty} c^{-t^2} \,.\, dt \,.\, \left\{s^{\frac{1}{2}}.\sqrt{2} + \tfrac{4}{3}\,t + \frac{1}{3.\sqrt{2}} \,.\, t^2 \,.\, s^{-\frac{1}{2}} + \&c.\right\}$$
$$= s^s \,.\, c^{-s} \,.\left\{s^{\frac{1}{2}} \,.\, \sqrt{2} \,.\, \int_{-\infty}^{\infty} c^{-t^2} .dt + \tfrac{4}{3} . \int_{-\infty}^{\infty} t dt \,.\, c^{-t^2} + \frac{1}{3.\sqrt{2}} .s^{-\frac{1}{2}} \,.\, \int_{-\infty}^{\infty} t^2 dt \,.\, c^{-t^2} + \&c.\right\}$$
$$= s^s \,.\, c^{-s} \,.\left\{s^{\frac{1}{2}} \,.\, \sqrt{2} \,.\, \sqrt{\pi} + \frac{1}{3.\sqrt{2}} \,.\, s^{-\frac{1}{2}} \,.\, \tfrac{1}{2} \,.\, \sqrt{\pi} + \&c.\right\}$$

[1534w]
$$= s^{s+\frac{1}{2}} \,.\, c^{-s} \,.\, \sqrt{2\pi} \,.\left\{1 + \frac{1}{12s} + \&c.\right\}.$$

determine the functions $Y^{(s)}$, $Y^{(s-1)}$, $Y^{(s-2)}$, &c., whose sum is equal to S.

When s is a very great number, all the terms of the factor $1+\frac{1}{12s}+\&c.$, except the first, are divided by s and its powers, and may therefore be neglected; hence we shall have, very nearly,

Theorem when s is very great.

$$1.2.3\ldots\ldots s = s^{s+\frac{1}{2}}.c^{-s}\sqrt{2\pi}. \qquad [1534x]$$

When s exceeds 10, the factor $1+\frac{1}{12s}$ [1534w], expresses very nearly, the ratio of the first member of [1534x], to the second member of the same equation; as is easily verified by direct calculation for $s=10$, 20, &c. If s be put successively equal to 10, 100, 1000, &c.; this ratio will be respectively 1,008, 1,0008, 1,00008, &c., nearly; hence the accuracy of the formula, when s is a very great number, is manifest.

We shall now proceed to the investigation of the value of $Q^{(i)}$, when i is a great number. If we put $\theta'=0$, in [1532c, f, e], we shall get $p=\cos.\theta$;

$$(1-2\alpha.\cos.\theta+\alpha^2)^{-\frac{1}{2}}=1+\alpha.Q^{(1)}+\alpha^2.Q^{(2)}+\&c.; \qquad [1534y]$$

$$Q^{(i)}=\frac{1.3.5\ldots\ldots(2i-1)}{1.2.3\ldots\ldots i}.\left\{\cos.^i\theta-\frac{i.(i-1)}{2.(2i-1)}.\cos.^{i-2}\theta+\frac{i.(i-1).(i-2).(i-3)}{2.4.(2i-1).(2i-3)}.\cos.^{i-4}\theta-\&c.\right\}. \qquad [1534z]$$

Dividing [1442a] by $\sqrt{\{i.(i+1)\}}$, which for brevity we shall put $=a$; we shall get, [1534α] by neglecting the term depending on ϖ, because the preceding value of $Q^{(i)}$ does not contain that quantity,

$$0=\frac{1}{a}.\left(\frac{d\,d\,Q^{(i)}}{d\theta^2}\right)+\frac{\cos.\theta}{a.\sin.\theta}.\left(\frac{dQ^{(i)}}{d\theta}\right)+a.Q^{(i)}. \qquad [1534\beta]$$

We shall now suppose,

$$Q^{(i)}=u.\cos.a\theta+u'.\sin.a\theta; \qquad [1534\gamma]$$

u, u', being functions of θ. Taking the differentials of this value of $Q^{(i)}$ relatively to θ, and multiplying by the coefficients of these differentials in [1534β], we get,

$$\frac{1}{a}.\frac{\cos.\theta}{\sin.\theta}.\left(\frac{d\,Q^{(i)}}{d\theta}\right)=\frac{1}{a}.\frac{\cos.\theta}{\sin.\theta}.\left\{\begin{array}{l}\left(\frac{du}{d\theta}\right).\cos.a\theta-au.\sin.a\theta\\ +\left(\frac{du'}{d\theta}\right).\sin.a\theta+au'.\cos.a\theta\end{array}\right\}; \qquad [1534\delta]$$

$$\frac{1}{a}.\left(\frac{d\,d\,Q^{(i)}}{d\theta^2}\right)=\frac{1}{a}.\left\{\begin{array}{l}\left(\frac{d\,d\,u}{d\theta^2}\right).\cos.a\theta-2a.\left(\frac{du}{d\theta}\right).\sin.a\theta-a^2u.\cos.a\theta\\ +\left(\frac{d\,d\,u'}{d\theta^2}\right).\sin.a\theta+2a.\left(\frac{du'}{d\theta}\right).\cos.a\theta-a^2u'.\sin.a\theta\end{array}\right\};$$

$$a.Q^{(i)}=au.\cos.a\theta+au'.\sin.a\theta.$$

Adding these three equations together, the first member becomes nothing, by means of [1534β]; and in order to make the second member vanish, we shall put the coefficients of $\sin.a\theta$, $\cos.a\theta$, separately equal to nothing, which will produce the two following equations [1534ε], by which the arbitrary quantities u, u', can be determined. In finding the sum of the terms of the second members of [1534δ], we may observe, that the seventh is destroyed by the eleventh, and the tenth by the twelfth.

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17. We shall now resume the equation of § 15 [1507],

[1534ε]
$$2\cdot\left(\frac{du}{d\theta}\right)+u\cdot\frac{\cos.\theta}{\sin.\theta}=\frac{1}{a}\cdot\left\{\left(\frac{ddu'}{d\theta^2}\right)+\left(\frac{du'}{d\theta}\right)\cdot\frac{\cos.\theta}{\sin.\theta}\right\},$$
$$2\cdot\left(\frac{du'}{d\theta}\right)+u'\cdot\frac{\cos.\theta}{\sin.\theta}=-\frac{1}{a}\cdot\left\{\left(\frac{ddu}{d\theta^2}\right)+\left(\frac{du}{d\theta}\right)\cdot\frac{\cos.\theta}{\sin.\theta}\right\}.$$

If we neglect the second members of these equations, because they are divided by the very large number a, they will become integrable by multiplying them by $\frac{1}{2}d\theta.\sqrt{(\sin.\theta)}$. For then the first of these equations will become $du.\sqrt{(\sin.\theta)}+u.\frac{d\theta.\cos.\theta}{2.\sqrt{(\sin.\theta)}}=d.\{u.\sqrt{(\sin.\theta)}\}=0$, and its integral is $u.\sqrt{(\sin.\theta)}=H$; H being a constant quantity. If we take H' for the constant quantity, depending on the integral of the second equation, we shall get

[1534ζ]
$$u=H.(\sin.\theta)^{-\frac{1}{2}}, \qquad u'=H'.(\sin.\theta)^{-\frac{1}{2}}.$$

Substituting these values in the second members of [1534ε], they will become functions of θ; then multiplying, as before, by $\frac{1}{2}d\theta.\sqrt{(\sin.\theta)}$, and integrating, we shall find,

[1534η]
$$u=H.(\sin.\theta)^{-\frac{1}{2}}+\frac{f(\theta)}{a}, \qquad u'=H'.(\sin.\theta)^{-\frac{1}{2}}+\frac{F(\theta)}{a};$$

$f(\theta)$, $F(\theta)$, being functions of θ. In this way we may obtain u, u', in a series of terms, proceeding according to the powers of $\frac{1}{a}$; therefore $Q^{(i)}$ [1534γ] will be expressed in a similar series. It will be sufficiently exact for the present object to neglect wholly the powers of $\frac{1}{a}$, using the values [1534ζ], and putting $H=c.\cos.\varepsilon$, $H'=-c.\sin.\varepsilon$; c, ε, being constant quantities. Substituting these in [1534γ], we shall get,

[1534θ]
$$Q^{(i)}=c.(\sin.\theta)^{-\frac{1}{2}}.\{\cos.a\theta.\cos.\varepsilon-\sin.a\theta.\sin.\varepsilon\}=c.(\sin.\theta)^{-\frac{1}{2}}.\cos.(a\theta+\varepsilon);$$

and we must now determine the constant quantities c, ε.

For this purpose we shall observe, that from a [1534α], we get $a=\sqrt{(i^2+i)}=i+\frac{1}{2}$, neglecting terms of the order $\frac{1}{a}$; hence

[1534ι]
$$Q^{(i)}=c.(\sin.\theta)^{-\frac{1}{2}}.\cos.\{(i+\tfrac{1}{2}).\theta+\varepsilon\}.$$

If we put $\theta=\frac{1}{2}\pi$, we shall have $\cos.\theta=0$, and when i is an odd number, every term of [1534z] will be multiplied by $\cos.\theta$, and will therefore vanish; so that for this case we ought to have $Q^{(i)}=0$. Therefore we must take the constant quantity ε, so that $\cos.\{(i+\frac{1}{2}).\frac{1}{2}\pi+\varepsilon\}=0$, and this is evidently obtained by putting $\varepsilon=-\frac{1}{4}\pi$, by which means the cosine becomes $\cos.\frac{1}{2}i\pi=0$. Substituting this value of ε in [1534ι], we get,

[1534κ]
$$Q^{(i)}=c.(\sin.\theta)^{-\frac{1}{2}}.\cos.\{(i+\tfrac{1}{2}).\theta-\tfrac{1}{4}\pi\}.$$

If we now suppose $\theta=\frac{1}{2}\pi$, and i equal to the even number $2s$, the expression [1534z], will be reduced to its last term, corresponding to $\cos.^{i-i}\theta=1$, and we shall have,

[1534λ]
$$Q^{(i)}=\pm\frac{1.3.5.....(2i-1)}{1.2.3.....i}\cdot\frac{i.(i-1).(i-2).....1}{(2.4.6.....i).(2i-1).(2i-3).....(i+1)}=\pm\frac{1.3.5.....(2i-1)}{(2.4.6.....i).(2i-1).(2i-3).....(i+1)}$$
$$=\pm\frac{1.3.5.....(i-1)}{(2.4.6.....i)}=\pm\frac{1.3.5.....(2s-1)}{2.4.6.....2s}=\pm\frac{1.3.5.....(2s-1)}{2^s.(1.2.3.....s)}.$$

$$U^{(i)} = \int \rho \cdot R^{i+2} \, dR \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)}. \qquad [1531]$$

Multiplying the numerator by $2 \cdot 4 \cdot 6 \ldots\ldots 2s$, and the denominator by its equivalent value $2^s \cdot (1 \cdot 2 \cdot 3 \ldots\ldots s)$, it becomes,

$$Q^{(i)} = \pm \frac{1 \cdot 2 \cdot 3 \ldots\ldots 2s}{2^{2s} \cdot (1 \cdot 2 \cdot 3 \ldots\ldots s)^2}\,; \qquad [1534\mu]$$

the upper sign being used when s is even, the lower when s is odd, as is evident by the inspection of [1534z]. Substituting in this, the value [1534x],

$$1 \cdot 2 \cdot 3 \ldots\ldots s = s^{s+\frac{1}{2}} \cdot c^{-s} \cdot 2^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}}\,; \qquad [1534\nu]$$

and the similar expression deduced from this, by changing s into $2s$, namely,

$$1 \cdot 2 \cdot 3 \ldots\ldots 2s = (2s)^{2s+\frac{1}{2}} \cdot c^{-2s} \cdot 2^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}} = 2^{2s+1} \cdot s^{2s+\frac{1}{2}} \cdot c^{-2s} \cdot \pi^{\frac{1}{2}},$$

we get,

$$Q^{(i)} = \pm \frac{2^{2s+1} \cdot s^{2s+\frac{1}{2}} \cdot c^{-2s} \cdot \pi^{\frac{1}{2}}}{2^{2s+1} \cdot s^{2s+1} \cdot c^{-2s} \cdot \pi} = \pm \frac{1}{s^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}}} = \pm \left(\frac{2}{i\,\pi}\right)^{\frac{1}{2}}. \qquad [1534\xi]$$

Now the value of $Q^{(i)}$ [1534κ] becomes, when $\theta = \frac{1}{2}\pi$, $Q^{(i)} = c \cdot \cos. i\theta = \pm c$. Comparing this with [1534ξ] we get $c = \left(\frac{2}{i\,\pi}\right)^{\frac{1}{2}}$; hence the expression [1534κ] becomes, for any value of θ,

$$Q^{(i)} = \left(\frac{2}{i\,\pi \cdot \sin. \theta}\right)^{\frac{1}{2}} \cdot \cos. \{(i + \tfrac{1}{2}) \cdot \theta - \tfrac{1}{4}\pi\}. \qquad [1534\xi']$$

We have, in this calculation of $Q^{(i)}$, used the symbol θ, to conform to the notation in [1532a, &c.]; but it is evident that we may, in [1534y], change θ into γ, using for γ its value [1532c]; and then the subsequent calculations will become the same functions of γ, that they are now of θ; hence [1534ξ'] will become generally,

$$Q^{(i)} = \left(\frac{2}{i\,\pi \cdot \sin. \gamma}\right)^{\frac{1}{2}} \cdot \cos. \{(i + \tfrac{1}{2}) \cdot \gamma - \tfrac{1}{4}\pi\} \qquad [1534\pi]$$

This expression of $Q^{(i)}$ decreases as i increases, on account of the divisor $\sqrt{(i\,\pi)}$. Therefore the series [1534y] $1 + \alpha \cdot Q^{(1)} + \alpha^2 \cdot Q^{(2)} + \&c.$, when α does not exceed unity, is converging, at intervals.

The terms $Q^{(i)}$ form a converging series.

In the development of $f(\theta, \varpi) = y$ [1533i], the general term of the series $Y^{(i)}$ [1533l] is multiplied by $2i + 1$. It is therefore important to ascertain, whether this factor does not render the series diverging, when i becomes very great; since this would make the computation inconvenient in practice, and sometimes unsafe in its application. This point [1534ρ]
has been examined by Mr. Poisson, who has shown that this series is generally converging. We shall here give the substance of his demonstration, and for greater simplicity we shall restrict ourselves to the examination of the terms of the series in its usual form, without attending to the minute circumstances of excepted cases in discontinuous functions; we shall therefore suppose $y' = f(\theta', \varpi')$ to be a continuous function, developed in the form [1533k], having the same value when $\varpi' = 0$, as when $\varpi' = 2\pi$; and its first and second [1534σ]
differentials, relative to θ', ϖ', always finite, within the proposed limits.

[1531′] We shall suppose R to be a function of μ', ϖ', and of a parameter a, which

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The function $Q^{(i)}$ [1441″] being symmetrical in θ, θ', and in ϖ, ϖ', we may, in [1442b], change θ, ϖ, into θ', ϖ', and the contrary, and we shall get,

[1534τ] $$Q^{(i)} = -\frac{1}{i.(i+1)} \cdot \left\{ \frac{1}{\sin.\theta'} \cdot \left(\frac{d.\left\{ \sin.\theta' . \left(\frac{d\,Q^{(i)}}{d\theta'}\right) \right\}}{d\theta'} \right) + \frac{1}{\sin.^2\theta'} \cdot \left(\frac{d\,d\,Q^{(i)}}{d\varpi'^2} \right) \right\}.$$

[1534υ] Substituting this in [1533h], we get, by using for brevity $f(\theta',\psi') = y'$,

[1534φ] $$Y^{(i)} = -\frac{(2i+1)}{4\pi.i.(i+1)} \cdot \left\{ \begin{array}{l} \int_0^{\pi} \int_0^{2\pi} \left\{ \frac{d.\left\{ \sin.\theta' . \left(\frac{d\,Q^{(i)}}{d\theta'}\right) \right\}}{d\theta'} \right\} . y' . d\theta' . d\varpi' \\ + \int_0^{\pi} \int_0^{2\pi} \left(\frac{d\,d\,Q^{(i)}}{d\varpi'^2} \right) . \frac{y'}{\sin.\theta'} . d\theta' . d\varpi' \end{array} \right\}.$$

Integrating by parts, we obtain the following, which is easily proved by differentiation and reduction,

$$\int \left\{ \frac{d.\left\{ \sin.\theta' . \left(\frac{d\,Q^{(i)}}{d\theta'}\right) \right\}}{d\theta'} \right\} . y' . d\theta'$$

[1534χ] $$= y' . \sin.\theta' . \left(\frac{d\,Q^{(i)}}{d\theta'}\right) - Q^{(i)} . \sin.\theta' . \left(\frac{dy'}{d\theta'}\right) + \int \left\{ \frac{d.\left\{ \sin.\theta' . \left(\frac{dy'}{d\theta'}\right) \right\}}{d\theta'} \right\} . Q^{(i)} . d\theta'.$$

At the two limits $\theta' = 0$, $\theta' = \pi$, the terms without the sign $\int$ vanish, on account of the factor $\sin.\theta' = 0$, and we then have,

[1534ψ] $$\int_0^{\pi} \left\{ \frac{d.\left\{ \sin.\theta' . \left(\frac{d\,Q^{(i)}}{d\theta'}\right) \right\}}{d\theta'} \right\} . y' . d\theta' = \int_0^{\pi} \left\{ \frac{d.\left\{ \sin.\theta' . \left(\frac{dy'}{d\theta'}\right) \right\}}{d\theta'} \right\} . Q^{(i)} . d\theta'.$$

In like manner, if we integrate by parts, we shall obtain the following expression, which may also be proved by differentiation,

[1534ω] $$\int \left(\frac{d\,d\,Q^{(i)}}{d\varpi'^2}\right) . y' . d\varpi' = \left(\frac{d\,Q^{(i)}}{d\varpi'}\right) . y' - Q^{(i)} . \left(\frac{dy'}{d\varpi'}\right) + \int \left(\frac{d\,d\,y'}{d\varpi'^2}\right) . Q^{(i)} . d\varpi'.$$

At the two limits $\varpi' = 0$, $\varpi' = 2\pi$, the quantity p [1532c], $Q^{(i)}$, and $\left(\frac{d\,Q^{(i)}}{d\varpi'}\right)$ [1532e], have the same values, also y' and $\left(\frac{dy'}{d\theta'}\right)$ [1531σ]; therefore the terms of [1534w], without the sign of integration, destroy each other, when we take the integrals within these limits. Hence

[1535a] $$\int_0^{2\pi} \left(\frac{d\,d\,Q^{(i)}}{d\varpi'^2}\right) . y' . d\varpi' = \int_0^{2\pi} \left(\frac{d\,d\,y'}{d\varpi'^2}\right) . Q^{(i)} . d\varpi'.$$

Substituting [1534ψ, 1535a] in [1531φ], and putting for brevity,

[1535b] $$\left\{ \frac{d.\left\{ \sin.\theta' . \left(\frac{dy'}{d\theta'}\right) \right\}}{d\theta'} \right\} + \frac{1}{\sin.\theta'} . \left(\frac{d\,d\,y'}{d\varpi'^2}\right) = F(\theta',\varpi') . \sin.\theta',$$

is constant for all parts of a stratum of the same density, but variable from [1531']

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we shall get,

$$Y^{(i)} = \frac{-(2i+1)}{4\pi i.(i+1)} \cdot \int_0^{\pi} \int_0^{2\pi} Q^{(i)} . F(\theta', \varpi') . \sin.\theta' . d\theta' . d\varpi'. \qquad [1535c]$$

The quantity $F(\theta', \varpi') . \sin.\theta'$ [1535b] is finite between the limits of the integral $\theta' = 0$, $\theta' = \pi$. For the differential coefficients $\left(\frac{dy'}{d\theta'}\right)$, $\left(\frac{ddy'}{d\theta'^2}\right)$, $\left(\frac{ddy'}{d\varpi'^2}\right)$, do not become [1535d] infinite in the case we have under consideration [1534σ]; therefore no term of the first member of [1535b] could become infinite, except the term $\frac{1}{\sin.\theta'} \cdot \left(\frac{ddy'}{d\varpi'^2}\right)$, which requires an examination in the case of $\theta' = 0$, and $\theta' = \pi$, because in both these cases the factor $\frac{1}{\sin.\theta'} = \infty$. But it appears, from the inspection of the formulas [1528b—e], that $\left(\frac{ddY'^{(i)}}{d\varpi'^2}\right)$ has the factor $\sqrt{(1-\mu'^2)} = \sin.\theta'$; therefore the term $\frac{1}{\sin.\theta'} . \left(\frac{ddy'}{d\varpi'^2}\right)$ will not become infinite. Moreover, $Q^{(i)}$ never exceeds ± 1 [1533χ]; hence $Q^{(i)}.F(\theta', \varpi').\sin.\theta'$ is a finite quantity; and if we suppose its greatest value independent of the sign to be A, we [1535e] shall get, from [1535c], $Y^{(i)} < \frac{(2i+1).A}{4\pi i.(2i+1)} \cdot \int_0^{\pi} \int_0^{2\pi} d\theta' . d\varpi'$; but

$$\int_0^{\pi} \int_0^{2\pi} d\theta' . d\varpi' = \int_0^{\pi} d\theta' . \int_0^{2\pi} d\varpi' = \int_0^{\pi} d\theta' . 2\pi = 2\pi . \int_0^{\pi} d\theta = 2\pi . \pi;$$

hence $Y^{(i)} < \frac{(2i+1).\pi}{2i.(2i+1)} . A$, independent of its sign. Therefore we may always take [1535f] i so large as to make $Y^{(i)}$ as small as we please.

The axis CX fig. 9, page 70, being arbitrary, we may suppose it to coincide with the line Cp, drawn through the attracted point p, and then θ' will become γ; and if we suppose the [1535g] angle ϖ' corresponding to this new axis to be represented by $\varpi_{,}$, the limits of γ, $\varpi_{,}$, will be the same as those of θ, ϖ, respectively. Then $F(\theta', \varpi')$ will become a function of γ, $\varpi_{,}$, which we shall represent by $F_{,}(\gamma, \varpi_{,})$, and the expression [1535c] will become

$$Y^{(i)} = \frac{-(2i+1)}{4\pi.i.(i+1)} \cdot \int_0^{\pi} \int_0^{2\pi} Q^{(i)} . F_{,}(\gamma, \varpi_{,}) . \sin.\gamma . d\gamma . d\varpi_{,}; \qquad [1535h]$$

The series $\Sigma Y^{(i)}$ converges as i increases.

substituting $Q^{(i)}$ [1534π], we get

$$Y^{(i)} = \frac{-(2i+1)}{(2\pi)^{\frac{3}{2}}.i.(i+1).\sqrt{i}} \cdot \int_0^{\pi} \int_0^{2\pi} \cos.\{(i+\tfrac{1}{2}).\gamma - \tfrac{1}{4}\pi\}.F_{,}(\gamma, \varpi_{,}) . \sin.^{\frac{1}{2}}\gamma . d\gamma . d\varpi_{,}. \qquad [1535i]$$

Hence it is evident that as i increases, the factor $\frac{2i+1}{i.(i+1).\sqrt{i}}$ will decrease, and when i is large, it will be very nearly equal to $\frac{2}{i.\sqrt{i}}$; so that when i is very great, the series of [1535k] terms $Y^{(i)}$, $Y^{(i+1)}$, &c., must be extremely small, and of a converging nature in successive intervals.

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one stratum to another.* The differential dR being taken, supposing μ' and

[1531''] ϖ' to be constant, we shall have, $dR = \left(\frac{dR}{da}\right) . da$; therefore

[1532]
$$U^{(i)} = \frac{1}{i+3} . \int \rho . \left(\frac{d.R^{i+3}}{da}\right) . da . d\mu' . d\varpi' . Q^{(i)}.$$

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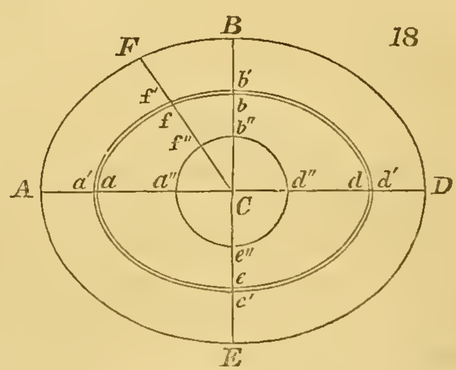
* (1051) The value of $U^{(i)}$ [1531] being substituted in [1459], gives V in a series which is generally *converging*, when the attracted point is situated so far without the spheroid, that *r is greater than the greatest value of R;* so that this expression of V may be used,
[1535l] even when the spheroid differs very much from a sphere, since it will not be liable to any difficulty from its want of convergency. We shall hereafter prove, in [1561b, &c.], that this value of V, developed so as to include the terms, as far as the order α^3, may be used, even when the attracted point is situated on the outer surface of the spheroid, where it becomes necessary to examine into the effect of the terms depending on the two signs of integration f', $f_{,}$, treated of in [1447d, &c.], and in the formulas [1447n, o]. Having made these preliminary remarks, on the nature of the series expressing the value of V, we shall now proceed to explain the manner in which the densities of the different strata are taken into consideration.



In the adjoined figure, $AFBDE$ represents the outer surface of the spheroid; $a\,b\,d\,e$, $a'b'd'e'$, the surfaces of an infinitely thin stratum having the same uniform density ρ; CB is a line drawn through the origin C to any given point B of the surface taken as its pole, and cutting the other surfaces in the points b, b'; so that
[1536a] we shall have $Cb = a$, $Cb' = a + da$, $CB = a_{,}$, for the parameters corresponding to the surfaces $a\,b\,c\,d$, $a'b'c'd'$, $A\,B\,C\,D$, respectively. Moreover, if we draw any radius CF, cutting these surfaces in f, f', F, we shall have $Cf = R$, $Cf' = R + dR$, $CF = R_{,}$; and as the angles θ', ϖ', depend entirely on the situation of the line
[1536b] CF, they will be the same for the points f, f', F. It is also evident that the radius $Cf = R$, may be considered as a function of μ', ϖ', a, which we shall denote by
[1536c] $R = \varphi(\mu', \varpi', a)$, and by changing a into $a + da$, we shall have the radius

$$Cf' = R + dR = \varphi(\mu', \varpi', a + da),$$

also, changing a into $a_{,}$, corresponding to the outer surface, we shall have the radius

[1536d]
$$CF = R_{,} = \varphi(\mu', \varpi', a_{,}).$$

Developing the preceding value of $R + dR$, according to the powers of da [617],

We shall also suppose R^{i+3} to be developed in a series of this form,*

$$R^{i+3}=Z'^{(0)}+Z'^{(1)}+Z'^{(2)}+Z'^{(3)}+\&c.\,; \qquad [1533]$$

$Z'^{(i)}$ being, for any integral value of i, a rational and integral function of μ', [1533']
$\sqrt{1-\mu'^2}\,.\,\sin.\varpi'$, and $\sqrt{1-\mu'^2}\,.\,\cos.\varpi'$, which satisfies the following equation of partial differentials,

$$0=\left\{\frac{d\,.\left\{(1-\mu'^2)\,.\left(\frac{dZ'^{(i)}}{d\mu'}\right)\right\}}{d\mu'}\right\}+\frac{\left(\frac{ddZ'^{(i)}}{d\varpi'^2}\right)}{1-\mu'^2}+i\,.\,(i+1)\,.\,Z'^{(i)}. \qquad [1534]$$

The differential of $Z'^{(i)}$, taken relative to a, will also satisfy this equation,

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neglecting the square and higher powers of da, we get

$$R+dR=\varphi(\mu',\varpi',a)+\left(\frac{d\,.\,\varphi(\mu',\varpi',a)}{da}\right).\,da\,;$$

hence $dR=\left(\frac{d\,.\,\varphi(\mu',\varpi',a)}{da}\right).\,da=\left(\frac{dR}{da}\right).\,da$, as in [1531'']. Substituting this in [1531], the part $R^{i+2}\,dR$ becomes $R^{i+2}\,.\left(\frac{dR}{da}\right).\,da=\frac{1}{i+3}\,.\left(\frac{d\,.\,R^{i+3}}{da}\right).\,da$, as is evident from development; hence the expression [1531] becomes as in [1532]; the constant quantity $i+3$ being brought from under the sign of integration.

* (1052) If a spherical surface $a''b''d''e''$ be described about the centre C, with a radius equal to unity; any point f'' of this surface will be determined by three rectangular co-ordinates x, y, z, drawn parallel to the axes x, y, z; and these will be given in functions of the angles μ', ϖ', by putting $R=1$ in [1529], and accenting the letters μ, ϖ, as in [1531']; by which means we shall have

$$x=\mu'\,; \qquad y=\sqrt{(1-\mu'^2)}\,.\,\cos.\varpi'\,; \qquad z=\sqrt{(1-\mu'^2)}\,.\,\sin.\varpi'. \qquad [1536e]$$

Therefore any point f'' of this spherical surface, corresponding to the angles μ', ϖ', or in other words, the *direction* of the line CF, will be determined by means of the three quantities μ', $\sqrt{(1-\mu'^2)}\,.\,\cos.\varpi'$, $\sqrt{(1-\mu'^2)}\,.\,\sin.\varpi'$; and it is evident, that when the *direction* is given, the length of the radius Cf will depend on the parameter a, corresponding to the surface $abde$; so that in general any radius $Cf=R$, or its power $i+3$, will [1536f]
be a function of the quantities μ', $\sqrt{(1-\mu'^2)}\,.\,\cos.\varpi'$, $\sqrt{(1-\mu'^2)}\,.\,\sin.\varpi'$, and the parameter a. Now the most general expression of an integral function of the quantities μ', $\sqrt{(1-\mu'^2)}\,.\,\cos.\varpi'$, $\sqrt{(1-\mu'^2)}\,.\,\sin.\varpi'$, may be reduced to the form [1533k], $Y'^{(0)}+Y'^{(1)}+Y'^{(2)}+\&c.$, which is similar to that in [1533]; each of the quantities $Z'^{(i)}$ satisfying the equation [1534], which is of the same form as [1525], changing $Y^{(i)}$ into $Z'^{(i)}$.

[1534′] consequently it is of the same form.* We must therefore, in conformity to the general theorem [1470], notice only the term $Z'^{(i)}$, in the development of R^{i+3},† and we shall then have,

[1535] $$U^{(i)}=\frac{1}{i+3}\cdot\int\rho\cdot\left(\frac{dZ'^{(i)}}{da}\right).\,da\,.\,d\mu'\,.\,d\varpi'\,.\,Q^{(i)}.$$

When the spheroid is homogeneous, and differs but little from a sphere, we
[1535′] may suppose $\rho=1$, and $R=a\,.\,(1+\alpha y')$; we shall then find, by taking the integral relative to a,‡

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* (1053) In the equation [1534], $Z'^{(i)}$ is a function of the independent variable quantities μ', ϖ', a; and if we take its differential relative to a, we shall get,

[1536g] $$0=\left\{\frac{d\,.\left\{(1-\mu'^2)\,.\left(\frac{ddZ'^{(i)}}{d\mu'.da}\right)\right\}}{d\mu'}\right\}+\left(\frac{ddZ'^{(i)}}{d\varpi'^2.da}\right)+i\,.\,(i+1)\,.\left(\frac{dZ'^{(i)}}{da}\right).$$

[1536h] If we put $\left(\frac{dZ'^{(i)}}{da}\right)=Z''^{(i)}$, and take its differentials relative to μ' and ϖ', we shall find $\left(\frac{ddZ'^{(i)}}{d\mu'.da}\right)=\left(\frac{dZ''^{(i)}}{d\mu'}\right)$, and $\left(\frac{ddZ'^{(i)}}{d\varpi'^2.da}\right)=\left(\frac{ddZ''^{(i)}}{d\varpi'^2}\right)$. Substituting these in [1536g], it will become of the same form as [1534], changing $Z'^{(i)}$ into $Z''^{(i)}$; therefore the quantity $Z''^{(i)}$, or $\left(\frac{dZ'^{(i)}}{da}\right)$, being substituted for $Z'^{(i)}$, in [1534], will satisfy that equation, as is observed in [1534′].

† (1054) Taking the partial differential of the equation [1533], relative to a, and using the abridged symbol $Z''^{(i)}$ [1536h], we shall get,

$$\left(\frac{d\,.\,R^{i+3}}{da}\right)=\left(\frac{dZ'^{(0)}}{da}\right)+\left(\frac{dZ'^{(1)}}{da}\right)+\left(\frac{dZ'^{(2)}}{da}\right)+\&c.=Z''^{(0)}+Z''^{(1)}+Z''^{(2)}+\&c.$$

Substituting this in [1532], it becomes

[1536i] $$U^{(i)}=\frac{1}{i+3}\cdot\int\rho\,.\,\{Z''^{(0)}+Z''^{(1)}+Z''^{(2)}+\&c.\}\,.\,da\,.\,d\mu'\,.\,d\varpi'\,.\,Q^{(i)}.$$

Now when i' differs from i, we have $\int.\,Z''^{(i')}\,.\,Q^{(i)}\,.\,d\mu'\,.\,d\varpi'=0$, [1470]; and as ρ is a function of a only [1503″″], the equation [1536i] becomes

$$U^{(i)}=\frac{1}{i+3}\cdot\int\rho\,.\,Z''^{(i)}\,.\,da\,.\,d\mu'\,.\,d\varpi'\,.\,Q^{(i)};$$

resubstituting the value of $Z''^{(i)}$ [1536h], we get [1535].

‡ (1055) $Q^{(i)}$ [1441″] is a function of μ, μ', $\varpi'-\varpi$, independent of a, so that if we put, as in [1535′], $\rho=1$, the quantity $Z'^{(i)}$ [1535] will be the only quantity depending on a;

$$U^{(i)} = \frac{1}{i+3} \cdot \int Z'^{(i)} \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)}. \qquad [1536]$$

If we suppose y' to be developed in a series of the form,

$$y' = Y'^{(0)} + Y'^{(1)} + Y'^{(2)} + Y'^{(3)} + \&c., \qquad [1537]$$

$Y'^{(i)}$ satisfying the same equation of partial differentials as $Z'^{(i)}$ [1534], we shall find, by neglecting quantities of the order α^2,* $Z'^{(i)} = (i+3) \cdot \alpha \cdot a^{i+3} \cdot Y'^{(i)}$; therefore we shall have, [1537']

$$U^{(i)} = \alpha \cdot a^{i+3} \cdot \int Y'^{(i)} \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)}. \qquad [1538]$$

If we denote by $Y^{(i)}$ what $Y'^{(i)}$ becomes when we change μ', ϖ', into μ, ϖ; we shall have, by [1466],† [1538']

$$U^{(i)} = \frac{4\alpha\pi \cdot a^{i+3}}{2i+1} \cdot Y^{(i)}; \qquad [1539]$$

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and since, by taking the integral relative to a, we have $\int \left(\frac{dZ'^{(i)}}{da}\right) \cdot da = Z'^{(i)}$; the expression [1535] will evidently become of the form [1536].

* (1056) From $R = a \cdot (1 + \alpha y')$ [1535'] we get, by neglecting α^2,

$$R^{i+3} = a^{i+3} \cdot \{1 + (i+3) \cdot \alpha y'\}.$$

Substituting this in [1533], and using the value of y' [1537], we get,

$$Z'^{(0)} + Z'^{(1)} + Z'^{(2)} + Z'^{(3)} + \&c. = a^{i+3} + (i+3) \cdot \alpha \cdot a^{i+3} \cdot \{Y'^{(0)} + Y'^{(1)} + Y'^{(2)} + Y'^{(3)} + \&c.\}. \qquad [1538a]$$

Comparing the similar terms of both members of this equation, we get generally, as in [1537'], $Z'^{(i)} = (i+3) \cdot \alpha \cdot a^{i+3} \cdot Y'^{(i)}$; and by using this value of $Z'^{(i)}$, the formula [1536] becomes as in [1538].

† (1057) In the calculations [1535'—1540'], the spheroid is supposed to be homogeneous, as in [1459—1466]; the radius $a \cdot (1 + \alpha y)$ [1461'], corresponds to the part of the *surface* of the spheroid, intersected by the line r, drawn to the attracted point, which is defined by means of the quantities μ, ϖ, a. In [1535'], the radius $R = a \cdot (1 + \alpha y')$ is supposed to correspond to any point whatever of the *surface* of the spheroid, defined by means of the quantities μ', ϖ', a. Hence it is evident. that if we change μ', ϖ', into μ, ϖ, respectively, y' will change into y, also $Y'^{(i)}$ into $Y^{(i)}$, [1537, 1464], and the values of $U^{(i)}$ [1466, 1538], must then become equal to each other; hence

$$\alpha \cdot a^{i+3} \cdot \int Y'^{(i)} \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)} = \frac{4\alpha\pi}{2i+1} \cdot a^{i+3} \cdot Y^{(i)}. \qquad [1539a]$$

therefore we shall obtain this remarkable result,

Important theorem.

[1540] $$\int_{-1}^{1}\int_{0}^{2\pi} Y'^{(i)} \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(i)} = \frac{4\pi \,.\, Y^{(i)}}{(2i+1)}; \qquad (1)$$

As this equation takes place, whatever be $Y'^{(i)}$, *we ought generally to conclude that the double integration of the function* $\int Z'^{(i)} \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(i)}$,

[1540′] *taken from* $\mu' = -1$ *to* $\mu' = 1$, *and from* $\varpi' = 0$ *to* $\varpi' = 2\pi$, *has the effect of changing* [any function whatever of the form] $Z'^{(i)}$ *into*

[1540″] $\frac{4\pi \,.\, Z^{(i)}}{2i+1}$;* that is

Very important theorem in definite integrals.

[1540‴] $$\int_{-1}^{1}\int_{0}^{2\pi} Z'^{(i)} \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(i)} = \frac{4\pi \,.\, Z^{(i)}}{2i+1};$$

Dividing by $\alpha \,.\, a^{i+3}$, we shall get the formula [1540]; observing that $Y^{(i)}$, in the second member, is deduced from $Y'^{(i)}$, by changing μ', ϖ', into μ, ϖ, respectively. This is a general formula, corresponding to *any function whatever* $Y'^{(i)}$, or $Y^{(i)}$, which satisfies the equation [1465]; since the term y' of the radius may be varied at pleasure; therefore $Y'^{(i)}$ [1537]

[1539b] *may be taken in its most general form, or in a more limited manner without any restriction.*

In finding the equation [1539a], terms of the order α^2 were neglected; we ought therefore, in that equation, to suppose the quantity α to be infinitely small. If we suppose these neglected terms to be represented by the quantity $\alpha^2 \,.\, a^{i+3} \,.\, Y''^{(i)}$, additive to the second

[1539c] member of [1539a], and divide this as above, by $\alpha \,.\, a^{i+3}$, it will produce, in the second member of [1540], the term $\alpha \,.\, Y''^{(i)}$; which, by putting the arbitrary quantity $\alpha = 0$, gives accurately the expression [1540]. This agrees with the demonstration given in [1533m].

* (1058) From the remarks made in [1539b], it is evident that we may change $Y'^{(i)}$ into $Z'^{(i)}$, and $Y^{(i)}$ into $Z^{(i)}$, in [1540], and it will become as in [1540‴]; by which means the integrals of the first member, relative to μ', ϖ', may be found between the limits therein expressed, by simply changing $Z'^{(i)}$ into $\frac{4\pi \,.\, Z^{(i)}}{2i+1}$.

For the purpose of illustrating the preceding formula, we shall calculate, in the usual manner, the integral in a simple case, in which $i = 1$; and shall compare the result of this direct operation, with that deduced from the formula [1540‴]. For this purpose, we

[1539d] shall suppose $Z'^{(1)} = A^{(1)} \,.\, \sqrt{(1-\mu'^2)} \,.\, \sin.\varpi'$, which is one of the terms of the general formula [1528b], satisfying the equation [1534]. Substituting this and

[1539e] $$Q^{(1)} = \mu\mu' + \sqrt{(1-\mu^2)} \,.\, \sqrt{(1-\mu'^2)} \,.\, (\cos.\varpi' - \varpi), \qquad [1441', 1441''],$$

in the first member of [1540‴], it becomes

$$\int Z^{(1)} \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(1)}$$

$$= \int A^{(1)} \,.\, d\mu' \,.\, d\varpi' \,.\, \{\mu\mu' \,.\, \sqrt{(1-\mu'^2)} \,.\, \sin.\varpi' + (1-\mu'^2) \,.\, \sqrt{(1-\mu^2)} \,.\, \sin.\varpi' \,.\, \cos.(\varpi'-\varpi)\}$$

[1539f] $$= \int A^{(1)} \,.\, d\mu' \,.\, d\varpi' \,.\, \{\mu\mu' \,.\, \sqrt{(1-\mu'^2)} \,.\, \sin.\varpi' + \tfrac{1}{2} \,.\, (1-\mu'^2) \,.\, \sqrt{(1-\mu^2)} \,.\, [\sin.(2\varpi'-\varpi) + \sin.\varpi]\}.$$

$Z^{(i)}$ being what $Z'^{(i)}$ becomes when μ', ϖ', are changed into μ, ϖ; therefore we shall have,* [1540'''']

$$U^{(i)} = \frac{4\pi}{(i+3).(2i+1)} . \int \rho . \left(\frac{d Z^{(i)}}{d a}\right) . d a ; \qquad [1541]$$

and the triple integration, on which $U^{(i)}$ [1531] depends, is reduced to a single integration, relative to a; from $a=0$, to its value at the surface of the spheroid. [1541']

The equation [1540] furnishes a very simple method of finding the integral of the function $\int Y^{(i)} . Z^{(i)} . d\mu . d\varpi$, from $\mu = -1$ to $\mu = 1$, and from $\varpi = 0$ to $\varpi = 2\pi$. For the part of $Y^{(i)}$, depending on the angle $n\varpi$, is, as has been shown in [1528], of the form [1541'']

$$Y^{(i)} = \lambda . \{ A^{(n)} . \sin. n\varpi + B^{(n)} . \cos. n\varpi \} ; \qquad [1541''']$$

λ being equal to the following expression,

$$\lambda = (1-\mu^2)^{\frac{n}{2}} . \left\{ \mu^{i-n} - \frac{(i-n).(i-n-1)}{2.(2i-1)} . \mu^{i-n-2} + \&c. \right\}, \qquad [1542]$$

therefore we shall have,†

Taking the integrals relative to ϖ', so as to vanish when $\varpi'=0$, and then putting $\varpi' = 2\pi$, we obtain $\int d\varpi' . \sin. (2\varpi' - \varpi) = -\tfrac{1}{2} \cos. (2\varpi' - \varpi) + \tfrac{1}{2} \cos \varpi = 0$; $\int d\varpi' = \varpi' = 2\pi$. Taking the integrals relative to μ', so as to vanish when $\mu' = -1$, and then putting $\mu'=1$, we get, $\int \mu' d\mu' . (1-\mu'^2)^{\frac{1}{2}} = -\tfrac{1}{3} . (1-\mu'^2)^{\frac{3}{2}} = 0$; and [1539$f$] becomes,

$$\begin{aligned} \int Z^{(1)} . d\mu' . d\varpi' . Q^{(1)} &= A^{(1)} . \pi . \sqrt{(1-\mu^2)} . \sin. \varpi . \int (1-\mu'^2) . d\mu' \\ &= A^{(1)} . \pi . \sqrt{(1-\mu^2)} . \sin. \varpi . (\mu' - \tfrac{1}{3} . \mu'^3 + \tfrac{2}{3}) \\ &= \frac{4\pi}{3} . A^{(1)} . \sqrt{(1-\mu^2)} . \sin. \varpi = \frac{4\pi}{3} . Z^{(1)} . \end{aligned}$$

$Z^{(1)}$ being deduced from $Z'^{(1)}$ [1539d], by changing μ', ϖ', into μ, ϖ, respectively. The result thus obtained agrees with the general formula [1540'''], supposing $i=1$.

* (1059) We may find the integral of [1535] relative to μ', ϖ', by means of the formula [1540'''], changing $\int \left(\frac{dZ'^{(i)}}{da}\right) . d\mu' . d\varpi' . Q^{(i)}$ into $\frac{4\pi}{2i+1} . \left(\frac{dZ^{(i)}}{da}\right)$, observing that ρ is a function of a independent of μ', ϖ'. Substituting this in [1535], it becomes as in [1541].

† (1060) This value of $Y'^{(i)}$ corresponds with that of $Y^{(i)}$ [1541'''], changing μ', ϖ', into μ, ϖ, as in [1538'].

[1543] $$Y'^{(i)} = \lambda' . \{ A^{(n)} . \sin. n \varpi' + B^{(n)} . \cos. n \varpi' \} ;$$

[1543′] λ' being what λ becomes by changing μ into μ'. The part of $Q^{(i)}$ depending on the angle $n \varpi$, is by § 15,* $\gamma . \lambda \lambda' . \cos. n . (\varpi - \varpi')$, or

[1543″] $$\gamma . \lambda \lambda' . \{ \cos. n \varpi . \cos. n \varpi' + \sin. n \varpi . \sin. n \varpi' \} ;$$

[1543‴] hence the part of the integral $\int Y'^{(i)} . d \mu' . d \varpi' . Q^{(i)}$, depending on the angle $n \varpi$, will be,†

[1544] $$\gamma \lambda . \sin. n \varpi . \int \lambda'^2 . d \mu' . d \varpi' . \sin. n \varpi' . \{ A^{(n)} . \sin. n \varpi' + B^{(n)} . \cos. n \varpi' \}$$
$$+ \gamma \lambda . \cos. n \varpi . \int \lambda'^2 . d \mu' . d \varpi' . \cos. n \varpi' . \{ A^{(n)} . \sin. n \varpi' + B^{(n)} . \cos. n \varpi' \}.$$

Performing the integrations relative to ϖ' this part becomes,‡

[1545] $$\gamma \lambda . \pi . \{ A^{(n)} . \sin. n \varpi + B^{(n)} . \cos. n \varpi \} . \int \lambda'^2 . d \mu' ;$$

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* (1061) The value of λ [1542], and the similar value of λ', found by changing μ into
[1542a] μ' in [1542], being substituted in [1514], we get $\beta = \gamma . \lambda \lambda'$; hence the part of $Q^{(i)}$ [1507″]
[1543a] depending on the angle $n . (\varpi - \varpi')$, is $\beta . \cos. n . (\varpi - \varpi') = \gamma . \lambda \lambda' . \cos. n . (\varpi - \varpi')$, as in [1543″].

† (1062) If the factor $Y'^{(i)} . Q^{(i)}$ of the integral expression $\int d \mu' . d \varpi' . Y'^{(i)} . Q^{(i)}$ [1543‴], contain any term of the form $b . \sin. m \varpi'$, or $b . \cos. m \varpi'$; m being an
[1543b] integer, and b independent of ϖ'; this term would vanish from the complete integral, taken relative to ϖ', from $\varpi' = 0$ to $\varpi' = 2 \pi$. For it is evident that

$$\int b . \sin. m \varpi' . d \varpi' = - \frac{b}{m} . \cos. m \varpi' + \frac{b}{m}, \qquad \text{and} \qquad \int b . \cos. m \varpi' . d \varpi' = \frac{b}{m} . \sin. m \varpi',$$

[1543c] vanish when $\varpi' = 0$, and $\varpi' = 2 \pi$. Now it follows from [17—20] Int., that if $Y'^{(i)}$ contain terms depending on $\sin. n \varpi'$, or $\cos. n \varpi'$; and $Q^{(i)}$ terms depending on $\sin. n' \varpi'$, or $\cos. n' \varpi'$; the product $Y'^{(i)} . Q^{(i)}$ will contain terms depending on the sine or cosine of the angle $(n \pm n') . \varpi' = m \varpi'$; putting for brevity $n \pm n' = m$; so that *if n' differ from*
[1543d] *n, the quantity m will be real, and the corresponding parts of the integral* [1543‴] *will vanish.* Therefore, in finding the product $Y'^{(i)} . Q^{(i)}$, it will only be necessary to notice
[1543e] those terms in which $n' = n$, making $m = 0$. These terms of $Y'^{(i)}$ are contained in [1543], and those of $Q^{(i)}$ in [1543″], and the corresponding terms produced by $Y'^{(i)} . Q^{(i)}$ are as in [1544]. The terms γ, λ, $\sin. n \varpi$, $\cos. n \varpi$, being independent of μ', ϖ', are brought from under the signs of integration.

‡ (1063) The expression [1544] contains these products $\sin.^2 n \varpi' = \frac{1}{2} - \frac{1}{2} \cos. 2 n \varpi'$, $\cos.^2 n \varpi' = \frac{1}{2} + \frac{1}{2} \cos. 2 n \varpi'$, $\sin. n \varpi' . \cos. n \varpi' = \frac{1}{2} \sin. 2 n \varpi'$, [1, 6, 31] Int.; and

but by means of the theorem [1540], this same part is equal to*

$$\frac{4\pi}{2i+1}\cdot\lambda\cdot\{A^{(n)}\cdot\sin.n\varpi.+B^{(n)}\cdot\cos.n\varpi\};$$ [1546]

therefore we shall have,

Theorem.

$$\int_{-1}^{1}\lambda'^2\cdot d\mu'=\frac{4}{(2i+1)\cdot\gamma}.$$ [1547]

We shall now represent the part of $Z^{(i)}$, depending on the angle $n\varpi$, in the following manner,

$$Z^{(i)}=\lambda\cdot\{A'^{(n)}\cdot\sin.n\varpi+B'^{(n)}\cdot\cos.n\varpi\}.$$ [1547']

This part only ought to be combined with the corresponding part of $Y^{(i)}$ [1541'''], because the terms depending on the sines and cosines of the angle ϖ and its multiples vanish [1543d],† by integration, from the function

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from [1543d] we may neglect the angle $2n\varpi'$, because the integral depending on it vanishes; and we shall have, between the limits $\varpi'=0$, $\varpi'=2\pi$,

Integrals.

$$\int_0^{2\pi}d\varpi'\cdot\sin.^2n\varpi'=\int_0^{2\pi}d\varpi'\cdot\tfrac{1}{2}=\pi;\qquad\int_0^{2\pi}d\varpi'\cdot\cos.^2n\varpi'=\int_0^{2\pi}d\varpi'\cdot\tfrac{1}{2}=\pi;$$ [1544a]

$$\int_0^{2\pi}d\varpi'\cdot\sin.n\varpi'\cdot\cos.n\varpi'=0.$$

Substituting these in [1544], it becomes as in [1545], the constant quantities π, $A^{(n)}$, $B^{(n)}$, being brought from under the sign $\int$.

* (1064) Substituting in [1540] the values $Y'^{(i)}$ [1543], and $Y^{(i)}$ [1541'''], it becomes as in [1546]. These two expressions [1545, 1546] of the part of $\int Y'^{(i)}\cdot d\mu'\cdot d\varpi'\cdot Q^{(i)}$, depending on the angle $n\varpi$, being put equal to each other, and divided by

$$\gamma\cdot\lambda\pi\cdot\{A^{(n)}\cdot\sin.n\varpi+B^{(n)}\cdot\cos.n\varpi\},$$

give the equation [1547].

† (1065) From what has been said in [1543b—d], it is evident that the terms of $Y^{(i)}$ depending on the angle $n\varpi$, must be combined only with those of $Z^{(i)}$ depending on [1547a] the same angle, all the other terms vanishing, the limits of the integrals being $\varpi=0$ and $\varpi=2\pi$. Therefore in finding the terms depending on the angle $n\varpi$, in $\int Y^{(i)}.Z^{(i)}.d\mu.d\varpi$, we must notice simply the terms of $Y^{(i)}$ [1541'''], and those of $Z^{(i)}$ [1547']. Substituting these in the first member of [1548], we shall obtain the second member of that expression. Multiplying the two factors of [1548], and putting, as in [1544a],

Integrals.

$$\int_0^{2\pi}d\varpi.\sin.^2\varpi=\pi;\qquad\int_0^{2\pi}d\varpi.\cos.^2\varpi=\pi;\qquad\int_0^{2\pi}d\varpi.\sin.\varpi.\cos.\varpi=0;$$ [1548a]

[1547''] $\int Y^{(i)} \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi,$ the limits being taken from $\varpi = 0$ to $\varpi = 2\pi$; therefore we shall have, by noticing only the part of $Y^{(i)}$ depending on the angle $n\varpi$,

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we shall get the first expression [1548']. If we now change μ' into μ in [1547], by which means λ' will become λ [1543']; the formula [1547] will become, by taking this integral from $\mu = -1$ to $\mu = 1$.

[1548b] $$\int_{-1}^{1} \lambda^2 \,.\, d\mu = \frac{4}{(2i+1)\,.\,\gamma};$$

γ being a constant quantity [1520, 1521]. Substituting this in the first expression [1548'],

[1548c] we get the last form [1548']. Now the function $Y^{(i)}$, or $Z^{(i)}$, contains all integral values of n, from $n=0$ to $n=i$, [1528']; and if we use the sign Σ_0^i of finite integrals [1373d], to denote the sum of $i+1$ terms of a series of quantities of the form

[1548d] $A_0,\ A_1,\ A_2, \ldots . A_{(n)} \ldots . A_i,$ we shall have the following system of equations, which are easily deduced from [1548', 1541''', 1547', 1542], putting, for the sake of symmetry in the notation, λ_n for λ; and using γ [1520, 1521].

Theorems in definite integrals.

[1548e] $$\int_{-1}^{1} \int_{0}^{2\pi} Y^{(i)} \,.\, Z^{(i)} \,.\, d\mu \,.\, d\varpi = \frac{4\pi}{(2i+1)\,.\,\gamma} \,.\, \Sigma_0^{(i)} \,.\, \{A^{(n)} \,.\, A'^{(n)} + B^{(n)} \,.\, B'^{(n)}\}$$

[1548f] $$Y^{(i)} = \Sigma_0^{(i)} \,.\, \lambda_n \,.\, \{A^{(n)} \,.\, \sin.\, n\varpi + B^{(n)} \,.\, \cos.\, n\varpi\};$$

[1548g] $$Z^{(i)} = \Sigma_0^{(i)} \,.\, \lambda_n \,.\, \{A'^{(n)} \,.\, \sin.\, n\varpi + B'^{(n)} \,.\, \cos.\, n\varpi\};$$

[1548h] $$\lambda_n = (1-\mu^2)^{\frac{n}{2}} \,.\, \left\{ \mu^{i-n} - \frac{(i-n)\,.\,(i-n-1)}{2\,.\,(2i-1)}\, \mu^{i-n-2} + \&c. \right\}.$$

Several integrals, somewhat similar to [1548], have been given, by Mr. Poisson, in forms which are very convenient for reference and use; one of them is the following double integral,

Theorem in definite integrals.

[1548i] $$P^{(i)} = \int_0^{\pi} \int_0^{2\pi} \frac{Y'^{(i)} \,.\, \sin.\, \theta' \,.\, d\theta' \,.\, d\varpi'}{(1 - 2\alpha \,.\, p + \alpha^2)^{\frac{1}{2}}};$$

in which α is a constant quantity; p is the same as in [1532c]; $Y^{(i)}$ is any function whatever,

[1548k] of the usual form [1524]; $Y'^{(i)}$ is the value of $Y^{(i)}$, when θ is changed into θ', and ϖ into ϖ'. Then developing the radical $(1 - 2\alpha \,.\, p + \alpha^2)^{-\frac{1}{2}}$ according to the powers of α, as in [1532g], we may, by [1476a], neglect all the terms of this development, except that depending on $Q^{(i)}$; and from [1533m], we shall have,

[1548l] $$P^{(i)} = \frac{4\pi \,.\, \alpha^i}{2i+1} \,.\, Y^{(i)}.$$

If the development be made according to the powers of $\frac{1}{\alpha}$, as in [1532g'], we shall have, by a similar calculation,

[1548m] $$P^{(i)} = \frac{4\pi}{(2i+1)\,.\,\alpha^{i+1}} \,.\, Y^{(i)}.$$

$$\int Y^{(i)} . Z^{(i)} . d\mu . d\varpi$$ Important definite integral.

$$= \int \lambda^2 . d\mu . d\varpi . \{A^{(n)} . \sin. n\varpi + B^{(n)} . \cos. n\varpi\} . \{A'^{(n)} . \sin. n\varpi + B'^{(n)} . \cos. n\varpi\}$$ [1548]

$$= \pi . \{A^{(n)} . A'^{(n)} + B^{(n)} . B'^{(n)}\} . \int \lambda^2 . d\mu = \frac{4\pi}{(2i+1).\gamma} . \{A^{(n)} . A'^{(n)} + B^{(n)} B'^{(n)}\}.$$ [1548′]

The first is to be used when $\alpha < 1$, the second when $\alpha > 1$, in order that the development of the radical may produce a converging series. [1548n]

Instead of limiting the value of $P^{(i)}$ to the term $Y'^{(i)}$, depending on a particular value of i, we may take the whole expression of $f(\theta', \varpi')$ [1533k], so as to include all the values of i, and put

Theorem in definite integrals.

$$P = \int_0^{\pi} \int_0^{2\pi} \frac{f(\theta', \varpi') . \sin. \theta' . d\theta' . d\varpi'}{(1 - 2\alpha . p + \alpha^2)^{\frac{1}{2}}} .$$ [1548o]

In this case we must, as above, develop the radical according to the powers of α if $\alpha < 1$, and according to the powers of $\frac{1}{\alpha}$ if $\alpha > 1$; in order to obtain a converging series. [1548p] Then substituting the development of $f(\theta', \varpi')$ [1533k], we shall obtain the value of P, in a series of terms of the form $P^{(0)} + P^{(1)} + P^{(2)} + \&c.$ [1548l, m]. This value of P will be composed of a finite number of terms, if $f(\theta', \varpi')$ consist of a finite number; but if P have an infinite number of terms, it will be necessary that they should form a converging series, otherwise it may not give the correct value of P. Moreover, it is requisite that the [1548q] series [1533k, 1532g, 1532g′], used in computing this value of P, should also be converging; since *the results obtained by the combination of two or more diverging series, will sometimes be inaccurate, even though the result be in a converging form;* for which reason it is proper to avoid wholly the use of such diverging series. To impress more strongly the necessity of [1548r] a strict attention to this point, we shall here give an example of this kind, being in substance the same as that published by Mr. Poisson.

Results of diverging series not to be relied upon.

Supposing, as in [1532f], $\rho = (1 - 2\alpha . p + \alpha^2)^{-\frac{1}{2}}$; $\rho' = (1 - 2\beta . p + \beta^2)^{-\frac{1}{2}}$; α, β, [1548s] being constant quantities, and p as in [1532c]; we may develop the quantities ρ, ρ', as in [1532g, g′], according to the powers of α, β, α^{-1}, β^{-1}, and we shall obtain the four [1548t] following series,

$$\rho = 1 + \alpha . Q^{(1)} + \alpha^2 . Q^{(2)} + \alpha^3 . Q^{(3)} + \&c.;$$ [1548u]

$$\rho' = 1 + \beta . Q^{(1)} + \beta^2 . Q^{(2)} + \beta^3 . Q^{(3)} + \&c.;$$ [1548v]

$$\rho = \alpha^{-1} + \alpha^{-2} . Q^{(1)} + \alpha^{-3} . Q^{(2)} + \alpha^{-4} . Q^{(3)} + \&c.;$$ [1548w]

$$\rho' = \beta^{-1} + \beta^{-2} . Q^{(1)} + \beta^{-3} . Q^{(2)} + \beta^{-4} . Q^{(3)} + \&c.$$ [1548x]

Now if it were required to find the double integral

$$z = \frac{1}{4\pi} . \int_0^{\pi} \int_0^{2\pi} \rho\rho' . \sin. \theta' . d\theta' . d\varpi',$$ [1548y]

[1548′′] Supposing now successively $n=0$, $n=1$, $n=2, \ldots\ldots n=i$ [1528′], in the second member; the sum of all these terms, will be the value of the integral $\int Y^{(i)}.Z^{(i)}.d\mu.d\varpi$.

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we might substitute either of the preceding values of ρ, ρ', and the product $\rho\rho'$ would be composed of terms of the form $\alpha^m.\beta^n.Q^{(i)}.Q^{(i')}$. These would produce in z, terms of the form $\frac{1}{4\pi}.\alpha^m.\beta^n.\int_0^{\pi}\int_0^{2\pi}Q^{(i)}.Q^{(i')}.\sin.\theta'.d\theta'.d\varpi'$, which would vanish if i

[1548z] differ from i', by the theorem [1476a]; so that it would only be necessary to retain terms of the product $\rho\rho'$, in which $i=i'$, of the form $\alpha^m.\beta^n.Q^{(i)}.Q^{(i)}$. These would produce in z terms of the form $\frac{1}{4\pi}.\alpha^m.\beta^n.\int_0^{\pi}\int_0^{2\pi}Q^{(i)}.Q^{(i)}.\sin.\theta'.d\theta'.d\varpi'$, which, by

[1548α] means of the formula [1533m], would become $\frac{1}{4\pi}.\alpha^m.\beta^n.Q_{\prime}^{(i)}.\frac{4\pi}{2i+1}$, $Q_{\prime}^{(i)}$ being the value of $Q^{(i)}$ when θ' is changed into θ, and ϖ' into ϖ. But by these changes, the value of p [1532c] becomes $p=\cos.\theta.\cos.\theta+\sin.\theta.\sin.\theta=1$, and the corresponding value

[1548β] of ρ [1548s], is $(1-2\alpha+\alpha^2)^{-\frac{1}{2}}$, which may be represented either by $(1-\alpha)^{-1}$, or $(\alpha-1)^{-1}$. The first of these expressions, being developed, is $\rho=1+\alpha+\alpha^2+\&c.$; the second, $\rho=\alpha^{-1}+\alpha^{-2}+\alpha^{-3}+\&c.$ Comparing the first with [1548u], or the second

[1548γ] with [1548w], we shall get generally $Q_{\prime}^{(i)}=1$. Substituting this in [1548α], we find that the term of the product $\rho\rho'$ represented by $\alpha^m.\beta^n.Q^{(i)}.Q^{(i)}$, produces in z, the term $\alpha^m.\beta^n.\frac{1}{2i+1}$. Therefore if we multiply ρ [1548u] by the values of ρ' [1548v, x], and

[1548δ] substitute the products in [1548y], we shall get the two first of the following values of z; and if we multiply ρ [1548w], by the values of ρ' [1548v, x], they will produce the other two formulas [1548η, θ].

[1548ε] $$z=1+\tfrac{1}{3}\alpha\beta+\tfrac{1}{5}\alpha^2\beta^2+\tfrac{1}{7}\alpha^3\beta^3+\&c.;$$

[1548ζ] $$z=\frac{1}{\beta}.\left\{1+\tfrac{1}{3}.\frac{\alpha}{\beta}+\tfrac{1}{5}.\left(\frac{\alpha}{\beta}\right)^2+\tfrac{1}{7}.\left(\frac{\alpha}{\beta}\right)^3+\&c.\right\};$$

[1548η] $$z=\frac{1}{\alpha}.\left\{1+\tfrac{1}{3}.\frac{\beta}{\alpha}+\tfrac{1}{5}.\left(\frac{\beta}{\alpha}\right)^2+\tfrac{1}{7}.\left(\frac{\beta}{\alpha}\right)^3+\&c.\right\};$$

[1548θ] $$z=\frac{1}{\alpha\beta}+\tfrac{1}{3}.\left(\frac{1}{\alpha\beta}\right)^2+\tfrac{1}{5}.\left(\frac{1}{\alpha\beta}\right)^3+\tfrac{1}{7}.\left(\frac{1}{\alpha\beta}\right)^4+\&c.$$

Errors in using diverging series.

Now if $\alpha<\beta$, $\beta<1$, the two first of these four series will be converging, and *it would seem that we might use either of them in computing z; but as they give different values of this quantity, as is very evident by supposing α to be infinitely small, one of them*

[1548ι] *must be erroneous.* It is the second which is not accurate, because it was computed by the combination of the *diverging* series [1548x] with the converging series [1548u]. On the contrary, the first series is accurate, because it was formed by the combination of the two

If the body be a spheroid of revolution, about that axis which forms, with the radius R, the angle θ, the angle ϖ will vanish from the expression of [1548‴]

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converging series [1548u, v]. Whatever be the values of α, β, two of the four series [1548ε—θ] will converge; but only one of them will be accurate, and it will be that which results from the converging developments of ρ, ρ'. *Thus we find, in this example, that the use of a function developed in a diverging series, leads to an inaccurate result, although this result is expressed in a converging series.* Hence we perceive the necessity of avoiding the use of diverging series in analytical calculations, or of using them with extreme caution. [1548κ] Important result. [1548λ]

If $f(\theta',\varpi')$ can be reduced to the form $f(p)$, one of the integrations of [1548o] may be made without developing $f(\theta',\varpi')$ in a series of the form [1533k], as has been shown by Mr. Poisson. For example, if g, h, k, be constant quantities, $a=\sqrt{(g^2+h^2+k^2)}$, and $f(\theta',\varpi')$ of the following form, [1548μ]

$$f(\theta',\varpi')=f(g\,.\cos.\theta'+h\,.\sin.\theta'\,.\sin.\varpi'+k\,.\sin.\theta'\,.\cos.\varpi'), \qquad [1548\nu]$$

we may put

$$g=a\,.\cos.\theta\,; \qquad h=a\,.\sin.\theta\,.\sin.\varpi\,; \qquad k=\sin.\theta\,.\cos.\varpi. \qquad [1548\xi]$$

Substituting these values of g, h, k, in $g\,.\cos.\theta'+h\,.\sin.\theta'\,.\sin.\varpi'+k\,.\sin.\theta'\,.\cos.\varpi'$, it becomes, by using [1532c],

$$a\,.\{\cos.\theta\,.\cos.\theta'+\sin.\theta\,.\sin.\theta'\,.(\sin.\varpi\,.\sin.\varpi'+\cos.\varpi\,.\cos.\varpi')\}$$

$$=a\,.\{\cos.\theta\,.\cos.\theta'+\sin.\theta\,.\sin.\theta'\,.\cos.(\varpi'-\varpi)=a\,.p. \qquad [1548\pi]$$

Hence [1548ν] becomes $f(\theta',\varpi')=f(a\,.p)$, or more simply $F(p)$, and then [1548o] changes into

$$P=\int_0^{\pi}\int_0^{2\pi}\frac{F(p)\,.\sin.\theta'\,.d\theta'\,.d\varpi'}{(1-2\alpha\,.p+\alpha^2)^{\frac{1}{2}}}\,; \qquad [1548\rho]$$

in which $\dfrac{F(p)}{(1-2\alpha\,.p+\alpha^2)^{\frac{1}{2}}}$ is a function of p, which, for brevity, we shall put equal to $\varphi(p)$. Substituting this, we shall get, $P=\int_0^{\pi}\int_0^{2\pi}\varphi(p)\,.\sin.\theta'\,.d\theta'\,.d\varpi'$. If we now use the variable quantities γ, $\varpi_{,}$, [1535g, &c.], instead of θ', ϖ', the preceding expression will become $P=\int_0^{\pi}\int_0^{2\pi}\varphi(p)\,.\sin.\gamma\,.d\gamma\,.d\varpi_{,}$. But from [1532$c$] we have $p=\cos.\gamma$; its differential gives $\sin.\gamma\,.d\gamma=-dp$; and if we substitute this in P, the limits of p, corresponding to $\gamma=0$ and $\gamma=\pi$, will be 1 and -1 respectively. If we change the order of these limits to -1 and 1, we may change the sign of the preceding value of $\sin.\gamma\,.d\gamma$, making it equal to dp, and then the value of P will become, [1548σ] [1548τ] [1548υ]

$$P=\int_{-1}^{1}\int_0^{2\pi}\varphi(p)\,.dp\,.d\varpi_{,}=\int_{-1}^{1}\varphi(p)\,.dp\,.\int_0^{2\pi}d\varpi_{,}=2\pi\,.\int_{-1}^{1}\varphi(p)\,.dp. \qquad [1548\varphi]$$

$Z^{(i)}$, which will then become of this form,*

[1549] $$\frac{1.3.5\ldots\ldots(2i-1)}{1.2.3\ldots\ldots i} \cdot A^{(i)} \cdot \left\{ \mu^{i} - \frac{i.(i-1)}{2.(2i-1)} \cdot \mu^{i-2} + \frac{i.(i-1).(i-2).(i-3)}{2.4.(2i-1).(2i-3)} \cdot \mu^{i-4} - \&c. \right\};$$

[1549'] $A^{(i)}$ being a function of a. We shall put $\lambda^{(i)}$ for the coefficient of $A^{(i)}$ in this function. The product

[1550] $$\left(\frac{1.3.5\ldots\ldots(2i-1)}{1.2.3\ldots\ldots i}\right)^2 \cdot \left\{ 1 - \frac{i.(i-1)}{2.(2i-1)} + \&c. \right\}^2,$$

[1548χ] Hence the double integral, relative to θ', ϖ', will be reduced to the single integral relative to p. Resubstituting the function f instead of φ, we finally obtain

Theorem in definite integrals.

[1548ψ] $$P = \int_0^{\pi}\int_0^{2\pi} \frac{f(g.\cos.\theta' + h.\sin.\theta'.\sin.\varpi' + k.\sin.\theta'.\cos.\varpi')}{(1-2\alpha.p+\alpha^2)^{\frac{1}{2}}} . \sin.\theta'.d\theta'.d\varpi' = 2\pi . \int_{-1}^{1} \frac{f(a.p)}{(1-2\alpha.p+\alpha^2)^{\frac{1}{2}}} . dp;$$

[1548ω] in which $p = \cos.\theta . \cos.\theta' + \sin.\theta . \sin.\theta' . \cos.(\varpi' - \varpi)$, and $a = \sqrt{(g^2 + h^2 + k^2)}$.

* (1066) If in fig. 9, page 70, we suppose CX to be the axis of revolution of a spheroid, whose centre is C, and m to be a point of the surface of the spheroid, corresponding to the angles θ', ϖ', [1531']; it is evident that the distance $Cm = R$ will not vary, while the point m makes a complete revolution about that axis, or while ϖ' increases from 0 to 2π; therefore R, R^{i+3}, and generally $Z'^{(i)}$ [1533], will be independent of ϖ', consequently $Z^{(i)}$ [1540''''] will be independent of ϖ. Now the only term of $Y^{(i)}$ or $Z^{(i)}$ [1528] independent of ϖ, is that corresponding to $B^{(0)}$, or $n = 0$, which is

[1549a] $$Z^{(i)} = \left\{ \mu^{i} - \frac{i.(i-1)}{2.(2i-1)} \cdot \mu^{i-2} + \&c. \right\} . B^{(0)}.$$

$B^{(0)}$ is independent of μ, ϖ, and must therefore be a function of a only. For

$$R = \varphi(\mu', \varpi', a), \qquad [1536c],$$

being, in this case, independent of ϖ', becomes $R = \varphi(\mu', a)$; therefore R^{i+3} and $Z'^{(i)}$ [1533], must be functions of μ', a,; consequently $Z^{(i)}$ must be a function of μ, a, [1540''''], represented in [1549a]; in which the factor of $B^{(0)}$ contains all the terms depending on μ; therefore $B^{(0)}$ must be a function of the remaining quantity a, which may be put under

[1549b] the form $B^{(0)} = \frac{1.3.5\ldots\ldots(2i-1)}{1.2.3\ldots\ldots i} . A^{(i)}$, and then [1549a] will become as in [1549].

This form of $B^{(0)}$ is used to render the calculations in [1549—1555] more symmetrical. If we now put, as in [1549'],

[1549c] $$\lambda^{(i)} = \frac{1.3.5\ldots\ldots(2i-1)}{1.2.3\ldots\ldots i} . \left\{ \mu^{i} - \frac{i.(i-1)}{2.(2i-1)} . \mu^{i-2} + \frac{i.(i-1).(i-2).(i-3)}{2.4.(2i-1).(2i-\;)} . \mu^{i-4} - \&c. \right\},$$

the expression of $Z^{(i)}$ [1549] will become,

[1549d] $$Z^{(i)} = \lambda^{(i)} . A^{(i)}.$$

is, by the preceding article, the coefficient of $\frac{R^i}{r^{i+1}}$, in the development of the radical *

$$[r^2 - 2rR . \{\mu\mu' + \sqrt{1-\mu^2} . \sqrt{1-\mu'^2} . \cos.(\varpi - \varpi')\} + R^2]^{-\frac{1}{2}} ; \qquad [1551]$$

when we suppose μ and μ' equal to unity. This coefficient is then equal to 1 ; therefore we shall have, [1551']

$$\frac{1.3.5 \ldots\ldots(2i-1)}{1.2.3\ldots\ldots i} . \left\{ 1 - \frac{i.(i-1)}{2.(2i-1)} + \&c. \right\} = 1 ; \qquad [1552]$$

that is, $\lambda^{(i)}$ [1549c] becomes equal to unity, when $\mu = 1$. Now we have†

$$U^{(i)} = \frac{4\pi . \lambda^{(i)}}{(i+3).(2i+1)} . \int \rho . \left(\frac{dA^{(i)}}{da}\right) . da ; \qquad [1553]$$

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* (1067) Putting $\mu = \mu' = 1$, in [1514], the factors $(1-\mu^2)^{\frac{n}{2}}$, $(1-\mu'^2)^{\frac{n}{2}}$, will vanish, except $n=0$; and if we put $n=0$, we shall have $(1-\mu^2)^{\frac{n}{2}} = 1$, $(1-\mu'^2)^{\frac{n}{2}} = 1$; and then the expression [1514] will become,

$$\beta = \gamma . \left\{ 1 - \frac{i.(i-1)}{2.(2i-1)} + \frac{i.(i-1).(i-2).(i-3)}{2.4.(2i-1).(2i-3)} - \&c. \right\}^2 ; \qquad [1551a]$$

which, by [1507'', 1508', 1509], represents the coefficient of $\frac{R^i}{r^{i+1}}$ in the development of T [1509], in the case of $\mu = 1$, $\mu' = 1$. Now this coefficient β is equal to unity, as is easily found by the substitution of $\mu = 1$, $\mu' = 1$, in [1509]; from which we get, [1551b]

$$T = (r^2 - 2Rr + R^2)^{-\frac{1}{2}} = (r-R)^{-1} = \frac{1}{r} . \left(1 - \frac{R}{r}\right)^{-1} = \frac{1}{r} + \frac{R}{r^2} + \frac{R^2}{r^3} + \ldots\ldots \frac{R^i}{r^{i+1}} + \&c. \qquad [1551c]$$

Substituting in [1551a], the preceding value of $\beta = 1$ [1551c], also γ [1521], the second member becomes as in [1550]. Extracting the square root, we get [1552]; the sign of the second member being positive, as will evidently appear, by applying the formula to any particular value of i; as for example, $i=1$, $i=2$, &c. The first member of [1552] is equal to the value of $\lambda^{(i)}$ [1549c], in the case of $\mu = 1$; therefore we shall have $\lambda^{(i)} = 1$, when $\mu = 1$. [1551d]

† (1068) Substituting in [1541] the value of $Z^{(i)}$ [1549d], it becomes as in [1553], the quantity $\lambda^{(i)}$ being brought from under the sign $\int$, which only affects the quantity a; and $\lambda^{(i)}$ [1549c] is a function of μ, which is not affected by this integration.

and if the attracted point be upon the axis of revolution, corresponding to $\mu=1$, this will become,*

[1554] $$U^{(i)}=\frac{4\pi}{(i+3)\cdot(2i+1)}\cdot\int\rho\cdot\left(\frac{dA^{(i)}}{da}\right)\cdot da;$$

therefore if we suppose that when the attracted point is situated upon the continuation of this axis, the function V is expressed by a series of the following form, arranged according to the powers of $\frac{1}{r}$, as in [1459],

[1555] $$V=\frac{B^{(0)}}{r}+\frac{B^{(1)}}{r^2}+\frac{B^{(2)}}{r^3}+\&c.;$$

we shall have the value of V, corresponding to another point placed at the
[1555'] same distance from the origin of the co-ordinates, but upon a radius which makes with the axis of revolution an angle whose cosine is μ, by multiplying
[1555''] the terms of this value respectively by $\lambda^{(0)}$, $\lambda^{(1)}$, $\lambda^{(2)}$, &c.

In case the spheroid is not formed by the revolution of a curve, this method

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* (1069) If the attracted point be situated in the axis of revolution, we shall have $\theta=0$ [1548''']; hence $\mu=\cos.\theta=1$ [1434'], and $\lambda^{(i)}=1$ [1551d]. In this case the value of $U^{(i)}$ [1553] will become as in [1554]; which, for the sake of distinction, we shall
[1554a] call $B^{(i)}$, so that $B^{(i)}=\frac{4\pi}{(i+3)\cdot(2i+1)}\cdot\int\rho\cdot\left(\frac{dA^{(i)}}{da}\right)\cdot da$. Multiplying this by $\lambda^{(i)}$, its
[1554b] second member becomes equal to the value of $U^{(i)}$ [1553]; hence $U^{(i)}=\lambda^{(i)}\cdot B^{(i)}$. Deducing from this the values of $U^{(0)}$, $U^{(1)}$, &c., and substituting them in the *general* value of V [1436], it becomes,

[1554c] $$V=\frac{\lambda^{(0)}\cdot B^{(0)}}{r}+\frac{\lambda^{(1)}\cdot B^{(1)}}{r^2}+\frac{\lambda^{(2)}\cdot B^{(2)}}{r^3}+\&c.$$

If we denote by $V_{,}$, the *particular* value of V, corresponding to an attracted point situated on the continuation of the axis of revolution, in which case $\lambda^{(0)}=1$, $\lambda^{(1)}=1$, &c., [1551d]; we shall have

[1554d] $$V_{,}=\frac{B^{(0)}}{r}+\frac{B^{(1)}}{r^2}+\frac{B^{(2)}}{r^3}+\&c.$$

This particular value $V_{,}$ is, in most cases, much more easy to be computed, by a direct operation, than the general value of V [1554c]; but when we have computed one of these expressions, the other may be easily deduced from it. For if $V_{,}$ [1554d] were known, we should obtain the general value of V [1554c], by changing $B^{(0)}$, $B^{(1)}$, $B^{(2)}$, &c., [1554d], into $\lambda^{(0)}\cdot B^{(0)}$, $\lambda^{(1)}\cdot B^{(1)}$, $\lambda^{(2)}\cdot B^{(2)}$, &c., respectively, according to the directions in [1555''].

will give the part of V, independent of the angle ϖ; we may determine the [1555''']
other part in the following manner. Supposing, for greater simplicity, that
the spheroid is such, that it may be divided into two equal and similar parts, [1555'''']
both by the equator, and by the meridian from which the angle ϖ is counted,
also by the meridian which is perpendicular to it; then V will be a function [1555^{v}]
of μ^2, $\sin.^2\varpi$, and $\cos.^2\varpi$;* or, in other words, it will be a function of
μ^2, and of the cosines of the angle 2ϖ, and of its multiples; $U^{(i)}$ will [1555^{vi}]

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* (1070) Let $ABCD$, in the plane of the present figure, be the equator of the body, and EE', perpendicular to it, the axis from which the angle θ is counted; BED the meridian, from which the angle ϖ is counted, and AEC perpendicular to it. Then if the attracted point be p, and we draw the line pP, perpendicular to the equator at P, we shall have the angle $E'Ep=\theta$, $BEP=\varpi$. Now by hypothesis, [1555''''], the body [1555a] is supposed to be divided into *eight* equal and similarly situated parts, by the meridians BED, AEC, and the plane of the equator; *four* being *above* the plane of the figure, and *four below* it. In each of these parts, we may find points, as p', p'', p''', π, π', π'', π''', in which the value of the function V, will be the same as at the point p. For if we continue the line pP to π, making $P\pi=Pp$, it is evident, that if the attracted point were at π, it would be situated, relative to the part of the spheroid falling below the plane BEA, in exactly the [1555b] same manner as the point p is relative to the upper part of the spheroid; so that the value of V [1428'''''] would be the same in both cases. This variation of place from p to π does not alter the value of ϖ, but changes the angle $E'Ep=\theta$ into $E'E\pi=\pi-\theta$, or μ into $-\mu$ [1434']. [1555c] Now draw the line $P'EP'''$, making the angle $BEP'=BEP$; continue PE to P'', making $EP=EP'=EP''=EP'''$; draw the lines $p'P'\pi'$, $p''P''\pi''$, $p'''P'''\pi'''$, parallel and equal to $pP\pi$, so that the points p, p', p'', p''', may be at equal elevations above the equator, and the points π, π' π'', π''', at equal depressions below it. Then it is evident, from the nature of the function V [1428'''''], and from the similarity of these eight sections of the spheroid [1555''''], that the value of V will be the same at each [1555d] of the eight points p, p', p'', p''', π, π', π'', π'''. From the construction we have

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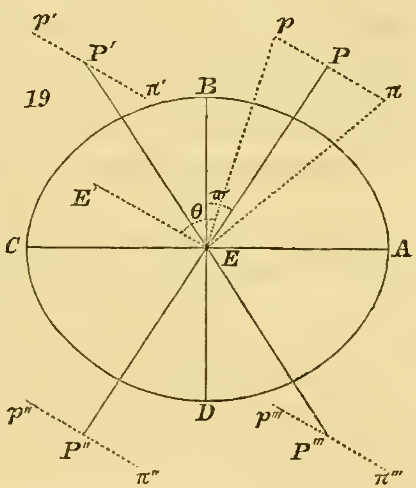

$$BEP=\varpi, \qquad BEP'''=\pi-\varpi, \qquad BEP''=\pi+\varpi, \qquad BEP'=2\pi-\varpi, \qquad [1555d']$$

therefore the function V will not vary by changing ϖ into $\pi-\varpi$, $\pi+\varpi$, or $2\pi-\varpi$; [1555e]

[1555vii] therefore be nothing when i is odd,* and if i be even, the term depending on the angle $2n\varpi$ will be of the form,†

[1555e'] or by changing μ into $-\mu$ [1555e]. V [1436, 1437'] is evidently a rational and integral function of μ, $\sqrt{(1-\mu^2)}.\sin.\varpi$, $\sqrt{(1-\mu^2)}.\cos.\varpi$, of the form

[1555f] $$G.\mu^a.\{\sqrt{(1-\mu^2)}.\cos.\varpi\}^b.\{\sqrt{(1-\mu^2)}.\sin.\varpi\}^c;$$

a, b, c, being integral numbers; and if we use the sign Σ of finite integrals, it becomes of the

[1555g] form $V=\Sigma G.\mu^a.(1-\mu^2)^{\frac{b+c}{2}}.\cos.^b\varpi.\sin.^c\varpi$. This must not change by putting $-\mu$ for μ [1555e'], without altering ϖ, which requires that the exponent a of μ, should be an *even* number. In like manner, in moving from the point p to p', the angle ϖ changes into $2\pi-\varpi$; by which means $\sin.\varpi$ changes into $-\sin.\varpi$, without changing the value of μ or $\cos.\varpi$; therefore the exponent c of $\sin.\varpi$ [1555g] must be *even*. Lastly, in moving from the point p to p''', the angle ϖ changes into $\pi-\varpi$, and $\cos.\varpi$ changes into $-\cos.\varpi$, without altering the sign of μ or $\sin.\varpi$; hence the exponent b of $\cos.\varpi$ [1555g] must also be *even*. Therefore in [1555g], we may change the exponents a, b, c, into the even integral numbers $2a$, $2b$, $2c$, and we shall get,

[1555h] $$V=\Sigma G.(\mu^2)^a.(1-\mu^2)^{b+c}.(\cos.^2\varpi)^b.(\sin.^2\varpi)^c,$$

making, as in [1555v], V an integral function of μ^2, $\cos.^2\varpi$, $\sin.^2\varpi$; and since, by [1, 6] Int., $\cos.^2\varpi=\tfrac{1}{2}+\tfrac{1}{2}.\cos.2\varpi$, $\sin.^2\varpi=\tfrac{1}{2}-\tfrac{1}{2}.\cos.2\varpi$, V will be an integral function of μ^2 and $\cos.2\varpi$. From [6—10, &c.] Int., the powers of $\cos.2\varpi$ may be expressed in terms of the cosines of the angle 2ϖ and its multiples; therefore V will

[1555i] be an integral function of these cosines, and of μ^2, as in [1555vi].

[1556a] * (1071) Using for brevity the values [1433a], P_0, P_1, P_2, we find, as in [1433k—l], that $U^{(i)}$ is composed of powers and products of those quantities, of the orders i, $i-2$, $i-4$, &c.; all of which will be *even* if i be even, but *odd* if i be odd. The general term

[1556b] of V [1555h] becomes, by the substitution of P_0, P_1, P_2, $G.P_0^{2a}.P_1^{2b}.P_2^{2c}$, which is of the *even* degree $2.(a+b+c)$, in P_0, P_1, P_2; hence all the terms of V, and therefore those of $U^{(i)}$ [1436], must be of an *even* degree in P_0, P_1, P_2; and there can be no term of $U^{(i)}$ which is of an odd degree; or in other words $U^{(i)}$ is nothing when i is odd.

† (1072) Since V and $U^{(i)}$ [1555i] are composed of terms depending only on the *cosines* of the multiples of 2ϖ, we must, in the general form of a function of this kind, [1528], put $A^{(n)}=0$, and change n into $2n$. The expression will then be,

[1557a] $$(1-\mu^2)^n.\left\{\mu^{i-2n}-\frac{(i-2n).(i-2n-1)}{2.(2i-1)}.\mu^{i-2n-2}+\&c.\right\}.B^{(2n)}.\cos.2n\varpi;$$

[1557b] which, by putting $B^{(2n)}=C^{(i)}$, becomes as in [1556].

$$C^{(i)}.(1-\mu\mu)^n.\left\{\mu^{i-2n}-\frac{(i-2n).(i-2n-1)}{2.(2i-1)}.\mu^{i-2n-2}+\&c.\right\}.\cos.2n\varpi. \qquad [1556]$$

With respect to an attracted point, situated in the plane of the equator, where $\mu=0$, the part of V depending on this term will be,* [1556′]

$$\pm\frac{C^{(i)}}{r^{i+1}}.\frac{1.3.5\ldots\ldots(i-2n-1)}{(i+2n+1).(i+2n+3)\ldots\ldots(2i-1)}.\cos.2n\varpi. \qquad [1557]$$

Hence it follows, that if we have developed V, in a series arranged according to the cosines of the angle 2ϖ and its multiples, in the case where the attracted point is situated in the plane of the equator, we may extend this solution to any other attracted point whatever, by multiplying the terms depending on $\frac{\cos.2n\varpi}{r^{i+1}}$ by the function† [1557′]

$$\pm\frac{(i+2n+1)\ldots..(2i-1)}{1.3.5\ldots..(i-2n-1)}.(1-\mu\mu)^n.\left\{\mu^{i-2n}-\frac{(i-2n).(i-2n-1)}{2.(2i-1)}.\mu^{i-2n-2}+\&c.\right\}; \qquad [1558]$$

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* (1073) If $\mu=0$, the factor $(1-\mu^2)^n$ [1556] becomes 1, and all the terms μ^{i-2n}, μ^{i-2n-2}, &c., vanish, except $\mu^0=1$, corresponding to the last term of the series [1556], which is equal to

$$\pm C^{(i)}.\frac{(i-2n).(i-2n-1).(i-2n-2)\ldots..1}{2.4.6\ldots..(i-2n).(2i-1).(2i-3)\ldots..(i+2n+1)}.\cos.2n\varpi.$$

Rejecting the terms $2.4.6\ldots..(i-2n)$, which are common to the numerator and denominator, and then inverting the order of the terms, it becomes

$$\pm C^{(i)}.\frac{1.3.5\ldots..(i-2n-1)}{(i+2n+1).(i+2n+3)\ldots..(2i-1)}.\cos.2n\varpi; \qquad [1557c]$$

which represents the part of $U^{(i)}$ depending on $\cos.2n\varpi$, in the case of $\mu=0$; and this produces in V [1436] the term [1557]. [1557d]

† (1074) If we suppose the coefficient of $\frac{\cos.2n\varpi}{r^{i+1}}$, in the function V, corresponding to an attracted point situated in the equator, to be computed and put equal to $C'^{(i)}$, we shall have, by [1557],

$$C'^{(i)}=\pm\frac{1.3.5\ldots..(i-2n-1)}{(i+2n+1).(i+2n+3)\ldots..(2i-1)}.C^{(i)}; \qquad [1557e]$$

hence $C^{(i)}=\pm C'^{(i)}.\frac{(i+2n+1)\ldots..(2i-1)}{1.3.5\ldots..(i-2n-1)}$. Substituting this in the general

we shall therefore have the complete value of V, when it has been determined in a series for the two following cases; *first*,* when the attracted point is
[1558′] situated upon the continuation of the polar axis;† *second*, when it is situated

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term of the value of $U^{(i)}$ [1556], we shall get the coefficient of $\frac{\cos. 2n\varpi}{r^{i+1}}$ in the value of V which will be found to be the same as the product of $C'^{(i)}$ by the function [1558], as in [1557′, &c.]

* (1075) It is unnecessary to notice the *first* of these cases, as it is comprised in the *second*. For having computed the terms [1557], for points situated in the plane of the equator, we can deduce from them, by the process [1558], the terms corresponding to any values of μ and n, [1556]. The part of this last expression corresponding to $n=0$, is

[1557f] $C^{(i)}.\left\{\mu^i-\frac{i.(i-1)}{2.(2i-1)}.\mu^{i-2}+\&c.\right\};$ but from [1557b, 1549b],

[1557g] $$C^{(i)}=B^{(2n)}=B^{(i)}=\frac{1.3.5.\ldots.(2i-1)}{1.2.3.\ldots.i}.A^{(i)};$$

therefore the preceding expression becomes as in [1549].

† (1076) The method used by the author, in determining the general form of V, by
[1558a] means of the values, computed for particular points, has been improved and generalized, by M. Biot, in an excellent paper, published in the Mémoires de l'Institut des Sciences, &c., Paris, Tom. 6. The principal results, he has obtained, will appear from the following application of his method, to the integration of the equation [1430].

Instead of assuming for a, b, c, the values [1431], of a given form in r, θ, ϖ, he represents, by functions of the most general nature, the values of r, θ, ϖ, in terms of a, b, c, so that

[1558b] $$r=F(a,b,c), \qquad \theta=F_1(a,b,c), \qquad \varpi=F_2(a,b,c);$$

F, F_1, F_2, representing functions of a, b, c. The values of a, b, c, deduced from these equations, may be put under the form,

[1558b′] $$a=f(r,\theta,\varpi), \qquad b=f_1(r,\theta,\varpi), \qquad c=f_2(r,\theta,\varpi);$$

the functions f, f_1, f_2, depending on F, F_1, F_2, and representing functions of r, θ, ϖ; observing also, that the quantities r, θ, ϖ, *are not restricted to the form assumed in* [1430′,
[1558c] 1430″], *but may be any quantities whatever, but wholly independent of each other*. Substituting these values of a, b, c, [1558b′] in V, it becomes a function of r, θ, ϖ, whose partial differentials, relative to a, b, c, may be found in the usual manner, as in [462], changing x into a; whence we shall get,

[1558d] $$\left(\frac{dV}{da}\right)=\left(\frac{dV}{dr}\right).\left(\frac{dr}{da}\right)+\left(\frac{dV}{d\theta}\right).\left(\frac{d\theta}{da}\right)+\left(\frac{dV}{d\varpi}\right).\left(\frac{d\varpi}{da}\right);$$

in the plane of the equator. By this means the investigation of the value of V is much simplified.

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and the values $\left(\frac{dV}{db}\right)$, $\left(\frac{dV}{dc}\right)$, are of a similar form, changing a into b or c. [1558e]
Substituting, in these expressions, the values of a, b, c, [1558b′], the coefficients of $\left(\frac{dV}{dr}\right)$, $\left(\frac{dV}{d\theta}\right)$, $\left(\frac{dV}{d\varpi}\right)$, will become functions of r, θ, ϖ, which, for brevity, we shall denote by A_0, A_1, A_2, B_0, B_1, B_2, C_0, C_1, C_2, and we shall have,

$$\left(\frac{dV}{da}\right)=A_0\cdot\left(\frac{dV}{dr}\right)+A_1\cdot\left(\frac{dV}{d\theta}\right)+A_2\cdot\left(\frac{dV}{d\varpi}\right);$$
$$\left(\frac{dV}{db}\right)=B_0\cdot\left(\frac{dV}{dr}\right)+B_1\cdot\left(\frac{dV}{d\theta}\right)+B_2\cdot\left(\frac{dV}{d\varpi}\right);$$ [1558f]
$$\left(\frac{dV}{dc}\right)=C_0\cdot\left(\frac{dV}{dr}\right)+C_1\cdot\left(\frac{dV}{d\theta}\right)+C_2\cdot\left(\frac{dV}{d\varpi}\right);$$

which corresponds to that in [464], and to the similar expressions in y, z. In like manner, by changing successively in [465g], x into a, b, c, we shall obtain the values of

$$\left(\frac{ddV}{da^2}\right),\qquad \left(\frac{ddV}{db^2}\right),\qquad \left(\frac{ddV}{dc^2}\right),$$ [1558f′]

expressed in the partial differentials of V, of the first and second orders, relative to r, θ, ϖ; [1558g]
and if we substitute in the factors, with which these quantities are connected, the values of a, b, c, [1558b′], as was done in [1558d, e], these factors will become functions of r, θ, ϖ; and the equation [1430], corresponding to an external attracted point will become of the following form,

$$0=D\cdot\left(\frac{ddV}{dr^2}\right)+E\cdot\left(\frac{ddV}{d\theta^2}\right)+F\cdot\left(\frac{ddV}{d\varpi^2}\right)+G\cdot\left(\frac{ddV}{dr.d\theta}\right)+H\cdot\left(\frac{ddV}{dr.d\varpi}\right)+I\cdot\left(\frac{ddV}{d\theta.d\varpi}\right)$$
$$+K\cdot\left(\frac{dV}{dr}\right)+L\cdot\left(\frac{dV}{d\theta}\right)+M\cdot\left(\frac{dV}{d\varpi}\right),$$ [1558h]

D, E, F, &c., being functions of r, θ, ϖ. This equation corresponds with that in [465t], or [465].

We shall now suppose V to be expressed in a series, proceeding according to the powers of any one of the variable quantities, as for example r, of the following form,

$$V=v^{(0)}+r\cdot v^{(1)}+\frac{r^2}{1.2}\cdot v^{(2)}+\frac{r^3}{1.2.3}\cdot v^{(3)}+\frac{r^4}{1.2.3.4}\cdot v^{(4)}+\&c.;$$ [1558i]

$v^{(0)}$, $v^{(1)}$, $v^{(2)}$, &c., being independent of r. Substituting this in [1558h], and arranging [1558i′]
the terms according to the powers of r, it will become of the form,

$$0=N^{(0)}+N^{(1)}\cdot r+N^{(2)}\cdot\frac{r^2}{1.2}+N^{(3)}\cdot\frac{r^3}{1.2.3}+N^{(4)}\cdot\frac{r^4}{1.2.3.4}+\&c.;$$ [1558k]

The spheroid which we have just considered, comprises the ellipsoid. If
[1558''] the attracted point be situated in the polar axis, which we shall suppose to

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the general value of the coefficient of $\frac{r^n}{1.2.3\ldots\ldots n}$ being

[1558l]
$$\mathcal{N}^{(n)}=D.v^{(n+2)}+E.\left(\frac{ddv^{(n)}}{d\theta^2}\right)+F.\left(\frac{ddv^{(n)}}{d\varpi^2}\right)+G.\left(\frac{dv^{(n+1)}}{d\theta}\right)+H.\left(\frac{dv^{(n+1)}}{d\varpi}\right)+I.\left(\frac{ddv^{(n)}}{d\theta.d\varpi}\right) +K.v^{(n+1)}+L.\left(\frac{dv^{(n)}}{d\theta}\right)+M.\left(\frac{dv^{(n)}}{d\varpi}\right).$$

Now the equation [1558k] ought to be satisfied for all values of r, which requires that we
[1555m] should have generally $\mathcal{N}^{(n)}=0$, and then the equation [1558l], being divided by D, becomes,

[1558n]
$$v^{(n+2)}=-\frac{1}{D}.\left\{\begin{array}{c}E.\left(\frac{ddv^{(n)}}{d\theta^2}\right)+F.\left(\frac{ddv^{(n)}}{d\varpi^2}\right)+G.\left(\frac{dv^{(n+1)}}{d\theta}\right)+H.\left(\frac{dv^{(n+1)}}{d\varpi}\right)+I.\left(\frac{ddv^{(n)}}{d\theta.d\varpi}\right)\\ +K.v^{(n+1)}+L.\left(\frac{dv^{(n)}}{d\theta}\right)+M.\left(\frac{dv^{(n)}}{d\varpi}\right)\end{array}\right\};$$

which, in the case of $n=0$, becomes

[1558o]
$$v^{(2)}=-\frac{1}{D}.\left\{\begin{array}{c}E.\left(\frac{ddv^{(0)}}{d\theta^2}\right)+F.\left(\frac{ddv^{(0)}}{d\varpi^2}\right)+G.\left(\frac{dv^{(1)}}{d\theta}\right)+H.\left(\frac{dv^{(1)}}{d\varpi}\right)+I.\left(\frac{ddv^{(0)}}{d\theta.d\varpi}\right)\\ +K.v^{(1)}+L.\left(\frac{dv^{(0)}}{d\theta}\right)+M.\left(\frac{dv^{(0)}}{d\varpi}\right)\end{array}\right\}.$$

Hence we get $v^{(2)}$ in terms of $v^{(0)}$, $v^{(1)}$. If we now put $n=1$, in [1558n], we shall get $v^{(3)}$ in terms of $v^{(2)}$, $v^{(1)}$, and by substituting the value of $v^{(2)}$, [1558o], we shall get $v^{(3)}$ in terms of $v^{(0)}$, $v^{(1)}$. In like manner, by putting successively $n=2$, $n=3$, &c.,
[1558p] in [1558n], and substituting the values of $v^{(3)}$, $v^{(4)}$, &c., in terms of $v^{(0)}$, $v^{(1)}$, we shall obtain the general value of $v^{(n)}$, in terms of $v^{(0)}$, $v^{(1)}$, which two last quantities will remain wholly arbitrary, and may be considered as the two arbitrary functions required to complete the integral of the differential equation [1558h], of the second degree. This integral will be expressed by the series [1558i], after substituting the values of $v^{(2)}$, $v^{(3)}$, &c., in terms of
[1558q] $v^{(0)}$, $v^{(1)}$; and when the forms of these two arbitrary functions $v^{(0)}$, $v^{(1)}$, shall be given, the complete value of V [1558i] will be wholly determined. These two arbitrary quantities $v^{(0)}$, $v^{(1)}$, are functions of θ, ϖ, but do not contain the variable quantity r, according to the powers of which the function V [1558i] is developed. The same would be true in other partial differential equations, in which the number of variable quantities r, θ, ϖ, should be varied; so that in every case, these arbitrary quantities would be functions of all the variable quantities θ, ϖ, &c., excluding the quantity r, which is used in the development of V; and
[1558r] when the form of these arbitrary functions is given, the value of V can be determined. It is

be the axis of x, we shall have $b=0$, $c=0$, [1347′], and then the expression of V [1394] will be integrable relative to p. If the attracted [1558‴]

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also evident that the principles here used may be applied to differential equations of *all orders*, which are functions of *any number of variable* quantities, and it will be found generally, that if the differential equation be of the *first* degree, there will be but *one* arbitrary function $v^{(0)}$; if of the *second* degree, there will be *two* arbitrary functions $v^{(0)}$, $v^{(1)}$, as in the above example; if of the *third* degree, there will be *three* arbitrary functions $v^{(0)}$, $v^{(1)}$, $v^{(2)}$; and so on for the higher orders; each of these quantities $v^{(0)}$, $v^{(1)}$, &c., being functions of all the variable quantities θ, ϖ, &c., excepting r, which is used for the development of V. We may therefore, with M. Biot, conclude, "*That the general effect of a partial differential equation is, to determine completely the form of the function relative to all the variable quantities, when this form is given relative to all these quantities excepting one.*" So that we may consider partial differential equations as serving to determine the forms of functions, and not their absolute quantities. [1558s]

Biot's theorem on partial differential equations.

[1558t]

After this digression on the general properties of differential equations, we shall resume the value of V [1558i], representing the integral of the equation [1558h], on which the attraction of a spheroid depends. The partial differentials of [1558i], relative to r, θ, ϖ, are

$$\left(\frac{dV}{dr}\right)=v^{(1)}+r\,.\,v^{(2)}+\frac{r^2}{1.2}\,.\,v^{(3)}+\&c.;$$
$$\left(\frac{dV}{d\theta}\right)=\left(\frac{dv^{(0)}}{d\theta}\right)+r\,.\left(\frac{dv^{(1)}}{d\theta}\right)+\frac{r^2}{1.2}\,.\left(\frac{dv^{(2)}}{d\theta}\right)+\&c.; \qquad [1558u]$$
$$\left(\frac{dV}{d\varpi}\right)=\left(\frac{dv^{(0)}}{d\varpi}\right)+r\,.\left(\frac{dv^{(1)}}{d\varpi}\right)+\frac{r^2}{1.2}\,.\left(\frac{dv^{(2)}}{d\varpi}\right)+\&c.$$

The function $v^{(0)}$ enters only under a differential form in the value of $v^{(2)}$ [1558o], and the same is true relative to the values of $v^{(3)}$, $v^{(4)}$, &c., found as in [1558p]. Now these partial differentials of $v^{(0)}$, and the value of $v^{(1)}$, may be determined by putting $r=0$ in [1558u]; in which case we shall have the following particular values, corresponding to the case of $r=0$, [1558v]

$$\left(\frac{dV}{dr}\right)=v^{(1)}, \qquad \left(\frac{dV}{d\theta}\right)=\left(\frac{dv^{(0)}}{d\theta}\right), \qquad \left(\frac{dV}{d\varpi}\right)=\left(\frac{dv^{(0)}}{d\varpi}\right). \qquad [1558w]$$

Hence it follows, that if we have the *particular* values of $\left(\frac{dV}{dr}\right)$, $\left(\frac{dV}{d\theta}\right)$, $\left(\frac{dV}{d\varpi}\right)$, [1558w], corresponding to $r=0$, we may from them obtain the *general* values of the same quantities, when r has any value whatever. Now the equations [1558f] will give, by the usual operation of algebraic equations of the first degree, $\left(\frac{dV}{dr}\right)$, $\left(\frac{dV}{d\theta}\right)$, $\left(\frac{dV}{d\varpi}\right)$, in functions of $\left(\frac{dV}{da}\right)$, $\left(\frac{dV}{db}\right)$, $\left(\frac{dV}{dc}\right)$; and if these last three quantities are known, in [1558x]

[1558''''] point be situated in the plane of the equator, we shall have $a=0$, [1347'], and the same expression of V will become integrable relative to q, by known

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the case of $r=0$, they will then give the values of $\left(\frac{dV}{dr}\right)$, $\left(\frac{dV}{d\theta}\right)$, $\left(\frac{dV}{d\varpi}\right)$, for the same case of $r=0$; from these *particular* values we may obtain, as above, the general
[1558y] values [1558u]; substituting these in [1558f], we shall finally obtain the general values of $\left(\frac{dV}{da}\right)$, $\left(\frac{dV}{db}\right)$, $\left(\frac{dV}{dc}\right)$, corresponding to any value of r.

[1558z] Now when $r=0$, we shall have, from [1558b], $0=F(a,b,c)$, which is the equation of any surface whatever, whose co-ordinates are a, b, c. This is evident, by observing, that if we assume at pleasure any values of a, b, and deduce, from the equation $0=F(a,b,c)$, the corresponding value of c; these three quantities a, b, c, will represent the co-ordinates of a point in space; and if we connect together all points of this kind, corresponding to the equation [1558z], and to different values of a, b, c, it will form the surface abovementioned, whose equation is $0=F(a,b,c)$; and in the present case, this
[1558α] surface is wholly arbitrary, because the functions F, F_1, F_2, [1558b], can be varied at pleasure. If the attracted point be situated anywhere on this surface, whose co-ordinates are a, b, c, the attraction of the spheroid upon this point, resolved in directions parallel to the axes a, b, c, will be represented, as in [1387, 1388], by $-\left(\frac{dV}{da}\right)$, $-\left(\frac{dV}{db}\right)$, $-\left(\frac{dV}{dc}\right)$; therefore if the attraction of the spheroid, upon the points of this surface, be known, we shall
[1558β] obtain the corresponding values of $\left(\frac{dV}{da}\right)$, $\left(\frac{dV}{db}\right)$, $\left(\frac{dV}{dc}\right)$, in the case of $r=0$; from which we may obtain, as in [1558y], the general values of the same quantities,
Biot's theorem on the attraction of a spheroid. corresponding to any value of r whatever. From what has been said, we may deduce this general theorem, "*To find the attraction of any spheroid upon any point, it is only necessary to know the attraction of the same spheroid, upon all points of any surface taken*
[1558γ] *at pleasure.*"

We may, for example, take for $r=0$, or $F(a,b,c)=0$, the equation of the
[1558δ] surface of the attracting spheroid, and we shall then see that the attraction of this spheroid, on all points of space will be known, when we shall have found the attraction, upon all the points, situated upon this surface. This includes, as a particular case, the results found in [1379, 1422], for spheroids of the second order.

If we suppose the arbitrary surface to be a plane, as for example the plane of bc, for
[1558ε] which we have $a=0$, it will not be necessary to transform the quantities a, b, c; and instead of the equations [1558b], we may put $r=a$, $\theta=b$, $\varpi=c$; and in this case, the equation $r=0$ [1558β], or $a=0$, will correspond to the required plane. The attraction of the spheroid will then be known generally, when we shall have found its value, for all the points situated in this plane. This includes, as a particular case, the result found

methods, supposing $\text{tang.}\, q = t$. In these two cases, the integral being taken relative to one of the variable quantities, between its limits, it becomes possible relative to the other; and we shall find that, M being the mass of [1558v]

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in [1558′], for a symmetrical spheroid. As the co-ordinates are not transformed, if we substitute the values of r, θ, ϖ, [1558ε], in [1558h], it ought to become identical with the [1558ζ]
proposed equation [1430], which requires that we should have $D = E = F = 1$, $G = H = I = K = L = M = 0$, and the equation [1558$n$] becomes

$$v^{(n+2)} = -\left\{ \left(\frac{d\,d\,v^{(n)}}{d\,b^2}\right) + \left(\frac{d\,d\,v^{(n)}}{d\,c^2}\right) \right\}. \qquad [1558\eta]$$

Putting successively $n = 0$, $n = 1$, $n = 2$, &c., we get,

$$v^{(2)} = -\left(\frac{ddv^{(0)}}{d\,b^2}\right) - \left(\frac{ddv^{(0)}}{d\,c^2}\right); \qquad v^{(3)} = -\left(\frac{ddv^{(1)}}{d\,b^2}\right) - \left(\frac{ddv^{(1)}}{d\,c^2}\right);$$

$$v^{(4)} = -\left(\frac{ddv^{(2)}}{d\,b^2}\right) - \left(\frac{ddv^{(2)}}{d\,c^2}\right); \qquad v^{(5)} = -\left(\frac{ddv^{(3)}}{d\,b^2}\right) - \left(\frac{ddv^{(3)}}{d\,c^2}\right); \qquad \&c. \qquad [1558\theta]$$

The partial differentials of these values of $v^{(2)}$, $v^{(3)}$, &c., give

$$\left(\frac{ddv^{(2)}}{d\,b^2}\right) = -\left(\frac{d^4\,v^{(0)}}{d\,b^4}\right) - \left(\frac{d^4\,v^{(0)}}{db^2.dc^2}\right), \qquad \left(\frac{ddv^{(2)}}{d\,c^2}\right) = -\left(\frac{d^4\,v^{(0)}}{db^2.dc^2}\right) - \left(\frac{d^4\,v^{(0)}}{d\,c^4}\right), \qquad \&c.; \qquad [1558\theta']$$

substituting these in $v^{(4)}$, $v^{(5)}$, &c., we get,

$$v^{(4)} = \left(\frac{d^4\,v^{(0)}}{d\,b^4}\right) + 2\cdot\left(\frac{d^4\,v^{(0)}}{d\,b^2\,.\,d\,c^2}\right) + \left(\frac{d^4\,v^{(0)}}{d\,c^4}\right);$$

$$v^{(5)} = \left(\frac{d^4\,v^{(1)}}{d\,b^4}\right) + 2\cdot\left(\frac{d^4\,v^{(1)}}{d\,b^2\,.\,d\,c^2}\right) + \left(\frac{d^4\,v^{(1)}}{d\,c^4}\right); \qquad \&c.; \qquad [1558\iota]$$

and the expression of V [1558i] becomes of the following form, first given by La Grange,

Integral of the fundamental equation for the attraction of a spheroid, by LaGrange.

$$\begin{aligned} V = v^{(0)} + a\,.\,v^{(1)} &- \frac{a^2}{1.2}\cdot\left\{ \left(\frac{ddv^{(0)}}{d\,b^2}\right) + \left(\frac{ddv^{(0)}}{d\,c^2}\right) \right\} \\ &- \frac{a^3}{1.2.3}\cdot\left\{ \left(\frac{ddv^{(1)}}{d\,b^2}\right) + \left(\frac{ddv^{(1)}}{d\,c^2}\right) \right\} \\ &+ \frac{a^4}{1.2.3.4}\cdot\left\{ \left(\frac{d^4\,v^{(0)}}{d\,b^4}\right) + 2\cdot\left(\frac{d^4\,v^{(0)}}{db^2.dc^2}\right) + \left(\frac{d^4\,v^{(0)}}{d\,c^4}\right) \right\} + \&c. \end{aligned} \qquad [1558\kappa]$$

Taking the partial differentials of this expression, relative to a, b, c, we get, by observing that $v^{(0)}$, $v^{(1)}$, $v^{(2)}$, &c., are independent of a, or r [1558i'],

the spheroid, the value of $\frac{V}{M}$ is independent of the semi-axis k of the

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$$\left(\frac{d\,V}{d\,a}\right)=v^{(1)}-a\,.\left\{\left(\frac{d^2\,v^{(0)}}{d\,b^2}\right)+\left(\frac{d^2\,v^{(0)}}{d\,c^2}\right)\right\}-\frac{a^2}{1.2}\,.\left\{\left(\frac{d^2\,v^{(1)}}{d\,b^2}\right)+\left(\frac{d^2\,v^{(2)}}{d\,c^2}\right)\right\}$$
$$+\frac{a^3}{1.2.3}\,.\left\{\left(\frac{d^4\,v^{(0)}}{d\,b^4}\right)+2\,.\left(\frac{d^4\,v^{(0)}}{d\,b^2.d\,c^2}\right)+\left(\frac{d^4\,v^{(0)}}{d\,c^4}\right)\right\}+\&c.\,;$$

[1558λ]
$$\left(\frac{d\,V}{d\,b}\right)=\left(\frac{d\,v^{(0)}}{d\,b}\right)+a\,.\left(\frac{d\,v^{(1)}}{d\,b}\right)-\frac{a^2}{1.2}\,.\left\{\left(\frac{d^3\,v^{(0)}}{d\,b^3}\right)+\left(\frac{d^3\,v^{(0)}}{d\,b\,.d\,c^2}\right)\right\}-\&c.\,;$$
$$\left(\frac{d\,V}{d\,c}\right)=\left(\frac{d\,v^{(0)}}{d\,c}\right)+a\,.\left(\frac{d\,v^{(1)}}{d\,c}\right)-\frac{a^2}{1.2}\,.\left\{\left(\frac{d^3\,v^{(0)}}{d\,b^2.d\,c}\right)+\left(\frac{d^3\,v^{(0)}}{d\,c^3}\right)\right\}-\&c.$$

These values are common to all spheroids. For any particular body, they will depend on the values of the functions $v^{(1)}$, $\left(\frac{d\,v^{(0)}}{d\,a}\right)$, $\left(\frac{d\,v^{(0)}}{d\,b}\right)$, *which express the attractions on*
[1558μ] *the points situated in the plane of* $b\,c$, *corresponding to* $a=0$, *in* [1558λ]. If we have two different bodies for which these quantities are respectively in a constant ratio, independently of a, b, c, the general expressions of $\left(\frac{d\,V}{d\,a}\right)$, $\left(\frac{d\,V}{d\,b}\right)$, $\left(\frac{d\,V}{d\,c}\right)$; or, in other words, the attractions of the two bodies on any point whatever, will be in the same ratio.

This is precisely the case with an ellipsoid, or spheroid of the second order. The calculation of the attraction of a spheroid of this kind upon any point of the equator, was found, without much difficulty, by the methods of integration formerly used by Le Gendre
[1558ν] and others; which gave, for this particular case, V or $v^{(0)}=M\,v$, $v^{(1)}=0$; v being a function of the excentricities of the ellipsoid, wholly independent of the polar semi-axis k. This value of V is a particular case of the general expression which was proved in [1399, 1412′]; and the equation $v^{(1)}=0$, is a necessary consequence of the symmetrical form of the ellipsoid, on each side of the equator, which renders the attraction $-\left(\frac{d\,V}{d\,a}\right)=-v^{(1)}$, [1558λ], in the direction of the axis a, equal to nothing for any attracted point, situated in
[1558ξ] the plane of the equator; this being a property common to all such symmetrical spheroids. Substituting these values in the general development of V [1558κ], we shall get, for the attraction of an ellipsoid, upon any point whose co-ordinates are a, b, c,

Value of V for an ellipsoid, given by LaGrange.

[1558ξ′]
$$V=M\,.\left\{\begin{array}{l} v-\frac{a^2}{1.2}\,.\left\{\left(\frac{d\,d\,v}{d\,b^2}\right)+\left(\frac{d\,d\,v}{d\,c^2}\right)\right\} \\ +\frac{a^4}{1.2.3.4}\,.\left\{\left(\frac{d^4v}{d\,b^4}\right)+2\,.\left(\frac{d^4\,v}{d\,b^2.d\,c^2}\right)+\left(\frac{d^4v}{d\,c^4}\right)\right\}-\&c.\end{array}\right\}.$$

For another ellipsoid, having the same principal sections and foci, but whose magnitude is

spheroid, [1412', 1412'''], perpendicular to the equator, and depends only [1558vi]

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different, we shall have the same value of $v^{(0)}$ [1558v], M will change into M', and V into V'; therefore we shall get,

$$V' = M' \cdot \left[v - \frac{a^2}{1.2} \cdot \left\{ \left(\frac{ddv}{db^2}\right) + \left(\frac{ddv}{dc^2}\right) \right\} + \&c. \right]; \qquad [1558\pi]$$

hence we obtain,

$$\frac{V}{V'} = \frac{M}{M'}. \qquad [1558\rho]$$

From this it follows generally, that two ellipsoids, whose principal axes have the same directions, and which have the same foci, attract an external point in proportion to the masses of the ellipsoids, as was before shown in [1412'''']. [1558σ]

The theorems we have given become more simple if the body be a spheroid of revolution, about either of the axes; as for example, the axis of a. In this case, if we put r for the distance of the attracted point from the axis of a, or $r = \sqrt{(b^2 + c^2)}$, it will be evident, that all the points, situated at the same distance from the axis, and at the same height above the plane of $b\,c$, will be equally attracted; since the spheroid is symmetrical relative to all these points. Then V will be a function of a, r; and its partial differentials, relative to b, considering r as a function of b, c, are [1558τ]

$$\left(\frac{dV}{db}\right) = \left(\frac{dV}{dr}\right) \cdot \left(\frac{dr}{db}\right), \qquad \left(\frac{ddV}{db^2}\right) = \left(\frac{ddV}{dr^2}\right) \cdot \left(\frac{dr}{db}\right)^2 + \left(\frac{dV}{dr}\right) \cdot \left(\frac{ddr}{db^2}\right). \qquad [1558\tau']$$

But from $r = \sqrt{(b^2 + c^2)}$, we get

$$\left(\frac{dr}{db}\right) = \frac{b}{(b^2+c^2)^{\frac{1}{2}}} = \frac{b}{r}, \qquad \left(\frac{ddr}{db^2}\right) = \frac{c^2}{(b^2+c^2)^{\frac{3}{2}}} = \frac{c^2}{r^3}; \qquad [1558\upsilon]$$

hence the preceding value of $\left(\frac{ddV}{db^2}\right)$ becomes,

$$\left(\frac{ddV}{db^2}\right) = \frac{b^2}{r^2} \cdot \left(\frac{ddV}{dr^2}\right) + \frac{c^2}{r^3} \cdot \left(\frac{dV}{dr}\right); \qquad [1558\varphi]$$

and in like manner, by changing b into c, and c into b, which does not alter r, we get,

$$\left(\frac{ddV}{dc^2}\right) = \frac{c^2}{r^2} \cdot \left(\frac{ddV}{dr^2}\right) + \frac{b^2}{r^3} \cdot \left(\frac{dV}{dr}\right). \qquad [1558\chi]$$

Substituting these in [1430], and reducing by putting $b^2 + c^2 = r^2$, we obtain,

$$0 = \frac{1}{r} \cdot \left(\frac{dV}{dr}\right) + \left(\frac{ddV}{dr^2}\right) + \left(\frac{ddV}{da^2}\right). \qquad [1558\psi]$$

This contains only the two variable quantities r and a; and it is evident that the preceding theorems, which exist with three dimensions a, b, c, in spheroids of a general form, have

upon the excentricities of the ellipsoid.* Therefore by multiplying the different terms of the value of $\frac{V}{M}$, relative to these two cases, reduced into a series, and arranged according to the powers of $\frac{1}{r}$, by the factors of which we have just spoken, [1555'', 1557'], we shall obtain the value
[1558vii] of $\frac{V}{M}$, relative to any attracted point whatever; the function which results will be independent of k, and will depend only upon the excentricities; this furnishes another demonstration of the theorem, which we have proved in [1412'', &c.]

If the attracted point be situated within the spheroid,† the attraction

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Biot's theorem for a spheroid of revolution. analogous ones with two dimensions a, r, in spheroids of revolution; so that *the attraction of these spheroids of revolution upon any point of space, will be generally determined, when we shall have found the same attractions, for all the points of any curve, described at pleasure, in*
[1558ω] *the plane of the meridian.*

* (1077) The calculations of the integrals [1558''', &c.] by the usual methods require very
[1559a] tedious operations; as may be seen, by referring to the *Mémoires de l'Académie des Sciences, de Paris*, 1788, where M. Le Gendre has given a detailed account of such integrals. This manner of considering the subject is now wholly superseded by that of Mr. Ivory, which we
[1559b] have already explained in [1428a, &c.]; and therefore it will not be necessary to enter into any further explanation, which would require much time, and would be of no real service; since we have already obtained two different solutions of the same question; the one given by La Place [1412'''', &c.], the other by Mr. Ivory, as in [1428σ].

† (1078) The formula [1559], for an internal attracted point, is deduced from [1447], by substituting $d\theta' \,.\, \sin.\theta' = -d\mu'$ [1492a]; and if we change the limits of the integral, as in [1492b], so as to have them from $\mu' = -1$ to $\mu' = 1$, we may put $d\theta' \,.\, \sin.\theta' = d\mu'$, and then [1447] becomes as in [1559]. Moreover,

[1559c]
$$\frac{dR}{R^{i-1}} = R^{1-i} dR = \frac{1}{2-i} \,.\, d \,.\, R^{2-i},$$

and if we suppose, as in [1531'], R to be a function of a, μ', ϖ', and dR to be taken on the supposition that μ', ϖ', are constant, we shall get $\frac{dR}{R^{i-1}} = \frac{1}{2-i} \cdot \left(\frac{d \,.\, R^{2-i}}{da}\right) . \, da$, and then [1559] will become as in [1560]. This last is of a similar form to the expression $U^{(i)}$ [1532], and becomes identical by writing $2-i$ for the exponent of R, instead of $i+3$; and in the same manner by which [1532] was changed into [1541], we may change

which it suffers, depends, as we have seen in [1444], upon the function $v^{(i)}$, [1558viii]

[1560] into [1563]. For [1533] will correspond with [1561], [1534] with [1562], and by making the abovementioned change of the factor $i+3$ into $2-i$, in [1541], it will become as in [1563].

In all the calculations of the author, in this section, if they be applied to the attraction upon a point, situated near the surface of the spheroid, or upon an internal point, it will be necessary [1559d] to examine into the convergency of the resulting series, for the reasons stated in [1447a, &c.] The result of this examination will generally be, to limit the applications of these calculations to bodies differing but little from a sphere, so that when R is expressed, as in [1535′], by $a.(1+\alpha y')$, the quantity α may be so small, that V can be expressed in a converging series, arranged according to the powers of α, as in the formulas we shall hereafter give [1560α, β]; which are sufficiently accurate for all practical purposes, in the investigation of the figures of the planetary bodies.

(1079) In the values of V [1467, 1496], the author has retained only the first power of α, supposing that to be sufficiently accurate for all practical purposes; but he has shown, in [1820″, &c.], how to carry on the approximation, so as to embrace the terms depending on the square and higher powers of α. This has also been done by Mr. Poisson, in a much [1560a] more symmetrical manner, by developing the formula [1447n]. We shall here give a full account of his method, as it is found in the *Connoisance des Tems* for the year 1829.

Poisson's method of noticing the higher powers of α, in the development of V.

Supposing, as in [1447d] that u is the radius of the surface of the spheroid, corresponding to the angles θ', ϖ', we shall have, as in [1461′]

$$u=a.(1+\alpha y'), \qquad [1560b]$$

α being a very small positive constant quantity; a the radius of a sphere differing but little from the spheroid; y' a function of a, θ', ψ', which is positive in those parts where $u>a$, [1560c] but negative where $u<a$; the spheroid being homogeneous, and its density ρ equal to unity.

If we suppose the distance r, of the attracted point from the origin, to exceed u, the integrals $\int_{\prime}$ will disappear from [1447n], and the terms depending on $\int'$ will become complete integrals $\int$; hence $V=\Sigma_0^{\infty}\left\{\frac{1}{r^{i+1}}\cdot\int\left(\int_0^u\rho\,.\,R^{i+2}\,dR\right).\,Q^{(i)}.\,dw\right\}$. We [1560$d$] shall divide the integral relative to r, into two parts; the one extending from $r=0$ to $r=a$, and the other from $r=a$ to $r=u$. The first of these integrals is a constant quantity; for $\int\rho\,.\,R^{i+2}\,dR=\frac{\rho\,.\,R^{i+3}}{i+3}$, is nothing when $R=0$; and when $R=a$, it becomes $\frac{\rho\,.\,a^{i+3}}{i+3}$, which is a constant quantity of the form $Y^{(0)}=\frac{\rho\,.\,a^{i+3}}{i+3}$, so that [1560$e$] this part of V is equal to $\Sigma_0^{\infty}\left\{\frac{1}{r^{i+1}}.\int Y'^{(0)}.Q^{(i)}.dw\right\}$. Now from the formula [1476a], it appears that all the integrals, of the form $\int Y'^{(0)}.Q^{(i)}.dw$, must vanish, except that corresponding

and we have, by [1447],

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to $i=0$; hence this part becomes $\frac{1}{r}\cdot\int Y'^{(0)}\cdot Q^{(0)}\cdot dw=\frac{1}{r}\cdot 4\pi\cdot Y^{(0)}$, [1533n];

and as the preceding value of $Y^{(0)}$, or $Y'^{(0)}$, is then equal to $\frac{\rho . a^3}{3}$, we have, for this part

[1560f] of the integral, $\frac{1}{r}\cdot 4\pi\cdot\frac{\rho . a^3}{3}=\frac{4\pi . \rho\, a^3}{3r}$. Therefore the complete value of V, corresponding

Attracted external point.

to this first case, is,

[1560g] $$V=\frac{4\pi\rho . a^3}{3r}+\rho\cdot\Sigma_0^{\infty}\left\{\frac{1}{r^{i+1}}\cdot\int\left(\int_a^u R^{i+2}\,dR\right)\cdot Q^{(i)}\cdot dw\right\}.$$

On the contrary, if r be less than the least value of u, the integrals $\int'$ will vanish, and $\int_{\prime}$ will change into $\int$. By this means the first term of [1447n] will vanish. The second term of

[1560h] [1447n] will be $\Sigma_0^{\infty}\left\{\frac{1}{r^{i+1}}\cdot\int\left(\int_0^r \rho\cdot R^{i+2}\,dR\right).Q^{(i)}\cdot dw\right\}$, and this can be computed

as in [1560e], changing the limit a into r, by which means the formula [1560f] will become

[1560i] $\frac{4\pi\rho . r^3}{3r}=\frac{4\pi\rho . r^2}{3}$, which represents the value of the second term of [1447n], in the case under consideration. The last term of [1447n] is $\Sigma_0^{\infty}\left\{r^i\cdot\int\left(\int_r^u\frac{\rho . dR}{R^{i-1}}.Q^{(i)}\cdot dw\right\}\right.$, in

which the integral, relative to R, may be divided into two parts; the one from $R=r$

[1560k] to $R=a$, the other from $R=a$ to $R=u$; this last part produces the term under the sign of integration, in the following formula [1560n]; the former part is

[1560l] $\Sigma_0^{\infty}\left\{r^i\cdot\left(\int_r^a\frac{\rho . dR}{R^{i-1}}\right)\cdot Q^{(i)}\cdot dw\right\}$. Now ρ being constant,

$$\int\frac{\rho . dR}{R^{i-1}}=\rho\cdot\int R^{1-i}dR=\frac{\rho}{2-i}\cdot(R^{2-i}-r^{2-i}),$$

which vanishes at the first limit, where $R=r$; and when $R=a$, it becomes

$\int_r^a\frac{\rho . dR}{r^{i-1}}=\frac{\rho}{2-i}\cdot(a^{2-i}-r^{2-i})$; this is independent of u, and may be considered, as in

[1560e], to be of the form $Y'^{(0)}$. Putting therefore for a moment $Y'^{(0)}=\frac{\rho}{2-i}\cdot(a^{2-i}-r^{2-i})$,

the expression [1560l] will become $\Sigma_0^{\infty}\, r^i\cdot\int Y'^{(0)}.Q^{(i)}\cdot dw$. Hence we may conclude, as in [1560e—f], that we need only notice the value $i=0$, which gives

[1560m] $$\int Y'^{(0)}\cdot Q^{(i)}\cdot dw=4\pi\cdot Y^{(0)}=4\pi\cdot\tfrac{1}{2}\rho\cdot(a^2-r^2)=2\pi\rho\cdot(a^2-r^2).$$

Connecting this part of the third term, with the second term, $\frac{4}{3}\pi\rho\cdot r^2$ [1560i], the sum becomes $2\pi\rho\cdot a^2-\frac{2}{3}\pi\rho\cdot r^2$. Adding this to the term under the sign of integration,

Attracted internal point.

mentioned in [1560k], we get, for the complete value of V, corresponding to an internal attracted point, the following expression,

[1560n] $$V=2\pi\rho\cdot a^2-\tfrac{2}{3}\pi\rho\cdot r^2+\rho\cdot\Sigma_0^{\infty}\left\{r^i\cdot\int\left(\int_a^u\frac{dR}{R^{i-1}}\right)\cdot Q^{(i)}\cdot dw\right\}.$$

To reduce either of the expressions [1560g, n] into a series, proceeding according to the powers of a, we may observe, that $\int R^{i+2}dR=\frac{1}{i+3}\cdot\{R^{i+3}-a^{i+3}\}$ vanishes,

$$v^{(i)} = \int \frac{\rho \,.\, dR \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(i)}}{R^{i-1}} \,; \qquad [1559]$$

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when $R = a$; and at the other limit, where $R = u = a \,.\, (1 + \alpha y')$, it becomes,

$$\int_a^u R^{i+2} dR = \frac{1}{i+3} \,.\, \{a^{i+3} \,.\, (1 + \alpha y')^{i+3} - a^{i+3}\} = \frac{a^{i+3}}{i+3} \,.\, \{(1 + \alpha y')^{i+3} - 1\}.$$

$$= \frac{a^{i+3}}{i+3} \,.\, \left\{ (i+3) \,.\, \alpha y' + \frac{(i+3) \,.\, (i+2)}{1 \,.\, 2} \,.\, \alpha^2 y'^2 + \&c. \right\}$$

$$= a^{i+3} \,.\, \left\{ \alpha y' + \frac{i+2}{2} \,.\, \alpha^2 y'^2 + \frac{(i+2) \,.\, (i+1)}{2 \,.\, 3} \,.\, \alpha^3 y'^3 + \&c. \right\}. \qquad [1560o]$$

Changing, in this last expression, i into $-i-1$, we get,

$$\int_a^u \frac{dR}{R^{i-1}} = a^{2-i} \,.\, \left\{ \alpha y' - \frac{(i-1)}{2} \,.\, \alpha^2 y'^2 + \frac{(i-1) \,.\, i}{2 \,.\, 3} \,.\, \alpha^3 y'^3 - \&c. \right\}. \qquad [1560p]$$

We shall put y for the value of y', when θ', ϖ', are changed into θ, ϖ, respectively; or in other words, y is the value of y', corresponding to the point where the radius r intersects the surface of the spheroid. We shall develop the powers of y in series of the form [1464], putting [1560q]

$$y = Y^{(0)} + Y^{(1)} + Y^{(2)} + Y^{(3)} + \&c. \qquad = \Sigma_0^\infty Y^{(i)} \,; \qquad [1560r]$$

$$y^2 = Y_1^{(0)} + Y_1^{(1)} + Y_1^{(2)} + Y_1^{(3)} + \&c. \qquad = \Sigma_0^\infty Y_1^{(i)} \,; \qquad [1560s]$$

$$y^{n+1} = Y_n^{(0)} + Y_n^{(1)} + Y_n^{(2)} + Y_n^{(3)} + \ldots\ldots Y_n^{(i)} + \&c. = \Sigma_0^\infty Y_n^{(i)} \,; \qquad [1560t]$$

and in like manner, with accented letters, we shall have,

$$y' = Y'^{(0)} + Y'^{(1)} + Y'^{(2)} + Y'^{(3)} + \&c. \,; \qquad [1560u]$$

$$y'^{n+1} = Y_n'^{(0)} + Y_n'^{(1)} + Y_n'^{(2)} + Y_n'^{(3)} + \&c. \ldots\ldots + Y_n'^{(i)} + \&c. \qquad [1560v]$$

If we substitute, in [1560o], the expressions of y', y'^2, y'^3, &c., deduced from [1560v], and then insert the result in the integral [1560g], $\int \left(\int_a^u R^{i+2} \, dR \right) . Q^{(i)} . dw$, it will be composed of terms of the form $\int Y_n'^{(i')} \,.\, Q^{(i)} \,.\, dw$. All the terms of this integral, in which i' differs from i, will vanish, by means of the formula [1476a], and the whole will be reduced to terms of the form $\int Y_n'^{(i)} \,.\, Q^{(i)} \,.\, dw = \frac{4\pi}{2i+1} \,.\, Y^{(i)}$ [1533n]; hence we finally obtain, [1560w]

$$\int \left(\int_a^u R^{i+2} dR \right) . Q^{(i)} . dw = \frac{4\pi \,.\, a^{i+3}}{2i+1} \,.\, \left\{ \alpha Y^{(i)} + \frac{i+2}{2} \,.\, \alpha^2 Y_1^{(i)} + \frac{(i+2) \,.\, (i+1)}{2 \,.\, 3} \,.\, \alpha^3 Y_2^{(i)} + \&c. \right\}. \qquad [1560x]$$

Proceeding in the same manner with the formulas [1560p, n], we get,

$$\int \left(\int_a^u \frac{dR}{R^{i-1}} \right) . Q^{(i)} \,.\, dw = \frac{4\pi \,.\, a^{2-i}}{2i+1} \,.\, \left\{ \alpha Y^{(i)} - \frac{(i-1)}{2} \,.\, \alpha^2 Y_1^{(i)} + \frac{(i-1) \,.\, i}{2 \,.\, 3} \,.\, \alpha^3 Y_2^{(i)} - \&c. \right\}. \qquad [1560y]$$

Hence the values of V [1560g, n], will become,

Attracted external point.

$$V = \frac{4\pi\rho \,.\, a^3}{3r} + \frac{4\pi\rho \,.\, a^3}{r} \,.\, \left\{ \begin{array}{l} \alpha \,.\, \Sigma_0^\infty \frac{1}{2i+1} \,.\, \frac{a^i}{r^i} \,.\, Y^{(i)} + \frac{\alpha^2}{2} \,.\, \Sigma_0^\infty \frac{i+2}{2i+1} \,.\, \frac{a^i}{r^i} \,.\, Y_1^{(i)} \\ \qquad + \frac{\alpha^3}{2.3} \,.\, \Sigma_0^\infty \frac{(i+2) . (i+1)}{2i+1} \,.\, \frac{a^i}{r^i} \,.\, Y_2^{(i)} + \&c. \end{array} \right\}; \qquad [1560\alpha]$$

Attracted internal point.

$$V = 2\pi\rho \,.\, a^2 - \frac{2\pi}{3} \,.\, \rho \,.\, r^2 + 4\pi\rho \,.\, a^2 \,.\, \left\{ \begin{array}{l} \alpha \,.\, \Sigma_0^\infty \frac{1}{2i+1} \,.\, \frac{r^i}{a^i} \,.\, Y^{(i)} - \frac{\alpha^2}{2} \,.\, \Sigma_0^\infty \frac{i-1}{2i+1} \,.\, \frac{r^i}{a^i} \,.\, Y_1^{(i)} \\ \qquad + \frac{\alpha^3}{2.3} \,.\, \Sigma_0^\infty \frac{(i-1) \,.\, i}{2i+1} \,.\, \frac{r^i}{a^i} \,.\, Y_2^{(i)} - \&c. \end{array} \right\}. \qquad [1560\beta]$$

which may be put under the form,

The sign Σ of finite integrals includes all the integral values of i, from $i=0$ to $i=\infty$; but it may be observed, that if the development of y contain only a finite number of terms, as
[1560γ] for example, $y=Y^{(0)}+Y^{(1)}+Y^{(2)}$, all the terms $Y^{(3)}$, $Y^{(4)}$, &c., will be nothing. The square and other powers of y, will then be limited to a finite number of terms, which
[1560δ] will however increase, according to the powers of y or α; so that the value of y^2, in the case just mentioned, might include terms as far as $Y_1^{(4)}$; y^3 as far as $Y_2^{(6)}$, &c.

The formulas [1560g, α], correspond to any external attracted point, and the formulas
[1560ε] [1560n, β] to an internal attracted point, without any exception. For though it might seem that some modification of these formulas would be necessary, when the attracted point is situated near the surface of the spheroid, on account of the changes in the limits of the integrals, relative to f', $f_{,}$ yet this is not the case, as Mr. Poisson has proved in the following manner, by means of the formula [1447o].

The spheroid being homogeneous, the first series contained in the formula [1447o], will be reduced, as in [1560d—f], to its first term, corresponding to $i=0$, and as the last
[1560ζ] limit of this integral is r, this term will become $\frac{4\pi\rho.r^3}{3r}$, or $\frac{4}{3}\pi\rho.r^2$, being the same as the first term of the following expression [1560ι]. Now if we represent by Q any quantity at pleasure, we shall have identically $f'Q.dw+f_{,}Q.dw=fQ.dw$; hence $f'Q.dw=fQ.dw-f_{,}Q.dw$. If we put Q equal to the factor of dw, in the
[1560η] second term of [1447o], the part depending on $fQ.dw$, will produce the second term of [1560ι], and the part depending on $-f_{,}Q.dw$ will be

[1560η′]
$$-\Sigma_0^{\infty}\left\{\frac{1}{r^{i+1}}.f_{,}\left(\int_r^u \rho.R^{i+2}dR\right).Q^{(i)}.dw\right\};$$

connecting this with the last term of [1447o], and using for brevity the following value of U,

[1560θ]
$$U=\frac{1}{r^{i+1}}.\int_r^u R^{i+2}dR-r^i.\int_r^u\frac{dR}{R^{i-1}},$$

the sum will become, $-\rho.\Sigma_0^{\infty}f_{,}U.Q^{(i)}.dw$, which is the last term of [1560ι]. In like manner, if we substitute $f_{,}Q.dw=fQ.dw-f'Q.dw$, in the last term of [1447o], and use the preceding value of U, it will become of the form [1560κ]. Hence we have

[1560ι]
$$V=\tfrac{4}{3}\pi\rho.r^2+\rho.\Sigma_0^{\infty}\left\{\frac{1}{r^{i+1}}.f\left(\int_r^u R^{i+2}dR\right).Q^{(i)}.dw\right\}-\rho.\Sigma_0^{\infty}f_{,}U.Q^{(i)}.dw;$$

Points near the surface.

[1560κ]
$$V=\tfrac{4}{3}\pi\rho.r^2+\rho.\Sigma_0^{\infty}\left\{r^i.f\left(\int_r^u\frac{dR}{R^{i-1}}\right).Q^{(i)}dw\right\}\qquad+\rho.\Sigma_0^{\infty}f'U.Q^{(i)}.dw.$$

It will suffice to take into consideration the first of these two formulas, as the same reasoning may easily be applied to the second. We shall now put,

[1560λ]
$$u=r-z',$$

z' being a function of θ' and ϖ', which we shall suppose to be of the same order as the factor α, in all those parts of the radius r, which we shall take into consideration. The integral $fR^{i+2}dR=\frac{1}{i+3}.(R^{i+3}-r^{i+3})$ vanishes when $R=r$; and when $R=u=r-z'$.

$$v^{(i)}=\frac{1}{2-i}\cdot\int\rho\cdot\left(\frac{d\,.\,R^{2-i}}{d\,a}\right).\,d\,a\,.\,d\,\mu'\,.\,d\,\varpi'\,.\,Q^{(i)}.$$ [1560]

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it becomes, $\int_r^u R^{i+2}dR=\frac{1}{i+3}\cdot\{(r-z')^{i+3}-r^{i+3}\}=-r^{i+2}.z'+\frac{i+2}{2}\cdot r^{i+1}.z'^2-\&c.$ [1560μ]

Hence $\frac{1}{r^{i+1}}\cdot\int_r^u R^{i+2}dR=-r\,.\,z'+\frac{i+2}{2}\cdot z'^2-\frac{(i+2)\,.\,(i+1)}{2\,.\,3\,.\,r}\cdot z'^3+\&c.$ Changing [1560ν]
in this, i into $-i-1$, we get,

$$r^i\cdot\int_r^u\frac{dR}{R^{i-1}}=-r\,.\,z'-\frac{(i-1)}{2}\cdot z'^2-\frac{(i-1)\,.\,i}{2.3.r}\cdot z'^3-\&c.$$ [1560ξ]

Subtracting this from the preceding expression, we obtain the value of U [1560θ],

$$U=\frac{2i+1}{2}\cdot z'^2-\frac{(2i+1)}{3r}\cdot z'^3+\&c.$$ [1560ξ']

The integrals relative to the characteristic $\int_{\prime}$ include only those values of z in which $r<u$ [1447e]. If we put ζ' for a discontinuous function of θ', ϖ', so that we may have $\zeta'=z'$ if $r<u$, and $\zeta'=0$ if $r>u$, we may change z' into ζ', and $\int_{\prime}$ into the complete [1560π]
integral $\int$. Therefore multiplying [1560ξ'], by $Q^{(i)}\,.\,dw$, and integrating, we get, by neglecting terms of the order z'^4, or α^4,

$$\int_{\prime}U\,.\,Q^{(i)}\,.\,d\,w=\frac{2i+1}{2}\cdot\int\zeta'^2\,.\,Q^{(i)}\,.\,dw-\frac{(2i+1)}{3r}\cdot\int\zeta'^3\,.\,Q^{(i)}\,.\,d\,w.$$ [1560ρ]

If ζ represent the value of ζ' when $\theta'=\theta$ and $\varpi'=\varpi$, this quantity ζ will correspond to the point where the radius r, produced if it be necessary, intersects the surface of the spheroid. It will be nothing if the attracted point be external, and equal to its distance from [1560σ]
the surface, in the direction of this radius, if the point be internal. Supposing now ζ^2, ζ^3, to be developed in series of the form [1532a], namely,

$$\zeta^2=X^{(0)}+X^{(1)}+X^{(2)}\,.....+X^{(i)}+\&c.=\Sigma_0^\infty X^{(i)}\,;$$ [1560τ]

$$\zeta^3=z^{(0)}+z^{(1)}+z^{(2)}\,.....+z^{(i)}+\&c.=\Sigma_0^\infty z^{(i)}\,;$$ [1560υ]

in which $X^{(i)}$, $z^{(i)}$, &c., are of the same nature as $Y^{(i)}$, satisfying an equation of the form [1464c]; we shall have in like manner, $\zeta'^2=\Sigma_0^\infty X'^{(i)}$, $\zeta'^3=\Sigma_0^\infty z'^{(i)}$. Substituting these [1560φ]
in [1560ρ], we may, by means of the formula [1476a], neglect all the terms of the form $X'^{(i')}$, $z'^{(i')}$, in which i' differs from i, and we shall get successively, by using [1533n, 1560τ, υ],

$$\begin{aligned}\int_{\prime}U.Q^{(i)}.dw&=\Sigma_0^\infty\left\{\frac{2i+1}{2}\cdot\int X'^{(i)}\,.\,Q^{(i)}\,.\,dw-\frac{(2i+1)}{3r}\cdot\int z'^{(i)}\,.\,Q^{(i)}\,.\,dw\right\}\\&=\Sigma_0^\infty\left\{\frac{2i+1}{2}\cdot\frac{4\pi}{2i+1}\cdot X^{(i)}-\frac{(2i+1)}{3r}\cdot\frac{4\pi}{2i+1}\cdot z^{(i)}\right\}\\&=\Sigma_0^\infty\left\{2\pi\,.\,X^{(i)}-\frac{4\pi}{3r}\cdot z^{(i)}\right\}=2\pi\,.\,\Sigma_0^\infty X^{(i)}-\frac{4\pi}{3r}\cdot\Sigma_0^\infty z^{(i)}\\&=2\pi\,.\,\zeta^2-\frac{4\pi}{3r}\cdot\zeta^3.\end{aligned}$$ [1560χ]

This last expression is nothing, by the nature of the function ζ [1560π], whenever the attracted point is without the spheroid. Therefore, for all *external points*, the equation [1560ι] will be reduced to

$$V=\tfrac{4}{3}\pi\rho\,.\,r^2+\rho\,.\,\Sigma_0^\infty\left\{\frac{1}{r^{i+1}}\cdot\int\left(\int_r^u R^{i+2}dR\right).\,Q^{(i)}.\,dw\right\}.$$ [1560ψ]

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We shall suppose R^{2-i} to be developed in a series of the following form,

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We may reduce this to the form [1560g], by separating the integral relative to R into two parts, and changing the limits r, u, into a, u, and a, r, in the following manner,

[1560ω] $$\int_r^u R^{i+2}\,dR=\int_a^u R^{i+2}dR-\int_a^r R^{i+2}\,dR.$$

Substituting this in [1560ψ], the first term of the second member will produce the integral given in [1560g], and the other terms of [1560ψ], will be represented by

[1561a] $$\frac{4\pi}{3}\cdot\rho\cdot r^2-\rho\cdot\Sigma_0^\infty\left\{\frac{1}{r^{i+1}}\cdot\int\left(\int_a^r R^{i+2}\,dR\right)\cdot Q^{(i)}\cdot dw\right\}.$$

But $\int R^{i+2}dR=\frac{1}{i+3}\cdot(R^{i+3}-a^{i+3})$ vanishes at the first limit $R=a$; and at the second limit $R=r$, it becomes $\int_a^r R^{i+2}\,dR=\frac{1}{i+3}\cdot(r^{i+3}-a^{i+3})$. This being a constant quantity, is of the form $Y'^{(0)}$, and we shall find, by proceeding as in [1560d, &c.], that the only value of i necessary to be retained in [1561a], is $i=0$. Hence we shall get,

$$\rho\cdot\Sigma_0^\infty\frac{1}{r^{i+1}}\cdot\int\left(\int_a^r R^{i+2}dR\right).Q^{(i)}.dw=\frac{\rho}{r}\cdot\int Y'^{(0)}.\,Q^{(0)}\,.\,dw=\frac{4\pi\rho}{3r}.Y^{(0)}=\frac{4\pi\rho}{3r}.(r^3-a^3)$$

[1561a'] $$=\frac{4\pi\rho.r^2}{3}-\frac{4\pi\rho.a^3}{3r}.$$

Substituting this in [1561a], it becomes $\frac{4\pi\rho.r^2}{3}-\frac{4\pi\rho.r^2}{3}+\frac{4\pi\rho.a^3}{3r}=\frac{4\pi.\rho\,a^3}{3r}$, which is the same as the part independent of the sign of integration in [1560g]. Therefore the

[1561b] expression [1560ι], corresponding to an *external attracted point*, is in all cases reduced to the form [1560g], or to the equivalent expression [1560α].

Proceeding in a similar manner, we may prove that $\Sigma_0^\infty\int' U.\,Q^{(i)}.\,dw$, in the formula [1560ϰ], is nothing, whenever the attracted point is situated within the spheroid. Therefore we shall have, for all *internal* points,

[1561c] $$V=\tfrac{4}{3}\pi\rho.r^2+\rho.\Sigma_0^\infty\left\{r^i.\int\left(\int_r^u\frac{dR}{R^{i-1}}\right).\,Q^{(i)}\,dw\right\}.$$

Separating this integral, relative to R, into two parts, between the limits r, a, and a, u, we have

[1561d] $$\int_r^u\frac{dR}{R^{i-1}}=\int_r^a\frac{dR}{R^{i-1}}+\int_a^u\frac{dR}{R^{i-1}}.$$

Substituting this in [1561c], we shall find that the last integral of the second member produces the integral given in [1560n], and the other terms of the function [1561c] become,

[1561e] $$\tfrac{4}{3}\pi\rho.r^2+\rho.\Sigma_0^\infty\left\{r^i.\int\left(\int_r^a\frac{dR}{r^{i-1}}\right).\,Q^{(i)}.\,dw\right\}.$$

Then we shall find, as in the preceding case, that the only term to be noticed is $i=0$, and as $\int_r^a\frac{dR}{R^{i-1}}=\int_r^a R\,dR=\tfrac{1}{2}.(a^2-r^2)=Y'^{(0)}$, $\int Y'^{(0)}.\,Q^{(i)}.\,dw=4\pi.\,Y^{(0)}$,

[1561f] [1533n], the expression [1561c] will become $\tfrac{4}{3}\pi\rho.r^2+\rho.4\pi.\tfrac{1}{2}.(a^2-r^2)=2\pi\rho.a^2-\tfrac{2}{3}\pi\rho.r^2$, which is the same as the part independent of the sign of integration, in [1560n]. Therefore the expression [1560ϰ], corresponding to an *internal attracted point*, is, in all cases, reduced

$$z'^{(0)}+z'^{(1)}+z'^{(2)}+z'^{(3)}+\&c.\,;$$ [1561]

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to the form [1560n], or to the equivalent expression [1560β]; which is what, in [1560ξ], was proposed to be proved.

If we deduce, from [1560α, β], the values of $-\left(\frac{dV}{dr}\right)=R'$, representing, as in [1811l], the attraction in the direction of the radius, we shall get, for an external attracted point, the formula [1561h]; and for an internal attracted point, the formula [1561i]; [1561g]

External point.

$$R'=-\left(\frac{dV}{dr}\right)=\frac{4\pi\rho.a^3}{3r^2}+\frac{4\pi\rho.a^3}{r^2}.\left\{\alpha.\Sigma_0^\infty\frac{i+1}{2i+1}.\frac{a^i}{r^i}.Y^{(i)}+\frac{\alpha^2}{2}.\Sigma_0^\infty\frac{(i+2).(i+1)}{2i+1}.\frac{a^i}{r^i}.Y_{,}^{(i)}+\&c.\right\};$$ [1561h]

Internal point.

$$R'=-\left(\frac{dV}{dr}\right)=\frac{4\pi\rho.r}{3}-\frac{4\pi\rho.a^2}{r}.\left\{\alpha.\Sigma_0^\infty\frac{i}{2i+1}.\frac{r^i}{a^i}.Y^{(i)}-\frac{\alpha^2}{2}.\Sigma_0^\infty\frac{i.(i-1)}{2i+1}.\frac{r^i}{a^i}.Y_{,}^{(i)}+\&c.\right\};$$ [1561i]

and in like manner, we might find the form in any other direction, as in [1811l].

If the attracted point be situated exactly on the surface of the spheroid, the two values of V [1560α, β], as well as those of $\left(\frac{dV}{dr}\right)$ [1561h, i], will coincide; as we may find, by putting $r=a.(1+\alpha y)$, and then developing these expressions according to the powers of α. As this development will furnish another demonstration of the theorem [1458], we shall go through the calculation, neglecting terms of the order α^3. [1561k]

The value of r [1561k] gives $\frac{a}{r}=1-\alpha y+\alpha^2y^2-\&c.$, $\frac{a^{i+1}}{r^{i+1}}=1-(i+1).\alpha y+\&c.$ [1561l]

Substituting these in the expression of V [1560α], for an *external* point, it becomes, by neglecting α^3 and using Σ for Σ_0^∞,

$$V=\frac{4\pi}{3}.\rho.a^2.(1-\alpha y+\alpha^2y^2)+4\pi\rho.a^2.\left\{\alpha.\Sigma\frac{1}{2i+1}.[1-(i+1).\alpha y].Y^{(i)}+\frac{\alpha^2}{2}.\Sigma\frac{i+2}{2i+1}.Y_{,}^{(i)}\right\}$$

$$=\frac{4\pi}{3}\rho.a^2-\frac{4\pi}{3}.\rho.a^2.\alpha.\left\{y-3.\Sigma\frac{1}{2i+1}.Y^{(i)}\right\}$$ [1561m]

$$-\frac{4\pi}{3}.\rho.a^2.\frac{\alpha^2}{2}.\left\{-2y^2+6y.\Sigma\frac{i+1}{2i+1}.Y^{(i)}-3.\Sigma\frac{i+2}{2i+1}.Y_{,}^{(i)}\right\};$$

in which the factors of the order α, α^2, may be reduced by means of the formulas [1560r, s], which give,

$$y-3.\Sigma\frac{1}{2i+1}.Y^{(i)}=\Sigma Y^{(i)}-3.\Sigma\frac{1}{2i+1}.Y^{(i)}=\Sigma\left(1-\frac{3}{2i+1}\right).Y^{(i)}=2.\Sigma\frac{i-1}{2i+1}.Y^{(i)};$$ [1561n]

also

$$6y.\Sigma\frac{i+1}{2i+1}.Y^{(i)}=6y.\Sigma\left(\tfrac{1}{2}+\frac{\frac{1}{2}}{2i+1}\right).Y^{(i)}=3y.\Sigma Y^{(i)}+y.\Sigma\frac{3}{2i+1}.Y^{(i)}=3y^2+y.\Sigma\frac{3}{2i+1}.Y^{(i)};$$ [1561o]

$$-3.\Sigma\frac{i+2}{2i+1}.Y_{,}^{(i)}=-\Sigma\frac{3i+6}{2i+1}.Y_{,}^{(i)}=-\Sigma\left(1+\frac{i+5}{2i+1}\right).Y_{,}^{(i)}=-\Sigma Y_{,}^{(i)}-\Sigma\frac{i+5}{2i+1}.Y_{,}^{(i)}$$

$$=-y^2-\Sigma\frac{i+5}{2i+1}.Y_{,}^{(i)}.$$ [1561o']

The expressions [1561o, o'], being added to $-2y^2$, we get,

$$-2y^2+6y.\Sigma\frac{i+1}{2i+1}.Y^{(i)}-3.\Sigma\frac{i+2}{2i+1}.Y_{,}^{(i)}=y.\Sigma\frac{3}{2i+1}.Y^{(i)}-\Sigma\frac{i+5}{2i+1}.Y_{,}^{(i)}.$$ [1561p]

$z'^{(i)}$ satisfying this equation of partial differentials,

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Attracted point at the surface.

Substituting this and $y = \Sigma Y^{(i)}$ in [1561m], using also the expression [1561n], we obtain,

[1561q] $$V = \frac{4\pi}{3}.\rho.a^2 - \frac{4\pi}{3}.\rho.a^2.2\alpha.\Sigma\frac{i-1}{2i+1}.Y^{(i)} - \frac{4\pi}{3}.\rho.a^2.\frac{\alpha^2}{2}.\left\{\Sigma Y^{(i)}.\Sigma\frac{3}{2i+1}.Y^{(i)} - \Sigma\frac{i+5}{2i+1}.Y_{,}^{(i)}\right\}.$$

Proceeding in like manner with the expression of V for an *internal* point, [1560β], we get,

$$V = 2\pi\rho.a^2 - \frac{2\pi}{3}\rho a^2.(1+2\alpha y+\alpha^2 y^2) + 4\pi\rho.a^2.\left\{\alpha.\Sigma\frac{1}{2i+1}.(1+i.\alpha y).Y^{(i)} - \frac{\alpha^2}{2}.\Sigma\frac{i-1}{2i+1}.Y_{,}^{(i)}\right\}$$

[1561r] $$= \frac{4\pi}{3}.\rho.a^2 - \frac{4\pi}{3}.\rho.a^2.\alpha.\left\{y - 3.\Sigma\frac{1}{2i+1}.Y^{(i)}\right\}$$
$$-\frac{4\pi}{3}.\rho.a^2.\frac{\alpha^2}{2}.\left\{y^2 - 6y.\Sigma\frac{i}{2i+1}.Y^{(i)} + 3.\Sigma\frac{i-1}{2i+1}.Y_{,}^{(i)}\right\}.$$

But we have

$$-6y.\Sigma\frac{i}{2i+1}.Y^{(i)} = -6y.\Sigma\left(\tfrac{1}{2} - \frac{\tfrac{1}{2}}{2i+1}\right).Y^{(i)} = -3y.\Sigma Y^{(i)} + y.\Sigma\frac{3}{2i+1}.Y^{(i)}$$
$$= -3y^2 + y.\Sigma\frac{3}{2i+1}.Y^{(i)};$$

[1561s]

$$3.\Sigma\frac{i-1}{2i+1}.Y_{,}^{(i)} = \Sigma\frac{3i-3}{2i+1}.Y_{,}^{(i)} = \Sigma\left(2 - \frac{(i+5)}{2i+1}\right).Y_{,}^{(i)} = 2.\Sigma Y_{,}^{(i)} - \Sigma\frac{i+5}{2i+1}.Y_{,}^{(i)}$$
$$= 2y^2 - \Sigma\frac{i+5}{2i+1}.Y_{,}^{(i)};$$

adding these two last expressions to y^2, we get, by observing that $y = \Sigma Y^{(i)}$,

[1561t] $$y^2 - 6y.\Sigma\frac{i}{2i+1}.Y^{(i)} + 3.\Sigma\frac{i-1}{2i+1}.Y_{,}^{(i)} = \Sigma Y^{(i)}.\Sigma\frac{3}{2i+1}.Y^{(i)} - \Sigma\frac{i+5}{2i+1}.Y_{,}^{(i)}.$$

Substituting this and [1561n] in [1561r], it becomes the same as in [1561q], so that both the values of V, developed in this manner, become identical.

Substituting also $r = a.(1+\alpha y)$, in [1561i], we get, by neglecting α^3, and putting $y = \Sigma Y^{(i)}$,

$$-\left(\frac{dV}{dr}\right) = \frac{4\pi}{3}.\rho.a.(1+\alpha y) - 4\pi\rho.a.\left\{\begin{array}{l}\alpha.\Sigma\frac{i}{2i+1}.[1+(i-1).\alpha y].Y^{(i)} \\ -\frac{\alpha^2}{2}.\Sigma\frac{i.(i-1)}{2i+1}.Y_{,}^{(i)}\end{array}\right\}$$

At the surface of the spheroid.

$$= \frac{4\pi}{3}.\rho.a - \frac{4\pi}{3}.\rho.a.\alpha.\left\{-y + \Sigma\frac{3i}{2i+1}.Y^{(i)}\right\}$$
$$+\frac{4\pi}{3}.\rho.a.\frac{\alpha^2}{2}.\left\{-y.\Sigma\frac{6i.(i-1)}{2i+1}.Y^{(i)} + \Sigma\frac{3i.(i-1)}{2i+1}.Y_{,}^{(i)}\right\}$$
$$= \frac{4\pi}{3}.\rho.a - \frac{4\pi}{3}.\rho.a.\alpha.\Sigma\frac{i-1}{2i+1}.Y^{(i)}$$

[1561u] $$+\frac{4\pi}{3}.\rho.a.\frac{\alpha^2}{2}.\left\{-\Sigma Y^{(i)}.\Sigma\frac{6i.(i-1)}{2i+1}.Y^{(i)} + \Sigma\frac{3i.(i-1)}{2i+1}.Y_{,}^{(i)}\right\}.$$

$$0 = \left\{ \frac{d \cdot \left\{ (1-\mu'^2) \cdot \left(\frac{d\, z'^{(i)}}{d\mu'} \right) \right\}}{d\mu'} \right\} + \frac{\left(\frac{d\,d\, z'^{(i)}}{d\varpi'^2} \right)}{1-\mu'^2} + i \cdot (i+1) \cdot z'^{(i)}. \qquad [1562]$$

The other value of $-\left(\frac{dV}{dr}\right)$ [1561h] will produce the same result. For by substituting $r = a \cdot (1 + \alpha y)$, and neglecting α^3, we get,

$$-\left(\frac{dV}{dr}\right) = \frac{4\pi}{3} \cdot \rho . a . (1 - 2\alpha y + 3\alpha^2 y^2) + 4\pi\rho . a . \left\{ \begin{array}{l} \alpha \cdot \Sigma \frac{i+1}{2i+1} \cdot [1-(i+2) \cdot \alpha y] \cdot Y^{(i)} \\ + \frac{\alpha^2}{2} \cdot \Sigma \frac{(i+2) \cdot (i+1)}{2i+1} \cdot Y_{\prime}^{(i)} \end{array} \right\}$$

$$= \frac{4\pi}{3} \cdot \rho \cdot a - \frac{4\pi}{3} \cdot \rho \cdot a . \alpha . \left\{ 2y - \Sigma \frac{3.(i+1)}{2i+1} \cdot Y^{(i)} \right\}$$

$$+ \frac{4\pi}{3} \cdot \rho \cdot a \cdot \frac{\alpha^2}{2} \cdot \left\{ 6y^2 - y \cdot \Sigma \frac{6.(i+1).(i+2)}{2i+1} \cdot Y^{(i)} + \Sigma \frac{3.(i+2).(i+1)}{2i+1} \cdot Y_{\prime}^{(i)} \right\}. \qquad [1561v]$$

The first term of this expression is the same as in [1561u], the coefficient of $-\frac{4}{3}\pi\rho . a . \alpha$ in the second term is

$$2y - \Sigma \frac{3.(i+1)}{2i+1} \cdot Y^{(i)} = 2\Sigma Y^{(i)} - \Sigma \frac{3.(i+1)}{2i+1} \cdot Y^{(i)} = \Sigma \left\{ 2 - \frac{3.(i+1)}{2i+1} \right\} \cdot Y^{(i)} = \Sigma \frac{i-1}{2i+1} \cdot Y^{(i)},$$

as in [1561u]. The coefficient of $\frac{4\pi}{3} \cdot \rho \cdot a \cdot \frac{\alpha^2}{2}$, is

$$6y^2 - y \cdot \Sigma \frac{6.(i+1).(i+2)}{2i+1} \cdot Y^{(i)} + \Sigma \frac{3.(i+2).(i+1)}{2i+1} \cdot Y_{\prime}^{(i)}; \qquad [1561w]$$

and if, for brevity, we put for a moment $n = i \cdot (i-1)$, $m = 2i+1$, we shall get $(i+2) \cdot (i+1) = i^2 + 3i + 2 = i \cdot (i-1) + 4i + 2 = n + 2m$; and the preceding coefficient will become,

$$6y^2 - y . \Sigma \frac{6.(n+2m)}{m} . Y^{(i)} + \Sigma \frac{3.(n+2m)}{m} . Y_{\prime}^{(i)}$$

$$= 6y^2 - 12y \cdot \Sigma Y^{(i)} . + 6 \cdot \Sigma Y_{\prime}^{(i)} - y \cdot \Sigma \frac{6n}{m} \cdot Y^{(i)} + \Sigma \frac{3n}{m} \cdot Y_{\prime}^{(i)}.$$

Substituting in the second and third terms the values $\Sigma Y^{(i)} = y$, $\Sigma Y_{\prime}^{(i)} = y^2$, the three first terms become $6y^2 - 12y^2 + 6y^2 = 0$, and the two remaining terms are, as in [1561u], $-y . \Sigma \frac{6n}{m} \cdot Y^{(i)} + \Sigma \frac{3n}{m} \cdot Y_{\prime}^{(i)}$.

If we add $\frac{2}{3}\pi\rho . a^2$ to half the value of V [1561q], we shall get,

$$\frac{2\pi\rho . a^2}{3} + \tfrac{1}{2}V = \frac{4\pi}{3} \cdot \rho \cdot a^2 - \frac{4\pi}{3} \cdot \rho \cdot a^2 . \alpha . \Sigma \frac{i-1}{2i+1} . Y^{(i)} \qquad [1561x]$$

$$- \frac{2\pi}{3} \cdot \rho \cdot a^2 \cdot \frac{\alpha^2}{2} \cdot \left\{ \Sigma Y^{(i)} . \Sigma \frac{3}{2i+1} . Y^{(i)} - \Sigma \frac{i+5}{2i+1} . Y_{\prime}^{(i)} \right\}.$$

Another demonstration of La Place's theorem on the attraction of spheroids.

Multiplying the expression of $-\left(\frac{dV}{dr}\right)$ [1561u] by a, and neglecting terms of the order α^2, we shall find that the product becomes equal to the second member of [1561x], and hence,

$$-a \cdot \left(\frac{dV}{dr}\right) = \frac{2\pi\rho . a^2}{3} + \tfrac{1}{2}V. \qquad [1561y]$$

Also if we put $z^{(i)}$ for what $z'^{(i)}$ becomes when we change μ' into μ, ϖ' into ϖ, we shall have, by what precedes,

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This is the same as the fundamental equation of La Place [1458], which was proved, in a different manner, in [1459a—x], neglecting terms of the order α^2, and putting the density $\rho=1$ [1457a]. The preceding demonstration could have been much abridged, if we had wholly neglected the consideration of terms of the order α^2, as was done by La Place in his first demonstration in the Mém. Acad. Paris, 1782.

[1561α] Retaining only the first power of α, in [1560α, β], we shall have, for a *spheroid*, whose radius is $a.(1+\alpha y)=a.(1+\alpha.\Sigma_0^\infty Y^{(i)})$, which we shall call the *first spheroid*,

Attracted external point.

[1561β] $$V=\frac{4\pi\rho.a^3}{3r}+\frac{4\pi\rho.\alpha.a^3}{r}.\Sigma_0^\infty\frac{1}{2i+1}.\frac{a^i}{r^i}.Y^{(i)};$$

Attracted internal point.

[1561γ] $$V=2\pi\rho.a^2-\frac{2\pi\rho.r^2}{3}+4\pi\rho.a^2.\alpha.\Sigma_0^\infty\frac{1}{2i+1}.\frac{r^i}{a^i}.Y^{(i)}.$$

[1561δ] For a *second spheroid*, whose radius is $a.(1+\alpha y+\alpha z)=a.(1+\alpha.\Sigma_0^\infty Y^{(i)}+\alpha.\Sigma_0^\infty z^{(i)})$, we shall get the values of V, by changing in the preceding expressions y into $y+z$, or

[1561ε] $\Sigma_0^\infty Y^{(i)}$ into $\Sigma_0^\infty Y^{(i)}+\Sigma_0^\infty z^{(i)}$. Subtracting the values corresponding to the first spheroid, from those of the second respectively, we shall get ΔV, or the increment of V, corresponding to the stratum included between the first and second spheroids, which will be,

Attracted external point.

[1561ζ] $$\Delta V=\frac{4\pi\rho.\alpha.a^3}{r}.\Sigma_0^\infty\frac{1}{2i+1}.\frac{a^i}{r^i}.z^{(i)};$$

Attracted internal point.

[1561η] $$\Delta V=4\pi\rho.a^2.\alpha.\Sigma_0^\infty\frac{1}{2i+1}.\frac{r^i}{a^i}.z^{(i)}.$$

The first of these expressions corresponds to an external attracted point, the second to an internal point, this last case being the same as is treated of by La Place in [1501]. If we put R_1 and R_2 for the corresponding values of $-\left(\frac{d.\Delta V}{dr}\right)$, representing the attraction of this stratum, in a direction towards the origin of the co-ordinates, we shall get the following values, similar to those in [1561h, i];

[1561θ] $$R_1=-\left(\frac{d.\Delta V}{dr}\right)=\frac{4\pi\rho.\alpha.a^3}{r^2}.\Sigma_0^\infty\frac{i+1}{2i+1}.\frac{a^i}{r^i}.z^{(i)};$$

[1561ι] $$R_2=-\left(\frac{d.\Delta V}{dr}\right)=-\frac{4\pi\rho.\alpha.a^2}{r}.\Sigma_0^\infty\frac{i}{2i+1}.\frac{r^i}{a^i}.z^{(i)}.$$

Neglecting α^2, we may put $r=a$, and they will become,

$$R_1=4\pi\rho.a.\alpha.\Sigma_0^\infty\frac{i+1}{2i+1}.z^{(i)},\qquad R_2=-4\pi\rho.a.\alpha.\Sigma_0^\infty\frac{i}{2i+1}.z^{(i)},$$

whose difference is,

[1561κ] $$R_1-R_2=4\pi\rho.a.\alpha.\Sigma_0^\infty\left\{\frac{i+1}{2i+1}+\frac{i}{2i+1}\right\}.z^{(i)}=4\pi\rho.a.\alpha.\Sigma_0^\infty z^{(i)}=4\pi\rho.a.\alpha z.$$

Difference of the internal and external attraction of a stratum.

Hence it appears, that if two points are situated upon the same radius, the one at the external surface of the stratum, and the other at the internal surface; *the difference of the action of the stratum upon these two points, in the direction of the radius, will be proportional to its*

$$v^{(i)}=\frac{4\pi}{(2i+1).(2-i)}.\int\rho.\left(\frac{dz^{(i)}}{da}\right).da;$$ [1563]

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thickness $a\alpha z$, *in the same direction; and is the same as if the stratum were spherical*, as [1561λ]
is evident, by supposing, in the preceding calculations [1561δ, &c.], $y=0$, and αz equal [1561μ]
to the thickness of the spherical stratum.

Having obtained the value of V [1560α, β], corresponding to a homogeneous spheroid, we may proceed, as in [1503″, &c.], to compute the attraction of a spheroid, composed of strata of variable densities, by supposing y and ρ to be functions of a; then taking the differential [1561ν]
of this value of V relative to a, we shall obtain the value of dV, corresponding to a stratum, whose thickness, in the direction of the radius, is equal to the differential of $a.(1+\alpha y)$, [1561ξ]
or $da+\alpha.d.(ay)$. Multiplying this by the variable density ρ, and then integrating relative to a, from $a=0$ to $a=\mathrm{a}$ at the surface, we shall get the value of V, corresponding to a heterogeneous spheroid. If the attracted point be situated without the spheroid, we must use the formula [1560α], which will produce the corresponding formula [1561τ]. But if the attracted point be within the spheroid, and correspond to a stratum, whose radius is $a.(1+\alpha y)$, we must use the formula [1560α], from $a=0$ to $a=a$; and the formula [1560β], from $a=a$ to $a=\mathrm{a}$. In this way, by using, for brevity, the expressions $A_n^{(i)}$, $B_n^{(i)}$, $C_n^{(i)}$, given below, which are quantities of the same nature as $Y^{(i)}$, [1561ξ]
satisfying the equation [1464c], we shall get the following values of V [1561τ, υ], of which the first corresponds to an external point, the second to an internal point.

Attraction of a spheroid of variable density,

$$A_n^{(i)}=\int_0^{\mathrm{a}}\rho.\left(\frac{d.(a^{i+3}.Y_n^{(i)})}{da}\right).da;$$ [1561π]

$$B_n^{(i)}=\int_0^{a}\rho.\left(\frac{d.(a^{i+3}.Y_n^{(i)})}{da}\right).da;$$ [1561ρ]

$$C_n^{(i)}=\int_a^{\mathrm{a}}\rho.\left(\frac{d.(a^{2-i}.Y_n^{(i)})}{da}\right).da.$$ [1561σ]

upon an external point;

$$V=\frac{4\pi}{3r}.\int_0^{\mathrm{a}}\rho.d.a^3+\frac{4\pi}{r}.\left\{\begin{array}{l}\alpha.\Sigma\frac{1}{(2i+1).r^i}.A^{(i)}+\frac{\alpha^2}{2}.\Sigma\frac{i+2}{(2i+1).r^i}.A_1^{(i)}\\ \qquad+\frac{\alpha^3}{2.3}.\Sigma\frac{(i+2).(i+1)}{(2i+1).r^i}.A_2^{(i)}+\&c.\end{array}\right\};$$ [1561τ]

upon an internal point.

$$V=\frac{4\pi}{3r}.\int_0^{a}\rho.d.a^3+\frac{4\pi}{r}.\left\{\alpha.\Sigma\frac{1}{(2i+1).r^i}.B^{(i)}+\frac{\alpha^2}{2}.\Sigma\frac{i+2}{(2i+1).r^i}.B_1^{(i)}+\&c.\right\}$$
$$+2\pi.\int_a^{\mathrm{a}}\rho.d.a^2+4\pi.\left\{\alpha.\Sigma\frac{r^i}{2i+1}.C^{(i)}-\frac{\alpha^2}{2}.\Sigma\frac{(i-1).r^i}{2i+1}.C_1^{(i)}+\&c.\right\}.$$ [1561υ]

We must, in this last value, put $r=a.(1+\alpha y)$, by which means it will become a [1561φ]
function of a, θ, ϖ.

The partial differentials of this last value of V [1561υ], relative to θ, ϖ, which are to be used in finding the attractions R'', R''', [1811*l*], in the directions perpendicular to the radius r, are to be taken before the substitution of the value $r=a.(1+\alpha y)$; and the same is [1561χ]
to be observed relative to the equations [1447*y*, π]. This is evident, because the independent variable quantities θ, ϖ, are also independent of r; and therefore the partial differentials of

hence we shall get the expression of V, corresponding to all the strata of

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V, relative to θ, ϖ, ought not to be affected by the functions of θ, ϖ, which enter implicitly in

Attraction upon an internal point, the expression of $r=a.(1+\alpha y)$, by means of the quantity y.

If we neglect terms of the order α^2 in [1561v], we shall get,

[1561ψ] neglecting terms of the order α^2.

$$V=\frac{4\pi}{3r}.\int_0^a \rho.d.a^3+2\pi.\int_a^{\mathrm{a}} \rho.d.a^2+4\alpha\pi.\Sigma\frac{r^{-i-1}.B^{(i)}}{2i+1}+4\alpha\pi.\Sigma\frac{r^i.C^{(i)}}{2i+1}.$$

We shall now, for the sake of illustrating more fully the calculations in which the limit a of the integrals is variable, show that this last value of V will produce the expression in the

[1561ω] second member of [1447π], which we have already investigated in a different manner. In making this calculation, we shall first compute the effect of the partial differentials relative to θ, ϖ, depending on the two first terms of the first member of the equation [1447π]; and then that depending on the third term of the same equation, or the differential relative to r.

[1562a] Now the first and second terms of V [1561ψ], being independent of θ, ϖ, evidently produce nothing in the two first terms of the first member of [1447π]; but the third term of V produces the following expressions,

[1562b]

$$\frac{1}{\sin.\theta}.\left(\frac{d.\left\{\sin.\theta.\left(\frac{dV}{d\theta}\right)\right\}}{d\theta}\right)=4\alpha\pi.\Sigma\frac{r^{-i-1}}{2i+1}.\frac{1}{\sin.\theta}.\left(\frac{d.\left\{\sin.\theta.\left(\frac{dB^{(i)}}{d\theta}\right)\right\}}{d\theta}\right);$$

$$\frac{1}{\sin.^2\theta}.\left(\frac{ddV}{d\varpi^2}\right)=4\alpha\pi.\Sigma\frac{r^{-i-1}}{2i+1}.\frac{1}{\sin.^2\theta}.\left(\frac{ddB^{(i)}}{d\varpi^2}\right).$$

The sum of these may be reduced by means of the following equation, which is like [1464c],

[1562c]

$$\frac{1}{\sin.\theta}.\left(\frac{d.\left\{\sin.\theta.\left(\frac{dB^{(i)}}{d\theta}\right)\right\}}{d\theta}\right)+\frac{1}{\sin.^2\theta}.\left(\frac{ddB^{(i)}}{d\varpi^2}\right)=-i.(i+1).B^{(i)};$$

observing that $B^{(i)}$ is a quantity of the same kind as $Y^{(i)}$ [1561ξ'], which must satisfy this equation. Hence this sum is $-4\alpha\pi.\Sigma\frac{r^{-i-1}}{2i+1}.i.(i+1).B^{(i)}$; if we neglect α^2,

[1562d] we may put $r=a$, and it will become $-4\alpha\pi.\Sigma\frac{i.(i+1)}{2i+1}.a^{-i-1}.B^{(i)}$. In like manner, the fourth term of V [1561ψ], depending on $C^{(i)}$, will produce the expression $-4\alpha\pi.\Sigma\frac{i.(i+1)}{2i+1}.a^i.C^{(i)}$. Adding this to the preceding term [1562d], we get,

[1562e]

$$\frac{1}{\sin.\theta}.\left(\frac{d.\left\{\sin.\theta.\left(\frac{dV}{d\theta}\right)\right\}}{d\theta}\right)+\frac{1}{\sin.^2\theta}.\left(\frac{ddV}{d\varpi^2}\right)=-4\alpha\pi.\Sigma\frac{i.(i+1)}{2i+1}.\{a^{-i-1}.B^{(i)}+a^i.C^{(i)}\}.$$

We shall now compute the effect of the third term of [1447π], depending on the partial differential of the other variable quantity r; in which it becomes necessary to notice the effect of the change of limits of the integrals, relative to a. Substituting $r=a.(1+\alpha y)$ in [1561ψ], and neglecting α^2, we get,

[1562f]

$$V=\frac{4\pi.(1-\alpha y)}{3a}.\int_0^a \rho.d.a^3+2\pi.\int_a^{\mathrm{a}} \rho.d.a^2+\frac{4\alpha\pi}{a}.\Sigma\frac{1}{(2i+1).a^i}.B^{(i)}+4\alpha\pi.\Sigma\frac{a^i}{2i+1}.C^{(i)}.$$

[1562g] If we take the differential of this expression relative to a, we shall find that the terms depending on the limits of the integrals, corresponding to this variable quantity, will destroy each other. For, by noticing only these terms, we shall have,

the spheroid, which fall without the attracted point. The value of V,

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$$\frac{d}{da}\cdot\int_0^a \rho\,.\,d\,.\,a^3=\frac{\rho.d.a^3}{da}=3\rho\,.\,a^2, \qquad \frac{d}{da}\cdot\int_a^{\text{a}} \rho\,.\,d\,.\,a^2=-\frac{\rho.d.a^2}{da}=-2\rho\,.\,a, \qquad [1562h]$$

$$\left(\frac{d\,.\,B^{(i)}}{da}\right)=\rho\,.\left(\frac{d.(a^{i+3}.Y^{(i)})}{da}\right)=(i+3)\,.\,\rho\,.\,a^{i+2}\,.\,Y^{(i)}+a^{i+3}\,.\,\rho\,.\left(\frac{dY^{(i)}}{da}\right), \qquad [1562i]$$

$$\left(\frac{d\,.\,C^{(i)}}{da}\right)=-\rho.\left(\frac{d.(a^{2-i}.Y^{(i)})}{da}\right)=(i-2)\,.\,\rho\,.\,a^{1-i}\,.\,Y^{(i)}-a^{2-i}\,.\,\rho\,.\left(\frac{dY^{(i)}}{da}\right); \qquad [1562k]$$

and the corresponding part of $\left(\frac{dV}{da}\right)$, deduced from [1562$f$], will be,

$$\frac{4\pi.(1-\alpha y)}{3a}\,.\,3\rho\,.\,a^2-2\pi\,.\,2\rho\,.\,a+4\alpha\pi\,.\,\rho\,.\,\Sigma\left\{\frac{i+3}{2i+1}.a\,.\,Y^{(i)}+\frac{a^2}{2i+1}\cdot\left(\frac{dY^{(i)}}{da}\right)\right\} \qquad [1562l]$$
$$+4\alpha\pi\,.\,\rho\,.\,\Sigma\left\{\frac{i-2}{2i+1}.a.Y^{(i)}-\frac{a^2}{2i+1}\cdot\left(\frac{dY^{(i)}}{da}\right)\right\}.$$

The first term of this expression $4\pi\rho\,.\,a$, is destroyed by the third $-4\pi\rho\,.\,a$; and those depending on $\left(\frac{dY^{(i)}}{da}\right)$ destroy each other. The terms depending on $Y^{(i)}$ are

$$4\alpha\pi.a.\rho.\Sigma\left\{\frac{i+3}{2i+1}+\frac{i-2}{2i+1}\right\}.Y^{(i)}=4\alpha\pi.a.\rho.\Sigma\frac{2i+1}{2i+1}.Y^{(i)}=4\alpha\pi\,.\,a\,.\,\rho\,.\,\Sigma Y^{(i)}=4\alpha\pi\,.\,a\,.\,\rho\,.\,y, \qquad [1562m]$$

and this sum is destroyed by the second term $-4\alpha\pi\,.\,a\,.\,\rho\,.\,y$; so that the whole expression becomes equal to nothing. Therefore, in finding the value of $\left(\frac{dV}{da}\right)$ from [1562f], we may neglect all the quantities, depending on the limits of the integrals; and we shall get, for the other terms,

$$\left(\frac{dV}{da}\right)=-\frac{4\pi}{3a^2}.\left\{1-\alpha y+\alpha a\,.\left(\frac{dy}{da}\right)\right\}.\int_0^a \rho.d.a^3-\frac{4\alpha\pi}{a^2}.\Sigma\frac{(i+1)}{(2i+1).a^i}.B^{(i)}+\frac{4\alpha\pi}{a}.\Sigma\frac{i.a^i}{2i+1}.C^{(i)}. \qquad [1562n]$$

Now we have $\left(\frac{dV}{da}\right)=\left(\frac{dV}{dr}\right)\cdot\left(\frac{dr}{da}\right)=\left(\frac{dV}{dr}\right)\cdot\left\{1+\alpha y+.\alpha a.\left(\frac{dy}{da}\right)\right\}$; because [1562$o$]

$r=a\,.\,(1+\alpha y)$ evidently gives $\left(\frac{dr}{da}\right)=1+\alpha y+\alpha a\left(\frac{dy}{da}\right)$. Putting these two values [1562p] of $\left(\frac{dV}{da}\right)$ equal to each other, and dividing by $1+\alpha y+\alpha a\,.\left(\frac{dy}{da}\right)$, we get, by always neglecting α^2,

$$\left(\frac{dV}{dr}\right)=-\frac{4\pi}{3a^2}\cdot(1-2\alpha y)\,.\int_0^a \rho.d.a^3-\frac{4\alpha\pi}{a^2}\,.\,\Sigma\frac{(i+1)}{(2i+1).a^i}.B^{(i)}+\frac{4\alpha\pi}{a}\,.\,\Sigma\frac{i.a^i}{2i+1}\,.\,C^{(i)}. \qquad [1562q]$$

This formula, taken with a contrary sign, will express as usual [1561g], for an internal point, the attraction in the direction of the origin of the co-ordinates. It is evidently the same as would have been found, by taking the differential of V [1561$\downarrow$], before the substitution of the value of r, and without making a variable; *but we should fall into an error, in finding* [1562r] *in this way the value of* $\left(\frac{ddV}{dr^2}\right)$, as we shall see by the following calculation.

Since $\left(\frac{d.rV}{dr}\right)=V+r\,.\left(\frac{dV}{dr}\right)=V+a\,.\,(1+\alpha y)\,.\left(\frac{dV}{dr}\right)$; if we substitute the [1562s] values [1562f, q], then reducing, and putting for brevity

corresponding to the strata within the attracted point, is found as we have before shown [1541].

[1562t] $$M = -\Sigma \frac{i \,.\, a^{-i-1}}{2i+1} \,.\, B^{(i)} + \Sigma \frac{(i+1) \,.\, a^{i}}{2i+1} \,.\, C^{(i)},$$ we shall get,

$$\left(\frac{d\,.\,r\,V}{d\,r}\right) = 2\,\pi \,.\, \int_{a}^{\mathrm{a}} \rho \,.\, d\,.\,a^{2} - \frac{4\,\alpha\,\pi}{a} \,.\, \Sigma \frac{i}{(2\,i+1)\,.\,a^{i}} \,.\, B^{(i)} + 4\,\alpha\,\pi \,.\, \Sigma \frac{(i+1)\,.\,a^{i}}{2\,i+1} \,.\, C^{(i)}.$$

[1562u] $$= 2\,\pi \,.\, \int_{a}^{\mathrm{a}} \rho \,.\, d\,.\,a^{2} + 4\,\alpha\,\pi \,.\, M.$$

The partial differential of this, relative to r, considered as a function of a, and noticing the limits of a, gives

$$\left(\frac{d\,d\,.\,r\,V}{d\,r^{2}}\right) = -4\,\pi\,\rho \,.\, a \,.\, \left(\frac{d\,a}{d\,r}\right) + 4\,\alpha\,\pi \,.\, \left(\frac{d\,M}{d\,a}\right) \,.\, \left(\frac{d\,a}{d\,r}\right)$$

[1562v] $$= -4\,\pi\,\rho \,.\, a \,.\, \left\{ 1 - \alpha y - \alpha\,a \,.\, \left(\frac{d\,y}{d\,a}\right) \right\} + 4\,\alpha\,\pi \,.\, \left(\frac{d\,M}{d\,a}\right).$$

Multiplying this by $r = a \,.\, (1 + \alpha\,y)$, we get,

[1562w] $$r \,.\, \left(\frac{d\,d\,.\,r\,V}{d\,r^{2}}\right) = -4\,\pi\,\rho \,.\, a^{2} \,.\, \left\{ 1 - \alpha\,a \,.\, \left(\frac{d\,y}{d\,a}\right) \right\} + 4\,\alpha\,a \,.\, \pi \,.\, \left(\frac{d\,M}{d\,a}\right),$$

[1562x] The partial differential of M [1562t], relative to a, using [1562i, k], is

$$\left(\frac{d\,M}{d\,a}\right) = \Sigma \frac{i.(i+1)}{2\,i+1} \,.\, \{a^{-i-2}.B^{(i)} + a^{i-1}\,.\,C^{(i)}\} - \Sigma \frac{i.a^{-i-1}}{2\,i+1} \,.\, \left(\frac{d\,B^{(i)}}{d\,a}\right) + \Sigma \frac{(i+1)\,.\,a^{i}}{2\,i+1} \,.\, \left(\frac{d\,C^{(i)}}{d\,a}\right)$$

$$= \Sigma \frac{i.(i+1)}{2\,i+1} \,.\, \{a^{-i-2}.B^{(i)} + a^{i-1}\,.\,C^{(i)}\} - \Sigma \left\{ \frac{i.(i+3)}{2\,i+1} - \frac{(i+1)\,.\,(i-2)}{2\,i+1} \right\} \,.\, \rho \,.\, a \,.\, Y^{(i)}$$

$$- \Sigma \left\{ \frac{i}{2\,i+1} + \frac{(i+1)}{2\,i+1} \right\} \,.\, \rho \,.\, a^{2} \,.\, \left(\frac{d\,Y^{(i)}}{d\,a}\right)$$

$$= \Sigma \frac{i.(i+1)}{2\,i+1} \,.\, \{a^{-i-2}.B^{(i)} + a^{i-1}\,.\,C^{(i)}\} - \Sigma\, 2\,\rho \,.\, a \,.\, Y^{(i)} - \Sigma\,\rho \,.\, a^{2} \,.\, \left(\frac{d\,Y^{(i)}}{d\,a}\right)$$

[1562y] $$= \Sigma \frac{i.(i+1)}{2\,i+1} \,.\, \{a^{-i-2}.B^{(i)} + a^{i-1}\,.\,C^{(i)}\} - 2\,\rho \,.\, a\,y - \rho \,.\, a^{2} \,.\, \left(\frac{d\,y}{d\,a}\right).$$

Substituting this in [1562w], and reducing, we get,

[1562z] $$r \,.\, \left(\frac{d\,d\,.\,r\,V}{d\,r^{2}}\right) = -4\,\pi\,\rho \,.\, a^{2}.(1 + 2\,\alpha\,y) + 4\,\alpha\,\pi.\,\Sigma \frac{i.(i+1)}{2\,i+1} \,.\, \{a^{-i-1} \,.\, B^{(i)} + a^{i} \,.\, C^{(i)}\},$$

which contains the quantity $-4\,\pi\,\rho \,.\, a^{2} \,.\, (1 + 2\,\alpha\,y)$, that would not have been found, if [1562α] we had taken the second differential of V [1561↓], relative to r, without varying a; as is evident by observing, that the first term of $r\,V$ would be independent of r, and therefore its differential would vanish; moreover, the second term of $r\,V$ would contain only the *first* power of r, and its second differential would vanish; and there would remain only the terms depending on $B^{(i)}$, $C^{(i)}$, which would produce the terms depending on these quantities in [1562z].

Adding together the expressions [1562e, z], we get,

[1562β] $$\frac{1}{\sin.\theta} \,.\, \left(\frac{d \,.\, \left\{ \sin.\theta \,.\, \left(\frac{d\,V}{d\,\theta}\right) \right\}}{d\,\theta} \right) + \frac{1}{\sin.^{2}\theta} \,.\, \left(\frac{d\,d\,V}{d\,\varpi^{2}}\right) + r \,.\, \left(\frac{d\,d\,.\,r\,V}{d\,r^{2}}\right) = -4\,\pi\,\rho.a^{2}.(1 + 2\,\alpha\,y),$$

[1562γ] in which the second member, neglecting α^{2}, is $-4\,\pi\,\rho \,.\, \{a \,.\, (1 + \alpha\,y)\}^{2} = -4\,\pi\,\rho \,.\, r^{2}$, being the same as is found, in another manner, in [1432m].

CHAPTER III.

ON THE FIGURE OF A HOMOGENEOUS FLUID MASS, IN EQUILIBRIUM, AND ENDOWED WITH A ROTATORY MOTION.

18. After having explained the theory of the attractions of spheroids* in the two preceding chapters, we shall now consider the figures which

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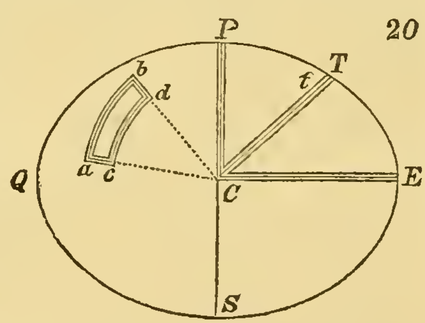
* (1080) In the Philosophical Transactions of the Royal Society of London for 1824, Mr. Ivory has published a paper, in which the principles used by the author, in this chapter, in finding the equilibrium of a fluid mass, have been objected to, as incomplete. For this reason, we shall give some account of the methods heretofore used, for the determination of the equilibrium of any fluid spheroid; by which means we shall be better able to judge of the difficulties of the subject, and of the sufficiency of the commonly received laws of equilibrium.

Newton, who first considered the form of a homogeneous fluid, revolving about its axis PCS, supposed, without any demonstration, the figure of the body to be an ellipsoid of [1563a]
revolution; and in this hypothesis, by the usual law of the gravitation of the particles, he computed the ratio of the polar semi-axis PC, to the equatorial semi-axis CE. The principle he uses, in making this calculation, is that *if an infinitely small fluid canal PCE, communicate from the centre C to the pole P and to the equator E, the pressure at C in the columns PC, EC, will exactly balance each other.* For the whole mass being, by hypothesis, in equilibrium, all parts of it



Newton's principle of equilibrium.

[1563b]

except this canal, may be supposed to become solid, without producing any change of situation, or pressure; and in this case it would be necessary that the fluid in the two [1563c]
branches of the canal should balance each other; and the same principle would apply to any two canals whatever, as CP, CT, proceeding from the centre C. The centrifugal force has no effect on the gravity of the particles situated in the axis of revolution, or canal CP, but it decreases the gravity in the canal CE, and this decrement of gravity is

they must assume, by means of the mutual action of their particles, and

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[1563d] balanced by the increased length of the canal CE. The manner in which this calculation was made by Newton, is given in Book XI, Chap. i, of this work; whence it will appear that this principle of the equilibrium in the two canals CP, CE, makes the ratio of the
[1563d′] polar to the equatorial semi-axis as 229 to 230, being nearly as it is found in this chapter, [1592″], from other principles.

Principle of Huygens. About the same time, Huygens remarked, that the figure of the surface ought to satisfy the condition, that *the combined action of the attraction of the spheroid, and the centrifugal*
[1563e] *force upon any point whatever of the surface, should be perpendicular to the surface at that point*, in conformity to the known laws of equilibrium of fluids at the surface of the sea. He afterwards computed the oblateness of the earth, upon the supposition that the whole force was concentrated, in the centre of gravity of the spheroid, and that it was of equal intensity at all distances from that centre. In computing the ratio of the axes, he also used Newton's hypothesis, of the equilibrium of the canals CP, CE. Upon these principles he found the ratio of the polar semi-axis of the earth to the equatorial, to be as 577 to 578. It will be seen hereafter, that this ratio is the same as would correspond, in the common theory of
[1563e′] gravitation, to a spheroid, in which the strata near the centre are infinitely dense, the whole of the fluid covering this central nucleus, or point, being infinitely rare.

Taking into consideration other laws of attraction, and supposing the bodies not to be homogeneous, it was soon perceived, that many cases might be found, in which these two principles would be inconsistent with each other. For example, if the particles of the fluid in the canal CtT, were acted upon more powerfully than in the canals CP, CE, so as to produce an effect similar to that of a greater density; it would be necessary to depress the surface of this canal, from T to t, in order to preserve the equality of pressures at the point C. In consequence of this, Bouguer proposed, as a more complete principle of equilibrium, the union of both these conditions; namely, *that the pressure, at the bottom of*
Bouguer's principle. *the canals CP, CE, should be equal, that the combined effect of the attraction of the*
[1563f] *body, and the centrifugal force, at any point, should be perpendicular to the surface at that point*, and that all forms of the surface, which did not satisfy both these conditions, should be excluded. The necessity of these conditions was very apparent, but it was soon found that even these were not sufficient; and that we might imagine such a law of attraction, as would satisfy both of them, without producing an equilibrium, but on the contrary an incessant tendency to motion, among the particles of the fluid. For example, if we do not restrict ourselves to the common law of attraction, but suppose the force of gravity to be directed
[1563g] towards the central point C, and that its action upon any particle does not depend wholly on the distance of the particle from that point, but is affected by the angle PCa, which the attracted particle a, makes with the axis of revolution PCS, or with any other line, taken at pleasure, &c. In this case, there might not be an equilibrium among the particles of the fluid. For if we suppose a canal $abdc$, of a very small uniform diameter, to be formed

the other forces acting on them. We shall first investigate the figure, [1563']

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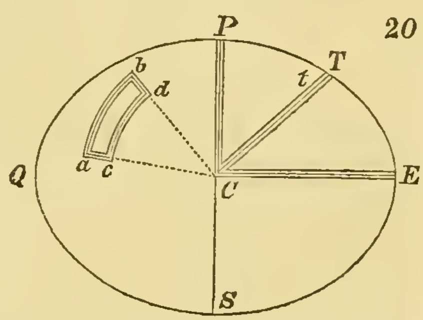
within the homogeneous spheroid $PESQ$, so that the two concentric and very near [1563h] branches ab, cd, may be circular arcs, described about the centre C; and ac, bd, two very short cylinders, directed towards the centre C; it is evident that the gravity would have no effect on the circular branches, because its direction is everywhere perpendicular to [1563i] the sides of these parts of the canal. Therefore, in order that there may be an equilibrium, it is necessary that the pressures of the two short and equal cylinders ac, bd, should be the same. This would generally require that the force of gravity should be the same at a as at b, which is contrary to the hypothesis in [1563g], where these forces are supposed not to depend wholly on the equal distances Ca, Cb; but to be affected by the different angles PCa, [1563k] PCb. Hence it appears, that in all hypotheses, where gravity is supposed to tend towards a centre, and the force not to depend wholly on its distance from that centre, it is possible that there may not be an equilibrium in the fluid mass. [1563l]

It is easy to imagine other cases, in which the preceding principles, of the equality of pressure in the canals CP, CT, CE, &c., and of the action of gravity being perpendicular to the surface, may both be satisfied, and yet the fluid not be in equilibrium. For instance, we may suppose a central force at C, to act upon any point T of the surface, with exactly [1563m] the requisite intensity, in comparison with the centrifugal force, to make the resultant perpendicular to the surface; by which means the second condition would be satisfied. Then in proceeding along any one of the canals TC, from the surface to the centre, we may suppose the force of gravity to vary, in an infinite number of ways, so as to produce however the same pressure at the bottom of the canal; in like manner as the same mass of heterogeneous [1563n] matter might be placed in an infinite number of ways, in the same canal, and produce an equal pressure at the bottom of the canal, without the particles being in equilibrium with each other. From these considerations, it is evident, that these two principles are not sufficient to embrace all the conditions necessary for the equilibrium of a fluid mass, in every case of attraction, which [1563o] might be arbitrarily assumed.



Clairaut finally proposed a more satisfactory principle, embracing both the preceding cases, as simple results of his general expression. It is founded upon the supposition that *a small canal* ORS, fig. 21, *of any form whatever, passing through the spheroid, and terminating at any points of its surface* O, S, *is in equilibrium, independently of the other parts of the spheroid.* For if we suppose, as in [1563c], that all parts of the spheroid [1563p] become solid, except those in the canal ORS, it will be evident that the fluid, in this [1563q] canal, must be in equilibrium, independently of the rest. Now this could not happen, unless

Clairaut's principle of equilibrium. First form.

which satisfies the equilibrium of a homogeneous fluid mass endowed with

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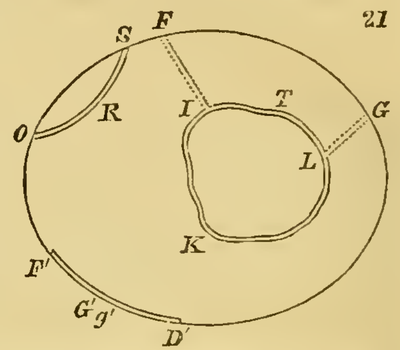
the efforts of the part OR to move towards S, should be equal to those of the part SR to move towards O. This principle evidently embraces that of Newton, supposing the canal ORS to be formed of the two cylindric branches
[1563r] PC, EC, passing through the centre C, as in fig. 20. It also includes the principle of Huygens, supposing the arbitrary canal to be placed along the surface of the spheroid $F'G'g'D'$ fig. 21. For this canal must, like all the others, be in equilibrium. But this can happen only in two ways. *First*, because the direction of gravity at each point G' is perpendicular
[1563s] to the direction $G'g'$ of the canal; or, *secondly*, because a part $F'G'$, tending to move towards D', is balanced by the other part $D'G'$, tending to move towards F'. Now this second condition must be rejected; for as there is no limit in the length of the canal, it is necessary that any part of it, as $F'G'$, should, of itself, be in equilibrium, as well as the whole of it $F'G'D'$; but this could not happen if $F'G'D'$ had
[1563t] been in equilibrium, in consequence of the equality of the opposite pressures of $F'G'$ and $D'G'$.



Another form of Clairaut's principle.

This principle of Clairaut may be more generally expressed by the condition, that *all the pressures of the fluid, in any oval or re-entering curve, of any figure whatever, and taken in any part of the spheroid, mutually destroy each other.* Thus, if $ITLK$ be such a canal,
[1563u] and we suppose, as in [1563e], that all the spheroid become solid, except this canal; it is evident, that if the whole fluid be in equilibrium, the canal must also be in equilibrium, after the rest has become solid; so that if we take, at pleasure, any two points I, L, in the canal,
[1563v] the pressures of the two parts IKL, ITL, against each other, must be equal; otherwise there would be a perpetual current in the canal. This result would also follow from the first supposition [1563p], of the equilibrium of a canal, which connects any two parts of the surface of the fluid; for if we suppose two canals IF, LG, to proceed, from the points I, L, of the canal, to the points F, G, at the surface, we shall find, from the first principle [1563p],
[1563w] that all parts of the canal $FITLG$, or of the canal $FIKLG$, will be in equilibrium. Now these two canals have the parts FI, GL, which are common; and if the pressures of both these parts be taken away from each of them, there will remain the two parts ITL, IKL, of the oval canal $ITLKI$, whose pressures mutually balance each other.

Effect of the centrifugal force.

In estimating the forces which act upon the particles of a fluid mass, which has a rotatory motion about an axis, it is necessary to notice the action of the centrifugal force. Thus if
[1563x] we suppose the spheroid to revolve uniformly about an axis, passing through the point C, perpendicular to the plane of the figure 22, so that a particle of fluid, which is at M at the

a rotatory motion, and we shall give a rigorous solution of this problem. [1563'']

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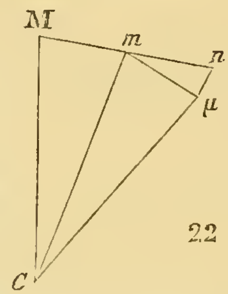
commencement of the motion, may be at m at the end of the time dt, and at μ at the end of the time $2\,dt$, &c.; we ought to have, in the case of a figure of equilibrium,

$$CM = Cm = C\mu, \qquad Mm = m\mu;$$ [1563y]



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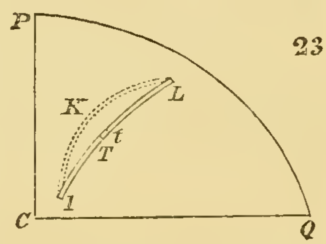
and if we continue Mm to n, making $mn = Mm$, it is evident, that if the particle which moved from M to m in the first instant, were left to itself, it would move in the second instant to n. But the actions of gravity, pressure, and the centrifugal force, would make it describe the space $n\mu$; therefore the resultant of these forces would make it describe the space $m\mu$, and by this means preserve the figure of equilibrium. In computing the figure of the surface of equilibrium, we may, for the simplicity of calculation, suppose that the body has no rotatory motion; but we must, in this case, in estimating the forces, which act on the fluid, take into consideration the centrifugal force arising from this motion. [1563z]

According to Clairaut's principle of equilibrium [1563p], if an oval or re-entering curve be drawn through two given points I, L, fig. 23, of the spheroid PCQ; the pressures, at these two points of the canal, will be the same, whatever be the form of the canal. Therefore the pressure in the canal ITL, at the points I, L, will be the same, as in any other canal IKL, passing through the same points I, L. Hence we may easily obtain an analytical expression of this principle of equilibrium. For if we refer the points of space to three [1563α]



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rectangular axes x, y, z; we may take x, y, z, for the co-ordinates of any point T fig. 23 of the canal ITL; x', y', z', for those of the first point I; and x'', y'', z'', for those of the last point L; the forces acting on any intermediate point T, from the mutual attraction of the particles of the fluid, the centrifugal force, and the action of foreign bodies, being represented, as in [1563'''], by P, Q, R, parallel to the ordinates x, y, z, and tending to decrease them respectively. The diameter of the canal being supposed constant, but infinitely small, and its length represented by s, we shall have the element $Tt = ds$. Then the force P, in the direction dx, parallel to the axis x, may be resolved into two forces, the one perpendicular to the side of the canal, which is destroyed by its reaction, and the other in the direction of the element ds, which will evidently be represented by $P.\frac{dx}{ds}$, as in note 34a. In like manner, the forces Q, R, in the directions dy, dz, will produce the forces $Q.\frac{dy}{ds}$; $R.\frac{dz}{ds}$, in the direction of the element of the canal Tt. The sum of these three forces, $P.\frac{dx}{ds} + Q.\frac{dy}{ds} + R.\frac{dz}{ds}$, represents the whole action [1563β] [1563γ] [1563δ] [1563ε]

Let a, b, c, be the rectangular co-ordinates of any point of the surface of

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on each particle of that element. If the density of the fluid, at the point T, be represented by ρ, ρ being a function of x, y, z, the number of particles in the element Tt, will be proportional to $\rho . ds$; and if we multiply this by the preceding expression of the force,

[1563ζ] the product $\rho . ds . \left\{ P . \frac{dx}{ds} + Q . \frac{dy}{ds} + R . \frac{dz}{ds} \right\}$, or $\rho . \{P.dx+Q.dy+R.dz\}$, will represent dp, the increment of the pressure p, corresponding to the element Tt; so that we shall have

[1563η] $$dp = \rho . \{P . dx + Q . dy + R . dz\};$$

and the integral being taken from the point I to the point L, will give p, the *variation of pressure* corresponding to the whole canal IL.

The quantities P, Q, R, ρ, are functions of x, y, z; and if the form of the canal were known, so that y, z, and therefore ρ, could be expressed in terms of x, the second member

[1563θ] of [1563η] would become a function of x only; in which case the integration relative to x, would be possible, and we should have $p = f(x) + \text{const.}$; $f(x)$ representing a function of x. Taking the constant quantity equal to $-f(x')$, so as to make p vanish when $x = x'$, we should finally get, at the other limit, where $x = x''$, $p = f(x'') - f(x')$; or as it may be expressed, more symmetrically, $p = f(x'', y'', z'') - f(x', y', z')$. Now Clairaut's theorem requires, that the value of p should be given, in this form, whatever be the

[1563ι] figure of the canal IL; leaving x, y, z, perfectly arbitrary; and it is evident, from a little consideration, that this is impossible, unless the second member of [1563η],

[1563κ] $$\rho . \{P . dx + Q . dy + R . dz\},$$

Clairaut's principle, expressed analytically.

be integrable of itself, without assigning any relation between the co-ordinates x, y, z. Hence we finally perceive, that *Clairaut's principle, expressed in an analytical form is nothing more than that the quantity* $\rho . \{P . dx + Q . dy + R . dz\}$, *should be an exact*

[1563λ] *differential*, as it is defined in note 13*e*, page 10, Vol. I; and this is the same as the condition of equilibrium given by La Place in [133, &c.] *Therefore the forces* P, Q, R, *must be*

[1563λ'] *such that they will satisfy the condition of integrability* [135], *otherwise the equilibrium will be impossible.*

Now it has been observed in note 13*f*, page 10, Vol. I; and it will be proved hereafter [1616viii], in a more full and satisfactory manner, that all the forces acting on the fluid, arising from the mutual attraction of the particles, the centrifugal force, and the action of foreign bodies, will render the quantity $P . dx + Q . dy + R . dz$ an exact differential, which may be represented, as in [137''], by $d\varphi$. Then the expression [1563η] will become

[1563μ] $dp = \rho . d\varphi$, as in [137''']; and as the second member of this equation must be an exact differential, by Clairaut's principle [1563λ], it follows, as in [137'''—138], that *p must be a function of ρ.* Now in a spheroid $PESQ$, fig. 24, composed of fluids of various densities, the heavier parts will subside, and when the whole has attained its state of equilibrium, the

this mass; and P, Q, R, the forces which act upon it, in directions parallel [1563″]

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particles having the same density ρ, will form *a level surface, or stratum* [138′], *extending around the whole body*, as $P'T'U'E'S'Q'$; and in proceeding from any point T' to another infinitely near point U' of this level surface, we shall have $d\rho=0$; and as ρ is a function of p [1563μ], we shall also have, for any point of the surface of this stratum, $dp=0$. Substituting this in [1563η], and dividing by ρ, we shall get, $0=P.dx+Q.dy+R.dz$; the elements dx, dy, dz, being supposed to correspond to this surface. From this equation we find, as in note 64, page 93, Vol. I, that the resultant of the forces P, Q, R, must be perpendicular to this level surface $P'E'S'Q'$, as has been stated by the author in [138′]. This is also evident from the consideration that if this resultant were not perpendicular to the surface, the fluid of that surface would descend towards the lowest place. [1563ν] [1563ξ]

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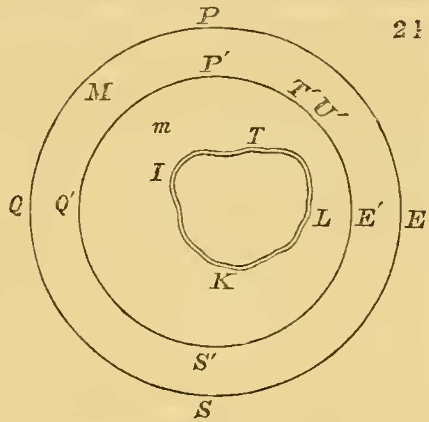

If the fluid be homogeneous, and its density $\rho=1$, the expression [1563η] will become $dp=P.dx+Q.dy+R.dz$; and by Clairaut's principle [1563λ], the second member $P.dx+Q.dy+R.dz$, must be an exact differential, when the fluid is in equilibrium. Upon the exterior surface of this fluid, where $dp=0$, the preceding equation becomes $0=P.dx+Q.dy+R.dz$; from which it follows, as in note 64, page 93, Vol. I, that the resultant of the forces P, Q, R, must be perpendicular to this surface, and it ought also to be directed, as in [138‴, 138^{iv}], towards the inner part of the fluid. [1563π] [1563ρ]

In computing the forces P, Q, R, which act upon any point I, fig. 24, of the interior part of a homogeneous fluid spheroid, the attraction of *the whole spheroid* is to be taken into account, together with the centrifugal force, and the attraction of foreign bodies. Then from what has been said [1563η, &c.], it appears, that if with these values of P, Q, R, the expression $P.dx+Q.dy+R.dz$ become an exact differential, the pressure at the point I, in any oval canal $ITLKI$, will be equal and in opposite directions, whatever be the direction of the canal; so that the fluid, at that point I, will remain in equilibrium; because the pressures, in opposite directions, will mutually balance each other. Moreover, if at any point of the surface of the fluid, the resultant of all the forces be perpendicular to the surface at that point, and directed inwards, that point will be in equilibrium. These principles seem plain and satisfactory, and they were used by mathematicians, during nearly a century, without any objection being made to them, and there was no doubt, in the mind of any one, that they comprised all the conditions necessary to the equilibrium of a fluid. But in the paper, published in the Transactions of the Royal Society of London, for the year [1563σ] [1563τ]

to these co-ordinates, *these forces being supposed to tend to decrease the*

1824, which is spoken of at the commencement of this note, another condition is proposed,
[1563υ] by Mr. Ivory, namely, that "every particle placed *within* a stratum bounded by two level surfaces, should be in equilibrium by the attraction of that stratum—or that every stratum of the exterior matter should be possessed of such a figure, as to attract all particles, in the inside, with equal force in opposite directions."

To illustrate this, we shall suppose $PESQ$, $P'E'S'Q'$, fig. 24, to be level surfaces, within which is situated the re-entering canal $ITLK$, of an infinitely small and constant
[1563φ] diameter, like that in the preceding figures. Putting m for the internal mass $P'E'S'Q'$, and M for the mass included between the level surfaces $PESQ$, $P'E'S'Q'$, we shall have $M+m$ for the whole mass of the spheroid. Then according to Clairaut's principle [1563u], the pressure at any point I of this canal, arising from foreign attractions, the centrifugal force, and the attraction of the *whole mass* $M+m$, would exactly balance each other, and this principle is expressed analytically, by means of the formulas [1563λ, &c.] Now it would seem as if nothing more than this were requisite; *all the forces, which act upon the point* I, *are taken into the calculation, and they are found to balance each other.* Mr. Ivory however requires, in addition to this, that the attraction of the mass M, upon any point
[1563χ] of the canal I, should be equal, in opposite directions; but the reasons he has given, in support of this new condition of equilibrium, have been generally considered unsatisfactory by mathematicians; and several papers have been published on the subject, in the Connoisance des Tems, Annales de Chimie et de Physique, The Philosophical Magazine, &c., by Mr. Poisson, Mr. Airy, Mr. Ivory, &c.

[1563ψ] Mr. Poisson has pointed out several examples, in which this new principle, when carried to its full extent, would lead to an erroneous result; and it is by such simple examples that the accuracy or inaccuracy of such a theory is most easily tested, without going into an elaborate examination of the subject, which becomes unnecessary if it can be shown to fail in one or two common cases. We shall here mention one of these examples, in which a homogeneous fluid mass $PE'SQ'=m$, revolving about its axis PS, in the same time as the earth, assumes the form of an ellipsoid of revolution [1574′], in equilibrium; the polar axis being to the equatorial, nearly as 230 to 231,

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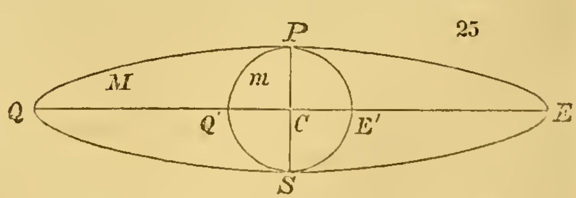

[1563ω] as will be seen hereafter [1592″]. Now if this mass be surrounded by a stratum M of the same fluid, revolving about the same axis, with the same angular velocity, so as to form the ellipsoid $PESQ$, in which the ratio of the polar axis PS, is to the equatorial EQ, as 1 to 680 nearly; the whole mass $PESQ=M+m$, will also be in equilibrium, as Mr. Ivory himself admits, and as we shall hereafter see, [1603″]. In this case, by the addition of the mass M, included between the two surfaces $PE'SQ'$, $PESQ$, we form a new

co-ordinates. It follows, from § 34 of the first book, [323], that to [1563'''']

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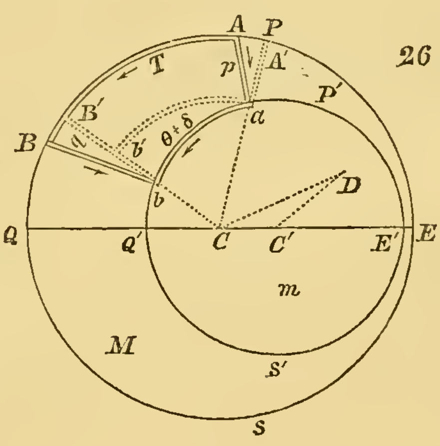
ellipsoid $PESQ$, in equilibrium, without disturbing the equilibrium of the particles of [1564a]
the ellipsoid $PE'SQ'$; although the action of the stratum M, upon any particle of m, must evidently be very powerful, instead of mutually balancing each other, as Mr. Ivory's principle requires. This is evident from the mere inspection of the figure 25, since a particle situated in the axis CP, near the point P, must be drawn downwards towards the point C, by the action of the mass M; and the increase of the attraction in the canal CP, will be so great as to balance the pressure in the very long canal CQ, instead of merely balancing that in the canal CQ', which is nearly equal to CP. It is not necessary, for the present purpose, to compute the exact amount of this force; though it might easily be done, by
taking the difference of the values of A, corresponding to these two ellipsoids, by means of [1564b]
the first of the formulas [1385]. We shall, in [1570r], give another of the excepted cases mentioned by Mr. Poisson.

In addition to these, I shall give the following extremely simple case, in which the rule of Mr. Ivory is defective. This consists in supposing the fluid $P'E'S'Q' = m$, fig. 26, to
be a homogeneous sphere, whose centre is C, at rest, and in equilibrium; it being very [1564c]
evident that the arguments of Mr. Ivory ought to apply to this extreme case, in which there is no rotatory motion, as well as to the more complex case, in which the rotation is finite. Then if this sphere be covered by a stratum M of the same fluid, included between the surfaces $PESQ$, $P'E'S'Q'$, so as to form a homogeneous spherical mass

$$PESQ = M + m,$$ [1564d]

whose centre is C; this whole mass will be in equilibrium, and the addition of this stratum M will not disturb the equilibrium of the internal sphere $P'E'S'Q'$. Moreover, it is evident that the attraction of the stratum M, upon any particle of the internal sphere m, is not balanced in every direction, as Mr. Ivory's principle would require; but on the contrary, the attraction of this stratum, upon any point D of this sphere, produces a force, in a direction parallel to the line $C'C$, joining the centres of the two spheres; and equal to the action of a sphere, described about the centre C, with the radius CC', upon the point C' of its surface. For the density of the sphere being put equal to unity, the attraction of the sphere $PESQ$,
upon the internal point D, is equal to $\frac{4}{3}\pi \,.\, DC$ [1430l], in the direction DC. This [1564e]
may be resolved into two forces, represented by $\frac{4}{3}\pi \,.\, C'C$, $\frac{4}{3}\pi \,.\, DC'$, in the directions parallel to the lines $C'C$, DC', respectively. The second of these forces, $\frac{4}{3}\pi \,.\, DC'$, is equal to the action of the sphere $P'E'S'Q'$, upon the same point D [1430l], therefore



maintain the equilibrium of the mass, it is only necessary we should have,

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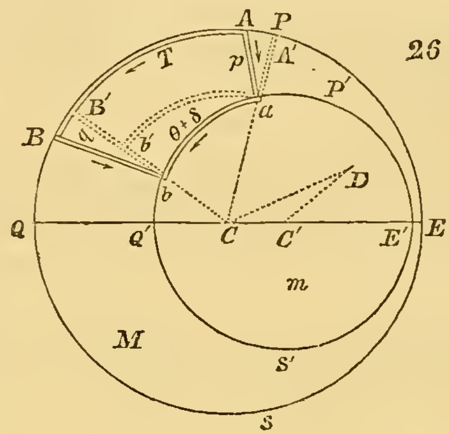
the other force $\frac{4}{3}\pi \cdot C'C$, will represent the difference of the actions of the two spheres; or, in other words, the action of the stratum M; and this is equal to the action of a sphere, of the radius CC', upon a point C' of its surface [1430*l*], as was stated above.

[1564*f*] If we suppose the mass m to be denser than M, the fluid M will not remain in equilibrium, until it has attained, at its surface, a spherical form, concentrical with that of the mass m. If we suppose, for the sake of simplicity, that the density of the mass m is double that of M; we may put $m=2m'$, and may consider the attraction of the whole mass $m+M$ to be equivalent
[1564*g*] to the action of two spheres; the one $PESQ=M+m'$, whose centre is C; and the other $P'E'S'Q'=m'$, whose centre is C'. The attraction of the sphere $PESQ$, towards the centre C, is perpendicular to the surface $PESQ$; but this action is disturbed by the attraction of the sphere $P'E'S'Q'=m'$, towards the centre C'; in consequence of this, the mass M will, by Clairaut's principle [1563ρ], not remain in equilibrium; but will
[1564*h*] move until it has attained, at its external surface, a spherical form, concentrical with the inner sphere. Finally it is evident, that in these cases, the principles furnished by Clairaut's method, afford all the necessary data for the determination of the equilibrium, without having recourse to the new, and sometimes contradictory, principle, mentioned in [1563*v*].

The pressure acquired in proceeding along any part of *the surface of the body* $P'E'S'Q'$, may be very much altered, by surrounding it with the stratum M; and on this subject, as it has been much discussed between Mr. Poisson and Mr. Ivory, we shall make a few remarks,
[1564*i*] referring to fig. 26, in which the surfaces are supposed to be spherical; observing that the same reasoning will apply to other surfaces. We shall suppose an oval or re-entering canal $ABbaA$, of an infinitely small and equal diameter, to be drawn, so that the parts AB, ab, may be on the surfaces $PESQ$, $P'E'S'Q'$, respectively; and the legs Aa, Bb, perpendicular to the surface $P'E'S'Q'$. We shall put T for the pressure acquired along
[1564*k*] the part AB, from A towards B; $\theta+\delta$ for the pressure acquired in the part ab, in proceeding from a towards b, θ being the similar pressure in the same canal, before it was surrounded by the stratum M; p for the pressure acquired in proceeding from A to a, along the canal Aa; lastly q the pressure in proceeding from B to b, along the canal Bb. The effort of the fluid Aab, to rise in the branch bB, is equal to $\theta+\delta+p$, and the contrary effort of the fluid ABb, to penetrate into the canal ba, is $T+q$; and since, by Clairaut's principle, [1563*u*], the fluid must be at rest, these two forces must
[1564*l*] destroy each other, and we shall have $\theta+\delta+p=T+q$. Now the mass $P'E'S'Q'$, before it was enveloped by the stratum M, was in equilibrium; therefore the resultant of all the forces, which then acted on any point of ab, must have been perpendicular to this curve; consequently the force in the direction of the curve must have been nothing, that is $\theta=0$. In like manner, after the envelopment, the whole mass $PESQ$ will be in equilibrium,
[1564*m*] and we shall have $T=0$. Substituting these values of θ, T, in the preceding equation, we shall get $\delta+p=q$, or $\delta=q-p$, in which δ represents the increment of the

$$0 = P \, . \, d a + Q \, . \, d b + R \, . \, d c \, ;^*$$ Equation of equilibrium. [1564]

pressure, acquired along the canal ab, in proceeding from a to b, after this canal is enveloped by the stratum M, and arising from the action of that stratum. This pressure is balanced by the difference $q - p$ of the pressures, at the two extremities b, a, of the canal ab.



In the preceding example, the value of δ may also be estimated very simply, by supposing two canals, $a A'$, $b B'$, to be formed on the continuation of the lines Ca, Cb, and then connecting these canals by a circular branch $a b'$, whose centre is C, and radius $Ca = Cb'$. In this case, the pressures in the canals $A'a$, $B'b'$, at the points a, b', are evidently equal to p; and the pressure at the point b, in the canal $B'b'b$, is equal to q; the difference of these pressures, $q - p$, or δ, is the same as is acquired in the canal $b'b$, in proceeding directly towards the centre of attraction C, through the space $b'b$; which is easily ascertained, because it depends on the attraction of the sphere whose centre is C. [1564n]

We may finally remark, that the surface $P' E' S' Q'$, which is a level surface before the addition of the mass M, ceases to be so after that addition. This is evident, because in the first case the attraction of the mass $P' E' S' Q'$ is in the direction towards the centre C', [1564o] perpendicular to the surface; and in the second case it is towards the centre C, in an oblique direction relative to the same surface.

* (1081) By formula [323], we have $0 = \delta V + n^2 . (y \delta y + z \delta z)$, in which [1564p] $\delta V = P . \delta x + Q . \delta y + R . \delta z$ [295'], and $n^2 . (y \delta y + z \delta z)$ [322'] is the product of the centrifugal force by the element of its direction, all these forces being supposed to tend to *increase* the co-ordinates [295a]; and as the forces P, Q, R, [1563'''], are supposed to tend to *decrease* the co-ordinates, we must change the signs of P, Q, R. If we also write a, b, c, respectively for x, y, z, to conform to the notation of this article, the equation [1564p] [1564q] will become $0 = -P . \delta a - Q . \delta b - R . \delta c + n^2 . (b \delta b + c \delta c)$, or

$$0 = P . da + (Q - n^2 b) . db + (R - n^2 c) . dc \, ; \qquad [1564r]$$

which is the same as [1564], supposing the terms $-n^2 b$, $-n^2 c$, arising from the [1564s] centrifugal force, to be included in Q, R, [1564']. The rotatory velocity, at the distance 1 from the centre, is n [320']; the centrifugal force g [54'], at that distance, is equal to the square of the velocity divided by the distance, which gives $n^2 = g$, hence [1564r] becomes [1564t] $0 = P . da + (Q - gb) . db + (R - gc) . dc$; therefore the effect of the centrifugal force is to decrease Q, R, by gb, gc, respectively. [1564u]

[1564'] taking care to include the centrifugal force, arising from the rotatory motion, in estimating the values of the forces P, Q, R.

To compute these forces, we shall suppose the figure of the fluid mass to
[1564''] be an ellipsoid of revolution, whose axis of rotation is the axis of revolution. If the forces P, Q, R, which result from this hypothesis, be substituted in
[1564'''] the preceding equation of equilibrium, and the result should become the differential equation of the ellipsoid, the preceding hypothesis would be legitimate, and the elliptical figure would satisfy the equilibrium of the fluid mass.

[1564''''] Supposing the axis of a to be the axis of revolution, the equation of the
Ellipsoid of revolution. surface of the ellipsoid will be of this form,*

[1565]
$$a^2+m\,.\,(b^2+c^2)=k^2\,;$$

[1565'] *the origin of the co-ordinates a, b, c, being at the centre of the ellipsoid*
[1565''] [1363*b*], *k will be the semi-axis of revolution ;* and if we put M for the mass of the ellipsoid, we shall get, by [1369*a*],†

[1566]
$$M=\frac{4\,\pi\,\rho\,.\,k^3}{3\,m}\,;$$

[1566'] ρ being the density of the fluid. If we put, as in [1377], $\frac{1-m}{m}=\lambda^2$, we shall have

[1566'']
$$m=\frac{1}{1+\lambda^2}\,;$$

Mass of the ellipsoid. therefore,

[1567]
$$M=\frac{4\,\pi\,\rho}{3}\,.\,k^3\,.\,(1+\lambda^2).$$

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* (1082) The equation of the ellipsoid [1363], changing x, y, z, into a, b, c, as in
[1565*a*] [1564*q*], becomes $a^2+m\,b^2+n\,c^2=k^2$; the three semi-axes [1363''], parallel to the
[1565*b*] co-ordinates a, b, c, being k, $\frac{k}{\sqrt{m}}$, $\frac{k}{\sqrt{n}}$, respectively; hence, from [1564''''], k must be the semi-axis of revolution, and $m=n$. If we use the value of m [1566''], the
[1565*c*] equatorial semi-axis will be $k\,.\sqrt{(1+\lambda^2)}$. Substituting $m=n$ in the preceding equation of the ellipsoid, it becomes as in [1565]. We may remark, that the symbol n in this note differs wholly from that used in notes 1081, 1086.

† (1084) Multiplying the mass of the ellipsoid [1369*a*] by the density ρ, we get [1566]; and by using m [1566''], it becomes as in [1567].

From this equation, we may determine k when λ is known. Now if we put

$$A'=\frac{4\pi\rho.(1+\lambda^2)}{\lambda^3}.\{\lambda-\text{arc. tang.}\,\lambda\};$$

[1568]

$$B'=\frac{4\pi\rho}{2\lambda^3}.\{(1+\lambda^2).\text{arc. tang.}\,\lambda-\lambda\};$$

we shall obtain, from [1385], by noticing only the attraction of the fluid mass,*

$$P=A'.a;\qquad Q=B'.b;\qquad R=B'.c.$$ [1569]

If we put g for the centrifugal force, at the distance 1 from the axis of rotation; this force, at the distance $\sqrt{b^2+c^2}$ from the same axis, will be $g.\sqrt{b^2+c^2}$. Resolving this in directions parallel to the axes b, c, it will produce, in Q, the term $-gb$,† and in R, the term $-gc$; we shall thus have, by noticing all the forces which act upon the particles of the surface, [1569'] [1569'']

Forces.

$$P=A'.a;\qquad Q=(B'-g).b;\qquad R=(B'-g).c;$$ [1570]

therefore the preceding equation of equilibrium [1564] will become,‡

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* (1085) Substituting M [1567] in [1385], and using the abridged expressions [1568], the quantities A, B, C, [1385], representing the attractions of the spheroid in the directions parallel to a, b, c, will become respectively equal to the values of P, Q, R, [1569].

† (1086) The distance of a point of the surface whose co-ordinates are a, b, c, from the axis of a, is equal to $\sqrt{(b^2+c^2)}$. The rotatory velocity of this point is therefore [322'] $n.\sqrt{(b^2+c^2)}$, and the centrifugal force [54'] being represented by the square of this velocity, divided by its distance from the axis $\sqrt{(b^2+c^2)}$, will be, as above, [1564t], $n^2.\sqrt{(b^2+c^2)}=g.\sqrt{(b^2+c^2)}$. This centrifugal force, in the direction of the radius $\sqrt{(b^2+c^2)}$, and tending to *increase* it, may be resolved, as in [138a], into two forces, gb, gc, parallel to the ordinates b, c, respectively, and tending to *increase* them; but as all the forces are supposed in [1563''''] to tend to *decrease* the co-ordinates, we must, to conform to the present notation, change the signs of these quantities, and they will become $-gb$, $-gc$, as in [1569'']. Connecting these with the other parts of Q, R, [1569], we obtain the expressions [1570]. These results are the same as those found in [1564p—u], in a somewhat different manner. [1569a] [1569b] [1569c] [1569d]

‡ (1087) Substituting P, Q, R, [1570] in [1564], and dividing by A', we get [1571]. Half the differential of [1565], is $0=ada+m.(bdb+cdc)$, and by using the [1569e]

[1571] $$0 = a\,d\,a + \frac{(B'-g)}{A'}.\{b\,d\,b + c\,d\,c\}.$$

[1571'] Substituting for m, its value $\frac{1}{1+\lambda^2}$, [1566''] in the differential equation

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[1570a] value of m [1566''], it becomes as in [1572]. This is identical with [1571], putting $\frac{(B'-g)}{A'} = \frac{1}{1+\lambda^2}$; which is easily reduced to the form [1573].

If the particles of a homogeneous fluid mass M have a rotatory motion about the axis of x,
[1570b] and a mutual attraction in the *direct ratio of the distance*, it will be easy to prove that the form of the external surface of the fluid, determined by the equation of equilibrium [1564], will necessarily become of the same form as in [1571], and will therefore be an ellipsoid of revolution. For by using the same notation as in [1346, 1356iv], we shall find that the
[1570c] attraction of the particle dM, upon a point whose co-ordinates are a, b, c, will thus be represented by $dM.r$; r being the distance of the particle from the attracted point [1356iv]. Resolving this force, in directions parallel to the axes x, y, z, they will become respectively, [1355b],

[1570d] $$dM.(a-x), \qquad dM.(b-y), \qquad dM.(c-z).$$

The integrals of these expressions, corresponding to the whole mass M, will represent the quantities which are named A, B, C, [1347']. Hence

$$A = \int dM.(a-x) = a.\int dM - \int x.dM = Ma - \int x.dM.$$

Now if we suppose the origin of the co-ordinates to be at the centre of gravity of the fluid, we shall have, as in [216], $\int x.dM = 0$, $\int y.dM = 0$, $\int z.dM = 0$; hence the preceding integrals will become,

[1570e] $$A = Ma, \qquad B = Mb, \qquad C = Mc.$$

Let R be the value of r, corresponding to a mass M collected in a single point at the centre
[1570f] of gravity of the fluid, where $x=0$, $y=0$, $z=0$; then we shall get, from [1355e], $R^2 = a^2 + b^2 + c^2$. The attraction of the concentrated mass M, upon the point whose co-ordinates are a, b, c, will, in this hypothesis, be represented by $M.R$; and if this be resolved, in the directions parallel to the axes x, y, z, it will evidently produce the forces Ma, Mb, Mc, respectively; being the same as were found in [1570e], for the whole spheroid. Hence it appears, that in this law of attraction, the force is wholly independent of the form of the spheroid; and *in computing the attraction, either on an internal or external*
[1570g] *point, we may suppose the whole mass to be collected at the centre of gravity of the spheroid.*

Attraction of a spheroid in the direct ratio of the distance;

To the two last of the forces, B, C, [1570e], we must add the terms $-gb$, $-gc$, [1569''], depending on the centrifugal force, and we shall obtain the whole forces P, Q, R, [1563'''], acting on the attracted particle,

[1570h] $$P = Ma, \qquad Q = (M-g).b, \qquad R = (M-g).c.$$

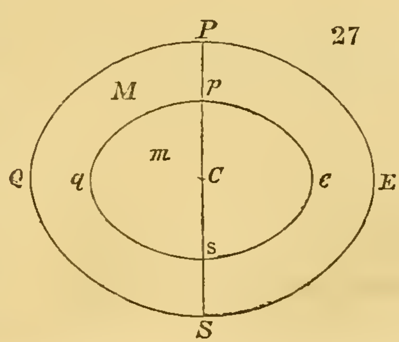
of the surface of the ellipsoid [1565], it becomes,

$$0 = a\,d\,a + \frac{b\,d\,b + c\,d\,c}{1+\lambda^2}. \qquad [1572]$$

Substituting these in the equation of equilibrium [1564], then dividing by $\frac{1}{2}M$, and putting for brevity $m = \frac{M-g}{M}$, we shall get $2\,a\,d\,a + m\,.\,(2\,b\,d\,b + 2\,c\,d\,c) = 0$, whose [1570i]
integral, using the constant quantity k^2, is $a^2 + m\,.\,(b^2+c^2) = k^2$; being the same as [1570k]
the equation of an ellipsoid of revolution [1565]. Now if M and g be given, we shall have $m = \frac{M-g}{M}$, and then k may be computed from [1566], ρ being unity. Therefore the polar semi-axis k, and the equatorial semi-axis $\frac{k}{\sqrt{m}}$ [1565b], will be determined, and the exact form of the body will be known. From what has been said, it is evident, that the [1570l]
external figure of this revolving homogeneous fluid, must necessarily be an ellipsoid of revolution, to the exclusion of all other forms; and in this particular the demonstration differs wholly from that in [1564″, &c.] For in this last case, it is *assumed*, that the figure is an [1570m]
ellipsoid, and the attractions are then computed for this assumed form, by the common theory of gravity; and it is found, in [1571, 1572], that the equation of equilibrium can be satisfied, with this form; but *it is not proved that this is the only figure which can satisfy that equation,* [1570n]
in the usual law of attraction.

In this case the fluid mass is an ellipsoid of revolution.

We may remark that if $M<g$, the value of $\sqrt{m}$ [1570i] will become imaginary, as well as the equatorial semi-axis $\frac{k}{\sqrt{m}}$ [1570l]; and it is evident, without any calculation, [1570o]
that this ought to be the case; because when the centrifugal force g exceeds the gravity M, the fluid will be dissipated. If $M=g$, we shall have $m=0$ [1570i], and the equatorial semi-axis $\frac{k}{\sqrt{m}} = \infty$; therefore the spheroid will then be infinitely flattened. If [1570p]
the mass M be considered as infinitely great, in comparison with the centrifugal force g, we shall have $m=1$, and the equatorial semi-axis $\frac{k}{\sqrt{m}}$, will become equal to the polar semi-axis; that is, the spheroid will change into a sphere.



In this law of attraction, if we suppose the ellipsoid $p\,e\,s\,q$, whose mass is m, to be in a state of equilibrium, while revolving about the axis $P\,p\,C\,s\,S$; and then add to it the stratum M, having the same centre of gravity C, and the same angular velocity, about the same axis $P\,C\,S$; so as that the whole mass may form another ellipsoid $P\,E\,S\,Q = m + M$, which may also be in a state of equilibrium; the attraction of the stratum M, upon any particle of the fluid, will be equal to the difference of the attractions of these two [1570q]

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Comparing this with the preceding equation, we shall get,

[1573] $$(1+\lambda^2)\,.\,(B'-g)=A'. \qquad (1)$$

[1573'] If we substitute in this the values of A', B', [1568], and put $\frac{g}{\frac{4}{3}\pi\rho}=q$, we shall find,*

Equation for the determination of λ.

[1574] $$0=\frac{9\lambda+2q\,.\,\lambda^3}{9+3\lambda^2}-\text{arc. tang.}\,\lambda\,; \qquad (2)$$

therefore, if we determine λ by this equation, which is independent of the
[1574'] co-ordinates a, b, c, we may make the equation of equilibrium [1571] coincide with that of the surface of the ellipsoid [1572]. Hence it follows, that the elliptical figure satisfies the equilibrium, unless the rotatory motion
[1574''] be such that the value of λ^2 is imaginary, or negative and equal to, or greater than, unity.† The case of λ^2 being imaginary, gives an imaginary solid;

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ellipsoids; and it will therefore be the same as the action of the mass M, concentrated at the
[1570r] point C. This is evident, from what has been proved in [1570g]; and it is directly contrary to the new principle of equilibrium, proposed by Mr. Ivory, [1563e], which would require that the action of the stratum M, upon any particle of the body m, should be wholly balanced, in ~~equal and~~ opposite directions.

[1571a] * (1088) Substituting in [1573], the values A', B', [1568], and putting $g=\frac{4}{3}\pi\rho\,.\,q$, [1573'], it becomes,

[1571b] $$(1+\lambda^2)\,.\left\{\frac{4\pi\rho}{2\lambda^3}\,.\,[(1+\lambda^2)\,.\,\text{arc. tang.}\,\lambda-\lambda]-\tfrac{4}{3}\pi\rho\,.\,q\right\}=\frac{4\pi\rho\,.\,(1+\lambda^2)}{\lambda^3}\,.\,\{\lambda-\text{arc. tang.}\,\lambda\}.$$

Dividing this by $\frac{2\pi\rho\,.\,(1+\lambda^2)}{3\lambda^3}$, it becomes,

$$\{3\,.\,[(1+\lambda^2)\,.\,\text{arc. tang.}\,\lambda-\lambda]-2q\,.\,\lambda^3\}=6\,.\,\{\lambda-\text{arc. tang.}\,\lambda\}.$$

Transposing the terms of the first member, and dividing by $9+3\lambda^2$, we obtain [1574].
[1571c] It may be observed, that as g and ρ are positive, q must also be positive, [1573'].

[1574a] † (1089) The semi-axis of revolution is k, and that of the equator $k\,.\sqrt{(1+\lambda^2)}$, [1565b, c], which are both real, while λ^2 has any positive value, from 0 to ∞, or negative value, from 0 to -1. In the first case the equatorial diameter will be the greatest, or the
[1574b] spheroid oblate; in the second case, the equatorial diameter will be the least, or the ellipsoid prolate. In the case of the prolate ellipsoid, we may put $\lambda^2=-e^2$, e^2 being a positive
[1574c] quantity; the semi-axis of revolution being k, that of the equator will be $k\,.\sqrt{(1-e^2)}$; the excentricity corresponding will be ke [378e, m], and the excentricity, divided by

$\lambda^2 = -1$ corresponds to a paraboloid; and if λ^2 be negative and greater [1574'''] than unity, it will be an hyperboloid.

19. If we put p for the force of gravity at the surface of the ellipsoid, [1574''''] we shall have,*

$$p = \sqrt{P^2 + Q^2 + R^2}. \qquad [1575]$$

Within the ellipsoid, the forces P, Q, R, are proportional to the co-ordinates, [1575'] a, b, c; for we have seen in § 3, [1379, 1379b], that the attractions of the ellipsoid, parallel to these co-ordinates, are respectively proportional to them; and the same takes place with the centrifugal force, resolved in directions [1575''] parallel to the same co-ordinates. Hence it follows, that the gravities, at the different points of the radius, drawn from the centre of the ellipsoid to its surface, are in parallel directions, and are proportional to the distances [1575''']

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the semi-axis k, will be represented by e, as in [377'']. But from [378b], we see that [1574d] $e < 1$ corresponds to an ellipsis; $e = 1$ to a parabola; and $e > 1$ to an hyperbola; and as $\lambda^2 = -e^2$, it follows that when λ^2 is negative, and independent of its sign is less, equal to, or greater than, unity, the figure will be an ellipsoid, paraboloid, or hyperboloid, [1574e] respectively. If λ^2 be imaginary, e will be imaginary, and the solid will become imaginary, as in [1574'''].

* (1090) P, Q, R, [1563''', 1564'] are the forces, acting upon a particle of the surface of the ellipsoid, in directions parallel to the three axes a, b, c. The resultant of these forces is, $\sqrt{(P^2 + Q^2 + R^2)}$ [11'], as in [1575]; which, by substituting the values of P, Q, R, [1574f] [1570], becomes as in [1576]. This represents the gravity at the *surface* of the spheroid, in the point whose co-ordinates are a, b, c, the distance of which from the centre of the spheroid is the radius $r = \sqrt{(a^2 + b^2 + c^2)}$. If upon this radius we take a point, whose [1575a] distance from the centre is $h.r$, the co-ordinates of this internal point, will evidently be $h.a$, $h.b$, $h.c$. Therefore by changing a, b, c, respectively into $h.a$, $h.b$, $h.c$, [1575b] in the expressions of the forces [1385, 1569, 1570, 1576], we shall obtain the corresponding forces for this *internal* point. Now if we put P', Q', R', p', for the forces at this internal point, corresponding to P, Q, R, p, at the surface; the formulas [1570, 1576], will give

$$P' = A'.ha; \qquad Q' = (B' - g).hb;$$
$$R' = (B' - g).hc; \qquad p' = \sqrt{\{A'^2 . h^2 a^2 + (B' - g)^2 . h^2 . (b^2 + c^2)\}}. \qquad [1575c]$$

Comparing these with [1570, 1576], we get $P' = h.P$; $Q' = h.Q$; $R' = h.R$; $p' = h.p$; therefore all the forces P', Q', R', p', are to P, Q, R, p, respectively, as the [1575d] radius $h.r$ to r, as in [1575'''].

from this centre; so that if we know the gravity at the surface, we shall have also the gravity at any point within the spheroid.

Expression of gravity p. If in the expression of p [1575], we substitute the values of P, Q, R, given in the preceding article [1570], we shall get,

[1576] $$p=\sqrt{A'^2\,a^2+(B'-g)^2\cdot(b^2+c^2)}.$$

Hence we deduce, by means of the preceding equation [1573],*

[1577] $$p=A'\cdot\sqrt{a^2+\frac{b^2+c^2}{(1+\lambda^2)^2}}.$$

[1577′] But the equation of the surface of the ellipsoid gives $\frac{b^2+c^2}{1+\lambda^2}=k^2-a^2$; therefore we shall have,

[1578] $$p=A'\cdot\frac{\sqrt{k^2+\lambda^2a^2}}{\sqrt{1+\lambda^2}}.$$

[1578′] a is equal to k at the pole [1565b], and it is nothing at the equator; hence it follows, that the gravity at the pole is to the gravity at the equator, as
[1578″] $\sqrt{1+\lambda^2}$ is to unity;† therefore it is in the same ratio as the diameter of the equator to the polar axis.

[1578‴] Let t be the perpendicular to the surface of the ellipsoid, continued till it meets the axis of revolution; we shall have,

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* (1091) Substituting $B'-g=\frac{A'}{1+\lambda^2}$ [1573], in [1576], we get [1577]. If we use the value of m [1566″], the equation of the ellipsoid [1565] will become

[1577a] $$a^2+\frac{b^2+c^2}{1+\lambda^2}=k^2; \qquad \text{hence} \qquad \frac{b^2+c^2}{1+\lambda^2}=k^2-a^2.$$

Substituting this in [1577], we get $p=A'.\left(a^2+\frac{k^2-a^2}{1+\lambda^2}\right)^{\frac{1}{2}}$; which, by reduction, becomes as in [1578].

† (1092) At the pole, where $a=k$, [1565b], the expression [1578] becomes

[1578a] $$p=A'k\cdot\frac{\sqrt{(1+\lambda^2)}}{\sqrt{(1+\lambda^2)}}=A'k;$$

and at the equator, where $a=0$, it becomes $p=\frac{A'k}{\sqrt{(1+\lambda^2)}}$. The first of these values of p is to the second as $\sqrt{(1+\lambda^2)}$ to 1, or as $k.\sqrt{(1+\lambda^2)}$ to k; which, by
[1578b] [1574a], is as the equatorial diameter to the polar axis.

Normal of an ellipsis.

$$t = \sqrt{(1+\lambda^2) \, . \, (k^2+\lambda^2 a^2)} \; ;^*$$ [1579]

First form.

hence

$$p = \frac{A' t}{1+\lambda^2} \; ;$$ [1580]

therefore gravity is proportional to t. [1580′]

If ψ be the complement of the angle, which t makes with the axis of [1580″]

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* (1093) In the annexed figure, the arc DA represents a quadrant of the meridian of the ellipsoid of revolution, whose plane passes through the attracted point; the centre of this ellipsoid is C, the polar semi-axis $CD = k$, equatorial semi-axis $CA = k \, . \sqrt{(1+\lambda^2)}$ [1565c], P any point of this arc, whose rectangular co-ordinates are $CH = a$, $HP = b' = \sqrt{(b^2+c^2)}$; and their differentials may be represented by the infinitely small lines $Pq = -da$, $pq = db'$, drawn parallel to CH, HP, respectively; lastly, $PG = t$ is drawn perpendicular to the arc Pp, meeting the equatorial axis CA in E, and the polar axis DCG in G. Then we have [1579a]

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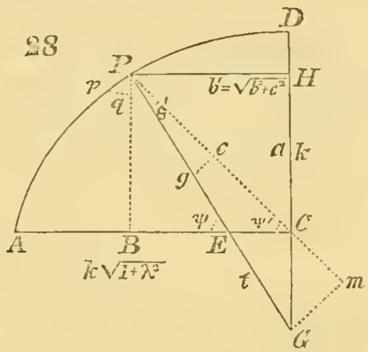

$$Pp = \sqrt{(Pq^2 + pq^2)} = \sqrt{(da^2 + db'^2)},$$ [1579b]

and in the similar triangles qPp, HPG, we have $Pq : Pp :: HP : PG$; or in symbols, $-da : \sqrt{(da^2+db'^2)} :: b' : t$; hence $t = -\frac{b' . \sqrt{(da^2+db'^2)}}{da}$. The [1579$c$] equation of the ellipsis [378n], changing x, y, a, b, into a, b', k, $k \, . \sqrt{(1+\lambda^2)}$, respectively, in order to conform to the present notation, is $\frac{a^2}{k^2} + \frac{b'^2}{k^2 . (1+\lambda^2)} = 1$. Hence

$b' = \sqrt{(1+\lambda^2)} \, . \sqrt{(k^2-a^2)}$, $\quad db' = -\frac{a\,da}{\sqrt{(k^2-a^2)}} \, . \sqrt{(1+\lambda^2)}$, and the arc [1579$d$]

$$Pp = \sqrt{(da^2+db'^2)} = -da \, . \left\{ 1 + \frac{(1+\lambda^2) \, . \, a^2}{k^2-a^2} \right\}^{\frac{1}{2}}$$

$$= -da \, . \sqrt{(k^2+a^2\lambda^2)} \, . \frac{1}{\sqrt{(k^2-a^2)}} = -da \, . \sqrt{(k^2+a^2\lambda^2)} \, . \frac{\sqrt{(1+\lambda^2)}}{b'} \, .$$ [1579e]

Substituting this in t [1579c], we get

$$t = \sqrt{(1+\lambda^2)} \, . \sqrt{(k^2+a^2\lambda^2)} = PG,$$ [1579f]

as in [1579]; hence $\sqrt{(k^2+a^2\lambda^2)} = \frac{t}{\sqrt{(1+\lambda^2)}}$. Substituting this in [1578], we [1579g] get [1580].

revolution; ψ will be the latitude of the point of the surface under consideration, and we shall have, by the nature of the ellipsis,*

[1579h] * (1094) Putting the complement of the angle $PGD = \psi =$ angle AEP, we shall have $PH = PG \,.\, \sin. PGD$; or, in symbols, $b' = t \,.\, \cos.\psi$; and by using the value

[1579i] of $PH = BC = b'$, [1579d], we get $PH = \surd(1+\lambda^2) \,.\, \surd(k^2 - a^2) = t \,.\, \cos.\psi$. Hence $a^2 = k^2 - \frac{t^2 \,.\, \cos.^2\psi}{1+\lambda^2}$, and $k^2 + \lambda^2 a^2 = k^2 \,.\, (1+\lambda^2) - \frac{\lambda^2 \,.\, t^2 \,.\, \cos.^2\psi}{1+\lambda^2}$. Multiplying this by $1+\lambda^2$, the first member will become equal to the value of t^2 [1579];

[1579k] hence $t^2 = k^2 \,.\, (1+\lambda^2)^2 - \lambda^2 \,.\, t^2 \,.\, \cos.^2\psi$, or $(1+\lambda^2 \,.\, \cos.^2\psi) \,.\, t^2 = k^2 \,.\, (1+\lambda^2)^2$. The square root of this gives t [1581]; and by substituting it in [1580], we get [1582].

We shall here investigate the values of CG, CE, and the sine of the angle CPG,

[1579l] which will be of use hereafter. Substituting the values of PG, PH, [1579f, i], in $GH^2 = PG^2 - PH^2$, we get

$$GH^2 = (1+\lambda^2) \,.\, (k^2 + \lambda^2 a^2) - (1+\lambda^2) \,.\, (k^2 - a^2) = (1+\lambda^2) \,.\, \{k^2 + \lambda^2 a^2 - k^2 + a^2\}$$
$$= (1+\lambda^2) \,.\, \{\lambda^2 a^2 + a^2\} = (1+\lambda^2)^2 \,.\, a^2,$$

whose square root is $GH = (1+\lambda^2) \,.\, a$, hence

[1579m] $$CG = GH - CH = (1+\lambda^2) \,.\, a - a = \lambda^2 a;$$

[1579n] and since $(1+\lambda^2) = \frac{CA^2}{CD^2}$ [1579a], we shall have $GH = \frac{CA^2}{CD^2} \,.\, CH$. If we change the axis CD into CA, and the contrary, we shall, in like manner, get,

[1579o] $$BE = \frac{CD^2}{CA^2} \,.\, CB.$$

[1579p] If we put $\psi' =$ complement of the angle $DCP =$ angle PCB,. and the angle $\theta' = \psi - \psi' =$ angle GPC; we shall evidently have, in the triangles PHC, PHG,

[1579q] $CH : GH :: \text{tang.}\,\psi' : \text{tang.}\,\psi$; hence $\text{tang.}\,\psi = \frac{GH}{CH} \,.\, \text{tang.}\,\psi' = (1+\lambda^2) \,.\, \text{tang.}\,\psi'$, [1579n]; therefore,

$$\lambda^2 \,.\, \text{tang.}\,\psi' = \text{tang.}\,\psi - \text{tang.}\,\psi' = \frac{\sin.\psi}{\cos.\psi} - \frac{\sin.\psi'}{\cos.\psi'} = \frac{\sin.\psi \,.\, \cos.\psi' - \cos.\psi \,.\, \sin.\psi'}{\cos.\psi \,.\, \cos.\psi'}$$
$$= \frac{\sin.(\psi - \psi')}{\cos.\psi \,.\, \cos.\psi'} = \frac{\sin.\theta'}{\cos.\psi \,.\, \cos.\psi'}.$$

[1579r] Multiplying this by $\cos.\psi \,.\, \cos.\psi'$, we get $\sin.\theta' = \lambda^2 \,.\, \cos.\psi \,.\, \sin.\psi'$. If in this we neglect λ^4, we may put $\psi = \psi'$, and $\cos.\psi' \,.\, \sin.\psi' = \frac{1}{2}\sin.2\psi'$; hence $\sin.\theta' = \frac{1}{2}\lambda^2 \,.\, \sin.2\psi'$, or

[1579s] $$\sin.GPC = \tfrac{1}{2} \,.\, \frac{CA^2 - CD^2}{CD^2} \,.\, \sin.2ACP = \tfrac{1}{2} \,.\, \frac{CA^2 - CD^2}{CD^2} \,.\, \sin.2DCP,$$

representing the sine of the angle formed by the radius PC, and the line PG, drawn perpendicular to the curve at P.

Second form.

$$t = \frac{(1+\lambda^2)\,.\,k}{\sqrt{1+\lambda^2\,.\,\cos.^2\psi}}\,;$$ [1581]

therefore we shall have,

$$p = \frac{A'k}{\sqrt{1+\lambda^2\,.\,\cos.^2\psi}}\,;$$ [1582]

Expression of gravity p.

and by substituting for A' its value [1568], we shall find,

$$p = \frac{4\,\pi\,\rho\,.\,k\,.\,(1+\lambda^2)\,.\,\{\lambda - \text{arc. tang.}\,\lambda\}}{\lambda^3\,.\,\sqrt{1+\lambda^2\,.\,\cos.^2\psi}}\,; \qquad (3)$$ [1583]

this equation gives the relation between the gravity and the latitude; but we must first determine the constant quantities which it contains.

Let T be the number of seconds in which the fluid mass makes one [1583']
revolution about its axis; then the centrifugal force g, at the distance 1 [1583'']
from the axis of rotation, will be, by § 9 of the first book, equal to $\frac{4\pi^2}{T^2}$;* [1583''']
therefore we shall have,

$$q = \frac{g}{\frac{4}{3}\pi\rho} = \frac{12\,\pi^2}{4\,\pi\,\rho\,.\,T^2} = \frac{3\,\pi}{\rho\,.\,T^2}\,;$$ [1584]

hence $4\,\pi\,\rho = \frac{12\,\pi^2}{q\,.\,T^2}$. The radius of curvature of the elliptical meridian is† [1584']

Radius of curvature of an ellipsis.

$$\frac{(1+\lambda^2)\,.\,k}{(1+\lambda^2\,.\,\cos.^2\psi)^{\frac{3}{2}}}.$$ [1585]

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* (1095) The centrifugal force [54'] is as the square of the velocity, divided by the radius. Now a point, at the distance 1 from the axis, describes the space $2\,\pi$ in one revolution; therefore its velocity, in one second, is $\frac{2\,\pi}{T}$; its square $\frac{4\pi^2}{T^2}$; divided by the [1584a]
radius 1, gives the centrifugal force equal to $\frac{4\pi^2}{T^2}$, as above. This is the quantity called g [1569'], and by substituting it in [1573'], we get the second expression of q [1584]. Multiplying this by $\frac{4\,\pi\,\rho}{q}$, we get $4\,\pi\,\rho = \frac{12\,\pi^2}{q\,.\,T^2}$ [1584']. [1584b]

† (1096) If in the expression of the radius of curvature r [53c], we change the co-ordinates y, z, into b', a, respectively, in order to conform to the present notation, we shall get $r = \frac{ds\,.\,db'}{dda}$. Now from [1579e], we have [1584c]

$$d\,s = -\,d\,a\,.\,(k^2+a^2\,\lambda^2)^{\frac{1}{2}}\,.\,(k^2-a^2)^{-\frac{1}{2}}\,;$$

[1585'] Putting therefore c for the length of a degree, in the latitude ψ, we shall have,*

[1586] $$\frac{(1+\lambda^2)\,.\,\pi\,.\,k}{(1+\lambda^2\,.\,\cos.^2\psi)^{\frac{3}{2}}}=200\,.\,c.$$

This equation, combined with the preceding, gives,

[1587] $$\frac{4\,\pi\,\rho\,.\,k\,.\,(1+\lambda^2)}{\sqrt{1+\lambda^2\,.\,\cos.^2\psi}}=200\,.\,c\,.\{1+\lambda^2\,.\,\cos.^2\psi\}\,.\,\frac{12\,\pi}{q\,.\,T^2};$$

hence we shall have,

[1588] $$p=200\,.\,c\,.\{1+\lambda^2\,.\,\cos.^2\psi\}\,.\,\frac{\{\lambda-\text{arc. tang.}\,\lambda\}}{\lambda^3}\,.\,\frac{12\,\pi}{q\,.\,T^2}\,.$$

[1588'] Let l be the length of a simple pendulum, which makes one oscillation

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and its differential, $d\,s$ being constant, is

$$0=-d\,d\,a\,.\,(k^2+a^2\lambda^2)^{\frac{1}{2}}\,.\,(k^2-a^2)^{-\frac{1}{2}}-a\,d\,a^2\,.\,\lambda^2\,.\,(k^2+a^2\lambda^2)^{-\frac{1}{2}}\,.\,(k^2-a^2)^{-\frac{1}{2}}$$
$$-a\,d\,a^2\,.\,(k^2+a^2\lambda^2)^{\frac{1}{2}}\,.\,(k^2-a^2)^{-\frac{3}{2}}.$$

Dividing this by the coefficient of $d\,d\,a$, we get

[1584d] $$d\,d\,a=-a\,d\,a^2\,.\left\{\frac{\lambda^2}{k^2+a^2\lambda^2}+\frac{1}{k^2-a^2}\right\}=-a\,d\,a^2\,.\,\frac{k^2\,.\,(1+\lambda^2)}{(k^2+a^2\lambda^2)\,.\,(k^2-a^2)}.$$

Substituting these values of $d\,s$, $d\,d\,a$, also $d\,b'=-a\,d\,a\,.\,(k^2-a^2)^{-\frac{1}{2}}\,.\,(1+\lambda^2)^{\frac{1}{2}}$,

[1584e] [1579d], in r [1584c], we get $r=\frac{-(k^2+a^2\lambda^2)^{\frac{3}{2}}}{k^2\,.\,(1+\lambda^2)^{\frac{1}{2}}}$. Putting the two expressions of t [1579, 1581] equal to each other, we get $k^2+\lambda^2a^2=\frac{(1+\lambda^2)\,.\,k^2}{1+\lambda^2\,.\,\cos.^2\psi}$; substituting this in

[1584f] [1584e], it becomes $r=\frac{(1+\lambda^2)\,.\,k}{(1+\lambda^2\,.\,\cos.^2\psi)^{\frac{3}{2}}}$; the sign of the radical being taken so as to make the expression positive, as in [1585].

* (1097) If the radius of the circle of curvature be r, its semi-circumference will be $\pi\,.\,r$, corresponding to 200 degrees of the centesimal division of the quadrant; hence $\pi\,.\,r=200\,.\,c$. Substituting r [1584f], it becomes as in [1586]. Multiplying this by

[1587a] $4\,\rho=\frac{12\pi}{q\,.\,T^2}$ [1584b], and then by $1+\lambda^2\,.\,\cos.^2\psi$, it becomes as in [1587]; again multiplying this last expression by $\frac{\lambda-\text{arc. tang.}\,\lambda}{\lambda^3}$, the first member becomes the same as the value of p [1583], which will therefore be as in [1588].

in a second of time; then it follows, from § 11 of the first book, that*

$$p = \pi^2 . l.$$ [1588'']

Comparing these two expressions of p, we obtain,

$$q = \frac{2400 . c . \{\lambda - \text{arc. tang.}\, \lambda\} . \{1 + \lambda^2 . \cos.^2 \psi\}}{\pi . l . T^2 . \lambda^3}. \qquad (4)$$ [1589]

This equation, and the equation [1574], will give the values of q and λ, by means of the length l of a pendulum, vibrating in a second of time, and the length c of a degree of the meridian, both being observed in the latitude ψ. [1589']

Supposing $\psi = 50°$, these equations will give,† [1589'']

$$q = \frac{800 . c}{\pi . l . T^2} - \tfrac{1}{4} . \left(\frac{800 . c}{\pi . l . T^2}\right)^2 + \&c.;$$ [1590]

Formulas for the ellipticity.

$$\lambda^2 = \tfrac{5}{2} . q + \tfrac{75}{14} . q^2 + \&c.;$$ [1590']

we have by observation, as we shall hereafter find,‡

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* (1098) To conform to the present notation, we must, in [86], put $r = l$, $g = p$, $T = 1$ second, and it will become $1 = \pi . \left(\frac{l}{p}\right)^{\frac{1}{2}}$; hence $p = \pi^2 . l$, as in [1588'']. Putting this equal to the expression [1588], and multiplying by $\frac{q}{\pi^2 . l}$, we get q [1589]. [1588a]

† (1099) Multiplying [1574] by $9 + 3\lambda^2$, and substituting [48] Int.

$$\text{arc. tang.}\, \lambda = \lambda - \tfrac{1}{3}\lambda^3 + \&c.,$$ [1589a]

we get $0 = 9\lambda + 2q . \lambda^3 - (9 + 3\lambda^2) . (\lambda - \frac{1}{3}\lambda^3 + \frac{1}{5}\lambda^5 - \&c.)$; dividing by $2\lambda^3$, we obtain, after reduction, $q = \frac{2}{5}\lambda^2 - \frac{12}{35}\lambda^4 + \&c.$ λ^2 being very small, if we neglect λ^4, we shall have $q = \frac{2}{5}\lambda^2$, or $\lambda^2 = \frac{5}{2}q$ nearly. If we retain the term depending on λ^4, we shall get $\lambda^2 = \frac{5}{2}q + \frac{6}{7}\lambda^4 = \frac{5}{2}q + \frac{6}{7} . (\frac{5}{2}q)^2 = \frac{5}{2}q + \frac{75}{14}q^2$, as in [1590']. If we put, as in [1589''], $\psi = 50° = 45^d$, we shall have $\cos.^2 50° = \frac{1}{2}$. Substituting this, and the value of arc. tang. λ [1589a], in [1589], neglecting q^3, or λ^6, and using $\lambda^2 = \frac{5}{2}q$ [1589b], we shall get, by putting for brevity $\frac{800 . c}{\pi . l . T^2} = c'$, [1589b] [1589c]

$$q = \frac{2400 . c . \{\frac{1}{3}\lambda^3 - \frac{1}{5}\lambda^5 + \&c.\} . \{1 + \frac{1}{2}\lambda^2\}}{\pi . l . T^2 . \lambda^3} = \frac{800 . c . \{1 - \frac{3}{5}\lambda^2 + \&c.\} . \{1 + \frac{1}{2}\lambda^2\}}{\pi . l . T^2}$$

$$= c' . \{1 - \tfrac{3}{5}\lambda^2 + \&c.\} . \{1 + \tfrac{1}{2}\lambda^2\} = c' . \{1 - \tfrac{1}{10}\lambda^2 + \&c.\} = c' - \tfrac{1}{4}c' . q + \&c.$$

$$= c' - \tfrac{1}{4}c'^2 + \&c., \qquad \text{as in [1590].}$$ [1589d]

‡ (1100) A metre is the ten millionth part of a quadrant of the meridian [2035], and this is nearly equal to $100.c$; c being the length of a degree of the meridian in the latitude [1590a]

[1591] $c = 100000^{\text{met.}}$; $l = 0^{\text{met.}},741608$.

[1591'] We also have $T = 99727''$;* hence we obtain,

[1592] $q = 0,00344957$; $\lambda^2 = 0,00868767$.†

[1592'] The ratio of the axis of the equator to that of the pole, being $\sqrt{1+\lambda^2}$, it becomes, in this case, $1,00433441$; therefore these two axes are nearly

[1592''] in the ratio of $231,7$ to $230,7$, and by what has been said, [1578''], the force of gravity at the pole, is to that at the equator, in the same ratio.

We shall have the polar semi-axis k, by means of the equation,

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$50^\circ = 45^d$, hence $c = 100000^{\text{met.}}$. Putting $\sin.^2\psi = \frac{1}{2}$, in the formula [2054], we get

[1590b] $l = 0,739502 + 0,002104 = 0,741606$, being nearly the same as in [1591]. This requires some correction, for the mistakes in computation mentioned [2039a, 2048a].

* (1101) In a year, which is nearly $365\frac{1}{4}$ days, the earth makes nearly $366\frac{1}{4}$ revolutions about its axis. Now the solar day is here supposed to be divided into 10 hours, or 100000

[1591a] seconds; therefore the time of one revolution will be $100000'' . \frac{365\frac{1}{4}}{366\frac{1}{4}} = 99727''$, as in [1591'].

† (1102) On account of the great importance of these quantities, in determining the figure of the earth, we shall give the calculation at full length, putting, as in [1589c], $c' = \frac{800 . c}{\pi . l . T^2}$.

Computation of the ellipticity of the earth.

$800 . c =$	80000000	log.	7.9030900
π		log. co.	9.5028501
l	0.741608	log. co.	0.1298256
T	99727	log. co.	5.0011872
T			5.0011872
$c' =$	0.00345255	log.	7.5381401
c'		log.	7.5381401
$\frac{1}{4}$		log.	9.3979400
$\frac{1}{4} . c'^2 =$	0.00000298	log.	4.4742202
$q =$	$0.00344957 = c' - \frac{1}{4} . c'^2$ [1589d]		

$q =$	0.00344957	log.	7.53776
q		log.	7.53776
75		log.	1.87506
14		log. co.	8.85387
$\frac{75}{14} . q^2 =$	0.00006375	log.	5.80445
$\frac{5}{2} . q =$	.00862392		
$\lambda^2 =$	0.00868767		
$1 + \lambda^2 =$	1.00868767	log.	0.00375672
$\sqrt{(1+\lambda^2)} =$	1.00433444	log.	0.00187836
$\sqrt{(1+\lambda^2)} - 1$	0.00433444	log.	7.6369330
[1591b] $\frac{\sqrt{(1+\lambda^2)}}{\sqrt{(1+\lambda^2)}-1} =$	231.7	log.	2.3649453

The values of λ^2, q, here computed, agree with those in [1592]. We have also found, in

[1592a] [1591b], that $\frac{\sqrt{(1+\lambda^2)}-1}{\sqrt{(1+\lambda^2)}} = 1 - \frac{1}{\sqrt{(1+\lambda^2)}} = \frac{1}{231.7}$; hence $\frac{1}{\sqrt{(1+\lambda^2)}} = \frac{230.7}{231.7}$; which represents the ratio of the polar to the equatorial diameter of the earth [1574a]; as in [1592''].

Polar semi-axis.

$$k = \frac{200.c.(1+\tfrac{1}{2}\lambda^2)^{\frac{3}{2}}}{\pi.(1+\lambda^2)} = \frac{200.c}{\pi}.\{1-\tfrac{1}{4}\lambda^2+\&c.\};*$$ [1593]

which gives,

$$k = 6352534^{\text{met.}}.$$ [1594]

To obtain the attraction of a sphere, of the radius k, and of any density whatever, we shall observe, that a sphere, whose radius is k, and density ρ, [1594′] acts upon a point, situated on its surface, with a force equal to $\frac{4}{3}\pi\rho.k$, [1594″] [1430*l*]; and this, by means of the equation [1583], is equal to

$$\frac{\lambda^3.p.\sqrt{1+\tfrac{1}{2}\lambda^2}}{3.(1+\lambda^2).(\lambda-\text{arc. tang. }\lambda)};\dagger \qquad \text{or} \qquad p.\{1-\tfrac{3}{20}\lambda^2+\&c.\};$$

[1594‴]

or lastly $0{,}998697.p$; p representing the force of gravity on the parallel

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* (1103) Putting $\cos.^2\psi=\frac{1}{2}$ in [1586], we get k [1593]; and by substituting the values of c, λ^2, [1591, 1592], it becomes, by the adjoined calculation, as in [1594]. The third member of [1593] is easily deduced from the first, by development in a series.

$1+\frac{1}{2}.\lambda^2=1.00434383$	log.	0.0018824
Its half		0.0009412
$200.c=20000000$	log.	7.3010300
π	log. co.	9.5028501
$1+\lambda^2=1.00868767$	log. co.	9.9962433
$k=6352534$	log.	6.8029470

[1592*b*]

† (1104) Putting $\cos.^2\psi=\frac{1}{2}$ in [1583], we shall get the value of p, corresponding to the latitude of $50^{\circ}=45^{d}$; and since,

$$\sqrt{(1+\lambda^2.\cos.^2\psi)}=\sqrt{(1+\tfrac{1}{2}\lambda^2)}=1+\tfrac{1}{4}\lambda^2-\tfrac{1}{32}\lambda^4+\&c.;$$

$$\lambda-\text{arc. tang. }\lambda=\tfrac{1}{3}\lambda^3-\tfrac{1}{5}\lambda^5+\tfrac{1}{7}\lambda^7-\&c.; \qquad [1589a],$$

it will become,

$$p=\frac{4\pi\rho.k.(1+\lambda^2).(\frac{1}{3}\lambda^3-\frac{1}{5}\lambda^5+\frac{1}{7}\lambda^7-\&c.)}{\lambda^3.(1+\frac{1}{4}\lambda^2-\frac{1}{32}\lambda^4+\&c.)}=\frac{4\pi\rho.k}{3}.\frac{(1+\lambda^2).(1-\frac{3}{5}\lambda^2+\frac{3}{7}\lambda^4-\&c.)}{1+\frac{1}{4}\lambda^2-\frac{1}{32}\lambda^4+\&c.};$$

hence

$$\tfrac{4}{3}\pi\rho.k=\frac{p.(1+\frac{1}{4}\lambda^2-\frac{1}{32}\lambda^4+\&c.)}{(1+\lambda^2).(1-\frac{3}{5}\lambda^2+\frac{3}{7}\lambda^4-\&c.)}=\frac{p.(1+\frac{1}{4}\lambda^2-\frac{1}{32}\lambda^4+\&c.)}{1+\frac{2}{5}\lambda^2-\frac{6}{35}\lambda^4+\&c.)}$$

$$=p.(1-\tfrac{3}{20}\lambda^2+\tfrac{1121}{5600}\lambda^4-\&c.)=p.(1-0{,}001303+0{,}000015-\&c.)$$

$$=0{,}998712.p \qquad [1592];$$ [1593*a*]

which represents the attraction of a sphere [1594″], of the radius k, and density ρ, upon a point of its surface. This differs from the value $0{,}998697.p$ [1594‴], on account of the neglect of the term $\frac{1121}{5600}.\lambda^4=0{,}000015$, by the author. [1593*b*]

[1594''''] of 50°. Hence it is easy to determine the attractive force of a sphere of any radius, and of any density, upon a point placed within or without its surface.

20. If the equation [1574] were susceptible of several real roots, there
[1594v] would be several figures of equilibrium, corresponding to the same rotatory motion; we must therefore examine, whether this equation has several real roots. For this purpose, we shall put,

Equation of the curve to determine the number of figures of equilibrium.

[1594vi]
$$\varphi = \frac{9\lambda + 2q.\lambda^3}{9+3\lambda^2} - \text{arc. tang.}\,\lambda\,;$$

so that this function φ, being put equal to nothing, will produce the equation [1574]. It is evident that if we increase the quantity λ, from nothing to infinity,
[1594vii] the expression of φ will begin and end with a positive value.* Therefore

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(1105) Putting the radius of the equator equal to k', the expression q [1573'], may be
[1593c] put under the form $q = \frac{g k'}{\frac{4}{3}\pi\rho . k'}$; in which gk' [1569'] represents the centrifugal force of a point, at the distance k' from the axis of rotation; and $\frac{4}{3}\pi\rho . k'$ [1594''] is equal to the attraction of a sphere, of the radius k', upon a point of its surface. Hence we have the following expression,

Centrifugal force.

[1594a]
$$q = \frac{\text{Centrifugal force at the equator of a sphere whose radius is } k' \text{ and time of revolution } T \text{ seconds}}{\text{Gravity of this sphere upon a point of its surface.}}.$$

This quantity for the earth is, by [1592], equal to $0,00344957 = \frac{1}{289}$, and is what, in [327f], is called $\frac{n^2}{g}$, n^2 being the same as g [1564t], and the quantity g, used in [327f], is the expression of the attraction of a sphere upon a point of its surface.

[1594b] * (1106) The expression $q = \frac{g}{\frac{4}{3}\pi\rho}$ [1584], which is a positive quantity, may be made to vary, by a change in the value of ρ, or g, and the corresponding values of λ^2 [1590'], and φ [1594vi], would partake of these changes. When λ is very small, we may substitute in [1594vi], the value arc. tang. λ [1589a], and we shall get,

[1594c]
$$\varphi = \frac{9\lambda+2q.\lambda^3}{9+3\lambda^2} - \lambda + \tfrac{1}{3}\lambda^3 - \tfrac{1}{5}\lambda^5 + \&c. = \tfrac{2}{9}q.\lambda^3.(1-\tfrac{1}{3}\lambda^2+\&c.) - \tfrac{4}{45}\lambda^5 + \&c. = \tfrac{2}{9}q.\lambda^3$$

nearly. This makes $\varphi = 0$, when $\lambda = 0$, which is also evident from [1594vi], without
[1594d] reducing it to a series. If λ be small and positive, $\varphi = \frac{2}{9}q.\lambda^3$ will be positive. If $\lambda = \infty$,
[1594e] arc. tang. $\lambda = \frac{1}{2}\pi$, and [1594vi] will become $\varphi = \frac{9\infty + 2q.\infty^3}{9+3\infty^2} - \frac{1}{2}\pi = \frac{2q}{3}\infty + \&c.$, which is positive and infinite, as in [1594vii].

by supposing a curve to be formed, whose absciss is λ, and ordinate φ, this curve will cut its axis, when $\lambda = 0$; afterwards the ordinates will be [1594^{viii}]
positive and increasing. When they have attained their *maximum*, they will decrease, and the curve will cut the axis a second time, at a point which will [1594^{ix}]
determine the value of λ, corresponding to the state of equilibrium of the fluid mass. The ordinates will then become negative; and since they are positive when $\lambda = \infty$, it is necessary that the curve should cut the axis a [1594^{x}]
third time, and this point determines a second value of λ, which satisfies the equilibrium.* Hence we see that for a given value of q, or for a given [1594^{xi}]
rotatory motion, there are several figures with which the equilibrium may subsist.

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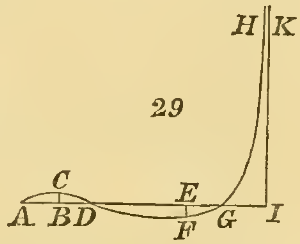
* (1107) Let AI fig. 29 be the axis of the positive values of λ, $ACDFGH$ the proposed curve, C the origin, corresponding to $\lambda = 0$; so that for any absciss $AB = \lambda$, the corresponding ordinate is $BC = \varphi$; the positive values of φ being taken above the axis AI, the negative ones below it; then the curve will cut the axis in the points where $\varphi = 0$; which is evidently the case when $\lambda = 0$ [1594d]. For any small values of λ, we have $\varphi = \frac{2}{9} q \cdot \lambda^3$ [1594d], which is positive; therefore the first part of the curve must fall above the axis AI. In the computation of the figure of the earth [1592], we have found that $\lambda^2 = 0{,}00868767$ satisfies the equation [1574], or, in other words, makes $\varphi = 0$ [1594^{vi}]; therefore the curve must cut the axis at another point D, which is very near to A, corresponding to this value of λ. At the point D the direction of the curve must be towards F, below the axis. For, [1594f]



by the value of $\frac{d\varphi}{d\lambda}$ [1595], which is investigated, in the following note, we have

$$\frac{d\varphi}{d\lambda} = \frac{6\lambda^2 \cdot q}{(3\lambda^2+9)^2 \cdot (1+\lambda^2)} \cdot \left\{ \lambda^4 + 10\lambda^2 - \frac{6\lambda^2}{q} + 9 \right\}, \qquad [1594g]$$

in which the factors $6\lambda^2 \cdot q$, $(3\lambda^2+9)^2$, $(1+\lambda^2)$, are positive; and by substituting in the other factor the value $\frac{6\lambda^2}{q} = 15 + \frac{225}{7} q + \&c.$ [1598], we shall get,

$$\lambda^4 + 10\lambda^2 - \frac{6\lambda^2}{q} + 9 = \lambda^4 + 10\lambda^2 - 6 - \tfrac{225}{7} q. \qquad [1594h]$$

This factor is negative for the earth, as is evident by using the values of λ^2, q, [1592]; therefore the value of $\frac{d\varphi}{d\lambda}$ must be negative at the point D, hence the direction of the curve will be downwards towards F, and the values of φ must become *negative*. But there must be a limit in these negative values, because we have seen, in [1594e], that if $\lambda = \infty$,

To determine the number of these figures, we shall observe, that we have

[1595] $$d\varphi = \frac{6\lambda^2 d\lambda . \{q.\lambda^4 + (10q-6).\lambda^2 + 9q\}}{(3\lambda^2+9)^2.(1+\lambda^2)}. *$$

[1595'] The supposition of $d\varphi = 0$, gives,

[1596] $$0 = q.\lambda^4 + (10q-6).\lambda^2 + 9q.$$

Hence, by noticing only the positive values of λ, we obtain,†

[1597] $$\lambda = \sqrt{\tfrac{3}{q} - 5 \pm \sqrt{\left(\tfrac{3}{q}-5\right)^2 - 9}}.$$

[1597'] From these values of λ we can determine the *maxima* and *minima* of the ordinate φ. Therefore there are but two such ordinates on the side of the positive abscisses. Hence it follows, that on this side, the

Only two figures of equilibrium of an ellipsoid.

[1594i] we shall have $\varphi = \infty$ and *positive;* and in passing from the negative to the positive values of φ, there must be at least one point G, where the ordinate φ will be nothing, and the curve will then again cut the axis, as in [1594x].

* (1108) Since $d\,.\,\text{arc. tang.}\,\lambda = \frac{d\lambda}{(1+\lambda^2)}$, [51] Int.; we shall have, for the differential

[1595a] of φ [1594vi], $d\varphi = d\lambda . \frac{(9+3\lambda^2).(9+6q.\lambda^2)-(9\lambda+2q.\lambda^3).6\lambda}{(9+3\lambda^2)^2} - \frac{d\lambda}{1+\lambda^2}$; which, by reducing to the same denominator, and connecting similar terms, becomes as in [1595].

† (1109) The maximum and minimum values of φ are found, as usual, by putting the expression of $d\varphi$ [1595] equal to nothing; which requires that the factor, given in [1596], should be equal to nothing. This factor, being divided by q, becomes

$$\lambda^4 - 2.\left(\frac{3}{q}-5\right).\lambda^2 + 9 = 0,$$

a quadratic equation in λ^2; hence,

[1596a] $$\lambda^2 = \frac{3}{q} - 5 \pm \left\{\left(\frac{3}{q}-5\right)^2 - 9\right\}^{\frac{1}{2}};$$

and we obtain the two positive values of λ [1597], corresponding to the maximum and minimum values of φ; the maximum being at the upper point C of the branch ACD, the minimum at the lower point F of the branch DFG. Moreover, as there are only two positive values of λ [1597] which give a maximum or minimum, there can be only two such branches; therefore after the curve crosses the axis at G, the ordinate φ must always increase, till it becomes infinite when $\lambda = \infty$ [1594e].

curve can cut the axis in three points only, including the point of origin; [1597″]
therefore the number of figures which satisfy the equilibrium is reduced to two.

The curve on the side of the negative abscisses, is of exactly the same form as on the side of the positive abscisses, except in the sign of the co-ordinates;* therefore it must cut the axis on each side in corresponding [1597‴]
points, equidistant from the origin of the co-ordinates. Hence the negative values of λ, which satisfy the equilibrium, are the same as the positive values, except in the signs; these negative values give the same elliptical [1597⁗]
figures, since the square only of λ enters in the determination of these figures.† It is therefore unnecessary to consider the curve, on the side of the negative abscisses.

If we suppose q to be very small, as is the case for the earth, we may satisfy the equation [1574], in either of the two hypotheses, of λ^2 being very [1597ᵛ]
small, or very great. In the first case, we shall have, by the preceding article [1590′].

Formula for λ^2, when the body is nearly spherical.

$$\lambda^2 = \tfrac{5}{2}\, q + \tfrac{75}{14}\, q^2 + \&c. \qquad [1598]$$

To obtain the value of λ^2, in the second hypothesis, we shall observe, that then $\text{arc. tang.}\,\lambda$ differs but very little from $\frac{1}{2}\pi$, and if we suppose [1598′]

$$\text{arc. tang.}\,\lambda = \tfrac{1}{2}\pi - \alpha, \qquad [1599]$$

α will be a very small angle, whose tangent is $\dfrac{1}{\lambda}$; we shall have,‡

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* (1110) If in the value of φ [1594ᵛⁱ] we write $-\lambda$ for λ, and put φ' for the corresponding value of φ, we shall get, by changing $\text{arc. tang.}\,(-\lambda)$ into $-\text{arc. tang.}\,\lambda$, $\varphi' = \dfrac{-9\lambda - 2q\,.\,\lambda^3}{9+3\lambda^2} + \text{arc. tang.}\,\lambda$, which is evidently equal to $-\varphi$ [1594ᵛⁱ]; therefore [1597a]
$\varphi' = -\varphi$, as in [1597‴].

† (1111) The semi-axes of the ellipsoid k, $k\,.\sqrt{(1+\lambda^2)}$ [1574a], by using the value of k [1593], become functions of λ^2; and as $(-\lambda)^2 = \lambda^2$, the negative values of λ will [1597b]
produce the same expressions of the semi-axes as its positive values.

‡ (1112) Putting for brevity $\lambda' = \text{arc. tang.}\,\lambda$, we get $\lambda' = \frac{1}{2}\pi - \alpha$, or $\alpha = \frac{1}{2}\pi - \lambda'$; therefore $\text{tang.}\,\alpha = \text{cot.}\,\lambda' = \dfrac{1}{\text{tang.}\,\lambda'} = \dfrac{1}{\lambda}$ [34′] Int. Hence

[1600] $$\alpha = \frac{1}{\lambda} - \frac{1}{3\lambda^3} + \frac{1}{5\lambda^5} - \&c.;$$

consequently,

[1601] $$\text{arc. tang. } \lambda = \frac{\pi}{2} - \frac{1}{\lambda} + \frac{1}{3\lambda^3} - \frac{1}{5\lambda^5} + \&c.$$

The equation [1574] will then become.

[1602] $$\frac{9\lambda + 2q.\lambda^3}{9 + 3\lambda^2} = \frac{\pi}{2} - \frac{1}{\lambda} + \frac{1}{3\lambda^3} - \&c.;$$

and by inverting the series, we shall obtain,*

Formula for the ellipticity in a very oblate ellipsoid.

$$\lambda = \frac{3\pi}{4q} - \frac{8}{\pi} + \frac{4q}{\pi} \cdot \left\{ 1 - \frac{64}{3\pi^2} \right\} + \&c.$$

[1603] $$= 2{,}356195 \cdot \frac{1}{q} - 2{,}546479 - 1{,}473885 \cdot q + \&c.$$

[1603'] We have seen, in the preceding article [1592], that for the earth we have

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[1600a] $\alpha = \text{arc. tang.} \frac{1}{\lambda} = \frac{1}{\lambda} - \frac{1}{3\lambda^3} + \frac{1}{5\lambda^5} - \&c.$ [48] Int., as in [1600]. Substituting this in [1599], we get [1601], by means of which the equation [1574] becomes as in [1602].

* (1113) Multiplying [1602] by $3\lambda^2 + 9$, and arranging the terms of the second member according to the decreasing powers of λ, it becomes,

$$9\lambda + 2q.\lambda^3 = \frac{3\pi}{2}.\lambda^2 - 3\lambda + \frac{9\pi}{2} - \frac{8}{\lambda} + \&c.$$

Transposing 9λ, and dividing by $2q.\lambda^2$, we get,

[1603a] $$\lambda = \frac{3\pi}{4q} - \frac{6}{q} \cdot \frac{1}{\lambda} + \frac{9\pi}{4q} \cdot \frac{1}{\lambda^2} - \frac{4}{q.\lambda^3} + \&c.;$$

from this the value of λ may be found, by means of La Grange's formulas [629c], which, by putting $x = \lambda$, $\psi(x) = x$, $\psi(t) = t$, $\psi'(t) = 1$, become,

[1603b] $$\lambda = t + F(\lambda);$$

[1603c] $$\lambda = t + F(t) + \frac{1}{1.2} \cdot \frac{d.\{F(t)^2\}}{dt} + \frac{1}{1.2.3} \cdot \frac{d.\{F(t)^3\}}{dt^2} + \&c.$$

Comparing the values of λ [1603a, b], we get,

$$t = \frac{3\pi}{4q}; \qquad F(\lambda) = -\frac{6}{q} \cdot \frac{1}{\lambda} + \frac{9\pi}{4q} \cdot \frac{1}{\lambda^2} - \&c.$$

$q = 0{,}00344957$; this value of q being substituted in the preceding [1603′]
expression, we get $\lambda = 680{,}49$. Thus the ratio of the equatorial diameter to the polar axis, which is represented by $\sqrt{1+\lambda^2}$ [1592′], is equal to [1603″]
$680{,}49$, when the spheroid is very oblate.

There is a limit in the value of q, beyond which the equilibrium is impossible with an elliptical figure. For if we suppose the curve to cut the [1603‴]
axis at the point of origin only, and to touch it at another point; we shall have, at this point of contact, $\varphi = 0$, $d\varphi = 0$;* the value of φ cannot [1603⁗]

Hence

$$F(t) = -\frac{6}{q}\cdot\frac{1}{t} + \frac{9\pi}{4q}\cdot\frac{1}{t^2} - \&c. = -\frac{8}{\pi} + \frac{4q}{\pi} - \&c.; \qquad \{F(t)\}^2 = \frac{36}{q^2}\cdot\frac{1}{t^2} - \&c.;$$

$$\frac{d.\{F(t)^2\}}{1.2.dt} = -\frac{36}{q^2}\cdot\frac{1}{t^3} + \&c. = -\frac{36}{q^2}\cdot\left(\frac{4q}{3\pi}\right)^3 = -\frac{256q}{3\pi^3} + \&c. \qquad [1603d]$$

Substituting these in [1603c], it becomes, $\lambda = \frac{3\pi}{4q} - \frac{8}{\pi} + \frac{4q}{\pi} - \frac{256q}{3\pi^3} + \&c.$, as in [1603]. Substituting in this $\pi = 3{,}141592$, $\frac{3\pi}{4} = 2{,}356195$, $\frac{8}{\pi} = 2{,}546479$, $\frac{4}{\pi}\cdot\left(1 - \frac{64}{3\pi^2}\right) = 1{,}478885$, nearly; we shall get the second expression of λ [1603]; and by using the value of q [1592], corresponding to the earth, we shall get,

$$\lambda = 683{,}040 - 2{,}546 - 0{,}005 = 680{,}49; \qquad [1603e]$$

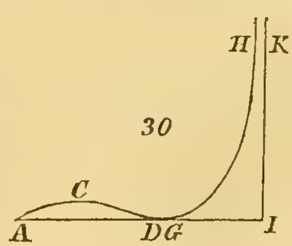
hence $\sqrt{(1+\lambda^2)} = 680{,}49$ nearly, which represents, by [1574a], the ratio of the equatorial diameter to the polar axis of the body, in this hypothesis.

* (1114) If we suppose the value of q to be increased by the positive quantity f, and [1603f]
that the expression of φ [1594vi] then becomes $\varphi + \delta\varphi$, we shall have,

$$\varphi + \delta\varphi = \frac{9\lambda + (2q+2f).\lambda^3}{9+3\lambda^2} - \text{arc. tang. } \lambda = \left\{\frac{9\lambda + 2q.\lambda^3}{9+3\lambda^2} - \text{arc. tang. } \lambda\right\} + \frac{2f.\lambda^3}{9+3\lambda^2}$$

$$= \varphi + \frac{2f.\lambda^3}{9+3\lambda^2}.$$

hence $\delta\varphi = \frac{2f.\lambda^3}{9+3\lambda^2}$, which is always positive. Therefore by increasing successively [1603g]
the value of q, we shall increase the positive values of φ, and decrease the negative values; so that the lower branch DFG of the curve, fig. 29, page 225, will approach towards the axis AI, till the points D, F, G, coincide, as in fig. 30. The ordinate corresponding to this point D, will then be $\varphi = 0$, and as this is also the point of minimum of that branch, we shall have $d\varphi = 0$, as in [1595′]; which is also evident from the



then become negative, on the side of the positive abscisses, which are the only ones necessary to be noticed. The value of q, determined by the two equations $\varphi = 0$, and $d\varphi = 0$, will therefore be the limit of q with which the equilibrium can subsist; and a greater value would render the
[1603v] equilibrium impossible. For q being supposed to be increased by the quantity f, the function φ will be increased by $\frac{2f.\lambda^3}{9+3\lambda^2}$ [1603g]; and as the value of φ, corresponding to q, never becomes negative, [1603''''], whatever be λ, the same function, corresponding to $q+f$, will always be
[1603vi] positive, and cannot vanish; therefore the equilibrium will then become impossible. It follows also, from this analysis, that there is but one real and
[1603vii] positive value of q, which can satisfy the two equations $\varphi = 0$, and $d\varphi = 0$. These equations produce the following,*

[1604]
$$q = \frac{6\lambda^2}{(1+\lambda^2).(9+\lambda^2)};$$
$$0 = \frac{7\lambda^5 + 30\lambda^3 + 27\lambda}{(1+\lambda^2).(3+\lambda^2).(9+\lambda^2)} - \text{arc. tang. } \lambda.$$

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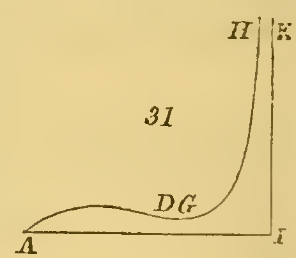
consideration that the ordinates φ, $\varphi + d\varphi$, corresponding to
[1603h] the two infinitely near points D, G, are both equal to nothing. Hence it follows, that in this case, there can be no negative value of φ, corresponding to the positive abscisses. If we still increase the value of q, the ordinate φ will be increased, by the positive quantity $\delta\varphi$, the curve, as in fig. 31, will not touch the axis, and the equilibrium will then be impossible with an ellipsoidal figure.



* (1115) The equation $d\varphi = 0$ gives the formula [1596], whence we get,

$$q = \frac{6\lambda^2}{9+10\lambda^2+\lambda^4} = \frac{6\lambda^2}{(1+\lambda^2).(9+\lambda^2)},$$

which is the first of the equations [1604]. Hence

[1604a] $$9\lambda + 2q.\lambda^3 = 9\lambda + \frac{12\lambda^5}{9+10\lambda^2+\lambda^4} = \frac{21\lambda^5+90\lambda^3+81\lambda}{9+10\lambda^2+\lambda^4} = \frac{3.(7\lambda^5+30\lambda^3+27\lambda)}{(1+\lambda^2).(9+\lambda^2)}.$$

Substituting this in [1574], it becomes as in the second of the equations [1604]. This may be simplified by rejecting the factor $3+\lambda^2$, which occurs in the numerator and denominator of the first term. Then it becomes $0 = \frac{\lambda.(9+7\lambda^2)}{(1+\lambda^2).(9+\lambda^2)} - \text{arc. tang. } \lambda$, or

[1604b] $\frac{7\lambda.(\frac{9}{7}+\lambda^2)}{(1+\lambda^2).(9+\lambda^2)} = \text{arc. tang. } \lambda.$ Multiplying the first member by the radius, expressed in

The value of λ, which satisfies this last equation, is $\lambda = 2,5292$; hence [1604']
we deduce $q = 0,337007$;* the quantity $\sqrt{1+\lambda^2}$, which expresses
the ratio of the axis of the equator to that of the pole [1574*a*], is, in this [1604'']
case, equal to $2,7197$.

The value of q, relative to the earth, is equal to $0,00344957$ [1592].
This corresponds to the time of rotation $0^{\text{day}},99727$ [1591']; now we

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sexagesimal seconds, $206264^s,8$, and then taking the tangent of the whole expression,
we get, $\text{tang.}\left\{\frac{1443854^s.\lambda.(\frac{9}{7}+\lambda^2)}{(1+\lambda^2).(9+\lambda^2)}\right\}=\lambda$; whence the value of $\lambda = 2,52932$, [1604*c*]
may be found, by approximation, being nearly as in [1604']. This last formula is adapted
to the use of logarithms; but if we have a table of natural tangents, as in Hutton's tables,
the first member of the expression [1604*b*] may be put under the form

$$\tfrac{1}{8}\cdot\frac{2\lambda}{1+\lambda^2}+\tfrac{9}{8}\cdot\frac{2.(\frac{1}{3}\lambda)}{1+(\frac{1}{3}\lambda)^2},$$

as is easily proved, by reducing to a common denominator, and connecting the terms. Now
from [30''] Int., we have

$$\frac{2\lambda}{1+\lambda^2}=\sin.2.(\text{arc. tang.}\,\lambda), \qquad \frac{2.(\frac{1}{3}\lambda)}{1+(\frac{1}{3}\lambda)^2}=\sin.2.(\text{arc. tang.}\,\tfrac{1}{3}\lambda);$$

hence the expression [1604*b*], becomes

$$\tfrac{1}{8}\sin.2.(\text{arc. tang.}\,\lambda)+\tfrac{9}{8}\sin.2.(\text{arc. tang.}\,\tfrac{1}{3}\lambda)=\text{arc. tang.}\,\lambda, \qquad [1604d]$$

from which we may easily obtain λ as above.

* (1116) The value $\lambda = 2,52932$ [1604*c*], being substituted in q [1604], gives
$q = 0,336998$; also $\sqrt{(1+\lambda^2)} = 2,7197$, nearly, as in [1604'']. [1604*e*]

From what has been said, it appears, that the equilibrium of an ellipsoid of revolution is
possible, in the case of a homogeneous fluid, when q falls anywhere between the extreme
limits $q = 0$, $q = 0,336998$. The case of $q = 0$ corresponds to a sphere at rest; [1604*f*]
the oblateness will increase with the rotatory velocity, or with q; and when $q = 0,336998$,
the ratio of the equatorial to the polar axis will become $2,7197$ [1604'']. If the oblateness
be still *increased*, the rotatory velocity must be *diminished;* and when q again becomes
infinitely small, the figure will be like an infinitely thin lens or plate. Moreover it is evident
that while the oblateness is thus increasing, from nothing, in a sphere, to its greatest value in
a thin plate, there is but one corresponding velocity with which an ellipsoid of a *given figure*
can be in equilibrium. But when the *time of revolution* is given, and q falls within the limits [1604*g*]
[1604*f*], there may be generally found *two figures* of the ellipsoid, which will satisfy the
equation of equilibrium [1597''].

On the limits of the two ellipsoids of revolution.

[1604'''] have generally $q = \frac{g}{\frac{4}{3}\pi\rho}$ [1584]. Hence, as it regards masses of the same density, q will be proportional to the centrifugal force g, arising from
[1604''''] the rotatory motion [1569']; therefore q will be inversely proportional to the square of the time of rotation;* and it follows also, relative to a mass of the same density as the earth, that the time of rotation which corresponds to $q = 0,337007$, will be $0^{\text{day}},10090$. We shall therefore obtain the two following theorems.

"*Any homogeneous fluid mass, of a density equal to the mean density of*
[1605] *the earth, cannot be in equilibrium with an elliptical figure, if the time of its rotation be less than* $0^{\text{day}},10090$. *If this time be greater, there will always be two elliptical figures, and no more, which will satisfy the equilibrium.*"

Important theorems on the limits of the equilibrium of the two ellipsoids of revolution.

"*If the density of the fluid mass be different from that of the earth, we shall have the time of rotation, in which the equilibrium ceases to be possible,*
[1605'] *with an elliptical figure; by multiplying* $0^{\text{day}},10090$ *by the square root of the ratio of the mean density of the earth to that of the fluid mass* [1604k]."

Therefore with a fluid mass whose density is a quarter part of that of the
[1605''] earth, which is nearly the case with the sun, this time would be $0^{\text{day}},20180$;† and if the earth were supposed to be fluid and homogeneous, with a density equal to a ninety-eighth part of its present value, the figure it must take, to
[1605'''] satisfy its present rotatory motion, would be the limit of all the elliptical

Limits in which the equilibrium is possible.

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* (1117) This follows from $q = \frac{3\pi}{\rho \,.\, T^2}$ [1584]. If this be supposed to refer to the
[1604h] earth, and we accent the letters q, r, T, for any other body we shall have $q' = \frac{3\pi}{\rho' . T'^2}$.
[1604i] Hence $\frac{q'}{q} = \frac{\rho . T^2}{\rho' . T'^2}$, and $T' = T . \left(\frac{\rho}{\rho'} . \frac{q}{q'}\right)^{\frac{1}{2}}$. If we substitute in this the values $q = 0,00344957$ [1592], $q' = 0,336998$ [1604e], $T = 0^{\text{day}},99727$, we shall
[1604k] get $T' = 0^{\text{day}},10090 . \left(\frac{\rho}{\rho'}\right)^{\frac{1}{2}}$. This represents the least value of T' with which the equilibrium is possible. For if q' exceed the value [1604e], $0,336998$, the equilibrium becomes impossible, and by increasing q' the value of T' [1604i] decreases. The expression of T' [1604k] produces the two theorems [1605, 1605'].

† (1118) Putting $\rho' = \frac{1}{4}\rho$, in [1604k], it becomes $T = 0^{\text{day}},20180$.

figures, with which the equilibrium could subsist.* The density of Jupiter is one fifth of that of the earth, and the duration of its rotation is [1605''''] $0^{day},41377$; hence it is evident that this duration is within the limits, in which the equilibrium is possible.†

It might be supposed that this limit of q, *is that in which the fluid would begin to fly off, because of its too rapid rotatory motion; but it is easy to prove that this is not the case,* observing that by § 19, [1578''], the gravity at the equator of the ellipsoid is to the gravity at the pole, in the ratio of the polar axis to the equatorial; ‡ which ratio, in the present case, is as 1 to 2,7197 [1604'']. The equilibrium therefore ceases to be possible, because [1605vi] with a more rapid rotatory motion, it would be impossible to give to the fluid mass an elliptical figure, such that the result of its attraction and the [1605vii] centrifugal force would be perpendicular to the surface.

[1605v]

The limit of q is not that where the fluid would fly off by the centrifugal force.

We have heretofore supposed λ^2 to be positive, which corresponds to a spheroid flattened at the poles. We shall now examine whether the equilibrium can take place with a figure lengthened at the poles. Supposing [1605viii] $\lambda^2 = -\lambda'^2$; λ'^2 ought, in this case, to be positive and less than unity,

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* (1119) Putting $\rho' = \frac{1}{98}\rho$, in [1604k], it becomes [1605a]

$$T' = 0^{day},10090 \,.\, \sqrt{98} = 0^{day},99886,$$

which exceeds the time of the earth's rotation $0^{day},99727$ [1591']; therefore the equilibrium, in this case, is impossible, [1605'].

† (1120) Putting $\rho' = \frac{1}{5}\rho$, in [1604k], it becomes [1605b]

$$T' = 0^{day},10090 \,.\, \sqrt{5} = 0^{day},22562,$$

which is less than $0^{day},41377$, the time of rotation of Jupiter [1605'''']. This is within the limits of the possible equilibrium, [1605'].

‡ (1121) The *whole* gravity at the pole, is to the combined effect of gravity and the attraction at the equator, as the equatorial to the polar axis [1578b]; which, in the case of the limiting figure abovementioned, is as 2,7197 to 1 [1604'']. This must therefore be [1605c] far from the limit, where the fluid would begin to fly off, which could happen only when the whole action at the equator is nothing; or when the centrifugal force becomes equal to the attraction of the spheroid.

[1605ix] otherwise the ellipsoid will become an hyperboloid.* The preceding value of $d\varphi$ [1595] gives, by integration,

[1606]
$$\varphi = \int \frac{6\lambda^2 . d\lambda . \{q.\lambda^4 + (10q - 6).\lambda^2 + 9q\}}{(1+\lambda^2).(9+3\lambda^2)^2};$$

the integral being taken from $\lambda = 0$. Substituting for λ its value $\pm\lambda'.\sqrt{-1}$ [1605viii], we shall get,†

[1607]
$$\varphi = \pm\sqrt{-1}.\int \frac{6\lambda'^2 . d\lambda' . \{q.(1-\lambda'^2).(9-\lambda'^2) + 6\lambda'^2\}}{(1-\lambda'^2).(9-3\lambda'^2)^2};$$

A homogeneous prolate ellipsoid cannot be in equilibrium.

Now it is evident, that the elements of this last integral, have all the same
[1607'] sign, from $\lambda'^2 = 0$, to $\lambda'^2 = 1$; hence the function φ cannot become nothing, in that interval; therefore the equilibrium cannot take place with a figure lengthened towards the poles.

21. *If the rotatory motion, first impressed upon a fluid mass, be more*
[1607''] *rapid than that which corresponds to the limit of* q, *we must not, for that reason, conclude that the equilibrium would be impossible, with an elliptical figure*; for it is evident, that as the body becomes more oblate, it will have
[1607'''] a less rapid rotatory motion. Supposing therefore, that there is a cohesive force among the particles, which is the case with all known fluids, this mass might, after a great number of revolutions, attain a rotatory motion, comprised within the limits of equilibrium, and maintain itself in that state. But the possibility of this result is merely a supposition, and to verify it will be
[1607''''] interesting. Moreover, it is important to ascertain, whether there can possibly be several figures of equilibrium, corresponding to one primitive force. For what we have demonstrated, on the possibility of two states of

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* (1122) λ' [1605viii] is the same as e [1574c], and the figures mentioned in [1574c]
[1606a] are the same as in [1605ix], including also a paraboloid.

† (1123) After making the substitution of $\pm\lambda'.\sqrt{-1}$, for λ, and connecting together the terms depending on q, the expression [1606] becomes as in [1607], in which each of
[1606b] the terms λ'^2, $1-\lambda'^2$, $9-\lambda'^2$, $6\lambda'^2$, $9-3\lambda'^2$, is positive, while λ' increases from 0 to 1; so that the quantity under the sign $\int$ cannot become nothing, during that interval. Therefore there cannot be any value of λ', within the limits [1607'], which will satisfy the equation of equilibrium $\varphi = 0$ [1574].

equilibrium, corresponding to the same rotatory motion, does not prove, that there are two figures of equilibrium, corresponding to one primitive force; [1607v] since the two figures of equilibrium correspond to two different primitive forces, or to such as are differently applied.*

We shall therefore consider a fluid mass, which is acted upon, in its primitive state, by any forces, and then left to itself, and to the mutual attractions of all its particles. If we suppose the centre of gravity to be at [1607vi] rest, and draw through it a plane, so situated that the sum of the areas, described by each particle, projected upon this plane, and multiplied respectively by the corresponding particles, may be a *maximum*, at the origin of the motion; this plane will always possess the same property, by [1607vii] § 21 and 22 of the first book [181″, 189‴], whatever be the manner in which the particles act upon each other; whether by their adhesion, attraction, or their mutual impact, even in cases where there are sudden and instantaneous losses of motion [167‴]. Hence, after a great number of oscillations, the fluid mass will attain a uniform rotatory motion about a [1607viii] fixed axis, which axis will be perpendicular to the plane just mentioned; and the plane itself will be that of the equator. The rotatory motion will be such, that the sum of the areas, described in the time dt, by the particles, projected upon that plane, *will be the same as at the origin of the motion*, [1607ix] [167″]; we shall denote this sum by $E \,.\, dt$.

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* (1124) Let $ADBE$, $adbe$, be the two figures of equilibrium corresponding to *equal* masses of fluid, having the same centre C, the same axis ACB, and the same [1607a] time of rotation; then the mass included between $FaGEHbID$ is common to both figures, and as the whole masses are equal, the sum of the parts near the poles, $AFaG$, $IbHB$, must be equal to the part $dFDI$, $cGEH$, towards the equator. But the angular motion of all the particles being equal, the momentum of the part near the equator must be very much greater than that of the equal masses near the poles.

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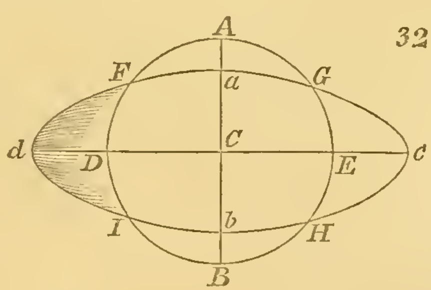

Hence it evidently follows, that the momentum of the mass $adbc$, must be much greater [1607b] than that of the mass $ADBC$; therefore the primitive forces, which produce these momenta, must be different if they be applied in the same manner, or at the same distance from the centre of gravity, &c.; and on the contrary, if the forces be equal, they must be applied at different distances, or in a different manner.

We shall observe, in this place, that the axis here treated of, is that, with respect to which, the sum of the momenta of the primitive forces of the
[1607x] system, is a maximum. It preserves this property during the motion of the system, and finally becomes the axis of rotation; for what we have demonstrated, about the plane of maximum of the projected areas, in the articles of the first book just mentioned [181″, 189‴], applies also to the
[1607xi] axis of the greatest momentum of the forces; since the elementary area, described by the projection of the radius vector of a body, upon a plane, and multiplied by its mass, is evidently proportional to the momentum of the finite force of this body, relative to the axis perpendicular to that plane.*

We shall put, as in [1569′], g for the centrifugal force arising from the
[1607xii] rotatory motion, at the distance 1 from that axis, and $\sqrt{g}$ will be the angular velocity of rotation [1564t]. We shall also put k for the semi-axis of rotation of the fluid mass, and $k.\sqrt{1+\lambda^2}$, for the semi-axis of the

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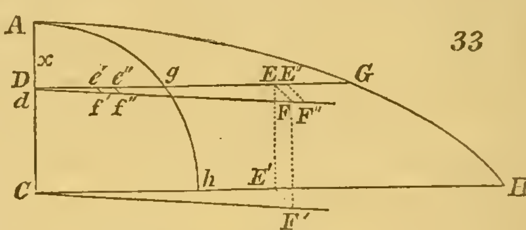
* (1125) We shall suppose, as in [1564s], that the angular velocity of rotation, about the axis AC, fig. 33, is represented by n, so that a point, at the distance 1 from this fixed axis, will describe the space $n\,dt$, in the time dt. Then if a particle dM be placed at E, at the distance $DE=r$ from the axis, it will describe, in the time
[1607c] dt, the arc $EF=n\,dt.r$. This arc, projected upon the plane of the maximum areas, or the plane of the equator HCF', which is perpendicular to AC, will be represented by



$$E'F'=EF=n\,dt.r;$$

and as the radius $CE'=DE=r$, the area $CE'F'$, described by the projection of this particle, will be represented by $\frac{1}{2}E'F'.CE'=\frac{1}{2}n\,dt.r^2$. This, being multiplied by dM, becomes $\frac{1}{2}n\,dt.r^2.dM$; and if the sum be taken for all the particles of the fluid, it will give the quantity represented by $E.dt$ [1607ix], so that

[1607d] $$E.dt=\tfrac{1}{2}n\,dt.\int r^2.dM;$$

n and dt being constant in this integration. Hence it evidently follows, that when $E.dt$
[1607e] is a maximum, $\frac{1}{2}n\,dt.\int r^2.dM$, and therefore $\int r^2.dM$, must also be a maximum. Now this last expression represents the momentum of inertia of the body about the axis AC [245‴], which is conformable to what is said in [1607xi]. Substituting $n=\sqrt{g}$ [1564t] in $E.dt$ [1607d], we get $E.dt=\frac{1}{2}.\sqrt{g}.dt.\int r^2.dM$; hence
[1607f] $E=\frac{1}{2}.\sqrt{g}.\int r^2.dM$.

equator [1574a]. It is easy to prove, that the sum of the areas described during the time dt, by all the particles projected upon the plane of the equator, and multiplied respectively by the corresponding particles, is [1607xiii]

$$\frac{4\pi\rho}{15}\cdot(1+\lambda^2)^2\cdot k^5\cdot dt\cdot\sqrt{g}\,;$$

hence we shall have,*

$$\frac{4\pi\rho}{15}\cdot(1+\lambda^2)^2\cdot k^5\cdot\sqrt{g}=E. \qquad [1608]$$

Then putting M for the mass of the fluid, we shall find, [1567],

$$\frac{4\pi\rho}{3}\cdot k^3\cdot(1+\lambda^2)=M. \qquad [1609]$$

Hence the quantity $\frac{g}{\frac{4}{3}\pi\rho}$, which we have denoted by q in [1573'], is [1609']

$q=q'\cdot(1+\lambda^2)^{-\frac{2}{3}}$, supposing q' to denote the following function,† [1609'']

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* (1126) The value of E [1608] may be easily deduced from that of C [275c, d], which represents the value of $fr^2\cdot dM$ for a sphere, whose radius is k, instead of R. [1607g] If in fig. 33 we put $AD=x$, $Dd=dx$, $De'=r$, angle $EDF=d\varpi$, the particle dm of the sphere AhC, situated near the point e', may be considered as a rectangular parallelopiped, whose length is $e'e''=dr$, width $e'f'=r\cdot d\varpi$, and depth equal to $Dd=dx$; the product of these three dimensions being $dm=r\,dr\cdot d\varpi\cdot dx$, and [1607h] the expression of $C=fr^2\cdot dm$ [275c], will become $C=fr^3\,dr\cdot d\varpi\cdot dx$. If [1607i] upon the continuation of the line De', we take the point E, so that

$$DE=\sqrt{(1+\lambda^2)}\cdot De'=\sqrt{(1+\lambda^2)}\cdot r, \qquad [1607k]$$

its differential will be the line $EE''=\sqrt{(1+\lambda^2)}\cdot dr$, the perpendicular arc $EF=r\cdot\sqrt{(1+\lambda^2)}\cdot d\varpi$, and the product of these two lines by the depth $Dd=dx$, may be considered as a particle dM of the spheroid, corresponding to the particle dm of the sphere, making $dM=(1+\lambda^2)\cdot r\,dr\cdot d\varpi\cdot dx$. Substituting this value of dM, [1607l] and $r\cdot\sqrt{(1+\lambda^2)}$, for r, in E [1607f], we shall get, for the spheroid,

$$E=\tfrac{1}{2}\sqrt{g}\cdot fr^2\cdot(1+\lambda^2)\cdot dM=\tfrac{1}{2}\sqrt{g}\cdot(1+\lambda^2)^2\cdot fr^3\,dr\cdot d\varpi\cdot dx=\tfrac{1}{2}\sqrt{g}\cdot(1+\lambda^2)^2\cdot C,$$

[1607i]; but by [275d], the quantity C, multiplied by the density ρ, and putting $R=k$, [1607m] [1607g], becomes $C=\frac{8\pi}{15}\cdot\rho\cdot k^5$; hence, $E=\frac{1}{2}\sqrt{g}\cdot(1+\lambda^2)^2\cdot\frac{8\pi}{15}\cdot\rho\cdot k^5$, as in [1608].

† (1127) From [1608, 1609], we get $25E^2=(\frac{4}{3}\pi\rho)^2\cdot(1+\lambda^2)^4\cdot k^{10}\cdot g$, and $M^{\frac{10}{3}}=(\frac{4}{3}\pi\rho)^{\frac{10}{3}}\cdot k^{10}\cdot(1+\lambda^2)^{\frac{10}{3}}$. Substituting these in q' [1610], it becomes,

[1610] $$q' = \frac{25\, E^2 \,.\, (\tfrac{4}{3}\,\pi\,\rho)^{\frac{1}{3}}}{M^{\frac{10}{3}}}.$$

The equation [1574] of the same article becomes*

[1611] $$0 = \frac{9\,\lambda + 2\,q' \,.\, \lambda^3 \,.\, (1+\lambda^2)^{-\frac{2}{3}}}{9 + 3\,\lambda^2} - \text{arc. tang.}\,\lambda.$$

This equation will determine λ; we shall then have k, by means of the preceding expression of M [1609].

We shall put φ for the function [1611], namely,

[1612] $$\varphi = \frac{9\,\lambda + 2\,q' \,.\, \lambda^3 \,.\, (1+\lambda^2)^{-\frac{2}{3}}}{9 + 3\,\lambda^2} - \text{arc. tang.}\,\lambda\,;$$

which ought to be equal to nothing, by the condition of equilibrium. This
[1612′] function is positive, at its commencement, when λ is small,† and is negative
when λ is infinite ;‡ consequently between $\lambda = 0$ and $\lambda = \infty$, there
[1612″] must be a value of λ, which will make this function equal to nothing.

$$q' = \frac{(\tfrac{4}{3}\,\pi\,\rho)^2 \,.\, (1+\lambda^2)^4 \,.\, k^{10} \,.\, g \,.\, (\tfrac{4}{3}\,\pi\,\rho)^{\frac{1}{3}}}{(\tfrac{4}{3}\,\pi\,\rho)^{\frac{10}{3}} \,.\, k^{10} \,.\, (1+\lambda^2)^{\frac{10}{3}}} = \frac{g \,.\, (1+\lambda^2)^{\frac{2}{3}}}{(\tfrac{4}{3}\,\pi\,\rho)} = q \,.\, (1+\lambda^2)^{\frac{2}{3}}, \qquad [1609'];$$

hence $q = q' \,.\, (1+\lambda^2)^{-\frac{2}{3}}$, as in [1609″].

* (1128) Substituting in [1574] the value of q [1609″], it becomes as in [1611]; from
[1611a] which λ may be determined; M, ρ and E being given. We may then obtain the polar semi-axis k, by means of the equation [1609], and the equatorial semi-axis $k \,.\, \sqrt{(1+\lambda^2)}$ [1574a] will also be known.

† (1129) Substituting in [1612] $-\text{arc. tang.}\,\lambda = -\lambda + \tfrac{1}{3}\,\lambda^3 - \tfrac{1}{5}\,\lambda^5 + \&c.$ [48] Int., and connecting the two terms $-\lambda + \tfrac{1}{3}\,\lambda^3$ with those having the denominator $9 + 3\,\lambda^2$, it becomes, by reduction, $\varphi = \dfrac{\lambda^5 + 2\,q' \,.\, \lambda^3 \,.\, (1+\lambda^2)^{-\frac{2}{3}}}{9 + 3\,\lambda^2} - \tfrac{1}{5}\,\lambda^5 + \&c.$ If we neglect
[1611b] terms of the order λ^5, it becomes $\varphi = \tfrac{2}{9}\,q' \,.\, \lambda^3$; now $q' = q \,.\, (1+\lambda^2)^{\frac{2}{3}}$ [1609″] is positive [1571c]; therefore when λ is small, φ must be positive, and it is nothing when $\lambda = 0$; as is also evident, by the mere inspection of the function [1612].

‡ (1130) Since $\lambda^3 \,.\, (1+\lambda^2)^{-\frac{2}{3}} = \lambda^{\frac{5}{3}} \,.\, \left(1 + \frac{1}{\lambda^2}\right)^{-\frac{2}{3}} = \lambda^{\frac{5}{3}} \,.\, \left(1 - \tfrac{2}{3} \,.\, \frac{1}{\lambda^2} + \&c.\right)$, the numerator of the first term of φ [1612], will contain no power of λ, greater than

Therefore, whatever be the value of q', there will always be one elliptical figure, with which the mass can be in equilibrium.

We may put the value of φ [1612] under this integral form,*

$$\varphi = 2 \cdot \int \frac{\lambda^4 d\lambda \cdot \left\{ \frac{27 q'}{\lambda^2} + 18 q' - \{ q' \cdot \lambda^2 + 18 \cdot (1+\lambda^2)^{\frac{2}{3}} \} \right\}}{(9+3\lambda^2)^2 \cdot (1+\lambda^2)^{\frac{5}{3}}}. \qquad [1613]$$

$\lambda^{\frac{5}{3}}$; whereas the denominator $9+3\lambda^2$ contains the power λ^2, or $\lambda^{\frac{6}{3}}$; therefore [1611c]
when $\lambda = \infty$, this first term will vanish, and the second term will be

$$-\text{arc. tang.}\,\lambda = -\text{arc. tang.}\,\infty = -\tfrac{1}{2}\pi.$$

Hence φ becomes negative and equal to $-\frac{1}{2}\pi$, when $\lambda = \infty$, as in [1612'].

* (1131) If we take the differential of φ [1612], make the usual reductions, and then prefix the sign of integration, we shall obtain the formula [1613]. To prove this, we may, for greater simplicity, compute the terms independent of q' separately. These terms are $\frac{9\lambda}{9+3\lambda^2} - \text{arc. tang.}\,\lambda$, whose differential, by [51] Int., is

$$\frac{9d\lambda.(9+3\lambda^2) - 54\lambda^2 d\lambda}{(9+3\lambda^2)^2} - \frac{d\lambda}{1+\lambda^2} = \frac{d\lambda.(81-27\lambda^2)}{(9+3\lambda^2)^2} - \frac{d\lambda}{1+\lambda^2}$$

$$= \frac{d\lambda}{(9+3\lambda^2)^2.(1+\lambda^2)} \cdot \{ 81 \cdot (1+\lambda^2) - 27\lambda^2 \cdot (1+\lambda^2) - (81 + 54\lambda^2 + 9\lambda^4) \}$$

$$= -\frac{36\lambda^4 d\lambda}{(9+3\lambda^2)^2.(1+\lambda^2)} = -\frac{36\lambda^4 d\lambda.(1+\lambda^2)^{\frac{2}{3}}}{(9+3\lambda^2)^2.(1+\lambda^2)^{\frac{5}{3}}}; \qquad [1613a]$$

which is the same as the term independent of q', in the differential of φ [1613]. The term q' [1610, 1611a], is given, or constant; and the coefficient of $2q'$, in [1612], is

$$\frac{\lambda^3}{9+3\lambda^2} \cdot (1+\lambda^2)^{-\frac{2}{3}}, \qquad [1613b]$$

whose differential is

$$\frac{27\lambda^2 d\lambda + 3\lambda^4 d\lambda}{(9+3\lambda^2)^2} \cdot (1+\lambda^2)^{-\frac{2}{3}} - \frac{\frac{4}{3}\lambda^4 d\lambda}{(9+3\lambda^2)} \cdot (1+\lambda^2)^{-\frac{5}{3}}$$

$$= \frac{\lambda^4 d\lambda}{(9+3\lambda^2)^2.(1+\lambda^2)^{\frac{5}{3}}} \cdot \left\{ \left(\frac{27}{\lambda^2} + 3 \right) \cdot (1+\lambda^2) - \tfrac{4}{3} \cdot (9+3\lambda^2) \right\}$$

$$= \frac{\lambda^4 d\lambda}{(9+3\lambda^2)^2.(1+\lambda^2)^{\frac{5}{3}}} \cdot \left\{ \frac{27}{\lambda^2} \cdot (1+\lambda^2) + 3 \cdot (1+\lambda^2) - \tfrac{4}{3} \cdot (9+3\lambda^2) \right\}$$

$$= \frac{\lambda^4 d\lambda}{(9+3\lambda^2)^2.(1+\lambda^2)^{\frac{5}{3}}} \cdot \left\{ \frac{27}{\lambda^2} + 18 - \lambda^2 \right\}; \qquad [1613c]$$

which is the same as the coefficient of $2q'$ in $d\varphi$ [1613].

When it becomes nothing, the function*

[1614] $$\frac{27\,q'}{\lambda^2}+18\,q'-\{q'\,.\,\lambda^2+18\,.\,(1+\lambda^2)^{\frac{2}{3}}\},$$

has passed through zero, to become negative. Now from the moment when this function becomes negative, it will remain so, as λ increases; because

[1614′] the part $\frac{27\,q'}{\lambda^2}+18\,q'$ decreases, while the negative part

[1614″] $$-\{q'\,.\,\lambda^2+18\,.\,(1+\lambda^2)^{\frac{2}{3}}\}$$

increases; the function φ cannot therefore vanish twice. *Hence it follows*

[1614‴] *that there is but one real and positive value of λ which can satisfy the equation of equilibrium; therefore the fluid can be in equilibrium only with one elliptical figure.*

There is only one figure of equilibrium of an ellipsoid for a given primitive force.

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* (1132) The factors $(9+3\lambda^2)$, $(1+\lambda^2)^{\frac{5}{3}}$, λ^4, [1613], being positive, as also φ, when λ is small, [1611*b*]; it is evident that φ cannot vanish as λ increases, until the remaining factor of φ, given in [1614], becomes negative, as is observed in [1614′].

CHAPTER IV.

ON THE FIGURE OF A SPHEROID, DIFFERING BUT LITTLE FROM A SPHERE, AND COVERED BY A FLUID STRATUM IN EQUILIBRIUM.

22. We have considered, in the preceding chapter, the conditions of the equilibrium of a homogeneous fluid mass, and we have found that the elliptical figure satisfies these conditions; but to obtain a complete solution of this problem, we ought to determine, *a priori*, all the figures of equilibrium, or prove that the elliptical figure is the only one which satisfies these conditions. Moreover, it is very probable that the heavenly bodies are not homogeneous masses, and that they are denser towards the centre than at [1614'''']
the surface; we ought not therefore, in the investigation of their figures, to limit ourselves to homogeneous bodies; and in this view of the subject, the investigation is very difficult. Fortunately it becomes more simple, from the circumstance, that the figures of the planets and their satellites vary but little [1614v]
from a spherical form, *which enables us to neglect the square of this difference*, and the quantities which depend upon it. Notwithstanding this simplification, the investigation of the figures of the planets is very complicated. To treat [1614vi]
the subject in the most general manner, we shall consider the equilibrium of a fluid mass, covering a body composed of strata of variable densities, endowed with a rotatory motion, and acted upon by the attractions of foreign bodies. For this purpose, we shall resume the laws of equilibrium of fluids, which we have demonstrated in the first book.

If we put

Symbols.

$\rho =$ the density of a particle of the fluid, [1614vii]

$\Pi =$ the pressure it suffers, [1614viii]

F, F', F'', &c., the forces acting upon it, [1614ix]

df, df', df'', &c., the elements of the directions of these forces, [1614x]

then the general equation of equilibrium of the fluid mass, will be, by § 17 of the first book, [133],*

Equation of equilibrium.

[1615] $$\frac{d\,\Pi}{\rho} = F\,.\,df + F'\,.\,df' + F''\,.\,df'' + \&c.$$

If we suppose the second member of this equation to be an exact differential, and denote it by

[1615'] $$d\,\varphi = F\,.\,df + F'\,.\,df' + F''\,.\,df'' + \&c.,$$

ρ must then necessarily be a function of Π and φ. The integral of this equation will give φ in terms of Π;† by means of which we can reduce ρ,
[1615''] to be a function of Π only; whence we may obtain Π in a function of ρ.

---

* (1134) If all the forces F, F', F', &c., are composed into one force V', in the
[1615a] direction of the line u, we shall have $V'.du = F.df + F'.df' + F''.df'' + \&c.$, [16]. Again, if this force V', in the direction u, be resolved into three others, P, Q, R, parallel to three rectangular co-ordinates x, y, z, we shall have, by the same theorem,

[1615b] $$V'\,.\,du = P\,.\,dx + Q\,.\,dy + R\,.\,dz\,;$$

hence $P\,.\,dx + Q\,.\,dy + R\,.\,dz = F\,.\,df + F'\,.\,df' + \&c.$ Substituting this in [133], we get $dp = \rho\,.\,\{F\,.\,df + F'\,.\,df' + F''\,.\,df'' + \&c.\}$; and by changing p into Π, to conform to the notation here used, and dividing also by ρ, we obtain [1615]. It may be
[1615c] observed, that the forces F, F', &c., *are here supposed to tend to increase the co-ordinates* f, f', &c., as in note 60, page 91, Vol. I.

† (1135) Cases in which $F\,.\,df + F'\,.\,df' + \&c.$ is an exact integral, are pointed out at the commencement of the following article [1616$^{\text{viii}}$, &c.]. Putting for

$$F\,.\,df + F'\,.\,df' + \&c.,$$

its assumed value $d\varphi$ [1615'], the equation [1615] becomes $\frac{d\Pi}{\rho} = d\varphi$, or $d\,\Pi = \rho\,.\,d\varphi$, and this cannot be integrated, except ρ be a function of Π and φ, as is
[1616a] observed in note 65, page 94, Vol. I. Supposing $\rho = \psi(\varphi, \Pi)$, the equation $d\Pi = \rho\,.\,d\varphi$ will become $d\,\Pi = d\,\varphi\,.\,\psi(\varphi,\Pi)$. This, being a function of two quantities only, may be integrated, and will give φ equal to a function of Π. Substituting it in $\rho = \psi(\varphi,\Pi)$, we get $\rho =$ function of Π; therefore Π will be equal to a function of ρ, which we shall
[1616b] denote by $\Pi = \Omega(\rho)$, whose differential is of the form $d\,\Pi = d\,\rho\,.\,\Omega'(\rho)$; and for a stratum whose density is constant, $d\,\rho$ will vanish, and then $d\,\Pi = 0$. Substituting this in [1615], we shall get [1616].

Hence as it regards a stratum of the same density, we shall have $d\Pi = 0$, consequently,

$$0 = F.df + F'.df' + F''.df'' + \&c. \qquad [1616]$$

This equation shows that the force, at the surface of any stratum, in the direction of the tangent of the surface, is nothing; consequently the [1616′]
resultant of all the forces F, F', F'', &c., is perpendicular to this surface;* Level surfaces.
so that such a stratum is also a *level surface*, [138′]. [1616″]

The pressure Π being nothing at the surface of the fluid, ρ must be [1616‴]
constant at that surface, and the resultant of all the forces which act on each particle of the surface must be perpendicular to it. This resultant is what is called *gravity*. Therefore the conditions of the equilibrium of a fluid mass Gravity.
are, *first*, that *the direction of gravity must be perpendicular to each point of* [1616⁗]
the external surface of the fluid; *second*, that *the direction of gravity of any* Conditions of equilibrium of a level surface.
particle, situated within the surface, must be perpendicular to the stratum, having the same density with the particle. And as we may, within the [1616ᵛ]
surface of a homogeneous fluid, select any stratum for the stratum of constant density, the second of the two preceding conditions of equilibrium [1616ᵛⁱ]
will always be satisfied; and it will then be sufficient to establish the equilibrium, to satisfy the first condition; that is, to make the resultant of [1616ᵛⁱⁱ]
all the forces, which act upon the external surface of the fluid, perpendicular to that surface.

23. In the theory of the figure of the heavenly bodies, the forces F, F', F'', &c., are produced by the attraction of their particles, by the centrifugal force arising from the rotatory motion, and by the attraction of [1616ᵛⁱⁱⁱ]
foreign bodies. It is easy to prove that the expression

$$F.df + F'.df' + \&c., \qquad [1616^{ix}]$$

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* (1136) Putting V' equal to the resultant of all the forces F, F', &c., and u for the direction of this force, we shall have, as in [1615a], $V'.du = F.df + F'.df' + \&c.$; [1616$c$]
then from [1616] we have $V'.du = 0$, for all the particles situated on a stratum of the same density. Hence if V' be supposed finite, we must have $du = 0$, which is the same as the expression $dr = 0$, or $\delta r = 0$, in the calculation, note 64, p. 93, Vol. I; [1616d]
and it is shown there, that $\delta r = 0$, or $du = 0$, corresponds to the case where the direction of the force V' is perpendicular to the surface of uniform density, or level surface, [138′]; $du = 0$ being the equation of this surface, or a multiple of it. [1616e]

$F df + F' df' + $&c., is an exact differential.

is then an exact differential; and we shall make it evident by the following analysis, in which we shall determine the part of the integral

$$\int (F . df + F' . df' + \&c.),$$

which corresponds to each of these causes.

If we put dM for any particle of the spheroid, and f its distance from

[1616x] the attracted particle; its action on this last particle will be $\frac{dM}{f^2}$. Multiplying this action by the element of its direction, which is $-df$, since it tends to diminish f, we shall have, for the action of the particle

[1616xi] dM,* $\int F . df = \frac{dM}{f}$. Hence it follows, that the part of the integral

Effect of the mass of the spheroid.

$\int (F . df + F' . df' + \&c.)$, which depends on the attraction of the

[1616xii] particles of the spheroid, is equal to the sum of all these particles, divided by their respective distances from the attracted particle. We shall represent this sum by V, as we have heretofore done [1385‴].

In the theory of the figure of the planets, it is required to determine the laws of the equilibrium of all their parts, about their common centre of

[1616xiii] gravity. We must therefore transfer to the attracted particle, in a contrary direction, all the forces which act on that centre, arising from the reciprocal action of all the parts of the spheroid. But we have shown, in the first book, [155″—158], that by the property of this centre, the resultant of all

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* (1137) The negative sign is used here, because in the formulas [1615, &c.], the force

[1616f] F is supposed to increase f, as is observed in [1615c]. Hence $F . df = - dM . \frac{df}{f^2}$; and its integral relative to f, corresponding to any given particle dM, is

[1616f′] $$\int F . df = dM . \int \frac{-df}{f^2} = \frac{dM}{f}.$$

If we take the sum of the similar expression for the other particles, so as to include the whole mass of the spheroid, we shall get $\int \frac{dM}{f}$ for the part of $\int (F . df + F' . df' + \&c.)$ corresponding to the mutual attraction of the particles. Now by [1393], we have $V = \int \frac{dM}{r}$, r being the distance which is here called f, so that we have, as above,

[1616g] $$V = \int \frac{dM}{f} = \int (F . df + F' . df' + \&c.)$$

these actions, upon that point, is nothing ;* therefore it will not be necessary [1616xiv]
to make any addition to V, to obtain the whole effect of the attraction of the spheroid upon the attracted particle.

To determine the effect of the centrifugal force ; we shall suppose that
the position of the particle is determined by three rectangular co-ordinates, [1616xv]
x', y', z', whose origin we shall fix at the centre of gravity of the spheroid.
We shall also suppose that the axis of x' is the axis of rotation, and that g
expresses the centrifugal force arising from the rotatory velocity, at the [1616xvi]
distance 1 *from the axis.* This force will be nothing in the direction of x',
and equal to gy', gz', in the directions of y' and z' ;† therefore by [1616xvii]
multiplying these two last forces respectively by the elements of their directions dy' and dz', we shall have $\frac{1}{2}g\cdot(y'^2+z'^2)$ for the part of the
integral $\int(F\,.\,df+F'\,.\,df'+\&c.)$ depending on the centrifugal force [1616xviii]
produced by the rotatory motion.

If we put, as above, [1430'], r for the distance of the attracted particle,
from the centre of gravity of the spheroid ; θ the angle which the radius r [1616xix]
makes with the axis of x' ; ϖ the angle formed by the plane $x'y'$ with the [1616xx]
plane which passes through the axis of x' and that particle ; finally, if we
put $\cos.\theta=\mu$, we shall have,‡ [1616xxi]

$$x'=r\,.\,\mu\,; \qquad y'=r\,.\,\sqrt{1-\mu^2}\,.\,\cos.\varpi\,; \qquad z'=r\,.\,\sqrt{1-\mu^2}\,.\,\sin.\varpi\,; \qquad [1617]$$

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* (1138) It appears from [155''—158], that the motion of the centre of gravity of a system of bodies is not affected by their mutual attraction ; therefore the effect of this
attraction may be neglected, as in [1616xiv]. [1616h]

† (1139) This is the same as in [1569b], changing the co-ordinates b, c, into y', z',
respectively ; by which means gb, gc, become gy', gz'. The forces, being multiplied
by the elements of their directions dy', dz', produce the quantities $g.y'dy'$, $g.z'dz'$, [1616i]
whose integrals are $\frac{1}{2}g\,.\,y'^2$, $\frac{1}{2}g\,.\,z'^2$. Their sum $\frac{1}{2}g\,.\,(y'^2+z'^2)$, represents the corresponding part of $\int(F\,.\,df+F'\,.\,df'+\&c.)$, as in [1616xviii].

‡ (1140) These are the same as [1529], changing R, x, y, z, into r, x', y', z',
respectively ; it being evident, from [1528'''], that by these changes we shall conform to the [1617a]
notation of the present article. The sum of the squares of y', z', gives, as in [1618], $y'^2+z'^2=r^2\,.\,(1-\mu^2)$. If we substitute in [1617], the values $\cos.\theta=\mu$ [1616xxi], $\sin.\theta=(1-\mu^2)^{\frac{1}{2}}$, we shall get the following expressions, which will be used hereafter,

$$x'=r\,.\,\cos.\theta, \qquad y'=r\,.\,\sin.\theta\,.\,\cos.\varpi, \qquad z'=r\,.\,\sin.\theta\,.\,\sin.\varpi. \qquad [1617b]$$

hence we deduce,

[1618] $$\tfrac{1}{2}g\,.\,(y'^2+z'^2)=\tfrac{1}{2}g\,r^2\,.\,(1-\mu^2).$$

Centrifugal force.

We shall put this quantity under the following form,*

[1619] $$\tfrac{1}{3}g\,r^2-\tfrac{1}{2}g\,r^2\,.\,(\mu^2-\tfrac{1}{3}),$$

in order to render the terms similar to those of the expression of V, which we have given in Chapter II, [1467, 1528a, &c.]; so that they may have the property of satisfying the equation of partial differentials [1525],

Differential equation in $Y^{(i)}$.

[1620] $$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}+\frac{\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,Y^{(i)},$$

[1620′] in which $Y^{(i)}$ is a rational and integral function of μ, $\sqrt{1-\mu^2}\,.\,\cos.\varpi$, and $\sqrt{1-\mu^2}\,.\,\sin.\varpi$, of the degree i. For it is evident that each of the two terms $\tfrac{1}{3}g\,r^2$ and $-\tfrac{1}{2}g\,r^2\,.\,(\mu^2-\tfrac{1}{3})$, satisfies the preceding equation, when substituted for $Y^{(i)}$ [1618a].

It now remains to determine the part of the integral

$$\int(F\,.\,df+F'\,.\,df'+\&c.)$$

[1620″] arising from the action of foreign bodies. Let S be the mass of one of these bodies, f its distance from the attracted particle, and s its distance from the centre of gravity of the spheriod. Multiplying the attraction by the element of its direction $-df$, and then taking the integral, we shall

[1620‴] obtain $\dfrac{S}{f}$ [1616^{xi}]. This is not the whole of the integral

Action of a foreign body.

$$\int(F\,.\,df+F'\,.\,df'+\&c.)$$

arising from the action of S; we must also transfer to the particle, in a contrary direction, the action of the body S upon the centre of gravity of

[1620⁗] this spheroid. For this purpose, we shall put v for the angle which the line s forms with the axis of x'; and ψ for the angle which the plane passing

[1620^{v}] through that axis and the body S, makes with the plane of $x'y'$. The

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* (1141) From [1528h] we have $1-\mu^2=\tfrac{2}{3}-(\mu^2-\tfrac{1}{3})=Y^{(0)}+Y^{(2)}$; in which

[1618a] $Y^{(0)}=\tfrac{2}{3}$; $Y^{(2)}=-(\mu^2-\tfrac{1}{3})$, both of these quantities satisfy the equation [1620], and the same equation would be satisfied, if $Y^{(0)}$, $Y^{(2)}$, were multiplied by $\tfrac{1}{2}gr^2$, as in [1619].

action $\frac{S}{s^2}$ of this body upon the centre of gravity of the spheroid, resolved in directions parallel to the axes of x', y', z', will produce the three [1620vi]
following forces,*

$$\frac{S}{s^2}\,.\cos.v\,;\qquad \frac{S}{s^2}\,.\sin.v\,.\cos.\psi\,;\qquad \frac{S}{s^2}\,.\sin.v\,.\sin.\psi\,. \qquad [1621]$$

Transferring these forces to the attracted particle, in opposite directions,

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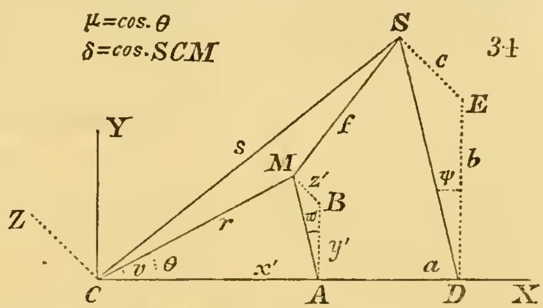
* (1142) In the annexed figure, C is the centre of the spheroid, CX, CY, CZ, the rectangular axes of x', y', z',; the two former being in the plane of the figure, and the third perpendicular to it, CX being the axis of rotation. M is the place of a particle dM of the spheroid, S the centre of the body S. The co-ordinates of the point M are



$$CA=x',\quad AB=y',\quad BM=z'\,; \qquad [1620a]$$

those of the body S are

$$CD=a,\quad DE=b,\quad ES=c, \qquad [1620a']$$

parallel to x', y', z', respectively. The angle $ACM=\theta$, $MAB=\varpi$, $DCS=v$, [1620b]
$SDE=\psi$, and the lines $SM=f$, $CS=s$, $CM=r$. Hence, as in [1431a'], [1620b']

$$CA=CM\,.\cos.ACM,\qquad AB=CM.\sin.ACM\,.\cos.MAB,$$
$$BM=CM\,.\sin.ACM\,.\sin.MAB, \qquad [1620c]$$

which correspond to the formulas [1617]. In precisely the same manner, we have, for the co-ordinates of the point S,

$$CD=CS\,.\cos.DCS,\qquad DE=CS\,.\sin.DCS\,.\cos.SDE,$$
$$ES=CS\,.\sin.DCS\,.\sin.SDE. \qquad [1620d]$$

Dividing these by CS, and using the symbols v, ψ, we get

$$\frac{CD}{CS}=\cos.v,\qquad \frac{DE}{CS}=\sin.v\,.\cos.\psi,\qquad \frac{ES}{CS}=\sin.v\,.\sin.\psi. \qquad [1620e]$$

The attraction of the body S upon the point C, is $\frac{S}{s^2}$, in the direction CS; and if we multiply this by the ratios $\frac{CD}{CS}$, $\frac{DE}{CS}$, $\frac{ES}{CS}$, we shall evidently obtain the values [1620f]
of this force, resolved in directions parallel to CD, DE, ES, respectively, as in [1621]. Now as the centre C is supposed to be at rest, we must apply the equal and opposite forces

$$-\frac{S}{s^2}\,.\cos.v,\qquad -\frac{S}{s^2}\,.\sin.v\,.\cos.\psi,\qquad -\frac{S}{s^2}\,.\sin.v\,.\sin.\psi, \qquad [1620g]$$

[1621'] which amounts to the same as to prefix the sign $-$; then multiplying them by the elements of their directions dx', dy', dz', and taking their integrals, we shall get, for the sum of these integrals,

[1622] $$-\frac{S}{s^2}\cdot\{x'\cdot\cos.v+y'\cdot\sin.v\cdot\cos.\psi+z'\cdot\sin.v\cdot\sin.\psi\}+\text{constant}.$$

Hence the whole of that part of the integral $\int(F\cdot df+F'\cdot df'+\&c.)$ arising from the action of the body S, will be,

[1623] $$\frac{S}{f}-\frac{S}{s^2}\cdot\{x'\cdot\cos.v+y'\cdot\sin.v\cdot\cos.\psi+z'\cdot\sin.v\cdot\sin.\psi\}+\text{constant}.$$

This quantity ought to be nothing at the centre of gravity of the spheroid,
[1623'] which is supposed to be at rest,* and at this point, f becomes s, and x', y', z', are nothing; therefore we shall have,

[1624] $$\text{constant}=-\frac{S}{s}.$$

Now we have†

[1625] $$f=\{(s\cdot\cos.v-x')^2+(s\cdot\sin.v\cdot\cos.\psi-y')^2+(s\cdot\sin.v\cdot\sin.\psi-z')^2\}^{\frac{1}{2}};$$

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to *every particle of the spheroid.* Therefore if we multiply these forces by the elements of their directions dx', dy', dz', when applied to the particle placed at M, they will produce the terms

[1620g'] $$-\frac{S}{s^2}\cdot dx'\cdot\cos.v,\qquad -\frac{S}{s^2}\cdot dy'\cdot\sin.v\cdot\cos.\psi,\qquad -\frac{S}{s^2}\cdot dz'\cdot\sin.v\cdot\sin.\psi,$$

in $F\cdot df+F'\cdot df'+\&c.$ To find the sum of all these terms, corresponding to the whole mass of the spheroid, we must take the integral of these expressions, supposing x', y', z', only to be variable; because the angles v, ψ, and the quantity s, are the same for
[1620h] every situation of the point M, or for every particle dM. The sum of all these integrals is given in [1622], and if this part be connected with the part $\frac{S}{f}$ [1620'''], arising from the direct attraction of the body S, upon the particle dM of the spheroid, we shall obtain the expression [1623], which represents the part of the integral $\int(F.df+F'.df'+\&c.)$ arising from the whole action of the body S.

* (1143) This assumed value of the constant quantity tends to simplify the second member of [1631], and conforms to the hypothesis [1620''''] that the centre C is at rest.
[1624a] We might also suppose a constant quantity to be included in the part of the integral $\int(F\cdot df+F'\cdot df'+\&c.)$ represented by V [1616^{xii}].

[1625a] † (1144) Accenting the letters x, y, z, in [1432a], in order to conform to the notation [1620a], we get $f=\{(a-x')^2+(b-y')^2+(c-z')^2\}^{\frac{1}{2}}$; in which a, b, c, are the

which gives, by substituting for x', y', z', their preceding values,

$$\frac{S}{f}=\frac{S}{\sqrt{s^2-2sr.\{\cos.v.\cos.\theta+\sin.v.\sin.\theta.\cos.(\varpi-\psi)\}+r^2}}. \qquad [1626]$$

If we reduce this function into a series, descending relative to the powers of s, and represent it by

$$\frac{S}{f}=\frac{S}{s}.\left\{P^{(0)}+\frac{r}{s}.P^{(1)}+\frac{r^2}{s^2}.P^{(2)}+\frac{r^3}{s^3}.P^{(3)}+\&c.\right\}, \qquad [1627]$$

we shall have generally, by § 15 and 17,*

Function $P^{(i)}$, or $Q^{(i)}$.

$$P^{(i)}=\frac{1.3.5.....(2i-1)}{1.2.3.....i}.\left\{\delta^i-\frac{i.(i-1)}{2.(2i-1)}.\delta^{i-2}+\frac{i.(i-1).(i-2).(i-3)}{2.4.(2i-1).(2i-3)}\delta^{i-4}-\&c.\right\}, \qquad [1628]$$

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co-ordinates of the point S, fig. 34, page 247, and x', y', z', the co-ordinates of the point M, f being equal to SM. Now if we substitute in [1620e] the values [1620a', b'], we shall get $a=s.\cos.v$, $b=s.\sin.v.\cos.\psi$, $c=s.\sin.v.\sin.\psi$; hence f [1625a] becomes as in [1625]. Substituting in this the values of x', y', z', [1617b], and inverting the order of the terms, which does not alter the value of f, because $(a-x')^2=(x'-a)^2$, &c., we get the following expression of f^2, [1625b]

$$f^2=\{r.\cos.\theta-s.\cos.v\}^2+\{r.\sin.\theta.\cos.\varpi-s.\sin.v.\cos.\psi\}^2$$
$$+\{r.\sin.\theta.\sin.\varpi-s.\sin.v.\sin.\psi\}^2. \qquad [1625c]$$

This is similar to the value of f^2, which follows immediately after [1432a], and may be derived from it, by changing R, θ', ϖ', into s, v, ψ, respectively; by which means [1432b] will become,

$$f^2=r^2-2sr.\{\cos.\theta.\cos.v+\sin.\theta.\sin.v.\cos.(\psi-\varpi)\}+s^2. \qquad [1625c']$$

Changing the order of the terms, and using the abridged symbol δ [1629], we get

$$f=\{s^2-2sr.\delta+r^2\}^{\frac{1}{2}}. \qquad [1625d]$$

Substituting this in $\frac{S}{f}$, it becomes, as in [1626, 1629],

$$\frac{S}{f}=\frac{S}{\sqrt{(s^2-2sr.\delta+r^2)}}=\frac{S}{s}.\left\{1-\frac{2r}{s}.\delta+\frac{r^2}{s^2}\right\}^{-\frac{1}{2}}. \qquad [1625e]$$

If we develop the radical according to the powers of r, it will become of the form [1627]; and as the two first terms of the development are $\frac{S}{s}+\frac{Sr}{s^2}.\delta$, we shall have $P^{(0)}=1$, $P^{(1)}=\delta$, which will be used hereafter. [1625f]

* (1145) $Q^{(i)}$ [1508'] is the coefficient of $\frac{R^i}{r^{i+1}}$, in the development of T [1509], according to the powers of $\frac{R}{r}$, and it is composed of terms of the form [1507″], [1628a]

putting*

[1629] $$\delta = \cos. v . \cos. \theta + \sin. v . \sin. \theta . \cos. (\varpi - \psi).$$

It is also evident, from § 9 [1442], that we shall have,†

$\beta . \cos. n . (\varpi - \varpi')$; the general value of β being given in [1514]. If we suppose $\mu' = 1$, the factor $(1 - \mu'^2)^{\frac{n}{2}}$ of this value of β becomes nothing, unless $n = 0$; so that all the terms of which $Q^{(i)}$ is composed will vanish, except that corresponding to $n = 0$; but when $n = 0$, we have generally

$$(1 - \mu'^2)^n = (1 - \mu'^2)^0 = 1, \qquad \cos. n . (\varpi - \varpi') = 1;$$

therefore $Q^{(i)}$ will be reduced to the value of β, corresponding to $n = 0$; and this, by means of the formula [1514], is

[1628b] $$\beta = \gamma . \left\{ 1 - \frac{i.(i-1)}{2.(2i-1)} + \&c. \right\} . \left\{ \mu^i - \frac{i.(i-1)}{2.(2i-1)} . \mu^{i-2} + \&c. \right\}.$$

Substituting the value of γ [1521] or [1524], also

$$\left\{ 1 - \frac{i.(i-1)}{2.(2i-1)} + \&c. \right\} = \frac{1.2.3.....i}{1.3.5.....2i-1}, \qquad [1559],$$

it becomes,

[1628c] $$\beta = \frac{1.3.5.....(2i-1)}{1.2.3.....i} . \left\{ \mu^i - \frac{i.(i-1)}{2.(2i-1)} . \mu^{i-2} + \&c. \right\} = Q^{(i)}.$$

In the case under consideration, where $\mu' = 1$, the value of T [1509] becomes

[1628d] $$T = \{r^2 - 2 R r . \mu + R^2\}^{-\frac{1}{2}} = \frac{1}{r} . \left\{ 1 - \frac{2R}{r} . \mu + \frac{R^2}{r^2} \right\}^{-\frac{1}{2}},$$

and β [1628c] represents the coefficient of $\frac{R^i}{r^{i+1}}$, in the development of T according to the powers of $\frac{R}{r}$. Now if in these expressions of β, T, [1628c, d], we change r, R, μ, into s, r, δ, respectively, β will become equal to $P^{(i)}$ [1628], and T [1628d] will

[1628e] change into $\frac{1}{s} . \left\{ 1 - \frac{2r}{s} . \delta + \frac{r^2}{s^2} \right\}^{-\frac{1}{2}}$. Multiplying these by S, we shall get $S . P^{(i)}$ for the coefficient of $\frac{r^i}{s^{i+1}}$ in the development of $\frac{S}{s} . \left\{ 1 - \frac{2r}{s} . \delta + \frac{r^2}{s^2} \right\}^{-\frac{1}{2}} = \frac{S}{f}$ [1625c], which agrees with the formulas [1627, 1628].

* (1145a) It is shown in [1432g, l], that δ is equal to $\cos. SCM$, fig. 34, page 247,

[1628e'] or the cosine of the angle formed by the lines drawn from the centre C, to the body S, and to the particle M.

† (1146) If we multiply the expression of T [1441] by S, and change r, R, θ', ϖ', into s, r, v, ψ, respectively, it will become identical with [1626]; and these changes being made

Differential equation in $P^{(i)}$.

$$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{dP^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddP^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,P^{(i)}\,;$$ [1630]

so that the terms of the preceding series [1627], have the property expressed [1630′]
by the equation [1630], in common with those of V [1436, 1437]. This being premised, we shall have,*

Action of a foreign body S.

$$\frac{S}{f}-\frac{S}{s}-\frac{S}{s^2}\,.\,(x'\,.\cos.\,v+y'\,.\sin.\,v\,.\cos.\,\psi+z'\,.\sin.\,v\,.\sin.\,\psi)$$ [1631]

$$=\frac{S\,.\,r^2}{s^3}\,.\left\{P^{(2)}+\frac{r}{s}\,.\,P^{(3)}+\frac{r^2}{s^2}P^{(4)}+\&c.\right\}.$$ [1631′]

If there be other bodies, S', S'', &c.; we may put s', v', ψ', $P'^{(i)}$, s'', v'', ψ'', $P''^{(i)}$, [1631″]

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in the general term of $S\,.\,T$ [1441″] namely, $S\,.\,Q^{(i)}\,.\,\frac{R^i}{r^{i+1}}$, must produce the [1628f]
corresponding term $S\,.\,P^{(i)}\,.\,\frac{r^i}{s^{i+1}}$, [1627]; therefore $Q^{(i)}$ must change into $P^{(i)}$, and [1442] will become as in [1630].

* (1147) Substituting the values of x', y', z', [1617b], in the factor of $\frac{S}{s^2}$ of the first member of [1631], we get,

$$x'\,.\cos.\,v+y'\,.\sin.\,v\,.\cos.\,\psi+z'\,.\sin.\,v\,.\sin.\,\psi$$
$$=r\,.\,\{\cos.\,v\,.\cos.\,\theta+\sin.\,v\,.\sin.\,\theta\,.\,(\cos.\,\varpi\,.\cos.\,\psi+\sin.\,\varpi\,.\sin.\,\psi)\}$$
$$=r\,.\,\{\cos.\,v\,.\cos.\,\theta+\sin.\,v\,.\sin.\,\theta\,.\cos.\,(\varpi-\psi)\}=r\,.\,\delta, \qquad [1629].$$ [1631a]

Hence the first member of [1631] is $\frac{S}{f}-\frac{S}{s}-\frac{S}{s^2}\,.\,r\,.\,\delta$; and by substituting the values of $\frac{S}{f}$ [1627], it becomes,

$$\frac{S}{s}\,.\,(P^{(0)}-1)+\frac{Sr}{s^2}\,.\,(P^{(1)}-\delta)+\frac{Sr^2}{s^3}\,.\left\{P^{(2)}+\frac{r}{s}\,.\,P^{(3)}+\frac{r^2}{s^2}\,.\,P^{(4)}+\&c.\right\};$$

the two first terms of which vanish, because $P^{(0)}-1=0$, $P^{(1)}-\delta=0$, [1625f], and the remaining terms are as in [1631′]. Connecting this and the similar terms depending on S', S'', &c., with the part depending on the centrifugal force [1619], and using the abridged values [1632], it becomes $\alpha r^2\,.\,\{Z^{(0)}+Z^{(2)}+r\,.\,Z^{(3)}+r^2\,.\,Z^{(4)}+\&c.\}$. To
this we must add the quantity V [1616^{xii}], depending on the mutual attraction of the [1631b]
particles, and we shall obtain the whole of the integral $\int(F\,.\,df+F'\,.\,df'+\&c.)$, as in [1633]. It is evident, from a slight examination of the abridged values $\alpha\,.\,Z^{(0)}$, $\alpha\,.\,Z^{(2)}$, $\alpha\,.\,Z^{(3)}$, &c., [1632, 1630, 1619], that *each separate term* of $Z^{(i)}$ satisfies the equation [1634]; therefore the sum of all the terms, or the whole value of $Z^{(i)}$, will satisfy it.

&c., for the quantities which correspond to s, v, ψ, $P^{(i)}$, in the body S; and we shall obtain the parts of the integral $\int(F.df+F'.df'+\&c.)$,
[1631'''] arising from their action, by marking the letters s, v, ψ, $P^{(i)}$, in the preceding expression of the part of this integral, depending upon the action of S with one accent, two accents, &c.

We shall now collect together all the parts of the integral

$$\int(F.df+F'.df'+\&c.),$$

putting

Values of $Z^{(0)}$, $Z^{(1)}$, &c., depending on all the disturbing forces.

$$\frac{g}{3}=\alpha.Z^{(0)}; \qquad [0=\alpha.Z^{(1)}];$$

$$\frac{S}{s^3}.P^{(2)}+\frac{S'}{s'^3}.P'^{(2)}+\&c.-\frac{g}{2}.(\mu^2-\tfrac{1}{3})=\alpha.Z^{(2)};$$

[1632]
$$\frac{S}{s^4}.P^{(3)}+\frac{S'}{s'^4}.P'^{(3)}+\&c. \qquad =\alpha.Z^{(3)};$$

&c.;

[1632'] α being a very small coefficient, since the condition [1614v], that the spheroid differs but little from a sphere, requires that the forces which make it vary from that figure, should be very small. We shall then have,

[1633]
$$\int(F.df+F'.df'+\&c.)=V+\alpha.r^2.\{Z^{(0)}+Z^{(2)}+r.Z^{(3)}+r^2.Z^{(4)}+\&c.\};$$

$Z^{(i)}$ satisfies the following equation of partial differentials, whatever be the value of i,

Differential equation in $Z^{(i)}$.

[1634]
$$0=\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dZ^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddZ^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i.(i+1).Z^{(i)}.$$

Therefore the general equation of equilibrium will be,*

Equation of equilibrium;

[1635]
$$\int\frac{d\Pi}{\rho}=V+\alpha.r^2.\{Z^{(0)}+Z^{(2)}+r.Z^{(3)}+r^2.Z^{(4)}\&c.\}, \qquad (1)$$

[1635'] If the external bodies are at a very great distance from the spheroid, we may neglect the quantities $r^3.Z^{(3)}$, $r^4.Z^{(4)}$, &c.; because the different terms of which these quantities are composed, are divided respectively by

$Z^{(3)}$, $Z^{(4)}$, &c., may be neglected;

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* (1148) Taking the integral of [1615], and substituting the expression [1633], we shall obtain [1635].

s^4, s^5, &c., s'^4, s'^5, &c.;* therefore these terms become very small, when s, s', &c., are very great in comparison with r. This is the case relative to the planets and satellites, with the single exception of Saturn's ring, which is too near the surface of Saturn to allow of the neglect of the preceding terms. Therefore we must, in the theory of the figure of this planet, include a greater number of terms of the second member of the equation [1635]. This equation has the advantage of producing a series, which is always converging; and as the particles of the ring, attracting the planet, are then infinite in number, the values of $Z^{(0)}$, $Z^{(2)}$, &c., will be given in definite integrals, depending on the figure, and on the internal constitution of the ring.

except in the case of Saturn's Ring. [1635''] [1635'''] [1635'''']

24. The spheroid may be wholly fluid; or a solid nucleus, covered with a fluid. In these two cases, the equation [1635] of the preceding article, will determine the figure of the strata of the fluid part; by taking into consideration that Π being a function of ρ,† *the second member of this equation must be constant at the outer surface, and upon every level surface; so that it can only vary from one surface to another.* [1635v] [1635vi]

The two preceding cases are reduced to one, when the spheroid is homogeneous. For it makes no difference relative to the equilibrium, whether we suppose the spheroid to be wholly fluid, or to contain a solid [1635vii]

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* (1149) If we substitute $\alpha.Z^{(3)}$, $\alpha.Z^{(4)}$, &c., [1632], in [1635], we shall find that the terms depending on $Z^{(3)}$, are of the order $\frac{r^3}{s^4}$, $\frac{r'^3}{s'^4}$, &c.; those depending on $Z^{(4)}$ are of the order $\frac{r^4}{s^5}$, $\frac{r'^4}{s'^5}$, &c.; which must be very small when s, s', &c., are much greater than r. [1635a]

† (1150) Since Π is a function of ρ [1616b], $\int\frac{d\Pi}{\rho}$, or the first member of the equation [1635], must also be a function of ρ. Now from [137''', &c.], the density ρ is constant for all the particles situated on any *level* stratum; therefore the first member of the equation [1635] will be constant, relative to all the particles of the fluid, situated upon any level stratum, or upon the external surface. Consequently the second member of [1635] must be constant relative to the same strata; and it will become of the form given in [1636], [1635b] [1635b']

$$\text{constant} = V + \alpha r^2.\{Z^{(0)} + Z^{(2)} + r.Z^{(3)} + \&c.\}. \qquad [1635c]$$

Equation of equilibrium. nucleus; since, by § 23 [1635c], it is only necessary to have, at the external surface,

[1636] $$\text{constant} = V + \alpha r^2 . \{ Z^{(0)} + Z^{(2)} + r . Z^{(3)} + r^2 . Z^{(4)} + \&c. \}.$$

If we substitute, in this equation, the value of V [1467], and observe, as in
[1636'] [1480'], that $Y^{(0)}$ vanishes, by putting a equal to the radius of a sphere, of the same magnitude as the spheroid, and that $Y^{(1)}$ [1483^{vi}] is nothing, when we fix the origin of the co-ordinates at the centre of gravity of the spheroid; we shall have,*

[1637] $$\text{constant} = \frac{4\pi . a^3}{3r} + \frac{4\alpha\pi . a^5}{r^3} . \left\{ \tfrac{1}{5} Y^{(2)} + \frac{a}{7r} . Y^{(3)} + \frac{a^2}{9r^2} . Y^{(4)} + \&c. \right\} + \alpha r^2 . \{ Z^{(0)} + Z^{(2)} + r . Z^{(3)} + r^2 . Z^{(4)} + \&c. \}.$$

[1637'] This is the equation of the surface of the spheroid; and by substituting for r its value, corresponding to that surface, $a . (1 + \alpha y)$, or†

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* (1151) Substituting $Y^{(0)} = 0$ [1480'], and $Y^{(1)} = 0$ [1483^{vi}], in [1467], we
[1637a] shall get, $V = \frac{4\pi . a^3}{3r} + \frac{4\alpha\pi . a^5}{r^3} . \left\{ \tfrac{1}{5} Y^{(2)} + \frac{a}{7r} . Y^{(3)} + \frac{a^2}{9r^2} . Y^{(4)} + \&c. \right\}$; hence [1636] becomes as in [1637].

† (1152) From [1461', 1464], we have

$$r = a . (1 + \alpha y) = a + a\alpha . \{ Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c. \};$$

and by putting, as in [1636'], $Y^{(0)} = 0$, $Y^{(1)} = 0$, it becomes as in [1638]. If we neglect terms of the order α^2, we shall get,

$$\frac{4\pi . a^3}{3r} = \frac{4\pi . a^3}{3a . (1 + \alpha y)} = \frac{4\pi . a^2}{3} . (1 - \alpha y) = \frac{4\pi . a^2}{3} - \frac{4\pi . a^2}{3} . \alpha y$$

[1637b] $$= \frac{4\pi . a^2}{3} - \frac{4\alpha\pi . a^2}{3} . \{ Y^{(2)} + Y^{(3)} + \&c. \}.$$

Substituting this in [1637], and instead of the factor $\frac{4\alpha\pi . a^5}{r^3}$, putting, $4\alpha\pi . a^2$, we shall get,

[1637c] $$\text{constant} = \frac{4\pi . a^2}{3} - \frac{4\alpha\pi . a^2}{3} . \{ Y^{(2)} + Y^{(3)} + Y^{(4)} + \&c. \} + 4\alpha\pi . a^2 . \left\{ \tfrac{1}{5} Y^{(2)} + \frac{a}{7r} . Y^{(3)} + \frac{a^2}{9r^2} . Y^{(4)} + \&c. \right\} + \alpha r^2 . \{ Z^{(0)} + Z^{(2)} + r . Z^{(3)} + \&c. \};$$

which is easily reduced to the form [1639], putting $r = a$, in the terms multiplied by α;
[1637d] by which means some quantities of the order α^2 are neglected.

$$r = a.(1+\alpha y) = a + \alpha a.\{Y^{(2)} + Y^{(3)} + Y^{(4)} + \&c.\},$$ [1638]

we obtain,

$$\text{constant} = \frac{4\pi}{3}.a^2 - \frac{8\alpha\pi.a^2}{3}.\{\tfrac{1}{5}Y^{(2)} + \tfrac{2}{7}Y^{(3)} + \tfrac{3}{9}Y^{(4)} + \&c.\}$$
$$+\alpha a^2.\{Z^{(0)} + Z^{(2)} + a.Z^{(3)} + a^2.Z^{(4)} + \&c.\}.$$ [1639]

We shall determine the arbitrary constant quantity of the first member of this equation, by means of the following,

$$\text{constant} = \tfrac{4}{3}\pi.a^2 + \alpha a^2.Z^{(0)};$$ [1640]

we shall then have, by comparing the similar functions;* or in other words, those subjected to the same equation of partial differentials,

$$Y^{(i)} = \frac{3.(2i+1)}{8.(i-1).\pi}.a^{i-2}.Z^{(i)},$$ $Y^{(i)}$. [1641]

i being greater than unity. The preceding equation may be put under the form,

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* (1152a) If we suppose the arbitrary constant quantity in the first member of [1639], to be equal to the constant terms of the second member, as in [1640]; and then transpose all the constant terms to the first member, it will vanish; and the whole equation, divided by $\frac{8}{3}\alpha\pi.a^2$, will become,

$$0 = -\tfrac{1}{5}Y^{(2)} - \tfrac{2}{7}Y^{(3)} - \tfrac{3}{9}Y^{(4)} - \&c. + \frac{3}{8\pi}.\{Z^{(2)} + a.Z^{(3)} + a^2.Z^{(4)} + \&c.\};$$

or

$$\tfrac{1}{5}Y^{(2)} + \tfrac{2}{7}Y^{(3)} + \tfrac{3}{9}Y^{(4)} + \&c. = \frac{3}{8\pi}.\{Z^{(2)} + a.Z^{(3)} + a^2.Z^{(4)} + \&c.\}.$$ [1640a]

The second member of this equation is a rational and integral function of

$$\mu, \qquad \sqrt{(1-\mu^2)}.\cos.\varpi, \qquad \sqrt{(1-\mu^2)}.\sin.\varpi,$$

satisfying the equation [1634]; and the first member is a like function [1620'], satisfying the similar equation [1620]. Now it has been proved, in [1479'], that there is but *one* way of developing a function in this form; therefore the similar terms of both members must be separately equal to each other. Hence $\tfrac{1}{5}Y^{(2)} = \frac{3}{8\pi}.Z^{(2)}$, $\tfrac{2}{7}Y^{(3)} = \frac{3a}{8\pi}.Z^{(3)}$, &c.; or

$$Y^{(2)} = \frac{3.5}{8.1.\pi}.Z^{(2)}; \qquad Y^{(3)} = \frac{3.7}{8.2.\pi}.a.Z^{(3)};$$
$$Y^{(4)} = \frac{3.9}{8.3.\pi}.a^2.Z^{(4)} \ldots\ldots Y^{(i)} = \frac{3.(2i+1)}{8.(i-1).\pi}.a^{i-2}.Z^{(i)},$$ [1640b]

as in [1641], the law of continuation being manifest.

[1642] $$Y^{(i)} = \frac{3}{4\pi} \cdot a^{i-2} \cdot Z^{(i)} + \frac{9}{8\,a\,\pi} \cdot \int_0^a r^{i-2} \cdot dr \cdot Z^{(i)};*$$

[1643] the integral being taken from $r=0$ to $r=a$. The radius $a.(1+\alpha y)$ of the surface of the spheroid, will by this means become,

General expression of the radius of a homogeneous spheroid in equilibrium.

[1644] $$a\,.(1+\alpha\,y) = a\,.\left\{ \begin{array}{l} 1 + \frac{3\,\alpha}{4\,\pi} \cdot \{Z^{(2)} + a\,.\,Z^{(3)} + a^2\,.\,Z^{(4)} + \&c.\} \\ + \frac{9\,\alpha}{8\,a\,\pi} \cdot \int_0^a dr\,.\{Z^{(2)} + r\,.\,Z^{(3)} + r^2\,.\,Z^{(4)} + \&c.\} \end{array} \right\}; \qquad (2)$$

We may put this equation under a finite form, by observing that we have, as in the preceding article,†

[1645] $$\alpha\,.\{Z^{(2)} + r\,.\,Z^{(3)} + r^2\,.\,Z^{(4)} + \&c.\} = -\frac{g}{2} \cdot (\mu^2 - \tfrac{1}{3}) - \frac{S}{s\,r^2} - \frac{S\,.\,\delta}{s^2\,r} + \frac{S}{r^2\,.\sqrt{s^2 - 2\,s\,r\,.\,\delta + r^2}} - \frac{S'}{s'r^2} - \&c.;$$

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* (1153) Putting for $\frac{2\,i+1}{i-1}$, its value $2 + \frac{3}{i-1}$ in [1641], we get,

[1641a] $$Y^{(i)} = \frac{3}{4\pi} \cdot a^{i-2} \cdot Z^{(i)} + \frac{9\,a^{i-2}}{8\pi\,.(i-1)} \cdot Z^{(i)}.$$

This may be put under another form, containing the sign of integration $\int$, as in [1642]; and though it is more complex than that of [1641], it will furnish a method of obtaining $a.(1+\alpha y)$, under a finite form, in [1644, 1645]. This reduction is made, by observing, that the value of
[1641b] $P^{(i)}$ [1628] is independent of r, therefore $Z^{(i)}$ [1632] must also be independent of r; and if we integrate $r^{i-2}\,dr\,.\,Z^{(i)}$, commencing the integral with $r=0$, we shall get, $\int r^{i-2} dr\,.\,Z^{(i)} = \frac{r^{i-1}}{i-1} \cdot Z^{(i)}$. If the integral terminate when $r=a$, we shall have, $\int r^{i-2} dr\,.\,Z^{(i)} = \frac{a^{i-1}}{i-1} \cdot Z^{(i)}$. Substituting this in the last term of $Y^{(i)}$ [1641a], it will become as in [1642]. Putting successively $i=2$, $i=3$, &c., we shall obtain,

[1641c] $$Y^{(2)} = \frac{3}{4\,\pi} \cdot Z^{(2)} + \frac{9}{8\,a\,\pi} \cdot \int dr\,.\,Z^{(2)}; \qquad Y^{(3)} = \frac{3}{4\,\pi}\,a\,.\,Z^{(3)} + \frac{9}{8\,a\pi} \cdot \int r\,dr\,.\,Z^{(3)}; \quad \&c.$$

Substituting these in [1638], we shall get [1644].

† (1154) Substituting the values of $Z^{(2)}$, $Z^{(3)}$, &c., [1632], in the first member of [1645], it becomes,

so that the integral $\int dr.\{Z^{(2)}+r.Z^{(3)}+\&c.\}$ can be easily determined [1645′]
by known methods.

25. The equation [1635] has the property, not only of showing the figure of the spheroid; but it gives also, by differentiation, the law of [1645″]
gravity at its surface. For it is evident, that the second member of this equation, being the integral of the sum of all the forces, acting upon each particle, multiplied by the elements of their respective directions;* we shall have the part of the resultant which acts in the direction of the radius r, by taking the differential of the second member of [1635] relative to r.† [1645‴]

$$-\tfrac{1}{2}g.(\mu^2-\tfrac{1}{3})+\left\{\frac{S}{s^3}.P^{(2)}+\frac{Sr}{s^4}.P^{(3)}+\frac{Sr^2}{s^5}.P^{(4)}+\&c.\right\}$$
$$+\left\{\frac{S'}{s'^3}.P'^{(2)}+\frac{S'r}{s'^4}.P'^{(3)}+\&c.\right\}+\&c. \qquad [1645a]$$

Transposing, to the first member of [1627], the terms depending on $P^{(0)}=1$, $P^{(1)}=\delta$, [1625f], we get $-\frac{S}{s}-\frac{Sr.\delta}{s^2}+\frac{S}{f}=\frac{Sr^2}{s^3}.\left\{P^{(2)}+\frac{r}{s}.P^{(3)}+\&c.\right\}$. Dividing this by r^2, we get, $-\frac{S}{sr^2}-\frac{S.\delta}{s^2r}+\frac{S}{r^2f}=\frac{S}{s^3}.P^{(2)}+\frac{Sr}{s^4}.P^{(4)}+\&c.$; accenting the letters, we obtain similar expressions for the bodies S', S'', &c. Substituting these in [1645a], it becomes,

$$-\tfrac{1}{2}g.(\mu^2-\tfrac{1}{3})+\left\{-\frac{S}{sr^2}-\frac{S.\delta}{s^2r}+\frac{S}{r^2f}\right\}+\left\{-\frac{S'}{s'r^2}-\frac{S'.\delta'}{s'^2r}+\frac{S'}{r^2f'}\right\}+\&c.; \qquad [1645b]$$

but from [1625d], $f=\surd(s^2-2sr.\delta+r^2)$, and in like manner, $f'=\surd(s'^2-2s'r.\delta+r^2)$, &c. Hence the preceding expression [1645b] becomes as in the second member of [1645]; in which the terms depending on each of the attracting bodies, are expressed in finite quantities, independently of series. Then the integration mentioned in [1645′] being made, the whole expression [1644] may be put under a finite form.

* (1155) This is evident by comparing the equations [1633, 1635], F, F', &c., being the forces, and f, f', &c., their directions, [1614^{ix}, &c.].

† (1156) We have, by using the symbols V', u, as in [1616c],

$$\int V'.du=\int(F.df+F'.df'+\&c.);$$

hence, from [1633], $\int V'.du=V+\alpha r^2.(Z^{(0)}+Z^{(2)}+r.Z^{(3)}+\&c.)$. Taking the differential relative to r, we get, $V'.\left(\frac{du}{dr}\right)=\left(\frac{dV}{dr}\right)+\frac{\alpha}{dr}.d.\{r^2.Z^{(0)}+r.^2Z^{(2)}+\&c.\}$. [1645$c$]

Hence if we put p for the force, by which a particle of the surface is urged, towards the centre of gravity of the spheroid, we shall get,

[1646] $$p = -\left(\frac{dV}{dr}\right) - \frac{\alpha}{dr} \cdot d \cdot \{r^2 \cdot Z^{(0)} + r^2 \cdot Z^{(2)} + r^3 \cdot Z^{(3)} + r^4 \cdot Z^{(4)} + \&c.\}.$$

[1646'] If, in this equation, we substitute the value of $-\left(\frac{dV}{dr}\right) = \frac{2}{3}\pi a + \frac{V}{2a}$, at the surface, given by the equation [1458], and for V its value, given by the equation [1635], we shall obtain,*

General expression of gravity at the surface of a homogeneous spheroid.

[1647] $$p = \tfrac{4}{3}\pi \cdot a - \tfrac{1}{2}\alpha a \cdot \{Z^{(2)} + a \cdot Z^{(3)} + a^2 \cdot Z^{(4)} + \&c.\} \qquad (3)$$
$$- \frac{\alpha}{dr} \cdot d \cdot \{r^2 \cdot Z^{(0)} + r^2 \cdot Z^{(2)} + r^3 \cdot Z^{(3)} + r^4 \cdot Z^{(4)} + \&c.\};$$

r ought to be changed into a, after taking the differentials, in the second

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Now in the same manner as we have found, in [28a, b], that if the force V, in the direction u, be resolved in the direction ρ, it will become $V \cdot \left(\frac{du}{d\rho}\right)$, we shall find that the force V', acting in the direction u, will produce the force $V' \cdot \left(\frac{du}{dr}\right)$, in the direction r; and this, by means of [1645c], is equal to $\left(\frac{dV}{dr}\right) + \frac{\alpha}{dr} \cdot d \cdot \{r^2 \cdot Z^{(0)} + r^2 \cdot Z^{(2)} + r^3 \cdot Z^{(3)} + \&c.\}$; the forces being supposed to increase the quantities u or r, as in [1615c]; and as this is contrary to the direction of gravity p, it must be put equal to $-p$; hence we obtain the expression [1646].

[1646a] * (1157) The equation [1458] gives $-\left(\frac{dV}{dr}\right) = \frac{2}{3}\pi a + \frac{V}{2a}$. To find $\frac{V}{2a}$, we must substitute the constant quantity [1640] in [1636], and we shall get,

$$\tfrac{4}{3}\pi a^2 + \alpha a^2 \cdot Z^{(0)} = V + \alpha a^2 \cdot \{Z^{(0)} + Z^{(2)} + a \cdot Z^{(3)} + a^2 \cdot Z^{(4)} + \&c.\},$$

the term a being put for r, in the quantities multiplied by α, as in [1637d]. This gives

$$\frac{V}{2a} = \tfrac{2}{3}\pi a - \tfrac{1}{2}\alpha a \cdot \{Z^{(2)} + a \cdot Z^{(3)} + a^2 \cdot Z^{(4)} + \&c.\};$$

hence [1646a] becomes

[1646b] $$-\left(\frac{dV}{dr}\right) = \tfrac{4}{3}\pi a - \tfrac{1}{2}\alpha a \cdot \{Z^{(2)} + a \cdot Z^{(3)} + a^2 \cdot Z^{(4)} + \&c.\};$$

substituting this in [1646], we get [1647]. After taking the differentials in the second member of [1647], we may put $r = a$, because all the terms containing r, are multiplied by α. It is also evident, from [1645], that the part of [1647] affected by the sign d, may be reduced to a finite function, as is observed in [1647'].

member of this equation, which, by the preceding article [1645], can always be reduced to a finite function. [1647']

This expression of p does not accurately represent the force of gravity, but only that part of it which is directed towards the centre of gravity of [1647''] the spheroid, supposing it to be resolved into two forces, of which the one is perpendicular to the radius r, and the other p in the direction of that radius. [1647'''] The first of these two forces is evidently a very small quantity, of the order α; therefore, if we denote it by $\alpha\gamma$, the gravity will be equal to [1647''''] $\sqrt{p^2+\alpha^2\gamma^2}$, which, by neglecting quantities of the order α^2, becomes p.* Hence we may suppose p to express the gravity at the surface of the [1647v] spheroid; and as the equations [1644, 1647] determine the figure of a homogeneous spheroid, in equilibrium, and the law of gravity upon its [1647vi] surface, they contain the complete theory of the equilibrium of such spheroids; upon the supposition that they differ but very little from a sphere.

If there be no foreign bodies S, S', &c., and the spheroid be urged only [1647vii] by the attraction of its particles, and the centrifugal force arising from its rotatory motion, which is the case with the earth, and all the primary planets, excepting Saturn; moreover, if we notice merely the permanent state of the figures, and put $\alpha\varphi$ *for the ratio of the centrifugal force to the* [1647viii] *gravity at the equator, this ratio being very nearly equal to* $\frac{g}{\frac{4}{3}\pi}$,† *supposing* [1647ix] *the density of the spheroid to be unity; we shall find,*

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* (1158) Let p' be the gravity at the point P, fig. 28, page 217, the direction of this force being on the line PgG, perpendicular to the curve at P; put $Pg=p'$, and upon PC, let fall the perpendicular gc; then the force p' may be resolved into the forces $Pc=p$, $gc=\alpha\gamma$; observing that the force gc is of the order α, in comparison with the whole force Pg, or Pc, because if there were no centrifugal, or other disturbing forces, the spheroid would be a sphere, and the line PG would fall on PC. Now in [1646c] the rectangular triangle Pcg, we have $Pg=\sqrt{(Pc^2+cg^2)}$, or in symbols, $p'=\sqrt{(p^2+\alpha^2\gamma^2)}$; and by neglecting the quantity of the order α^2, it becomes $p'=p$, as in [1647'''']. This is similar to the result obtained in [1458a—f].

† (1159) The quantity q [1594a] represents the ratio of the centrifugal force, to the gravity at the equator, and this by [1594b], is expressed by $\frac{g}{\frac{4}{3}\pi\rho}=\alpha\varphi$ [1647viii]. If we

Radius of an ellipsoid,

[1648] $$a \cdot (1 + \alpha y) = a \cdot \{1 - \tfrac{5}{4}\alpha\varphi \cdot (\mu^2 - \tfrac{1}{3})\};$$

[1648'] $$p = \tfrac{4}{3}\pi a \cdot \{1 - \tfrac{2}{3}\alpha\varphi + \tfrac{5}{4}\alpha\varphi \cdot (\mu^2 - \tfrac{1}{3})\};$$

and the gravity at its surface.

therefore *the spheroid is then an ellipsoid of revolution* [1647*h*], *upon which*
[1648''] *the increment of gravity, and the decrement of the radius, in proceeding from the equator to the poles, are nearly proportional to the square of the sine of*
[1648'''] *the latitude, μ being equal to that sine, neglecting quantities of the order α.*

[1647*a*] put $\rho = 1$, we shall get, as above, $q = \alpha\varphi = \frac{g}{\frac{4}{3}\pi}$; which, for the earth, is equal to $\frac{1}{289}$ [1594*a*]. Again, if we suppose $S = 0$, $S' = 0$, &c., in [1632], we shall
[1647*b*] have, $\alpha \cdot Z^{(0)} = \frac{1}{3}g$, $\alpha \cdot Z^{(2)} = -\frac{1}{2}g \cdot (\mu^2 - \frac{1}{3})$, $\alpha \cdot Z^{(3)} = 0$, &c. Substituting these in [1644], we get,

$$a \cdot (1 + \alpha y) = a \cdot \left\{1 + \frac{3\alpha}{4\pi} \cdot Z^{(2)} + \frac{9\alpha}{8a\pi} \cdot \int_0^a dr \cdot Z^{(2)}\right\} = a \cdot \left\{1 + \frac{3\alpha}{4\pi} \cdot Z^{(2)} + \frac{9\alpha}{8\pi} \cdot Z^{(2)}\right\}$$

[1647*c*] $$= a \cdot \left\{1 + \frac{15\alpha}{8\pi} \cdot Z^{(2)}\right\} = a \cdot \left\{1 - \frac{15g}{16\pi} \cdot (\mu^2 - \tfrac{1}{3})\right\}.$$

[1647*d*] Now the preceding value of $\alpha\varphi$ [1647*a*] gives $g = \frac{4}{3}\pi \cdot \alpha\varphi$; substituting this in [1647*c*], it becomes as in [1648]. Putting $Z^{(3)} = 0$, $Z^{(4)} = 0$, &c., in [1647], we get, $p = \frac{4}{3}\pi a - \frac{1}{2}\alpha a \cdot Z^{(2)} - \frac{\alpha}{dr} \cdot d \cdot (r^2 \cdot Z^{(0)} + r^2 \cdot Z^{(2)})$; and by developing the differential relative to r, then putting $r = a$, using the values of $Z^{(0)}$, $Z^{(2)}$, g, [1647*b*, *d*], we get successively,

$$p = \tfrac{4}{3}\pi a - \tfrac{1}{2}\alpha a \cdot Z^{(2)} - \alpha a \cdot (2Z^{(0)} + 2Z^{(2)}) = \tfrac{4}{3}\pi a - 2a\alpha \cdot Z^{(0)} - \tfrac{5}{2}a\alpha \cdot Z^{(2)}$$

$$= \tfrac{4}{3}\pi a - \tfrac{2}{3}ag + \tfrac{5}{4}ag \cdot (\mu^2 - \tfrac{1}{3}) = \tfrac{4}{3}\pi a - \tfrac{2}{3}a \cdot \tfrac{4}{3}\pi\alpha\varphi + \tfrac{5}{4}a \cdot \tfrac{4}{3}\pi\alpha\varphi \cdot (\mu^2 - \tfrac{1}{3})$$

[1647*e*] $$= \tfrac{4}{3}\pi a \cdot \{1 - \tfrac{2}{3}\alpha\varphi + \tfrac{5}{4}\alpha\varphi \cdot (\mu^2 - \tfrac{1}{3})\}, \qquad \text{as in [1648']}.$$

If we put r' for the radius of the equator, and p' for the gravity at the equator, which may be obtained from [1648, 1648'], by making $\mu = 0$, we shall get,

[1647*f*] $$r' = a \cdot (1 + \tfrac{5}{4}\alpha\varphi \cdot \tfrac{1}{3}); \qquad p' = \tfrac{4}{3}\pi a \cdot (1 - \tfrac{2}{3}\alpha\varphi - \tfrac{5}{4}\alpha\varphi \cdot \tfrac{1}{3}).$$

Subtracting these respectively from the expressions [1648, 1648'], we get,

[1647*g*] $$a \cdot (1 + \alpha y) - r' = -a \cdot \tfrac{5}{4}\alpha\varphi \cdot \mu^2; \qquad p - p' = \tfrac{4}{3}\pi a \cdot \tfrac{5}{4}\alpha\varphi \cdot \mu^2;$$

both of which are proportional to μ^2, or to the square of the sine of the latitude, nearly. Moreover, as the sign of the second member of the first equation [1647*g*] is negative, it indicates that the radius decreases as μ increases, in proceeding from the equator to the pole.
[1647*h*] The value of $p - p'$ [1647*g*] being positive, shows that gravity increases in proceeding from the equator to the pole. Lastly, as the radius $a \cdot \{1 - \frac{5}{4}\alpha\varphi \cdot (\mu^2 - \frac{1}{3})\}$ [1648], is independent of ϖ, the spheroid must be a figure of revolution.

It has been supposed, [1636′], that a is the radius of a sphere, the solidity [1648′′′′]
of which is equal to that of the spheroid. The gravity at the surface of this sphere, is $\frac{4}{3}\pi . a$.* Hence we can ascertain the point of the surface of the spheroid, where the gravity is the same as at the surface of the sphere, by determining μ from the equation

$$0 = -\tfrac{2}{3} + \tfrac{5}{4} . (\mu^2 - \tfrac{1}{3}) ; \qquad [1649]$$

which gives $\mu = \sqrt{\tfrac{13}{15}}$. [1649′]

26. From the preceding analysis, we have deduced the figure of a homogeneous fluid mass in equilibrium, without using any other hypothesis [1649′′]
than that of a figure differing but very little from a sphere; and we have shown that the elliptical figure, which by the preceding chapter, satisfies the equilibrium, is then the only one which will agree with it. But as the reduction of the radius of the spheroid, into a series of the form [1638],
$a . \{1 + \alpha . Y^{(0)} + \alpha . Y^{(1)} + \&c.\}$, might cause some difficulty, we shall [1649′′′]
demonstrate directly, and independently of this reduction, that the elliptical figure is the only figure of equilibrium of a fluid homogeneous mass, endowed [1649′′′′]
with a rotatory motion.† This will confirm the results of the preceding computation; and will serve, at the same time, to free it from any doubt [1649v]
that might arise, relative to the generality of this analysis.

We shall suppose, in the first place, the figure to be a spheroid of revolution,‡ and that *its radius is* $a . (1 + \alpha y)$, y being a function of μ, [1649vi]
or of the cosine of the angle θ, which this radius forms with the axis of

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* (1159a) This appears from [1564e], or by putting $\alpha = 0$; in [1648, 1648′], which gives, for a sphere of the radius a, the gravity $p = \frac{4}{3}\pi a$; putting this equal to the general expression of p [1648′], we shall obtain the value of μ which corresponds to the same gravity as in the sphere. This gives $\frac{4}{3}\pi a = \frac{4}{3}\pi a . \{1 - \frac{2}{3}\alpha\varphi + \frac{5}{4}\alpha\varphi . (\mu^2 - \frac{1}{3})\}$. If we reject the first term of each member, which mutually destroy each other, and then divide by $\frac{4}{3}\pi a . \alpha\varphi$, we shall obtain $0 = -\frac{2}{3} + \frac{5}{4} . (\mu^2 - \frac{1}{3})$, [1649]; whence we easily easily deduce $\mu = \sqrt{\frac{13}{15}} = 0{,}93094$ [1649′]; corresponding nearly to the latitude [1648a]
of $68^d\ 35^m$.

† (1160) In this article the spheroid is always supposed to differ but little from a sphere; [1648b]
so that terms of the order α^2 may be neglected, as in [1652′, 1655′].

‡ (1160a) The case of a spheroid which is not of revolution, is afterwards deduced from [1649a]
this in [1672, &c.].

revolution [1616^{xix}]. If we put f for the length of any right line, drawn
[1649^{vii}] from the extremity of this radius, to any point *within* the spheroid; p the complement of the angle formed by this line and by the plane passing through
[1649^{viii}] the radius $a.(1+\alpha y)$ and the axis of revolution; q the angle formed by the projection of f upon this plane, and by this radius; lastly V for the sum
[1649^{ix}] of all the particles of the spheroid, divided by their distances from the point placed at the extremity of the radius $a.(1+\alpha y)$; each particle being
[1650] equal to $f^2 df.dp.dq.\sin.p$,* we shall have,

[1651] $$V=\tfrac{1}{2}\int f'^2.dp.dq.\sin.p,$$

[1651′] f' being the value of f at the point where this line meets the surface of the spheroid. We must now find f' in terms of p and q.

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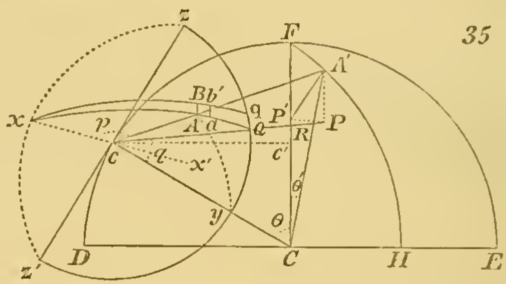
* (1161) We shall suppose CF to be the semi-axis of revolution of the spheroid $DcFEH$, which intersects the plane of the present figure in the meridian $DcFE$. The centre of this spheroid is C, its radius



[1651a] $Cc=a.(1+\alpha y)$, [1649^{vi}],

and the angle $FCc=\theta$. A is a point
[1651b] *within* the spheroid [1649^{vii}], whose distance from c is $cA=f$; this line cA continued, meets the surface of the spheroid
[1651c] in A', making the distance $cA'=f'$, the radius $CA'=a.(1+\alpha y')$, and the angle $FCA'=\theta'$ [1651″]. About c as a centre, with a radius equal to f, we shall describe a hemispherical surface $xzqQyz'x$; intersecting the plane DFE in the
[1651d] semicircle $z'yQqz$, whose diameter $z'cz$, is tangent to the meridian FcD, in the point c; and the axis xcx' is perpendicular to the plane of the figure. This hemisphere passes through the point A, and is marked with the same letters as fig. 2, page 6; the same symbols being also used, except that we must change r into f. Referring therefore occasionally to that figure, we shall have, as in [1355g, h], $cA=cy=cz=cz'=cx=f$,
[1651e] angle $Acx=p$, angle $ycQ=q$, $AB=f.dq.\sin.p$, $Aa=f.dp$. The product $AB.Aa$, by the differential df, will give, as in [1355i], the magnitude of a particle of the spheroid, whose base is $ABb'a$, and height, perpendicular to this base,
[1651f] df. This particle will therefore be represented by $f^2df.dp.dq.\sin.p$, as in [1650]. Substituting this in $V=\int\frac{dM}{f}$ [1616g], we get $V=\int fdf.dp.dq.\sin.p$.
[1651g] Integrating this from $f=0$ to $f=cA'=f'$, it becomes $V=\tfrac{1}{2}\int f'^2.dp.dq.\sin.p$, as in [1651]. This form of V is evidently the same, whether the spheroid be of revolution
[1651g′] or not; in other words, whether y be a function of μ only, or of μ,ϖ.

For this purpose, we shall observe, that if we put θ', for the value of θ, at the point where f' meets the surface, and $a.(1+\alpha y')$ for the corresponding [1651''] radius of the spheroid, y' being the same function of $\cos.\theta'$, or μ', that y is of μ; it is evident that the cosine of the angle, formed by the two right [1651'''] lines f' and $a.(1+\alpha y)$, is equal to $\sin.p.\cos.q$;* therefore, in the [1651''''] triangle formed by the three right lines f', $a.(1+\alpha y)$, and $a.(1+\alpha y')$, we shall have,†

$$a^2.(1+\alpha y')^2 = f'^2 - 2af'.(1+\alpha y).\sin.p.\cos.q + a^2.(1+\alpha y)^2. \qquad [1652]$$

This equation gives two values for f'^2; but one of them, being of the order α^2, vanishes when we neglect quantities of that order. The other [1652'] becomes,‡

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* (1162) The angle CcA', formed by the right lines $cA'=f'$, $Cc=a.(1+\alpha y)$, [1651h] is evidently equal to the angle ycA, which is measured by the spherical arch yA, forming the hypotenuse of the rectangular spherical triangle yQA, whose sides are

$$AQ = \tfrac{1}{2}\pi - p, \qquad yQ = q.$$

Now from [1345[27]] we have $\cos.CcA' = \cos.yA = \cos.AQ.\cos.yQ = \sin.p.\cos.q$, [1651i] as in [1651''''].

† (1163) In the plane triangle CcA', we have, by [62] Int.,

$$CA'^2 = cA'^2 - 2cA'.Cc.\cos.CcA' + Cc^2; \qquad [1651k]$$

and by substituting the values [1651h, i], it becomes as in [1652].

‡ (1164) The quadratic equation in f [1652], solved in the usual manner, and then developed in a series, according to the powers of α, neglecting α^2, becomes successively,

$$f' = a.(1+\alpha y).\sin.p.\cos.q \mp a.\{(1+\alpha y)^2.\sin.^2p.\cos.^2q + (1+\alpha y')^2 - (1+\alpha y)^2\}^{\frac{1}{2}}$$

$$= a.(1+\alpha y).\sin.p.\cos.q \mp a.\{(1+\alpha y)^2.\sin.^2p.\cos.^2q + 2\alpha.(y'-y) + \alpha^2.(y'^2-y^2)\}^{\frac{1}{2}}$$

$$= a.(1+\alpha y).\sin.p.\cos.q \mp a.\left\{(1+\alpha y).\sin.p.\cos.q + \frac{\alpha.(y'-y)}{\sin.p.\cos.q}\right\}. \qquad [1652a]$$

Hence the two values of f' are

$$f' = 2a.(1+\alpha y).\sin.p.\cos.q + \frac{a\alpha.(y'-y)}{\sin.p.\cos.q}, \qquad f' = \frac{-a\alpha.(y'-y)}{\sin.p.\cos.q}. \qquad [1652b]$$

This last value of f' makes f'^2 of the order α^2, which is neglected in [1652']. The other value of f', being squared, neglecting α^2, becomes as in [1653]. Substituting this in [1651], we get [1653].

[1653] $$f'^2 = 4a^2 . \sin.^2 p . \cos.^2 q . (1 + 2\alpha y) + 4\alpha a^2 . (y' - y) ;$$

which gives,

[1654] $$V = 2a^2 . \int dp . dq . \sin. p . \{(1 + 2\alpha y) . \sin.^2 p . \cos.^2 q + \alpha . (y' - y)\}.$$

[1654'] It is evident that the integrals must be taken from $p = 0$ to $p = \pi$, and from $q = -\frac{1}{2}\pi$ to $q = \frac{1}{2}\pi$.* Therefore we shall have,†

[1655] $$V = \tfrac{4}{3}\pi a^2 - \tfrac{4}{3}\alpha\pi . a^2 y + 2\alpha a^2 . \int y' . dp . dq . \sin. p,$$

[1655'] y' being a function of $\cos. \theta'$, and we must determine this cosine in terms of p and q. We may, in this computation, neglect quantities of the order α,

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* (1165) In finding the integral relative to p, we must consider the independent variable
[1653a] quantity q as constant; and then the integral is to be found, in the plane cxA fig. 35, page 262, while the point A moves through the great circle xAQ, till it meets the
[1653b] opposite pole at x'; so that the integral is to be taken from $p = 0$ to $p = \pi$. Then we may suppose the plane of the semicircle xAx' to revolve about the axis xx', so that the line cQ may commence its motion in the position cz', where the angle $q = -\frac{1}{2}\pi$,
[1653c] and finish it in the position cz, where the angle $q = \frac{1}{2}\pi$, during which motion the revolving plane will pass over the whole mass of the spheroid; these limits of p, q, are the same as in [1654']. In finding these limits, it is tacitly supposed, that the whole of the spheroid falls on one side of the tangent zz'; or in other words, that the surface of the
[1653d] spheroid, or the curve of the meridian, does not contain points of contrary flexure, near the point c. This restriction appertains to the calculation as far as [1701''].

† (1166) Connecting together the terms multiplied by y [1654], and bringing that quantity from under the sign of integration, because it is constant through the whole integral, we get,

[1654a] $$V = 2a^2 . \int dp . dq . \sin.^3 p . \cos.^2 q + 2a^2 . \alpha y . \int dp . dq . (2\sin.^3 p . \cos.^2 q - \sin. p) + 2a^2\alpha . \int y' . dp . dq . \sin. p.$$

The integrals relative to p, q, in the first terms, are found by means of the first of the following formulas, which is the same as [84a]; the second is derived from it, by changing θ into $\frac{1}{2}\pi + \theta$.

[1654b] $$\int d\theta . \sin.^n \theta = -\frac{1}{n} . \cos. \theta . \sin.^{n-1}\theta + \frac{n-1}{n} . \int d\theta . \sin.^{n-2}\theta ;$$

Integrals.

[1654c] $$\int d\theta . \cos.^n \theta = \frac{1}{n} . \sin. \theta . \cos.^{n-1}\theta + \frac{n-1}{n} . \int d\theta . \cos.^{n-2}\theta .$$

Now if n be any term of the series 2, 3, 4, &c., and the limits of the integral be any terms of the series 0, $\pm\frac{1}{2}\pi$, $\pm\pi$, $\pm\frac{3}{2}\pi$, $\pm 2\pi$, $\pm\frac{5}{2}\pi$, &c.; the quantities without

because y is already multiplied by α; this being supposed, we shall easily find,*

$$a.\cos.\theta'=(a-f'.\sin.p.\cos.q).\cos.\theta+f'.\sin.p.\sin.q.\sin.\theta\,; \qquad [1656]$$

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the signs of integration will vanish, and we shall have between the limits $\frac{1}{2}a\pi$, $\frac{1}{2}b\pi$; a, b, being any numbers of the series 0, 1, 2, 3, 4, &c., positive or negative,

Definite integrals.

$$\int_{\frac{1}{2}a\pi}^{\frac{1}{2}b\pi} d\theta.\sin.^{n}\theta=\frac{n-1}{n}.\int_{\frac{1}{2}a\pi}^{\frac{1}{2}b\pi} d\theta.\sin.^{n-2}\theta\,; \qquad [1654e]$$

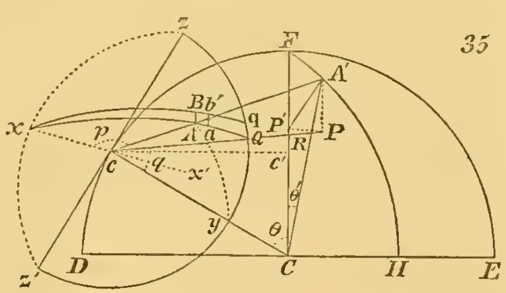
$$\int_{\frac{1}{2}a\pi}^{\frac{1}{2}b\pi} d\theta.\cos.^{n}\theta=\frac{n-1}{n}.\int_{\frac{1}{2}a\pi}^{\frac{1}{2}b\pi} d\theta.\cos.^{n-2}\theta. \qquad [1654f]$$

The last of these formulas, putting $\theta=q$, $n=2$, gives $\int dq.\cos.^2q=\frac{1}{2}\int dq=\frac{1}{2}q+\frac{1}{4}\pi$, which vanishes when $q=-\frac{1}{2}\pi$, and at the other limit $q=\frac{1}{2}\pi$ [1653c], becomes $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} dq.\cos.^2q=\frac{1}{2}\pi$; substituting this in [1654a], we get, [1654g]

$$V=\pi a^2.\int dp.\sin.^3p+2\pi a^2.\alpha y.\int dp.(\sin.^3p-\sin.p)+2a^2\alpha.\int y'.dp.dq.\sin.p. \qquad [1654h]$$

Again, $\int dp.\sin.p=-\cos.p+1$, which vanishes at the first limit of $p=0$ [1653b], and at the second limit, $p=\pi$, it becomes $\int_0^{\pi} dp.\sin.p=2$. If we now put [1654i] $n=3$, $\theta=p$, in [1654e], we shall get, $\int dp.\sin.^3p=\frac{2}{3}\int dp.\sin.p=\frac{2}{3}.2=\frac{4}{3}$, hence $\int dp.(\sin.^3p-\sin.p)=\frac{4}{3}-2=-\frac{2}{3}$; substituting these in [1654h], it becomes as in [1655]. In this nothing is determined relative to the quantity y, which [1654k] remains arbitrary, as in [1651g'].

* (1167) Upon the line $cQRP$, fig. 35, let fall the perpendicular $A'P$, upon the plane of the figure; and in this plane, draw the lines cc', PP', perpendicular to the axis of revolution $Cc'RP'F$. Then, in the rectangular plane triangles $Cc'c$, $CP'A'$, we have, by neglecting terms of the order α,



$$Cc'=Cc.\cos.cCc'=a.\cos.\theta,$$

$$CP'=CA'.\cos.P'CA'=a.\cos.\theta',$$

hence,

$$P'c'=CP'-Cc'=a.\cos.\theta'-a.\cos.\theta, \qquad [1656a]$$

In the similar right angled triangles $Rc'c$, $RP'P$, we have $Rc'=cR.\cos.cRC$, $RP'=RP.\cos.cRC$, whose sum is $Rc'+RP'=(cR+RP).\cos.cRC$, or $P'c'=cP.\cos.cRC$. In the rectangular triangle cPA', we have $cP=cA'.\cos.A'cP=f'.\sin.p$, hence

[1656'] and by substituting the value of $f' = 2a \,.\, \sin. p \,.\, \cos. q$, it gives,*

[1657] $$\mu' = \mu \,.\, \cos.^2 p - \sin.^2 p \,.\, \cos. (2q + \theta).$$

We must now observe, with respect to the integral $\int y' \,.\, dp \,.\, dq \,.\, \sin. p$,
[1657'] taken relative to q, from $2q = -\pi$ to $2q = \pi$ [1654'], that the result would be the same, if we were to take this integral from $2q = -\theta$ to
[1657''] $2q = 2\pi - \theta$, because the values of μ', and therefore those of y', are the same from $2q = -\pi$ to $2q = -\theta$, as from $2q = \pi$ to $2q = 2\pi - \theta$.†
[1658] Supposing therefore $2q + \theta = q'$, which gives,

[1659] $$\mu' = \mu \,.\, \cos.^2 p - \sin.^2 p \,.\, \cos. q',$$

[1656b] $$P'c' = f' \,.\, \sin. p \,.\, \cos. cRC;$$

but in the plane triangle CcR, we have the angle

[1656c] $$cRC = \pi - CcR - cCR = \pi - q - \theta, \qquad \text{hence,}$$

$$\cos. cRC = \cos. (\pi - q - \theta) = -\cos. (q + \theta) = -\cos. q \,.\, \cos. \theta + \sin. q \,.\, \sin. \theta.$$

Substituting this in $P'c'$ [1656b], and then putting it equal to the other expression of $P'c'$ [1656a], we get,

[1656d] $$a \,.\, \cos. \theta' - a \,.\, \cos. \theta = f' \,.\, \sin. p \,.\, \{-\cos. q \,.\, \cos. \theta + \sin. q \,.\, \sin. \theta\};$$

which is easily reduced to the form [1656].

* (1168) The first value of f' [1652b], neglecting terms of the order α, is, as in [1656'], $f' = 2a \,.\, \sin. p \,.\, \cos. q$. Substituting this in [1656], and dividing by a, we get, from [23, 20] Int.,

$$\cos. \theta' = (1 - 2\sin.^2 p \,.\, \cos.^2 q) \,.\, \cos. \theta + 2\sin.^2 p \,.\, \sin. q \,.\, \cos. q \,.\, \sin. \theta$$

$$= \cos. \theta - 2\sin.^2 p \,.\, \cos. q \,.\, (\cos. q \,.\, \cos. \theta - \sin. q \,.\, \sin. \theta) = \cos. \theta - 2\sin.^2 p \,.\, \cos. q \,.\, \cos. (q + \theta)$$

$$= \cos. \theta - \sin.^2 p \,.\, \{2\cos. q \,.\, \cos. (q + \theta)\} = \cos. \theta - \sin.^2 p \,.\, \{\cos. (2q + \theta) + \cos. \theta\}$$

[1657a] $$= \cos. \theta \,.\, (1 - \sin.^2 p) - \sin.^2 p \,.\, \cos. (2q + \theta) = \cos. \theta \,.\, \cos.^2 p - \sin.^2 p \,.\, \cos. (2q + \theta).$$

Substituting $\cos. \theta = \mu$, $\cos. \theta' = \mu'$, [1649vi, 1651'''], it becomes as in [1657]. In this value of μ', terms of the order α [1655'], are neglected; so that in fact it is the same as would be found upon the supposition that the spheroid is a sphere. It will be seen hereafter, that these neglected terms of the order α produce, in the equation of equilibrium [1663],
[1657b] only terms of the order α^2, which are neglected; and the result will be the same, whether these neglected terms of μ' are supposed to be a function of μ only, or a function of μ, ϖ.

† (1169) The spheroid [1649vi] is supposed to be formed by the revolution of the curve $FA'H$, fig. 35, page 265, about the axis CF; therefore the radius $CA' = a \,.\, (1 + \alpha y')$

we shall have,

$$V = \tfrac{4}{3}\pi a^2 - \tfrac{4}{3}\alpha\pi \cdot a^2 y + \alpha a^2 \cdot \textstyle\int y' \cdot dp \cdot dq' \cdot \sin. p\,; \qquad [1660]$$

the integrals being taken from $p=0$ to $p=\pi$, and from $q'=0$ to $q'=2\pi$. [1660']

Now if we put $a^2 \cdot N$ for the integral of all the forces, foreign from the attraction of the spheroid, multiplied by the elements of their directions respectively, we shall have, by [1636], in the case of equilibrium,* [1660'']

$$\text{constant} = V + a^2 \cdot N\,; \qquad [1661]$$

and by substituting the value of V [1660], we shall have, Equation of equilibrium.

$$\text{constant} = \tfrac{4}{3}\alpha\pi \cdot y - \alpha \cdot \textstyle\int y' \cdot dp \cdot dq' \cdot \sin. p - N\,; \qquad [1662]$$

which is evidently the equation of equilibrium [1636], reduced to another

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[1651''] is a function of the angle FCA', or of its cosine μ', [1651''']; and as a, α, are given, y' must be a function of μ', which we shall denote by [1659a]

$$y' = \varphi(\mu') = \varphi(\mu \cdot \cos.^2 p - \sin.^2 p \cdot \cos.[2q+\theta]) \qquad [1657].$$

Substituting this in the expression under the sign of integration in [1655], we get,

$$\textstyle\int y' \cdot dp \cdot dq \cdot \sin. p = \int dp \cdot dq \cdot \sin. p \cdot \varphi(\mu \cdot \cos.^2 p - \sin.^2 p \cdot \cos.[2q+\theta])\,; \qquad [1659b]$$

in which the integral relative to q is to be taken from $q = -\frac{1}{2}\pi$ to $q = \frac{1}{2}\pi$ [1654']. Now it is evident, that within these limits, the term $\cos.(2q+\theta)$ passes through all possible values, corresponding to the *whole* circumference of the circle; the other quantities μ, p, θ, being considered as constant, in the integration relative to q. It is therefore of no importance from what point we compute the angle q. For instead of commencing the integral with $2q = -\pi$, and ending with $2q = \pi$, [1654'], we may commence with $2q = -\theta$, [1659c] and terminate with $2q = 2\pi - \theta$, since the number, value, and signs of the elements, of which the integral [1659b] is composed, are the same in both cases. If we now put, as in [1658], $2q+\theta = q'$, $2dq = dq'$, the expression [1655] will become as in [1660]; [1659d] also μ' [1657] will become as in [1659]; and the limits of q' [1659c, d] may be taken from $q'=0$ to $q'=2\pi$, as in [1660'].

* (1170) Putting $\alpha r^2 \cdot \{Z^{(0)} + Z^{(2)} + r \cdot Z^{(3)} + \&c.\} = a^2 \cdot N$, in [1636], it [1662a] becomes as in [1661]; substituting the value of V [1660], we obtain,

$$\text{constant} = \tfrac{4}{3}\pi a^2 - \tfrac{4}{3}\alpha\pi \cdot a^2 y + \alpha a^2 \cdot \textstyle\int y' \cdot dp \cdot dq' \cdot \sin. p + a^2 \cdot N. \qquad [1662a']$$

Connecting the quantity $\frac{4}{3}\pi a^2$ with the constant term of the first member, and dividing by $-a^2$, we get [1662].

[1662′] form.* This equation being linear, it follows that if there be any number i of radii $a.(1+\alpha y)$, $a.(1+\alpha v)$, &c., which satisfy it, the radius $a.\left\{1+\frac{\alpha}{i}.(y+v+\&c.)\right\}$ will also satisfy it.†

If we suppose that no external force acts upon the body, the whole force [1662″] will be reduced to the mutual action of the particles, and to the centrifugal force, arising from the rotatory motion of the spheroid. We shall put g for the centrifugal force, at the distance 1 from the axis of rotation [1616^{xvi}]; [1662‴] and we shall have, by [1632, 1662a], $N=\frac{1}{2}g.(1-\mu^2)$;‡ therefore the equation of equilibrium [1662] will be,

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* (1170a) It is evident, from the remarks in [1651g', 1657b, &c.], that the values y, y', in the equation of equilibrium [1662], correspond to a spheroid of any form, differing but [1662b] little from a sphere; and are not restricted to the case of a spheroid of revolution, assumed in [1649^{vi}]; so that in [1662], y may be generally considered as a function of θ,ϖ, and y' as a function of θ',ϖ'. This remark will be of use hereafter.

† (1171) Supposing there were i radii, $a.(1+\alpha y_1)$, $a.(1+\alpha y_2)$, &c.,.....$a.(1+\alpha y_i)$, which would satisfy the equation [1662]; we should get, by accenting y_1, y_2, &c., to obtain the values of y, corresponding in each case to the point A', in fig. 35, page 265, the following system of equations, satisfying [1662]; observing that N [1662a] is supposed to be the same in these expressions, neglecting terms of the order α^2; we shall however see, in [1676^{v}, &c.], that there is a case in which N is variable at different points of the surface.

[1663a]

$$\text{constant}=\tfrac{4}{3}\alpha\pi.y_1-\alpha.\textstyle\int y_1'.dp.dq.\sin.p-N;$$
$$\text{constant}=\tfrac{4}{3}\alpha\pi.y_2-\alpha.\textstyle\int y_2'.dp.dq.\sin.p-N;$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\text{constant}=\tfrac{4}{3}\alpha\pi.y_i-\alpha.\textstyle\int y_i'.dp.dq.\sin.p-N.$$

Dividing the sum of these equations by i, and putting

[1663a'] $$Y=\frac{1}{i}.(y_1+y_2.....+y_i),\qquad Y'=\frac{1}{i}.(y_1'+y_2'.....+y_i'),$$

we shall get,

[1663b] $$\text{constant}=\tfrac{4}{3}\alpha\pi.Y-\alpha\textstyle\int Y'.dp.dq.\sin.p-N;$$

which is the same as would be obtained from the equation of equilibrium [1662], by putting Y, Y', for y, y', respectively; therefore the radius $a.(1+\alpha Y)$ will satisfy this equation of equilibrium.

‡ (1171a) In this case we have, from [1632],

$$\alpha.Z^{(0)}=\tfrac{1}{3}g,\qquad \alpha.Z^{(2)}=-\tfrac{1}{2}g.(\mu^2-\tfrac{1}{3}),\qquad Z^{(3)}=0,\qquad \&c.$$

$$\text{constant} = \tfrac{4}{3}\alpha\pi\,.\,y - \alpha\,.\int y'\,.\,dp\,.\,dq'\,.\,\sin.p - \tfrac{1}{2}g\,.\,(1-\mu^2). \qquad [1663]$$

Taking the differential of this relative to μ, three times in succession, observing also that $\left(\frac{d\mu'}{d\mu}\right) = \cos.^2 p$, as is evident from the equation [1659], [1663']

$$\mu' = \mu\,.\,\cos.^2 p - \sin.^2 p\,.\,\cos.\,q'; \qquad [1664]$$

we shall get,*

$$0 = \tfrac{4}{3}\pi\,.\left(\frac{d^3 y}{d\mu^3}\right) - \int dp\,.\,dq'\,.\,\sin.p\,.\,\cos.^6 p\,.\left(\frac{d^3 y'}{d\mu'^3}\right). \qquad [1665]$$

Now we have† $\int dp\,.\,dq'\,.\,\sin.p\,.\,\cos.^6 p = \tfrac{4}{7}\pi$; therefore we may put [1665'] this expression under the following form,

$$0 = \int dp\,.\,dq'\,.\,\sin.p\,.\,\cos.^6 p\,.\left\{\tfrac{7}{3}\,.\left(\frac{d^3 y}{d\mu^3}\right) - \left(\frac{d^3 y'}{d\mu'^3}\right)\right\}. \qquad [1666]$$

Hence [1662a] becomes $a^2\,.\,N = r^2\,.\,\{\tfrac{1}{3}g - \tfrac{1}{2}g\,.\,(\mu^2 - \tfrac{1}{3})\} = \tfrac{1}{2}g\,r^2\,.\,(1-\mu^2)$, as in [1618]; and as the radius r [1649^{vi}] differs from a only by quantities of the order α, we may, by neglecting α^2, divide by a^2 or r^2, and we shall get N, as in [1662''']. Substituting this in [1662], we get [1663], from which the value of y is to be determined by the usual principles of analysis, considering y as a function of μ, and the curve of the meridian to be subject to the restriction mentioned in [1653d].

* (1172) The quantity μ being independent of p, q', we shall, from [1659], evidently obtain the expression $\left(\frac{d\mu'}{d\mu}\right) = \cos.^2 p$, [1663']. The first member of [1663] being [1663b'] constant, its differentials vanish; and if we take the third differential of this equation relative to μ, the term depending on $-\tfrac{1}{2}g\,.\,(1-\mu^2)$ will vanish; the term depending on $\tfrac{4}{3}\alpha\pi\,.\,y$ will give the first term of [1665]. The term depending on y' will produce the second term of [1665], using the formula [1663'], and observing that

$$\left(\frac{dy'}{d\mu}\right) = \left(\frac{dy'}{d\mu'}\right)\cdot\left(\frac{d\mu'}{d\mu}\right) = \left(\frac{dy'}{d\mu'}\right)\,.\,\cos.^2 p\,; \quad \left(\frac{d^2y'}{d\mu^2}\right) = \left(\frac{d^2y'}{d\mu'^2}\right)\cdot\left(\frac{d\mu'}{d\mu}\right)\,.\,\cos.^2 p. = \left(\frac{d^2y'}{d\mu'^2}\right)\,.\,\cos.^4 p\,;$$
$$\left(\frac{d^3y'}{d\mu^3}\right) = \left(\frac{d^3y'}{d\mu'^3}\right)\cdot\left(\frac{d\mu'}{d\mu}\right)\,.\,\cos.^4 p = \left(\frac{d^3y'}{d\mu'^3}\right)\,.\,\cos.^6 p. \qquad [1663c]$$

† (1173) $\int dq' = q'$ vanishes at the first limit of $q = 0$, and at the second limit, $q' = 2\pi$, it is $\int_0^{2\pi} dq' = 2\pi$. Hence the integral [1665'] becomes, by substitution, [1665a] and taking the integral relative to p,

$$\int dp.dq'.\sin.p\,.\,\cos.^6 p = 2\pi\,.\int dp.\sin.p\,.\,\cos.^6 p = -2\pi\,.\int d.\cos.p\,.\,\cos.^6 p = -\tfrac{2}{7}\pi\,.\,\cos.^7 p + \tfrac{2}{7}\pi\,; \qquad [1666a]$$

This equation must be satisfied, whatever be the value of μ. Now it is
[1666'] evident, that among all the values comprised between $\mu = -1$ and
[1666''] $\mu = 1$, there exists one, that we shall denote by h, which is such that, independent of its sign, none of the values of $\left(\frac{d^3 y}{d\mu^3}\right)$ exceeds that
[1666'''] corresponding to h. Denoting this last value by H, we shall have,

[1667] $$0 = \int dp \,.\, dq' \,.\, \sin. p \,.\, \cos.^6 p \,.\, \left\{ \tfrac{7}{3} H - \left(\frac{d^3 y'}{d\mu'^3}\right) \right\}.$$

The quantity $\frac{7}{3} H - \left(\frac{d^3 y'}{d\mu'^3}\right)$ has evidently the same sign as H,* and
[1667'] the factor $\sin. p \,.\, \cos.^6 p$ is always positive through the whole limits of the
[1667''] integral; therefore all the elements of this integral have the same sign as H. Hence it follows, that the whole integral cannot vanish, unless H be
[1667'''] equal to nothing; which requires that we should have generally $0 = \left(\frac{d^3 y}{d\mu^3}\right)$; and then, by integration, we obtain,†

[1668] $$y = l + m \,.\, \mu + n \,.\, \mu^2,$$

[1668'] l, m, n, being arbitrary constant quantities.

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which vanishes at the first limit $p = 0$ [1660'], and at the second limit $p = \pi$, becomes $\frac{4}{7}\pi$, as in [1665']; from which we get, $\frac{4}{3}\pi = \frac{7}{3}\int dp \,.\, dq' \,.\, \sin. p \,.\, \cos.^6 p$. Substituting this in the first term of [1665], we obtain [1666]. In the particular case of
[1666b] $\mu = h$, and $\left(\frac{d^3 y}{d\mu^3}\right) = H$, it becomes as in [1667].

* (1174) H being the greatest value of $\left(\frac{d^3 y}{d\mu^3}\right)$, or of $\left(\frac{d^3 y'}{d\mu'^3}\right)$, independent of its sign [1666'''], the quantity $\frac{7}{3} H - \left(\frac{d^3 y'}{d\mu'^3}\right)$ must evidently be of the same sign as H, and between the limits of p [1660'], the factor $\sin. p \,.\, \cos.^6 p$ is always positive; hence *each* of the elements $dp \,.\, dq' \,.\, \sin. p \,.\, \cos.^6 p \,.\, \left\{ \frac{7}{3} H - \left(\frac{d^3 y'}{d\mu'^3}\right) \right\}$ must have the same sign as H; therefore their sum, or the *whole* integral, must have the same sign as H, and
[1667a] cannot vanish, unless $H = 0$, which, being a maximum value, independent of its sign, requires that we should have generally $\left(\frac{d^3 y}{d\mu^3}\right) = 0$, as in [1667''']; corresponding to any point whatever of the surface of the spheroid, or to any value whatever of μ. The manner in which this equation is obtained is deserving of attention for its singularity.

† (1175) Multiplying the equation $\left(\frac{d^3 y}{d\mu^3}\right) = 0$ by $d\mu$, integrating, and adding

If we fix the origin of the radii in the middle point of the axis of revolution, and take half of this axis for a; y will be nothing, when $\mu = 1$, or [1668″]
$\mu = -1$; from which we shall get $m = 0$, and $n = -l$, and the value of y will become,*

$$y = l \cdot (1 - \mu^2). \qquad [1668‴]$$

Substituting this in the equation of equilibrium [1663],

$$\text{constant} = \tfrac{4}{3} \alpha \pi \cdot y - \alpha \cdot \int y' \cdot dp \cdot dq' \cdot \sin. p - \tfrac{1}{2} g \cdot (1 - \mu^2), \qquad [1669]$$

we shall find,†

$$\alpha l = \frac{15 g}{16 \pi} = \tfrac{5}{4} \alpha \varphi ; \qquad [1670]$$

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the constant quantity $2n$, we get $\left(\frac{d^2 y}{d\mu^2}\right) = 2n$. Again multiplying by $d\mu$, and taking the integral, we get $\left(\frac{dy}{d\mu}\right) = m + 2n \cdot \mu$. Another integration gives, as in [1668],

$$y = l + m \cdot \mu + n \cdot \mu^2. \qquad [1667b]$$

* (1176) The general expression of the radius is $a \cdot (1 + \alpha y)$ [1649^{vi}]; and this, by hypothesis [1668″], is equal to a at the extremities of the axis of revolution; hence we must have $y = 0$ at these points, where $\mu = 1$ or $\mu = -1$. Substituting these values of μ in $y = 0$ [1668], we get $0 = l + m + n$, $0 = l - m + n$. Half the difference of these equations is $m = 0$, and then either of the equations gives $n = -l$. Hence [1668] becomes generally, $y = l - l \cdot \mu^2 = l \cdot (1 - \mu^2)$, as in [1668a]
[1668‴]. For any other point, corresponding to y', μ', it becomes $y' = l \cdot (1 - \mu'^2)$. [1668b]

† (1177) Substituting, in the term under the sign $\int$ [1669], the value of y' [1668b], and μ' [1659], we get,

$$\int y' \cdot dp \cdot dq' \cdot \sin. p = l \cdot \int dp \cdot dq' \cdot \sin. p \cdot \{1 - (\mu \cdot \cos.^2 p - \sin.^2 p \cdot \cos. q')^2\}$$
$$= l \cdot \int dp \cdot dq' \cdot \sin. p \cdot \{(1 - \mu^2 \cdot \cos.^4 p) + 2 \mu \cdot \cos.^2 p \cdot \sin.^2 p \cdot \cos. q' - \sin.^4 p \cdot \cos.^2 q'\}. \qquad [1670a]$$

But $\int dq' = q'$, $\int dq' \cdot \cos. q' = \sin. q'$; which vanish at the first limit of $q' = 0$, [1660′], and at the second limit they become $\int_0^{2\pi} dq' = 2\pi$, $\int_0^{2\pi} dq' \cdot \cos. q' = 0$; [1670b]
moreover, if we put $n = 2$, and $\theta = q'$, in [1654f], we shall get,

$$\int_0^{2\pi} dq' \cdot \cos.^2 q' = \tfrac{1}{2} \cdot \int_0^{2\pi} dq' = \tfrac{1}{2} \cdot 2\pi = \pi. \qquad [1670c]$$

Substituting these in [1670a] it becomes,

$$\int y' \cdot dp \cdot dq' \cdot \sin. p = 2 \pi l \cdot \int dp \cdot \sin. p \cdot \{(1 - \mu^2 \cdot \cos.^4 p) - \tfrac{1}{2} \sin.^4 p\}$$
$$= 2 \pi l \cdot \int dp \cdot \sin. p - 2 \pi l \cdot \mu^2 \cdot \int dp \cdot \sin. p \cdot \cos.^4 p - \pi l \cdot \int dp \cdot \sin.^5 p. \qquad [1670d]$$

[1670'] $\alpha\varphi$ *being the ratio of the centrifugal force to the gravity at the equator* [1647^{viii}], which ratio is very nearly equal to $\frac{3g}{4\pi}$ [1647d]; therefore the radius of the spheroid will be,

Radius of an ellipsoid of revolution.

[1671] $$a \cdot \{1 + \tfrac{5}{4}\alpha\varphi \cdot (1-\mu^2)\}.$$

[1671'] Hence it follows, that this spheroid is an ellipsoid of revolution, which is conformable to what has been heretofore found, [1648, &c.].

[1671''] We have thus obtained, in a direct manner, independently of any series, the figure of a homogeneous spheroid of revolution, revolving about its axis;
[1671'''] and we have shown that it must necessarily be an ellipsoid,* which becomes a sphere when $\varphi = 0$; therefore *a sphere is the only figure of revolution*
[1671''''] *which satisfies the equilibrium of a homogeneous fluid mass that has no rotatory motion.*

Now $\int dp \cdot \sin. p = -\cos. p + 1$ vanishes when $p=0$; and when $p=\pi$, it becomes $\int_0^\pi dp \cdot \sin. p = 2$; also $\int dp \cdot \sin. p \cdot \cos.^4 p = -\frac{1}{5}\cos.^5 p + \frac{1}{5}$, which
[1670e] vanishes when $p=0$, and when $p = 2\pi$, it becomes $\int_0^\pi dp \cdot \sin. p \cdot \cos.^4 p = \frac{2}{5}$.
If we put successively $n=5$, $n=3$, $\theta=p$, in [1654c], we shall get,

[1670f] $$\int_0^\pi dp \cdot \sin.^5 p = \tfrac{4}{5} \cdot \int_0^\pi dp \cdot \sin.^3 p = \tfrac{4}{5} \cdot \tfrac{2}{3} \cdot \int_0^\pi dp \cdot \sin. p = \tfrac{4}{5} \cdot \tfrac{2}{3} \cdot 2 = \tfrac{16}{15};$$

hence [1670d] becomes

$$\int y' \cdot dp \cdot dq' \cdot \sin. p = 2\pi l \cdot 2 - 2\pi l \cdot \mu^2 \cdot \tfrac{2}{5} - \pi l \cdot \tfrac{16}{15} = \tfrac{44}{15}\pi l - \tfrac{4}{5}\pi l \cdot \mu^2.$$

Substituting this, and y [1668'''], in [1669], it becomes,

[1670g] $$\text{const.} = \tfrac{4}{3}\pi.al.(1-\mu^2) - \tfrac{44}{15}\pi.al + \tfrac{4}{5}\pi.al.\mu^2 - \tfrac{1}{2}g.(1-\mu^2) = -\tfrac{8}{5}\pi.al - \tfrac{1}{2}g + \mu^2.(\tfrac{1}{2}g - \tfrac{8}{15}\pi.al).$$

This equation cannot be satisfied, for all values of μ, unless the coefficient of μ^2 be equal to
[1670h] nothing, which gives $\frac{1}{2}g - \frac{8}{15}\pi \cdot \alpha l = 0$; hence $\alpha l = \frac{15g}{16\pi}$, as in [1670].
[1670i] Substituting $g = \frac{4}{3}\pi \cdot \alpha\varphi$ [1647d], we get $\alpha l = \frac{5}{4}\alpha\varphi$, or $l = \frac{5}{4}\varphi$, as in [1670]; hence y [1668'''] becomes $y = \frac{5}{4}\varphi \cdot (1-\mu^2)$, and the expression of the radius [1649^{vi}],
[1670k] $a \cdot (1 + \alpha y)$, gives [1671].

* (1178) In all the computations in this article, quantities of the order α^2 are neglected [1648b], therefore *the results must be considered as being proved to that degree of accuracy only.* When the body is at rest, the centrifugal force $\alpha\varphi$ [1670'] is nothing, and the general expression of the radius [1671] becomes a, corresponding to a sphere, as in [1671'''']; the
[1671a] origin of this radius being at the centre of gravity of this homogeneous sphere.

Hence we may prove generally, that if the fluid mass be acted upon only [1671v]
by very small forces, there can be but one possible figure of equilibrium ; or
in other words, there can be but one radius $a.(1+\alpha y)$, which can satisfy
the equation of equilibrium [1662].

There is but one figure of equilibrium of a homogeneous fluid, differing but little from a sphere.

$$\text{constant} = \tfrac{4}{3}\alpha\pi.y - \alpha.\int y'.dp.dq'.\sin.p - N; \qquad [1672]$$

y being a function of θ *and of the longitude* ϖ [1662b] ; and y' representing
the value of y when we change θ, ϖ, into θ', ϖ', respectively. For if we [1672']
suppose that there are two different radii, $a.(1+\alpha y)$ and $a.(1+\alpha y+\alpha v)$, [1672'']
which will satisfy this equation ; we shall have,*

$$\text{constant} = \tfrac{4}{3}\alpha\pi.(y+v) - \alpha.\int (y'+v').dp.dq'.\sin.p - N. \qquad [1673]$$

Subtracting from this, the preceding equation [1672], we shall get,

$$\text{constant} = \tfrac{4}{3}\pi.v - \int v'.dp.dq'.\sin.p. \qquad [1674]$$

This equation is evidently that of a homogeneous spheroid, in equilibrium,
whose radius is $a.(1+\alpha v)$, and which is not acted upon by any force, [1674']
except that arising from the attraction of its particles. As the angle ϖ must
vanish from this equation,† the radius $a.(1+\alpha v)$ will also satisfy it, if [1674'']

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* (1179) If the radius be $a.(1+\alpha y+\alpha v)$, instead of $a.(1+\alpha y)$, assumed in [1649vi], we must make, in the equation [1662], or [1672], a corresponding change, by writing $y+v$ for y, and $y'+v'$ for y'; by which means it will become as in [1673]. [1673a] Subtracting from this the expression [1672], and dividing by α, we get [1674]. Comparing this last equation with the general equation of equilibrium [1672], divided by α, it will be found, that they will become identical, if we change y into v, y' into v', and put $N=0$. Then the radius $a.(1+\alpha y)$ [1649vi], will become $a.(1+\alpha v)$ [1674'], corresponding to the case of $N=0$ *when all the disturbing forces cease*, [1660'']. We may observe, as [1673b] in [1662b], that y, v, in this equation, may be generally considered as functions of θ,ϖ; and y', v', as functions of θ',ϖ'. We may also remark, that in these calculations, the value of N is supposed to be the same in [1672] as in [1673], which in general is very nearly the case [1673c] in homogeneous spheroids. There may however be cases where there is a difference, as will be shown in [1676v, &c.].

† (1180) Supposing, in fig. 35, page 262, that Cc represents the radius $a.(1+\alpha v)$, [1674'], v will be a function of θ,ϖ [1673b], depending on the form of the surface of the spheroid, and we shall denote it by

$$v = \varphi(\theta,\varpi). \qquad [1674a]$$

The radius CA' of the same figure will be $CA'=(1+\alpha v')$, and $v'=\varphi(\theta',\varpi')$. Now

we change successively ϖ into $\varpi + d\varpi$, $\varpi + 2\,d\varpi$, &c. Hence it follows, that if we put v_1, v_2, &c., for what v becomes by means of these changes, the radius

[1675] $$a.\{1 + \alpha v . d\varpi + \alpha v_1 . d\varpi + \alpha v_2 . d\varpi + \&c.\},$$

or $a.(1 + \alpha . \int v . d\varpi)$, will satisfy the preceding equation. If we take the integral $\int v . d\varpi$, from $\varpi = 0$ to $\varpi = 2\pi$ [1674h], the radius

[1675′] $a.(1 + \alpha . \int v . d\varpi)$ *becomes that of a spheroid of revolution,* which, by*

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it is evident, from [1651b, &c.], that θ', ϖ', are functions of θ, ϖ, p, q; and if these values be substituted in v', the expression $-\int v' . dp . dq' . \sin. p$ [1674], taking the integral between the limits [1660′], will become a function of θ, ϖ. Moreover, the first term of the second member of [1674], $\frac{4}{3}\pi . v$, is a function of θ, ϖ [1674a]; therefore the whole of the second member of [1674] is a function of θ, ϖ, which we shall denote by $\psi(\theta, \varpi)$; and this equation of equilibrium will become

[1674b] $$\text{constant} = \psi(\theta, \varpi).$$

This must be satisfied, for all values of θ, ϖ, in a manner similar to that by which the equation [1670g] was satisfied, in [1670h]; namely, by making the coefficients of the terms, affected by any powers or products of the sines and cosines of the variable quantities θ, ϖ, equal to nothing. In this way the angle ϖ will vanish from the equation [1674b], as is observed in [1674″]; and it would therefore vanish, if we were to change ϖ into $\varpi + i\,d\varpi$,

[1674c] by which means it would become $\text{constant} = \psi(\theta,\ \varpi + i\,d\varpi)$; so that the equation of equilibrium [1674b] would be satisfied, if ϖ were changed into $\varpi + i d\varpi$, and v [1674a]

[1674d] into $v = \varphi(\theta,\ \varpi + i\,d\varpi)$; i being any number whatever.

If we now put successively $i = 0$, $i = 1$, $i = 2, \ldots\ldots i = n$, and suppose the corresponding values of v [1674d], to be $v, v_1, v_2, \ldots\ldots v_n$; the radii, or values of

[1674e] Cc, will become $a.(1 + \alpha v)$, $a.(1 + \alpha v_1), \ldots\ldots a.(1 + \alpha v_n)$, the number of which is $n + 1$; and as all these values of the radius satisfy the equation of equilibrium

[1674f] [1674, 1674b], the radius $a.\left\{1 + \frac{\alpha}{n+1}.(v + v_1 + v_2 \ldots\ldots + v_n)\right\}$, will also satisfy it, as is evident from the method of reasoning in [1663a, a′, b]. If we suppose the quantity $n + 1$ to be infinite, and put $\frac{1}{n+1} = \frac{d\varpi}{2\pi}$, changing also the constant quantity α into $2\alpha\pi$, this last expression of the radius will become,

$$a.\{1 + \alpha . d\varpi . (v + v_1 + v_2 \ldots\ldots + v_\infty)\},$$

as in [1675]; and this, according to the usual notation of the integral calculus, is expressed

[1674g] by $a.\{1 + \alpha . \int v . d\varpi\}$ [1675′], v being the general term of the series $v, v_1, v_2, \ldots\ldots v_\infty$: and if we suppose the first limit of this integral to be $\varpi = 0$, the second will be

[1674h] $(n + 1) . d\varpi = 2\pi$, which are the limits assumed in [1675′].

* (1181) Substituting, in $\int v . d\varpi$, the value of v [1674a], it becomes $\int d\varpi . \varphi(\theta, \varpi)$. Taking the integral of this expression, between the limits $\varpi = 0$, $\varpi = 2\pi$, [1675′], the

what has been said [1671''''], *must be a sphere.* We shall now investigate [1675'']
the value of v which results from this condition.

We shall suppose a to be the shortest distance from the centre of gravity
to the surface of the spheroid, whose radius is $a \cdot (1 + \alpha v)$; and we shall [1675''']
fix the pole, or the origin of the angle θ, at the extremity of a. v will be
nothing at the pole, and positive in all other places, and it will be the same [1675'''']
with the integral $\int v \cdot d\varpi$. Now since the centre of gravity of the spheroid, [1676]
whose radius is $a \cdot (1 + \alpha v)$, is the centre of the sphere,* whose radius
is a, this point will also be the centre of gravity of the spheroid, whose [1676']
radius is $a \cdot (1 + \alpha \cdot \int v \cdot d\varpi)$;† the different radii, drawn from the centre [1676'']

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quantity ϖ will disappear from the integral, and it will become a function of θ, which we shall represent by $\int v \cdot d\varpi = \int d\varpi \cdot \varphi(\theta, \varpi) = F(\theta)$, and the radius [1675'] will be [1675a]

$$a \cdot (1 + \alpha \cdot \int v \cdot d\varpi) = a \cdot \{1 + \alpha \cdot F(\theta)\}.$$

This radius does not vary with ϖ, but depends solely on θ; it must therefore be a spheroid of revolution, as in [1675'].

* (1182) The origin of the radius of the *first* spheroid, $a \cdot (1 + \alpha v)$, is fixed at the centre of gravity of this spheroid [1675''']; and if we suppose a sphere to be described about the same origin as a centre, with the radius a, it is evident that the centres of gravity of the spheroid and sphere will be at the same point of origin, which is the centre of the sphere. The plane drawn through this origin, perpendicular to the axis from which the angle θ is counted, may be considered as the equator of the first spheroid, and sphere; and [1675b]
the common centre of gravity of both these bodies, being in this origin, will also be in the plane of the equator, as is mentioned in the following note.

† (1183) The radius of this *second* spheroid, $a \cdot (1 + \alpha \cdot \int v \cdot d\varpi)$, is that of a [1676a]
spheroid of revolution [1675'], about the axis a; therefore the centre of gravity of this spheroid must be in this axis of revolution, which passes also through the centre of gravity of [1676a']
the sphere [1675b]. The *first* spheroid, whose radius is $a \cdot (1 + \alpha v)$, may be considered as being composed of two parts; the one a sphere, whose radius is a; the other a shell, [1676b]
whose thickness, in the direction of the radius, is $a \cdot \alpha v$. Now if we suppose θ to remain constant, and ϖ to vary from 0 to 2π, the values of v, in the first spheroid, will become, as in [1674g, h], $v, v_1, v_2, \ldots\ldots v_\infty$, corresponding to the whole circumference of the shell 2π. If we neglect terms of the order α, the distance D of these parts from the equator of the sphere, or from the plane passing through its centre of gravity, perpendicular [1676c]
to the axis, will be constant. The products of this distance, by the thickness of the successive points of this shell, $a \cdot \alpha v$, $a \cdot \alpha v_1$, &c., neglecting quantities of the order α^2, will therefore, when added together, be represented by $D \cdot a\alpha \cdot (v + v_1 \ldots\ldots + v_\infty)$; or, as in [1676d]

to the surface of this last spheroid, will not therefore be equal to each other,
[1676‴] except v be nothing; it cannot therefore be a sphere unless $v = 0$;* thus we are assured that a homogeneous spheroid, acted upon by any very small
[1676⁗] forces, can be in equilibrium in one manner only.

27. We have supposed N to be independent of the figure of the spheroid
[1676^v] [1673c],† which is nearly the case when the forces, exclusive of those depending on the attractions of the particles of the fluid, arise from the centrifugal force, produced by the rotatory motion, and from the attractions of foreign bodies. But if we suppose, that there is, at the centre of the
[1676^{vi}] spheroid, a finite force, whose action on any body varies with its distance, the effect on a particle, situated at the surface of the fluid, will depend on the nature of that surface; therefore N will depend on y. This is the case with a homogeneous fluid mass, which covers a sphere of a density different
[1676^{vii}] from that of the fluid. For we may consider this sphere as being of the

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[1674g], by $D \,.\, a\alpha \,.\, fv \,.\, d\varpi$. Now this sum is exactly the same as would be found from
[1676e] multiplying the distance D by the quantity $a\alpha \,.\, fv \,.\, d\varpi$, which corresponds to the thickness of the shell of the second spheroid [1676a]; and as these products are equal, in both spheroids, for any assumed value of θ, they must be the same for the whole of both spheroidal shells, and therefore for the whole of both spheroids. But from [1675b], the
[1676f] centre of gravity of the first spheroid, whose radius is $a \,.\, (1 + \alpha v)$, is in the plane of the equator of the sphere; therefore the sum of the products $D \,.\, a\alpha \,.\, fv \,.\, d\varpi$, computed in [1676$d$] for the whole of the first shell, or for the whole of the first spheroid, must be nothing, by the usual formula for the centre of gravity [124]; consequently the equal expression, found in [1676e], for the second spheroid, must also be nothing. Hence, from the same formula [124], it follows, that the centre of gravity of the second spheroid must be in the equator of the sphere; and as it is also in the axis of the sphere [1676a'], it must be in the centre of the sphere, as in [1676″].

* (1184) By [1675″], the radius of the spheroid of revolution, $a \,.\, (1 + \alpha \,.\, fv \,.\, d\varpi)$ must be a sphere, whose polar radius, counted from the centre of gravity, is a [1675‴];
[1676g] therefore, to make the radii equal, as in a sphere, we must have $a\alpha \,.\, fv \,.\, d\varpi$ or $fv \,.\, d\varpi$ equal to nothing; and v being, by hypothesis [1675⁗] positive, we must necessarily have $v = 0$, as in [1676‴].

† (1185) This supposition has been tacitly made in formulas [1672, 1673], where the value of N is supposed to be the same, when the radius is $a \,.\, (1 + \alpha y)$, as when it is $a \,.\, (1 + \alpha y + \alpha v)$; therefore N must be supposed independent of these radii.

same density as the fluid, and place at its centre a force, varying inversely as the squares of the distances; so that if we put c for the radius of the sphere, [1676viii]
and ρ for its density, that of the fluid being taken for unity; this force, at the distance r will be equal to $\frac{4}{3}\pi \cdot \frac{c^3 \cdot (\rho - 1)}{r^2}$.* If we multiply it by [1676ix]
the element of its direction $-dr$, and take the integral of the product, we shall get, $\frac{4}{3}\pi \cdot \frac{c^3 \cdot (\rho - 1)}{r}$. This quantity must be added to $a^2 . N$,
and as we have, at the surface, $r = a \cdot (1 + \alpha y)$ [1637'], we must, in [1676x]
the equation of equilibrium of the preceding article, add to N the quantity $\frac{4}{3}\pi \cdot \frac{(\rho - 1) \cdot c^3}{a^3} \cdot (1 - \alpha y)$. This equation will then become,

Equation of equilibrium.

$$\text{constant} = \tfrac{4}{3}\alpha\pi \cdot \left\{ 1 + (\rho - 1) \cdot \frac{c^3}{a^3} \right\} \cdot y - \alpha \cdot \int y' \cdot dp \cdot dq' \cdot \sin. p - N. \qquad [1677]$$

If we put $a \cdot (1 + \alpha y + \alpha v)$ for another expression of the radius of the [1677']

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* (1186) Each particle of the sphere, whose density is ρ, may be supposed to be divided into two parts; the one with the same density as the fluid 1, the other with the density $\rho - 1$. The former, being connected with the mass of the surrounding fluid, will produce a spheroid of the same density as the fluid; and its attraction being computed, and then combined with the attraction of a *sphere*, of the radius c and density $\rho - 1$, will give the whole attraction. The mass of this internal sphere is $\frac{4}{3}\pi \cdot c^3 \cdot (\rho - 1)$ [1430k]. Dividing this by the square [1677a]
of the distance of the attracted point r, we get the attraction of this sphere, or of the mass collected at the centre [1416'], equal to $\frac{4}{3}\pi \cdot \frac{c^3 \cdot (\rho - 1)}{r^2}$, [1676ix]. This attraction tends [1678a]
to decrease the distance r; therefore, by [1615c], the negative sign must be given to dr, in finding the corresponding element of $a^2 . N$ [1660''], which becomes

$$\tfrac{4}{3}\pi \cdot c^3 \cdot (\rho - 1) \cdot \frac{-dr}{r^2}; \qquad [1678b]$$

whose integral is $\frac{4}{3}\pi \cdot c^3 \cdot (\rho - 1) \cdot \frac{1}{r}$. This is the part of $a^2 . N$, arising from the attraction of the sphere; and the corresponding part of $-N$ [1662] is

$$-\tfrac{4}{3}\pi \cdot \frac{(\rho - 1) \cdot c^3}{a^2 . r} = -\tfrac{4}{3}\pi \cdot \frac{(\rho - 1) \cdot c^3}{a^3 . (1 + \alpha y)} = -\tfrac{4}{3}\pi \cdot (\rho - 1) \cdot \frac{c^3}{a^3} + \tfrac{4}{3}\alpha\pi \cdot (\rho - 1) \cdot \frac{c^3}{a^3} \cdot y, \qquad [1678c]$$

neglecting α^2. Adding this to the second member of [1662], transposing the term

$$-\tfrac{4}{3}\pi \cdot (\rho - 1) \cdot \frac{c^3}{a^3},$$

and connecting it with the constant term of the first member, we obtain [1677].

spheroid when in equilibrium, we shall obtain the following equation, to determine v,*

[1678] $$\text{constant} = \tfrac{4}{3}\pi . \left\{ 1 + (\rho - 1) . \frac{c^3}{a^3} \right\} . v - \int v' . dp . dq' . \sin. p\,;$$

[1678'] which is the same as the equation of equilibrium [1677], supposing the spheroid to be at rest, and neglecting all external forces.

If the spheroid be of revolution, v will be a function of $\cos. \theta$ or μ
[1678''] only; now we may, in this case, determine v by the analysis of the preceding article. For if we take the differential of this equation, $i+1$ times in succession, considering μ only as variable, we shall get,†

[1679] $$0 = \tfrac{4}{3}\pi . \left\{ 1 + (\rho - 1) . \frac{c^3}{a^3} \right\} . \left(\frac{d^{i+1} v}{d\mu^{i+1}}\right) - \int \left(\frac{d^{i+1} v'}{d\mu'^{i+1}}\right) . dp . dq' . \sin. p . \cos.^{2i+2} p.$$

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* (1187) If we suppose the radius to be $a . \{1 + \alpha . (y + v)\}$, instead of $a . (1 + \alpha y)$, it will be necessary to change y into $y + v$, in [1677], and it will become,

$$\text{constant} = \tfrac{4}{3}\alpha\pi . \left\{ 1 + (\rho - 1) . \frac{c^3}{a^3} \right\} . (y + v) - \alpha . \int (y' + v') . dp . dq' . \sin. p - N.$$

Subtracting [1677] from this, and dividing the remainder by α, we get [1678]. If we now suppose the spheroid to be at rest, and not to be acted upon by any external force, we shall
[1678d] have $N = 0$ [1660'']. Substituting this in [1677], and dividing by α, we get, for the equation of equilibrium of this last spheroid,

$$\text{constant} = \tfrac{4}{3}\pi . \left\{ 1 + (\rho - 1) . \frac{c^3}{a^3} \right\} . y - \int y' . dp . dq' . \sin. p,$$

which is of the same form as [1678]; therefore this last equation [1678] is the same as that of the equation of equilibrium of a spheroid, at rest, and not acted upon by any external force.

† (1188) This equation is deduced from [1678], by taking its differential $i+1$ times relatively to μ, observing that v is a function of μ, v' a function of μ', the other quantities being considered as constant. The differential of v, of the order $i+1$, being divided
[1679a] by $d\mu^{i+1}$, is evidently $\left(\dfrac{d^{i+1} v}{d\mu^{i+1}}\right)$. In taking the differentials of v' relative to μ, we must consider v' as a function of μ', and μ' as a function of μ [1664]; which gives, as in [1663'], $\left(\dfrac{d\mu'}{d\mu}\right) = \cos.^2 p$; hence we shall get successively, in like manner as in [1663c],

[1679b] $$\left(\frac{dv'}{d\mu}\right) = \left(\frac{dv'}{d\mu'}\right) \left(\frac{d\mu'}{d\mu}\right) = \left(\frac{dv'}{d\mu'}\right) . \cos.^2 p\,; \quad \left(\frac{d^2 v'}{d\mu^2}\right) = \left(\frac{d^2 v'}{d\mu'^2}\right) . \left(\frac{d\mu'}{d\mu}\right) . \cos.^2 p . = \left(\frac{d^2 v'}{d\mu'^2}\right) . \cos.^4 p\,;$$

But we have*

Theorem in definite integrals.

$$\int_0^{\pi}\int_0^{2\pi} dp\,.\,dq'\,.\,\sin.p\,.\,\cos.^{2i+2}p=\frac{4\pi}{2i+3}\,;$$ [1680]

the preceding equation may therefore be put under the following form,

$$0=\int dp.dq'.\sin.p.\cos.^{2i+2}p.\left\{\left(\frac{2i+3}{3}\right).\left[1+(\rho-1).\frac{c^3}{a^3}\right].\left(\frac{d^{i+1}v}{d\mu^{i+1}}\right)-\left(\frac{d^{i+1}v'}{d\mu'^{i+1}}\right)\right\}.$$ [1681]

We can take i of such a magnitude, that by neglecting the sign, we may have,

$$\left(\frac{2i+3}{3}\right).\left\{1+(\rho-1).\frac{c^3}{a^3}\right\}>1.$$ [1682]

Now if we suppose that i is the least integral positive number which renders this quantity greater than unity, we may prove, as in the last article [1667'''], [1682']
that this equation cannot be satisfied, unless we suppose $\left(\frac{d^{i+1}v}{d\mu^{i+1}}\right)=0$,† [1682'']
from which we get,

$$v=\mu^i+A.\mu^{i-1}+B.\mu^{i-2}+\&c.$$ [1683]

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and generally $\left(\frac{d^{i+1}v'}{d\mu^{i+1}}\right)=\left(\frac{d^{i+1}v'}{d\mu'^{i+1}}\right).\cos.^{2i+2}p$; this value being substituted in the [1679c]
differential of [1678], of the order $i+1$, we shall get [1679].

* (1189) The limits of the integral [1680] are as in [1660']. Integrating relative to q', we get, as in [1665a], $\int dq'=2\pi$; and the first member of [1680] becomes,

$$2\pi.\int dp.\sin.p.\cos.^{2i+2}p=-2\pi.\int d.\cos.p.\cos.^{2i+2}p=-\frac{2\pi}{2i+3}.\cos.^{2i+3}p+\frac{2\pi}{2i+3};$$ [1680a]

the constant quantity being taken, so as to make it vanish at the first limit $p=0$ [1660']; and at the second limit $p=\pi$, it becomes as in [1680]. From this we get,

$$\tfrac{4}{3}\pi=\left(\frac{2i+3}{3}\right).\int dp'.dq'.\sin.p.\cos.^{2i+2}p\,;$$ [1680b]

substituting it in [1679], it becomes as in [1681].

† (1190) If we put m for the value of $\left(\frac{2i+3}{3}\right).\left\{1+(\rho-1).\frac{c^3}{a^3}\right\}$ mentioned in [1682'], which first exceeds unity; the equation [1681], corresponding to this value of i, will be $0=\int dp.dq'.\sin.p.\cos.^{2i+2}p.\left\{m.\left(\frac{d^{i+1}v}{d\mu^{i+1}}\right)-\left(\frac{d^{i+1}v'}{d\mu'^{i+1}}\right)\right\}$; which is

Substituting this value of v in the preceding equation of equilibrium, and also the following similar value of v',

[1684] $$v' = \mu'^{i} + A \,.\, \mu'^{i-1} + B \,.\, \mu'^{i-2} + \&c. ;$$

[1684'] μ' being, by what precedes, equal to $\mu \,.\, \cos.^2 p - \sin.^2 p \,.\, \cos.\, q'$, [1664], we shall find, in the first place,

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similar to [1666], y being changed into v. Proceeding as in [1666', &c.], we shall find that
[1682a] this equation exists for all values of μ, from $\mu = -1$ to $\mu = 1$; also that there must be a value $\mu = h$, which will render $\left(\frac{d^{i+1} v}{d\,\mu^{i+1}}\right)$ a maximum, independent of its sign; and if this maximum value be put equal to H, we shall have,

$$0 = \int d\,p \,.\, d\,q' \,.\, \sin.\, p \,.\, \cos.^{2i+2} p \,.\, \left\{ m \,.\, H - \left(\frac{d^{i+1} v'}{d\mu'^{i+1}}\right) \right\} ;$$

which is similar to [1667]. Now between the limits of p [1660'], the factor $\sin. p. \cos.^{2i+2} p$ is always positive, and the quantity $m \,.\, H - \left(\frac{d^{i+1} v'}{d\mu'^{i+1}}\right)$ must evidently have the same
[1682a'] sign as H; because by hypothesis, $m > 1$, and H is the greatest value of $\left(\frac{d^{i+1} v}{d\,\mu^{i+1}}\right)$ independent of its sign. Hence we shall get, as in [1667a, &c.], $H = 0$, and generally $\left(\frac{d^{i+1} v}{d\,\mu^{i+1}}\right) = 0$. Multiplying this by $d\,\mu$, and integrating, we get $\left(\frac{d^{i} v}{d\mu^{i}}\right) = A_0$; again multiplying by $d\,\mu$, and integrating, we find $\left(\frac{d^{i-1} v}{d\,\mu^{i-1}}\right)$, &c. Proceeding in this way, we obtain the finite integral $v = A' \,.\, \mu^{i} + B' \,.\, \mu^{i-1} + \&c.$; or, as it may be written, $v = A' \,.\, (\mu^{i} + A \,.\, \mu^{i-1} + B \,.\, \mu^{i-2} + \&c.)$; in which the factor A' may be rejected, supposing it to be included in the value of α, of the term $\alpha\, v$, in the radius [1677']. Hence v becomes as in [1683], and by accenting the letters, we obtain v' as in [1684]. The above reasoning will apply, if we take for i *any* integer *greater* than that above assumed,
[1682b] as i'; for we shall then have $\left(\frac{d^{i'+1} v}{d\,\mu^{i'+1}}\right) = 0$; which is the same as the $i' - i$ differential of the equation $\left(\frac{d^{i+1} v}{d\,\mu^{i+1}}\right) = 0$, [1682''], and may be considered as included in it. To satisfy the equation [1681], in the most simple manner, it will only be necessary to put $\left(\frac{d^{i+1} v}{d\,\mu^{i+1}}\right) = 0$; taking i according to the directions in [1682'].

$$1+(\rho-1)\cdot\frac{c^3}{a^3}=\frac{3}{2i+1}.*$$ [1685]

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* (1191) Substituting the values of v, v', [1683, 1684], in [1678], it becomes,

$$\text{const.}=\tfrac{4}{3}\pi\cdot\left\{1+(\rho-1)\cdot\frac{c^3}{a^3}\right\}\cdot\{\mu^i+A\cdot\mu^{i-1}+\&c.\}-\int dp\cdot dq'\cdot\sin.p\cdot\{\mu'^i+A\cdot\mu'^{i-1}+\&c.\};$$ [1684a]

in which we must substitute the value of μ' [1664], and take the integrals, observing that μ is constant in these integrations. This cannot be satisfied, for all values of μ, except by putting the coefficients of μ^i, μ^{i-1}, μ^{i-2}, &c., separately equal to nothing, as in [1670h]; the constant term of the second member being equal to the arbitrary constant quantity of the first member. In this way we can obtain the values of the constant coefficients A, B, &c., [1684b] [1686]; and then from [1683] we shall obtain the general value of v [1687], as we shall prove in [1686m, n], by a shorter method. For the purpose of demonstrating [1685], it will be sufficient to compute the coefficient of μ^i in [1684a], and then we need only retain the term $\mu^i\cdot\cos.^{2i}p$ of μ'^i [1664]. Putting this coefficient equal to nothing, and dividing by μ^i, we obtain $0=\frac{4}{3}\pi\cdot\left\{1+(\rho-1)\cdot\frac{c^3}{a^3}\right\}-\int dp\cdot dq'\cdot\sin.p\cdot\cos.^{2i}p$. Changing [1684c] i into $i-1$ in [1680], we get, $\int_0^{\pi}\int_0^{2\pi}dp\cdot dq'.\sin.p.\cos.^{2i}p=\frac{4\pi}{2i+1}$. Substituting [1684d] this in the preceding equation, and dividing by $\frac{4}{3}\pi$, we get

$$0=1+(\rho-1)\cdot\frac{c^3}{a^3}-\frac{3}{2i+1},$$

as in [1685]; by means of which the equation [1684a] becomes,

$$\text{constant}=\frac{4\pi}{2i+1}\cdot\{\mu^i+A\cdot\mu^{i-1}+B\cdot\mu^{i-2}+\&c.\}-\int dp\cdot dq'.\sin.p\cdot\{\mu'^i+A\cdot\mu'^{i-1}+\&c.\}.$$ [1684e]

If the quantities c, a, be so related to each other as to produce integral values of i in [1685], the expression v [1683] will be real, and we shall have two forms of the radius of the spheroid [1677'], namely $a.(1+\alpha y)$ and $a.(1+\alpha y+\alpha v)$. We may neglect the case of [1684f] $i=0$, because the expression [1683] is then reduced to its first term $v=1$, and the corresponding increment of the radius $a\cdot\alpha v$ becomes constant, and may be considered as included in the radius a. If $i=1$, the expression [1683] becomes $v=\mu+A$, and [1684g] the corresponding variable part of the radius [1684f] is $a\cdot\alpha\mu$, of which we shall treat in [1687a]; in this case the equation [1685] becomes

$$1+(\rho-1)\cdot\frac{c^3}{a^3}=\tfrac{3}{3},\qquad\text{hence}\qquad(\rho-1)\cdot\frac{c^3}{a^3}=0;$$ [1684h]

which is satisfied by putting either $\rho=1$ or $c=0$. The last of these conditions does, in fact, include the first, because if we put $c=0$, it renders the whole spheroid of the same density as the fluid [1676$^{\text{viii}}$]; making the solid sphere or nucleus infinitely small, or nothing, always supposing ρ to be finite.

[1685] This requires that ρ should be equal to, or less than, unity ;* and when a, c, ρ, have such values, that this equation is not satisfied, i being an integral positive number, the fluid can be in equilibrium only in one manner.† In the next place we shall have,

[1686] $$A=0, \qquad B=-\frac{i.(i-1)}{2.(2i-1)}, \qquad \&c.;$$

so that,‡

[1687] $$v=\mu^i-\frac{i.(i-1)}{2.(2i-1)}.\mu^{i-2}+\frac{i.(i-1).(i-2).(i-3)}{2.4.(2i-1).(2i-3)}.\mu^{i-4}-\&c.;$$

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* (1192) From [1685] we get $(1-\rho)=\frac{2i-2}{2i+1}.\frac{a^3}{c^3}$; and the second member
[1685a] being positive when $i>1$, the first member must also be positive; therefore $\rho<1$ when $i>1$, ρ being equal to unity when $i=1$, as in [1684h].

† (1193) When the equation [1685] is not satisfied, we must suppose $v=0$, $v'=0$, in order to satisfy the equation [1678]; and then the increment of the radius $a.\alpha v$ [1677']
[1685b] becomes nothing, and the radius [1677'] becomes $a.(1+\alpha y)$, which is the only form that will satisfy the equation of equilibrium [1677].

‡ (1194) The values of A, B, [1686], and the resulting expression of v [1683, 1687], may be computed, by the method in [1684b]; but the general solution or value of v is obtained much more simply in [1694'], or [1701]. The same values may also be found in the following manner; in which, for the purpose of illustrating these calculations, we have separately taken into consideration, the case of curves of revolution, before entering upon the general solution [1695—1701], where the same results are obtained in their most general form, by a similar computation.

If the spheroid be homogeneous, and of the density 1, the value of V, arising from the mutual attraction of its particles, will be as in [1467], which, in the case of a sphere, where $\alpha=0$, becomes $\frac{4\pi.a^3}{3r}$. Changing a into c, and multiplying by the density
[1686a] $\rho-1$, we shall obtain the quantity $\frac{4\pi.c^3}{3r}.(\rho-1)$ [1676^ix], to be added to the value of V, computed for a homogeneous spheroid in [1467], to obtain the whole value of V corresponding to the spheroid treated of in [1676^vii—1687'''']. Putting, as in [1638],

[1686b] $$r=a.\{1+\alpha.(Y^{(2)}+Y^{(3)}+Y^{(4)}+\&c.)\},$$

we get, by neglecting α^2, $\frac{1}{r}=\frac{1}{a}.\{1-\alpha.(Y^{(2)}+Y^{(3)}+\&c.)\}$, and the preceding
[1686c] expression [1686a] becomes $\frac{4\pi.c^3}{3a}.(\rho-1).\{1-\alpha.(Y^{(2)}+Y^{(3)}+\&c.)\}$. If the value

therefore there are, in general, two figures of equilibrium, since αv is susceptible of two values, one of which is given by the supposition of $v=0$, [1687′]
another by the supposition of v equal to the preceding function of μ.

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of V [1467] be increased by this quantity, before substitution in [1636], it will augment the second members of [1637, 1639], by the same quantity, and the expression [1639] will become,

$$\text{constant} = \frac{4\pi}{3}.a^2 - \frac{8\alpha\pi.a^2}{3}.\{\tfrac{1}{5}Y^{(2)} + \tfrac{2}{7}Y^{(3)} + \tfrac{3}{9}Y^{(4)} + \&c.\}$$
$$+\frac{4\pi.c^3.(\rho-1)}{3a}.\{1-\alpha.Y^{(2)}-\alpha.Y^{(3)}-\alpha.Y^{(4)}-\&c.\}+\alpha a^2.\{Z^{(0)}+a.Z^{(2)}+\&c.\} \quad [1686d]$$

Supposing, as in [1647vii], $S=0$, $S'=0$, &c.; we shall get, from [1632],

$$\alpha.Z^{(2)} = -\tfrac{1}{2}g.(\mu^2-\tfrac{1}{3}), \qquad \alpha.Z^{(3)}=0, \qquad \alpha.Z^{(4)}=0, \qquad \&c. \quad [1686e]$$

Substituting these in [1686d], and putting the arbitrary constant quantity equal to the constant terms of the second member, $\frac{4\pi}{3}.a^2+\frac{4\pi.c^3}{3a}.(\rho-1)+\alpha a^2.Z^{(0)}$, we shall get, by connecting the terms depending on $Y^{(2)}$, $Y^{(3)}$, &c.,

$$0=-\frac{4\alpha\pi.a^2}{3}.\left\{\begin{array}{c}\left[\tfrac{2}{5}+\frac{c^3}{a^3}.(\rho-1)\right].Y^{(2)}+\left[\tfrac{4}{7}+\frac{c^3}{a^3}.(\rho-1)\right].Y^{(3)} \\ \ldots\ldots+\left[\frac{2i-2}{2i+1}+\frac{c^3}{a^3}.(\rho-1)\right].Y^{(i)}+\&c.\end{array}\right\}-\tfrac{1}{2}a^2.g.(\mu^2-\tfrac{1}{3}). \quad [1686f]$$

Dividing this by the coefficient of $Y^{(2)}$, and putting $\tfrac{5}{3}\alpha\pi.\varphi=\dfrac{\tfrac{1}{2}g}{\tfrac{2}{5}+\frac{c^3}{a^3}.(\rho-1)}$, it will become of the following form,

$$Y^{(2)}+\frac{\tfrac{4}{7}+\frac{c^3}{a^3}.(\rho-1)}{\tfrac{2}{5}+\frac{c^3}{a^3}.(\rho-1)}.Y^{(3)}\ldots\ldots+\frac{\frac{2i-2}{2i+1}+\frac{c^3}{a^3}.(\rho-1)}{\tfrac{2}{5}+\frac{c^3}{a^3}.(\rho-1)}.Y^{(i)}\ldots\ldots+\&c.=-\tfrac{5}{4}\varphi.(\mu^2-\tfrac{1}{3}). \quad [1686g]$$

Both members of this equation are functions of μ, $\sqrt{(1-\mu^2)}.\cos.\varpi$, $\sqrt{(1-\mu^2)}.\sin.\varpi$, satisfying the equation [1620]; therefore each of those members must be of the same form, as is proved in [1479′]. Hence the similar functions of μ in both members must be equal, and we shall have $Y^{(2)}=-\tfrac{5}{4}\varphi.(\mu^2-\tfrac{1}{3})$, $Y^{(3)}=0$, $Y^{(4)}=0$, and generally [1686h] $Y^{(i)}=0$, i being any integral number greater than 2. We must however except the case where the factor of $Y^{(i)}$, in the first member of [1686g], is equal to nothing; for then we shall have $\frac{2i-2}{2i+1}+\frac{c^3}{a^3}.(\rho-1)=0$, and the expression depending on $Y^{(i)}$ will vanish from [1686f], and leave $Y^{(i)}$ arbitrary. Therefore the general expression of the radius [1686b] will become $a.\{1+\alpha.Y^{(2)}+\alpha.Y^{(i)}\}$, whenever the quantities a, c, ρ, are so [1686i]

If the spheroid have no rotatory motion, and be not acted upon by any
[1687''] force, except that arising from the attraction of its particles, the first of these two figures will be a sphere, and the second will have for its meridian, a curve of the order i.* These two curves become identical when $i=1$,

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dependent on each other as to satisfy this equation, which is the same as [1685],

[1686k] $$0=\frac{2i-2}{2i+1}+\frac{c^3}{a^3}\cdot(\rho-1)=1-\frac{3}{2i+1}+\frac{c^3}{a^3}\cdot(\rho-1).$$

If this equation be not satisfied, we must also put $Y^{(i)}=0$, and the radius [1686i] will be reduced to $a.(1+\alpha.Y^{(2)})$; substituting the value of $Y^{(2)}$ [1686h], it becomes

[1686l] $$r=a.(1+\alpha.Y^{(2)})=a.\{1-\tfrac{5}{4}\alpha\varphi.(\mu^2-\tfrac{1}{3})\};$$

which is of the same form as in [1648]. If the equation [1685] be satisfied, the value of v may be taken equal to $Y^{(i)}$ [1528], and if we suppose, as in [1678''], that this corresponds to a spheroid of revolution, in which $Y^{(i)}$ is independent of ϖ, its value must be found by putting $n=0$. Hence $Y^{(i)}=\left\{\mu^i-\frac{i.(i-1)}{2.(2i-1)}.\mu^{i-2}+\&c.\right\}.B$; and the radius

[1686m] $a.\{1+\alpha.Y^{(i)}\}=a.\left\{1+\alpha.B.\left(\mu^i-\frac{i.(i-1)}{2.(2i-1)}.\mu^{i-2}+\&c.\right)\right\}$. If we put $\alpha.B=\alpha'$,

[1686n] and use v [1687], it becomes $a.(1+\alpha'v)$. Comparing [1683, 1687], we get A, B, &c., [1686].

* (1195) If the spheroid have no rotatory motion, we shall have the centrifugal force $\alpha\varphi=0$ [1647viii], hence $Y^{(2)}=0$ [1686h]; and the value of the radius [1686l] will be reduced to the constant quantity a, corresponding to a sphere. Then the general expression of the radius [1686i] will be

$$a.(1+\alpha.Y^{(2)}+\alpha.Y^{(i)})=a.(1+\alpha.Y^{(i)})=a.\left\{1+\alpha'.\left(\mu^i-\frac{i.(i-1)}{2.(2i-1)}.\mu^{i-2}+\&c.\right)\right\},$$

as in [1686n]. If $i=1$, this radius will become $a.(1+\alpha'\mu)$; and by neglecting the accent on α', it will be, as in [1687'''],

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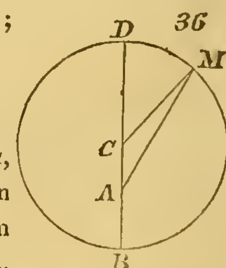

[1687a] $$a.(1+\alpha\mu)=a.(1+\alpha.\cos.\theta);$$

corresponding to a sphere DMB, whose radius is $CD=CM=a$, and centre C. The origin of $r=AM$, being at the point A, in the axis of revolution DCB, at the distance $CA=a.\alpha$ from the centre C; the angle $CAM=\theta$, $\cos.\theta=\mu$. Then by [62] Int.,

[1687b] $$CM=\{AM^2-2AM.CA.\cos.CAM+CA^2\}^{\frac{1}{2}}=AM-CA.\cos.ACM,$$

neglecting α^2. Hence $AM=CM+CA.\cos.\theta$, or in symbols, $r=a+a\alpha.\mu$, as in [1687'''].

because the radius $a.(1+\alpha\mu)$ is that of a sphere, in which the origin of the radius is at the distance $a.\alpha$ from its centre [1687*b*]; but then it is evident that $\rho=1$ [1684*g*, &c.]; therefore the spheroid must be homogeneous; which is conformable to the result of the preceding article [1671''', &c.]. [1687'''] [1687'''']

28. When we have figures of revolution, which satisfy the equilibrium, it is easy to deduce, from them, figures which are not of revolution, by the following method. Instead of fixing the origin of the angle θ, at the extremity of the axis of revolution, we shall suppose that it is placed at the distance γ from that extremity, putting θ' for the distance of any point from the extremity of the axis of revolution, and θ for the distance of the same point from the new origin of the angle θ; also $\varpi-\beta$ for the angle contained between the two arcs θ and γ; then we shall have,* [1687^{v}] [1687^{vi}] [1687^{vii}]

$$\cos.\theta'=\cos.\gamma.\cos.\theta+\sin.\gamma.\sin.\theta.\cos.(\varpi-\beta). \qquad [1688]$$

Putting therefore the following function equal to $\Gamma.(\cos.\theta')$,

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* (1196) Let CB fig. 37, be the axis of revolution, corresponding to any figure of revolution [1687^{v}]; CA the new axis from which the polar distances γ, θ, are counted; ABD a spherical surface, described about the centre C, with the radius 1, and cutting the radius of the spheroid CDR in the point D. Then in the spherical triangle BAD, we have $BD=\theta'$, the polar distances $AB=\gamma$, $AD=\theta$; and if AEF be the meridian, from which the longitudes β, ϖ, are counted, we shall have the angles $EAB=\beta$, $EAD=\varpi$, whose difference is the angle $BAD=\varpi-\beta$. Hence from [63] Int., we get [1688]. Now the radius $a.(1+\alpha v)$ of a spheroid, in equilibrium, and *having no rotatory motion*, is given in [1686*m*, *n*], and if we change α' into α, to conform to the notation [1690], [1688*a*] [1688*b*] [1688*c*]

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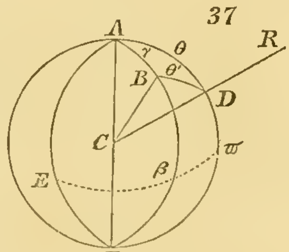

it becomes $a.\left\{1+\alpha.\left(\mu^{i}-\frac{i.(i-1)}{2.(2i-1)}.\mu^{i-2}+\&c.\right)\right\}$, in which $\mu=\cos.BD=\cos.\theta'$ of the present notation [1687^{vi}]. Substituting this value of μ in the preceding expression of the radius, and using the abridged symbol [1689], it becomes $a.\{1+\alpha.\Gamma.(\cos.\theta')\}$, as in [1689]. This, by means of [1688], is easily reduced to the form [1690]; which represents the value of the radius $a.(1+\alpha v)$, satisfying the equation of equilibrium [1674], and makes $\alpha v=\alpha.\Gamma.\{\cos.\gamma.\cos.\theta+\sin.\gamma.\sin.\theta.\cos.(\varpi-\beta)\}$. As the situation of the assumed point B may be varied at pleasure, without altering the point D, the quantities γ, β, as well as α, may be considered as wholly arbitrary, and be taken at pleasure. [1688*d*] [1688*e*] [1688*f*] [1688*g*]

[1689] $$\Gamma.(\cos.\theta') = \cos.^{i}\theta' - \frac{i.(i-1)}{2.(2i-1)}.\cos.^{i-2}\theta' + \&c.;$$

the radius of a spheroid, at rest, and in equilibrium, which we have just
[1689'] proved to be equal to $a.\{1+\alpha.\Gamma.(\cos.\theta')\}$, will become,

[1690] $$a+\alpha a.\Gamma.\{\cos.\gamma.\cos.\theta+\sin.\gamma.\sin.\theta.\cos.(\varpi-\beta)\};$$

and although this is a function of ϖ, it appertains to a solid of revolution, in
[1690'] which the angle θ has not, for its origin, the extremity of the axis of revolution.

Since this radius satisfies the equation of equilibrium [1674], whatever be the values of α, β, γ, it will also satisfy it, if we change these quantities into
[1690''] α', β', γ'; α'', β'', γ'', &c.; and as this equation is linear in v, v', it will follow that the radius*

[1691]
$$\begin{aligned} &a+\alpha a.\Gamma.\{\cos.\gamma.\cos.\theta+\sin.\gamma.\sin.\theta.\cos.(\varpi-\beta)\} \\ &\quad +\alpha' a.\Gamma.\{\cos.\gamma'.\cos.\theta+\sin.\gamma'.\sin.\theta.\cos.(\varpi-\beta')\} \\ &\quad +\&c. \end{aligned}$$

will also satisfy it. The spheroid to which this radius appertains, is no
[1691'] longer a spheroid of revolution, but is formed of a sphere of the radius a,

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* (1197) The value αv [1688f] is a function of $\theta, \varpi, \gamma, \beta$, which for brevity we shall denote by $\alpha v = \alpha.\Gamma'(\theta,\varpi,\gamma,\beta)$; hence the corresponding value of $\alpha v'$ [1672', &c.] will be $\alpha v' = \alpha.\Gamma'(\theta',\varpi',\gamma,\beta)$. These values satisfy the equation of equilibrium [1674] of a body having no rotatory motion, leaving the quantities α, β, γ, arbitrary [1688g]; and this equation will be represented by

[1691a] $$\text{constant} = \tfrac{4}{3}\pi\alpha.\Gamma'(\theta,\varpi,\gamma,\beta) - \alpha.\int\Gamma'(\theta',\varpi',\gamma,\beta).dp.dq'.\sin.p.$$

In like manner, if we change α, β, γ, into α', β', γ', $\alpha'', \beta'', \gamma''$, &c., it will be evident that $\alpha' v = \alpha'.\Gamma'(\theta,\varpi,\gamma',\beta')$, $\alpha'' v = \alpha''.\Gamma'(\theta,\varpi,\gamma''.\beta'')$, &c., will also satisfy the equation of equilibrium [1674], which will become of the forms,

[1691b]
$$\begin{aligned} &\text{constant} = \tfrac{4}{3}\pi\alpha'.\Gamma'(\theta,\varpi,\gamma',\beta') - \alpha'.\int\Gamma'(\theta',\varpi',\gamma',\beta').dp.dq'.\sin.p; \\ &\text{constant} = \tfrac{4}{3}\pi\alpha''.\Gamma'(\theta,\varpi,\gamma'',\beta'') - \alpha'.\int\Gamma'(\theta',\varpi',\gamma'',\beta'').dp.dq'.\sin.p; \\ &\&c. \end{aligned}$$

Adding together the equations [1691a, b], putting

[1691c]
$$\begin{aligned} \alpha_1 v_1 &= \alpha.\Gamma'(\theta,\varpi,\gamma,\beta) + \alpha'.\Gamma'(\theta,\varpi,\gamma',\beta') + \&c., \\ \alpha_1 v'_1 &= \alpha.\Gamma'(\theta',\varpi',\gamma,\beta) + \alpha'.\Gamma'(\theta',\varpi',\gamma',\beta') + \&c., \end{aligned}$$

and of any number whatever of strata, similar to that formed by the excess of the spheroid of revolution, whose radius is $a+\alpha a \,.\, \Gamma(\mu)$, above the sphere whose radius is a; these strata being placed above each other, in any arbitrary manner. [1691'']

If we compare the expression of $\Gamma \,.\, (\cos.\theta')$ [1689], with that of $P^{(i)}$ [1628], we shall perceive that these two functions are similar, and that they differ only by the quantities γ, β, which in $P^{(i)}$ are v and ψ, and by a factor independent of μ, ϖ; therefore we shall have,* [1691'''] [1691'''']

$$0=\left\{\frac{d\,.\,(1-\mu^2)\,.\left\{\frac{d\,.\,\Gamma\,.\,(\cos.\theta')}{d\mu}\right\}}{d\mu}\right\}+\frac{\left(\frac{dd\,.\,\Gamma.(\cos.\theta')}{d\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,\Gamma\,.\,(\cos.\theta'). \qquad [1692]$$

Hence it is evident, that if we represent by $\alpha \,.\, Y^{(i)}$ the following function, namely,

$$\begin{aligned}\alpha \,.\, Y^{(i)} = \alpha \,.\, \Gamma \,.\, &\{\cos.\gamma \,.\, \cos.\theta+\sin.\gamma \,.\, \sin.\theta \,.\, \cos.(\varpi-\beta)\}\\ +\alpha' \,.\, \Gamma \,.\, &\{\cos.\gamma' \,.\, \cos.\theta+\sin.\gamma' \,.\, \sin.\theta \,.\, \cos.(\varpi-\beta')\}\\ +\,&\&c.\,;\end{aligned} \qquad [1693]$$

$Y^{(i)}$ will be a rational and integral function of μ, $\sqrt{1-\mu^2}.\cos.\varpi$, $\sqrt{1-\mu^2}.\sin.\varpi$, which will satisfy the following equation of partial differentials,† [1693']

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and dividing the sum by α_1, we shall get,

$$\text{constant}=\tfrac{4}{3}\pi \,.\, v_1-\int v'_1 \,.\, dp \,.\, dq' \,.\, \sin.p\,; \qquad [1691d]$$

which is the same as the equation of equilibrium [1674], in which v is changed into v_1, and v' into v'_i. Therefore the radius $a \,.\, (1+\alpha_1 v_1)$ will also satisfy the equation of equilibrium, as in [1691]. [1691e]

* (1198) Putting $v=\gamma$, $\psi=\beta$, in δ [1629], it will become like the value of $\cos.\theta'$ [1688]; therefore we shall have $\delta=\cos.\theta'$. Substituting this in [1628], we get, by using [1689],

$$P^{(i)}=\frac{1.3.5.....(2i-1)}{1.2.3.....i}\,.\left\{\cos.^i\theta'-\frac{i.(i-1)}{2.(2i-1)}.\cos.^{i-2}\theta'+\&c.\right\}=\frac{1.3.5.....(2i-1)}{1.2.3.....i}\,.\,\Gamma\,.\,(\cos.\theta'). \qquad [1693a]$$

Substituting this value of $P^{(i)}$ in [1630], and rejecting the common factor $\frac{1.3.5.....(2i-1)}{1.2.3.....i}$, we obtain [1692].

† (1199) Substituting $\cos.(\varpi-\beta)=\cos.\varpi \,.\, \cos.\beta+\sin.\varpi \,.\, \sin.\beta$ [24] Int., in [1688], and putting $\cos.\theta=\mu$, $\sin.\theta=\sqrt{(1-\mu^2)}$, we get,

$$\cos.\theta'=\cos.\gamma \,.\, \mu+(\sin.\gamma \,.\, \cos.\beta) \,.\, \sqrt{(1-\mu^2)} \,.\, \cos.\varpi+(\sin.\gamma \,.\, \sin.\beta) \,.\, \sqrt{(1-\mu^2)} \,.\, \sin.\varpi\,; \qquad [1693b]$$

Differential equation in $Y^{(i)}$.

[1694] $$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}+\frac{\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,Y^{(i)}.$$

Therefore if we take for $Y^{(i)}$ the most general function of this nature, the
[1694′] function $a\,.\,(1+\alpha\,.\,Y^{(i)})$ will be the most general expression of the radius of the spheroid, at rest and in equilibrium.

We may obtain the same result, by means of the expression of V in series, given in [1467]. For the equation of equilibrium being, by the preceding article [1661],

[1695] $$\text{constant}=V+a^2\,.\,N\,;$$

if we suppose all the forces, excepting those arising from the mutual action of the particles, to be reduced to a single attractive force, equal to

[1695′] $$\tfrac{4}{3}\pi\,.\,\frac{(\rho-1)\,.\,c^3}{r^2},\ *$$

placed at the centre of the spheroid, and then multiply this force by the element of its direction $-d\,r$, and take the integral, we shall obtain,

[1696] $$\tfrac{4}{3}\pi\,.\,\frac{(\rho-1)\,.\,c^3}{r}=a^2\,.\,N\,;$$

[1696′] and since at the surface, $r=a\,.\,(1+\alpha y)$ [1676x], the preceding equation of equilibrium [1695] will become,

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which contains the first power of the three co-ordinates

$$\mu,\qquad \surd(1-\mu^2)\,.\,\cos.\,\varpi,\qquad \surd(1-\mu^2)\,.\,\sin.\,\varpi\,;$$

therefore, as i is a positive integer, the expression of $\Gamma\,.\,(\cos.\,\theta')$ [1689] will evidently be a rational and integral function of the same three co-ordinates, which will, as has already been seen, satisfy the equation [1692]. The same may be proved of the functions multiplied by α', α'', &c., in [1693], and the sum of all of them, or the whole function [1693], may be
[1693c] expressed by $\alpha\,.\,Y^{(i)}$. This quantity satisfies the equation [1694], which is similar to [1692]; and as the number of the factors α, α', α'', &c., [1693], is unlimited, we may take the most general function of the form $\alpha\,.\,Y^{(i)}$, and then the most general expression of the radius [1691, 1693] will become $a\,.\,(1+\alpha\,.\,Y^{(i)})$, as in [1694′].

* (1200) This quantity and the first member of [1696], which is deduced from it, are found as in [1676ix, &c.]; this last being taken for $a^2\,.\,N$ [1660″, 1695′].

$$\text{constant} = V + \tfrac{4}{3}\,\alpha\,\pi\,.\,\frac{c^3}{a}\,.\,(1-\rho)\,.\,y.\ *$$ [1697]

We shall substitute in this equation, the value of V given by the formula [1467], putting $r = a\,.\,(1+\alpha\,y)$, and using y [1464], namely,

$$y = Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$$ [1698]

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* (1201) Substituting the value of r [1696′] in [1696], we get,

$$a^2\,.\,N = \tfrac{4}{3}\,\pi\,.\,\frac{(\rho-1)\,.\,c^3}{a}\,.\,\frac{1}{1+\alpha\,y} = \tfrac{4}{3}\,\pi\,.\,\frac{(\rho-1)\,.\,c^3}{a}\,.\,(1-\alpha\,y),$$

neglecting α^2; hence [1695] becomes

$$\text{constant} = V + \tfrac{4}{3}\,\pi\,.\,(\rho-1)\,.\,\frac{c^3}{a} - \tfrac{4}{3}\,\alpha\,\pi\,.\,\frac{c^3}{a}\,.\,(\rho-1)\,.\,y.$$

Transposing the term $\tfrac{4}{3}\,\pi\,.\,(\rho-1)\,.\,\frac{c^3}{a}$, and connecting it with the constant term of the first member, we find $\text{constant} = V - \tfrac{4}{3}\,\alpha\,\pi\,.\,\frac{c^3}{a}\,.\,(\rho-1)\,.\,y$, as in [1697]. Substituting the value of V [1467], putting $r = a$, in terms of the order α, and also $\frac{4\pi\,.\,a^3}{3r} = \frac{4\pi\,.\,a^3}{3a\,.\,(1+\alpha\,y)} = \tfrac{4}{3}\,\pi\,.\,a^2\,.\,(1-\alpha\,y)$, we get,

$$\text{const.} = \frac{4\pi.a^3}{3r} + 4\,\alpha\pi\,.\,a^2\,.\left\{Y^{(0)} + \tfrac{1}{3}\,Y^{(1)} + \tfrac{1}{5}\,Y^{(2)}\,\ldots\ldots + \frac{1}{2i+1}\,.\,Y^{(i)} + \&c.\right\} + \tfrac{4}{3}\alpha\pi.\frac{c^3}{a}.(1-\rho)\,.\,y$$

$$= \tfrac{4}{3}\,\pi\,.\,a^2\,.\,(1-\alpha\,y) + 4\,\alpha\,\pi\,.\,a^2\,.\left\{Y^{(0)} + \tfrac{1}{3}\,Y^{(1)} + \tfrac{1}{5}\,Y^{(2)}\,\ldots\ldots + \frac{1}{2i+1}\,.\,Y^{(i)} + \&c.\right\}$$ [1697a]

$$+ \tfrac{4}{3}\,\alpha\,\pi\,.\,\frac{c^3}{a}\,.\,(1-\rho)\,.\,y.$$

Transposing the term $\tfrac{4}{3}\,\pi\,.\,a^2$, and supposing it to be equal to the constant term of the first member, then dividing by $\tfrac{4}{3}\,\alpha\,\pi\,.\,a^2$, we get, [1697b]

$$0 = -y + 3\,.\left\{Y^{(0)} + \tfrac{1}{3}\,Y^{(1)} + \tfrac{1}{5}\,Y^{(2)}\,.\,.\,.\,.\,. + \frac{1}{2i+1}\,.\,Y^{(i)} + \&c.\right\} + \frac{c^3}{a^3}\,.\,(1-\rho)\,.\,y.$$

Substituting for y its value [1698], and reducing, we obtain,

$$0 = \left\{2\,Y^{(0)} - \tfrac{2}{5}\,Y^{(2)} - \tfrac{4}{7}\,Y^{(3)}\,.\,.\,.\,.\,. - \frac{(2i-2)}{2i+1}\,.\,Y^{(i)} - \&c.\right\}$$ [1697c]

$$+ \frac{c^3}{a^3}\,.\,(1-\rho)\,.\,\{Y^{(0)} + Y^{(1)} + Y^{(2)}\,.\,.\,.\,.\,. + Y^{(i)} + \&c.\},$$

which is easily reduced to the form [1699].

Hence we shall obtain,

[1699]
$$0=\left\{(1-\rho)\cdot\frac{c^3}{a^3}+2\right\}.Y^{(0)}+(1-\rho)\cdot\frac{c^3}{a^3}\cdot Y^{(1)}+\left\{(1-\rho)\cdot\frac{c^3}{a^3}-\tfrac{2}{5}\right\}.Y^{(2)}$$
$$\ldots\ldots+\left\{(1-\rho)\cdot\frac{c^3}{a^3}-\left(\frac{2i-2}{2i+1}\right)\right\}.Y^{(i)}+\&c.\,;$$

the quantity a being taken of such magnitude as to make the constant term in the first member of [1697] equal to $\frac{4}{3}\pi\,.\,a^2$ [1697b]. This equation
[1699′] gives $Y^{(0)}=0$, $Y^{(1)}=0$, $Y^{(2)}=0$, &c.; unless one of the coefficients, as for example that of $Y^{(i)}$, becomes nothing; from which we shall get,*

[1700]
$$(1-\rho)\cdot\frac{c^3}{a^3}=\frac{2i-2}{2i+1};$$

i being an integral positive number. In this case all these quantities vanish,
[1701] except $Y^{(i)}$, and we shall have $y=Y^{(i)}$, which agrees with what we have just found, [1694′].

Hence it is evident that the results obtained by the reduction of V in a
[1701′] series, are as general as possible; and there is no fear that any figure will be omitted, in using the analysis depending on this reduction. This agrees with what we have seen *a priori* in the analysis of § 11 [1465′], where we have proved that the form we have given to the radius of the spheroid is
[1701″] not arbitary,† but follows necessarily from the nature of the attractions of such spheroids.

29. We shall now resume the equation [1635]; and if we substitute the value of V, given by formula [1506], we shall get, for the different strata of the fluid,‡

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* (1202) Proceeding in the same manner as in [1686g, h], we shall obtain, from
[1700a] [1699], $Y^{(0)}=0$, $Y^{(1)}=0$, &c. But if the coefficient of $Y^{(i)}$ be nothing, it will leave $Y^{(i)}$ arbitrary, and [1698] will become $y=Y^{(i)}$. The coefficient of $Y^{(i)}$, put equal to nothing, gives [1700], which may be put under the form [1685].

† (1203) The subject of this development has already been very fully discussed in note 1050, pages 136—157.

‡ (1204) Substituting the value of V [1506] in [1635], it produces [1702]. For the first, second, third and fourth integrals of [1506] produce respectively the third, fourth, first

$$\int\frac{d\Pi}{\rho}=2\pi.\int_a^1\rho.d.a^2+4\alpha\pi.\int_a^1\rho.d.\Big\{a^2.Y^{(0)}+\frac{ar}{3}.Y^{(1)}+\frac{r^2}{5}.Y^{(2)}+\frac{r^3}{7a}.Y^{(3)}+\&c.\Big\}$$ Equation of equilibrium of any stratum.

$$+\frac{4\pi}{3r}.\int_0^a\rho.d.a^3+\frac{4\alpha\pi}{r}.\int_0^a\rho.d.\Big\{a^3.Y^{(0)}+\frac{a^4}{3r}.Y^{(1)}+\frac{a^5}{5r^2}.Y^{(2)}+\frac{a^6}{7r^3}.Y^{(3)}+\&c.\Big\}$$

$$+\alpha r^2.\{Z^{(0)}+Z^{(2)}+r.Z^{(3)}+r^2.Z^{(4)}+\&c.\}. \qquad (1)$$ [1702]

The differentials and integrals refer to the variable quantity a; the two first integrals of the second member of this equation must be taken from $a=a$ to $a=1$; *a being the value of a, corresponding to the level stratum under consideration, and this value at the surface being taken for unity.* The two last integrals must be taken from $a=0$ to $a=a$. Lastly the radius r must be changed into $a.(1+\alpha y)$ [1503''''], after taking all the differentials and integrals. In the terms multiplied by α, it will suffice to change r into a; but in the term $\frac{4\pi}{3r}.\int_0^a\rho.d.a^3$, we must substitute $a.(1+\alpha y)$ for r, and then it becomes $\frac{4\pi}{3a}.(1-\alpha y).\int_0^a\rho.d.a^3$, which is equal to the following expression,* [1702'] [1702''] [1702'''] [1702'''']

$$\frac{4\pi}{3a}.\{1-\alpha.Y^{(0)}-\alpha.Y^{(1)}-\alpha.Y^{(2)}-\&c.\}.\int_0^a\rho.d.a^3.$$ [1703]

This being premised, if we compare the similar functions in the equation [1702], we shall find, in the first place,

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and second of the second member of [1702]. The greatest value of a, which in [1505'] is put equal to a, is here [1702''] put equal to unity, or $a=1$, corresponding to the outer surface of the fluid. [1701a]

* (1205) If we put, as in [1702''''], $r=a.(1+\alpha y)$, the term $\frac{4\pi}{3r}.\int_0^a\rho.d.a^3$, neglecting α^2, and using y [1698], will become as in [1703]. Substituting this in [1702], we get,

$$\int\frac{d\Pi}{\rho}=2\pi.\int_a^1\rho.d.a^2+4\alpha\pi.\int_a^1\rho.d.\Big\{a^2.Y^{(0)}+\frac{ar}{3}.Y^{(1)}+\frac{r^2}{5}.Y^{(2)}\ldots\ldots+\frac{r^i}{(2i+1).a^{i-2}}.Y^{(i)}+\&c.\Big\}$$

$$+\frac{4\pi}{3a}.\int_0^a\rho.d.a^3-\frac{4\alpha\pi}{3a}.\{Y^{(0)}+Y^{(1)}+Y^{(2)}\ldots\ldots+Y^{(i)}+\&c.\}.\int_0^a\rho.d.a^3$$

$$+\frac{4\alpha\pi}{r}.\int_0^a\rho.d.\Big\{a^3.Y^{(0)}+\frac{a^4}{3r}.Y^{(1)}\ldots\ldots+\frac{a^{i+3}}{(2i+1).r^i}.Y^{(i)}+\&c.\Big\}$$ [1702a]

$$+\alpha r^2.\{Z^{(0)}+Z^{(2)}+r.Z^{(3)}+r^2.Z^{(4)}\ldots\ldots+r^{i-2}.Z^{(i)}+\&c.\}.$$

Equation to determine $Y^{(0)}$.

$$\int \frac{d\Pi}{\rho} = 2\pi . \int_a^1 \rho . d . a^2 + 4\alpha\pi . \int_a^1 \rho . d . (a^2 . Y^{(0)}) + \frac{4\pi}{3a} . \int_0^a \rho . d . a^3$$

[1704] $$- \frac{4\alpha\pi}{3a} . Y^{(0)} . \int_0^a \rho . d . a^3 + \frac{4\alpha\pi}{a} . \int_0^a \rho . d . (a^3 . Y^{(0)}) + \alpha a^2 . Z^{(0)} ;*$$

the two first integrals of the second member of this equation must be taken
[1704′] from $a = a$ to $a = 1$; the three other integrals of this second member

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* (1206) The first member of [1702a] is the same as the integral of the first member of [1615], representing the integral of the *sum* of the forces F, F', &c., acting on the fluid, multiplied respectively by the elements of their directions df, df', &c.; this sum is integrable in the case of nature [1616^{ix}, &c.], therefore the first member of [1702a], or $\int \frac{d\Pi}{\rho}$, is also integrable; and as ρ is a function of a [1503′′′′], the quantity $\int \frac{d\Pi}{\rho}$ must also be a function of ρ or a, which is *constant* for the whole of the level stratum corresponding to a; that is, it is independent of the variable quantities μ, ϖ, corresponding to the different points of the surface of this stratum. Transposing all the terms of [1702a], except those depending on $Z^{(0)}$, $Z^{(2)}$, &c., to the first member, and putting, in the factor αr^2 of these last terms, $r = a$, because they are of the order α, it will become of this form,

[1702b] $$A_0 . Y^{(0)} + A_1 . Y^{(1)} + A_2 . Y^{(2)} + \&c. = \alpha a^2 . \{Z^{(0)} + Z^{(2)} + a . Z^{(3)} + \&c.\}.$$

Now the second member is given, in [1632], in functions of μ, $\sqrt{(1-\mu^2)} . \cos. \varpi$, $\sqrt{(1-\mu^2)} . \sin. \varpi$, which satisfy the equation [1634]; and as this can be done only in one manner [1479′], the similar terms of [1702b] in each member must be equal; hence, for any integral value of i, we have $A_i . Y^{(i)} = \alpha a^i . Z^{(i)}$. When $i = 0$, it becomes $A_0 . Y^{(0)} = \alpha a^2 . Z^{(0)}$, which denotes that all the quantities of [1702a], independent of μ, ϖ; or, in other words, independent of $Y^{(1)}$, $Y^{(2)}$, &c., $Z^{(1)}$, $Z^{(2)}$, &c., are to be put equal to each other, in the two members; and this gives the equation [1704]. The other equation $A_i . Y^{(i)} = \alpha a^2 . Z^{(i)}$, indicates also, that the quantities depending on $Z^{(i)}$, $Y^{(i)}$, in [1702a], destroy each other; and by putting all these terms equal to nothing, we obtain,

[1702c] $$0 = 4\alpha\pi . \int_a^1 \rho . d . \left(\frac{r^i . Y^{(i)}}{(2i+1) . a^{i-2}}\right) - \frac{4\alpha\pi}{3a} . Y^{(i)} . \int_0^a \rho . d . a^3 + \frac{4\alpha\pi}{r} . \int_0^a \rho . d . \left(\frac{a^{i+3} . Y^{(i)}}{(2i+1) . r^i}\right) + \alpha r^i . Z^{(i)} ;$$

in which the sign d does not affect r, but only a; therefore r may be brought from under the sign of integration; and as all these quantities are multiplied by α, we may afterwards, by neglecting α^2, put $r = a$; and the equation [1702c], divided by α, will become as in [1705].

from $a=0$ to $a=a$. This equation does not determine either a or $Y^{(0)}$, but gives merely the relation between these two quantities; so that [1704''] the value of $Y^{(0)}$ will be arbitrary, and may be taken at pleasure. We [1704'''] shall then have, *when* i *is equal to, or greater than, unity,*

Equation to determine $Y^{(i)}$. First form.

$$0=\frac{4\pi . a^i}{2i+1}\cdot\int_a^1 \rho . d . \left(\frac{Y^{(i)}}{a^{i-2}}\right)-\frac{4\pi}{3a}\cdot Y^{(i)}\cdot\int_0^a \rho . d . a^3 \qquad (2)$$
$$+\frac{4\pi}{(2i+1) . a^{i+1}}\cdot\int_0^a \rho . d . (a^{i+3} . Y^{(i)})+a^i . Z^{(i)}; \qquad [1705]$$

the first integral being taken from $a=a$ to $a=1$, and the other two from $a=0$ to $a=a$. This equation will give the value of $Y^{(i)}$, [1705'] corresponding to each fluid stratum, when the law of the densities ρ shall be known.

To reduce these integrals to the same limits, we shall put,

$$\frac{4\pi}{2i+1}\cdot\int_0^1 \rho . d . \left(\frac{Y^{(i)}}{a^{i-2}}\right)+Z^{(i)}=\frac{4\pi}{2i+1}\cdot Z'^{(i)}, \qquad [1706]$$

the integral being taken from $a=0$ to $a=1$; $Z'^{(i)}$ will be a quantity [1706'] independent of a, and the equation [1705] will become,*

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* (1207) Separating the integral $\int_0^1 \rho . d . \left(\frac{Y^{(i)}}{a^{-2}}\right)$ into two parts, the one between the limits $a=0$, $a=a$, the other between $a=a$ and $a=1$, the sum of these two parts will be equal to the whole integral, or

$$\int_0^1 \rho . d . \left(\frac{Y^{(i)}}{a^{i-2}}\right)=\int_0^a \rho . d . \left(\frac{Y^{(i)}}{a^{i-}}\right)+\int_a^1 \rho . d . \left(\frac{Y^{(i)}}{a^{i-2}}\right). \qquad [1707a]$$

Substituting this in [1706], we get, by transposition,

$$\frac{4\pi}{2i+1}\cdot\int_a^1 \rho . d . \left(\frac{Y^{(i)}}{a^{i-2}}\right)=\frac{4\pi}{2i+1}\cdot Z'^{(i)}-Z^{(i)}-\frac{4\pi}{2i+1}\cdot\int_0^a \rho . d . \left(\frac{Y^{(i)}}{a^{i-2}}\right).$$

Multiplying by a^i, we obtain,

$$\frac{4\pi . a^i}{2i+1}\cdot\int_a^1 \rho . d . \left(\frac{Y^{(i)}}{a^{i-2}}\right)=\frac{4\pi . a^i}{2i+1}\cdot Z'^{(i)}-a^i . Z^{(i)}-\frac{4\pi . a^i}{2i+1}\cdot\int_0^a \rho . d . \left(\frac{Y^{(i)}}{a^{i-2}}\right);$$

substituting this in [1705], it becomes,

$$0=\frac{4\pi . a^i}{2i+1}\cdot Z'^{(i)}-\frac{4\pi . a^i}{2i+1}\cdot\int_0^a \rho . d . \left(\frac{Y^{(i)}}{a^{i-2}}\right)-\frac{4\pi}{3a}\cdot Y^{(i)}\cdot\int_0^a \rho . d . a^3$$
$$+\frac{4\pi}{(2i+1) . a^{i+1}}\cdot\int_0^a \rho . d . (a^{i+3} . Y^{(i)}); \qquad [1707b]$$

multiplying by $-\frac{(2i+1) . 3}{4\pi} . a^{i+1}$, and changing the order of the terms, we get [1707].

Second form.

[1707]
$$0=(2i+1)\,.\,a^i\,.\,Y^{(i)}\int_0^a\rho\,.\,d\,.\,a^3+3\,a^{2i+1}\,.\int_0^a\rho\,.\,d\,.\left(\frac{Y^{(i)}}{a^{i-2}}\right)$$
$$-3\,.\int_0^a\rho\,.\,d\,.\,(a^{i+3}\,.\,Y^{(i)})-3\,a^{2i+1}\,.\,Z'^{(i)},$$

all the integrals being taken from $a=0$ to $a=a$.

We may make the signs of integration disappear, by taking the differentials
[1707′] relative to a. Hence we shall obtain the following differential equation of the second order,*

[1708]
$$\left(\frac{d\,d\,Y^{(i)}}{d\,a^2}\right)=\left\{\frac{i\,.\,(i+1)}{a^2}-\frac{6\,\rho\,.\,a}{\int\rho\,.\,d\,.\,a^3}\right\}.\,Y^{(i)}-\frac{6\,\rho\,.\,a^2}{\int\rho\,.\,d\,.\,a^3}\,.\left(\frac{d\,Y^{(i)}}{d\,a}\right).$$

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* (1208) $Z^{(i)}$ [1632] is independent of a; $Y^{(i)}$ [1464, 1463] is a function of a; and if we take the differential of [1707] relative to a, and divide by da, we shall get, by putting $d\,.\int\rho\,.\,d\,.\,a^3=\rho\,.\,d\,.\,a^3=3\,a^2\,d\,a$, &c.,

[1707c]
$$0=(2i+1)\,.\,i\,.\,a^{i-1}\,.\,Y^{(i)}\,.\int\rho\,.\,d\,.\,a^3+(2i+1)\,.\,a^i\,.\left(\frac{d\,Y^{(i)}}{d\,a}\right).\int\rho\,.\,d\,.\,a^3+(2i+1)\,.\,a^i\,.\,Y^{(i)}\,.\,(\rho\,.\,3a^2)$$
$$+3\,.\,(2i+1)\,.\,a^{2i}\,.\int\rho\,.\,d\,.\,(Y^{(i)}\,.\,a^{2-i})+3\,a^{2i+1}\,.\,\rho\,.\left\{\left(\frac{d\,Y^{(i)}}{d\,a}\right).\,a^{2-i}+(2-i)\,.\,Y^{(i)}\,.\,a^{1-i}\right\}$$
$$-3\,\rho\,.\left\{(i+3)\,.\,a^{i+2}\,.\,Y^{(i)}+a^{i+3}\,.\left(\frac{d\,Y^{(i)}}{d\,a}\right)\right\}-3\,.\,(2\,i+1)\,.\,a^{2i}\,.\,Z'^{(i)}.$$

Reducing this, by neglecting those terms free from the sign of integration, depending on $Y^{(i)}$, $dY^{(i)}$, which mutually destroy each other, by the vanishing of their coefficients, we get,

[1707d]
$$0=(2\,i+1)\,.\,i\,.\,a^{i-1}\,.\,Y^{(i)}\,.\int\rho\,.\,d\,.\,a^3+(2\,i+1)\,.\,a^i\,.\left(\frac{dY^{(i)}}{d\,a}\right).\int\rho\,.\,d\,.\,a^3$$
$$+3\,.\,(2\,i+1)\,.\,a^{2i}\,.\int\rho\,.\,d\,.\,(Y^{(i)}\,.\,a^{2-i})-3\,.\,(2\,i+1)\,.\,a^{2i}\,.\,Z'^{(i)}.$$

Dividing this by $(2\,i+1)\,.\,a^{2i}$, we obtain,

$$0=i\,.\,a^{-i-1}\,.\,Y^{(i)}\,.\int\rho\,.\,d\,.\,a^3+a^{-i}\,.\left(\frac{d\,Y^{(i)}}{d\,a}\right).\int\rho\,.\,d\,.\,a^3+3\,.\int\rho\,.\,d\,.\,(Y^{(i)}\,.\,a^{2-i})-3\,Z'^{(i)}.$$

Taking the differential of this equation, and putting $d\,.\int\rho\,.\,d\,.\,a^3=3\,a^2\,d\,a$, and $d\,.\int\rho\,.\,d\,.\,(Y^{(i)}\,.\,a^{2-i})=\rho\,.\,d\,.\,(Y^{(i)}\,.\,a^{2-i})=\rho\,.\left\{\left(\frac{d\,Y^{(i)}}{d\,a}\right).\,a^{-i+2}+(2-i)\,.\,Y^{(i)}\,.\,a^{-i+1}\right\}$, we shall get, by observing that $Z'^{(i)}$ is independent of a [1706′],

[1708a]
$$0=-i\,.\,(i+1)\,.\,a^{-i-2}\,.\,Y^{(i)}\,.\int\rho\,.\,d\,.\,a^3+i\,.\,a^{-i-1}\,.\left(\frac{d\,Y^{(i)}}{d\,a}\right).\int\rho\,.\,d\,.\,a^3+3\,i\,.\,a^{-i+1}\,.\,Y^{(i)}\,.\,\rho$$
$$-i\,.\,a^{-i-1}\,.\left(\frac{d\,Y^{(i)}}{d\,a}\right).\int\rho\,.\,d\,.\,a^3+a^{-i}\,.\left(\frac{d\,d\,Y^{(i)}}{d\,a^2}\right).\int\rho\,.\,d\,.\,a^3+3\,a^{-i+2}\,.\left(\frac{dY^{(i)}}{d\,a}\right).\,\rho$$
$$+3\,\rho\,.\left\{\left(\frac{dY^{(i)}}{d\,a}\right).\,a^{-i+2}+(2-i)\,.\,Y^{(i)}\,.\,a^{-i+1}\right\};$$

The integral of this equation will give the value of $Y^{(i)}$, with two arbitrary constant quantities. These constant quantities are rational and integral functions of μ, $\sqrt{1-\mu^2}\,.\,\cos.\varpi$, $\sqrt{1-\mu^2}\,.\,\sin.\varpi$, of the order i;* and if we represent them by $U^{(i)}$, they will satisfy the following equation of partial differentials, [1708′]

Differential equation in $U^{(i)}$.

$$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{dU^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddU^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,U^{(i)}. \qquad [1709]$$

One of these functions is determined by means of the function $Z'^{(i)}$, which vanished in taking the differentials, and it is evident that it will be a multiple of that function.† With respect to the other function, if we suppose the [1709′]

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of which the second and fourth terms destroy each other; and if we connect the third with the eighth, and the sixth with the seventh, we get,

$$\begin{aligned}0=&-i\,.\,(i+1)\,.\,a^{-i-2}\,.\,Y^{(i)}\,.\int\rho\,.\,d\,.\,a^3+6\,a^{-i+1}\,.\,Y^{(i)}\,.\,\rho\\&+6\,\rho\,.\,a^{-i+2}\,.\left(\frac{dY^{(i)}}{da}\right)+a^{-i}\,.\left(\frac{d\,d\,Y^{(i)}}{da^2}\right).\int\rho\,.\,d\,.\,a^3.\end{aligned} \qquad [1708a']$$

Dividing this by $a^{-i}\,.\int\rho\,.\,d\,.\,a^3$, we get [1708].

* (1209) It follows from [1466, 1433k—l], that $Y^{(i)}$ may contain the powers of μ, $\sqrt{(1-\mu^2)}\,.\,\cos.\varpi$, $\sqrt{(1-\mu^2)}\,.\,\sin.\varpi$, not exceeding i, but some of the terms may be of the order of the positive integral numbers $i-2$, $i-4$, &c.; as may also be perceived by the formulas [1528a—e]; and we may incidentally remark, that the integrals or differentials of these expressions, relative to a, do not alter the nature of these functions, which must satisfy [1708b] an equation similar to [1460, 1465], as in [1709]. The partial differentials in [1708] are relative to a; and μ, ϖ, are considered as constant; therefore the two arbitrary quantities, added to complete the integral, must be arbitrary functions of μ, ϖ, or rather of

$$\mu, \qquad \sqrt{(1-\mu^2)}\,.\,\cos.\varpi, \qquad \sqrt{(1-\mu^2)}\,.\,\sin.\varpi\,;$$

because it is only under this form these quantities enter in the expression of $Y^{(i)}$ [1708′].

† (1210) Supposing $U^{(i)}$ to be connected with the factor A, in the general value of $Y^{(i)}$, we may consider $Y^{(i)}=A\,.\,U^{(i)}$ as a particular value of $Y^{(i)}$, satisfying the equation [1707]; $U^{(i)}$, being independent of a [1708′], may be brought from under the sign of integration, and the expression [1707] will become, by considering A as a function of a,

$$\begin{aligned}0=&\left\{(2i+1)\,.\,a^i\,.\,A\,.\int_0^a\rho.d.a^3+3a^{2i+1}\,.\int_0^a\rho.d.(a^{2-i}\,.\,A)-3\,.\int_0^a\rho\,.\,d\,.\,(a^{i+3}\,.\,A)\right\}.\,U^{(i)}\\&-3\,a^{2i+1}\,.\,Z'^{(i)}.\end{aligned} \qquad [1709a]$$

fluid to cover a solid nucleus, it may be determined by means of the equation of the surface of the nucleus, observing that the value of $Y^{(i)}$, corresponding
[1709''] to the fluid stratum contiguous to this surface, is the same as that of the surface. Therefore the figure of the spheroid will depend on that of the internal nucleus, and on the forces which act upon the fluid.

Figure of a heterogeneous fluid spheroid.

30. If the spheroid were entirely fluid, we should have nothing to determine one of the arbitrary constant quantities, and it would seem as if there ought then to be an infinite number of cases of equilibrium. We shall
[1709'''] examine, in a particular manner, this case, which is the more interesting, because it appears to have been the primitive state of the heavenly bodies.

We shall observe, in the first place, *that the strata of the spheroid must*
[1709''''] *decrease in density, in proceeding from the centre to the surface.* For it is evident, that if a denser stratum were placed above a rarer one, the particles of the upper stratum would penetrate into the lower, in the same manner as a heavy body sinks in a lighter fluid; therefore the spheroid would not be in equilibrium. But whatever be the density, at the centre it must be finite; therefore by reducing the expression of ρ to a series, ascending relative to the powers of a, it will be of the form

Density.

[1709v] $$\rho = \beta - \gamma \,.\, a^n - \&c.\,;^*$$

Dividing by the coefficient of $U^{(i)}$, we get its value, represented by a multiple of $Z'^{(i)}$; but it is necessary that the values A, ρ, &c., should be so adjusted, as to make this value of
[1709b] $U^{(i)}$ conform to the hypothesis of being independent of a. Several examples of finding such values of $U^{(i)}$ are given in this chapter, as for example in [1726].

* (1211) In this formula, the density at the centre is β, corresponding to $a=0$, and
[1710a] the value of ρ [1709v] must decrease as a increases, supposing n to be positive. This value of ρ gives $\rho \,.\, d \,.\, a^3 = 3a^2 da \,.\, \rho = 3a^2 da \,.\, \beta - 3\gamma \,.\, a^{n+2} da - \&c.$; the integral of which is $\int \rho \,.\, d \,.\, a^3 = a^3 \,.\, \beta - \frac{3}{n+3} \,.\, \gamma \,.\, a^{n+3} - \&c.$ Also $a^3 \,.\, \rho = a^3 \,.\, \beta - \gamma \,.\, a^{n+3} - \&c.$; therefore,

$$\frac{a^3 \,.\, \rho}{\int \rho \,.\, d \,.\, a^3} = \frac{a^3 \,.\, \beta - \gamma \,.\, a^{n+3} - \&c.}{a^3 \,.\, \beta - \frac{3}{n+3} \,.\, \gamma \,.\, a^{n+3} - \&c.} = \frac{1 - \frac{\gamma}{\beta} \,.\, a^n - \&c.}{1 - \frac{3}{n+3} \,.\, \frac{\gamma}{\beta} \,.\, a^n - \&c.}$$

[1710b] $$= 1 - \frac{n}{n+3} \,.\, \frac{\gamma}{\beta} \,.\, a^n - \&c.\,;$$

β, γ, and n, being positive; hence we shall have, [1709vi]

$$\frac{a^3 \,.\, \rho}{\int \rho \,.\, d \,.\, a^3} = 1 - \frac{n \,.\, \gamma \,.\, a^n}{(n+3) \,.\, \beta} - \&c.\,; \qquad [1710]$$

and the differential equation in $Y^{(i)}$ will become

Differential equation in $Y^{(i)}$.

$$\left(\frac{ddY^{(i)}}{d\,a^2}\right) = \left\{ (i-2) \,.\, (i+3) + \frac{6n \,.\, \gamma \,.\, a^n}{(n+3) \,.\, \beta} + \&c. \right\} . \frac{Y^{(i)}}{a^2}$$
$$- \frac{6}{a} . \left\{ 1 - \frac{n \,.\, \gamma \,.\, a^n}{(n+3) \,.\, \beta} - \&c. \right\} . \left(\frac{d\,Y^{(i)}}{d\,a}\right). \qquad [1711]$$

To integrate this equation, we shall suppose $Y^{(i)}$ to be developed in a series, ascending relative to the powers of a, of the following form, [1711']

General value of $Y^{(i)}$.

$$Y^{(i)} = a^s \,.\, U^{(i)} + a^{s'} \,.\, U'^{(i)} + \&c. \qquad [1712]$$

The preceding differential equation will give,*

$$(s+i+3) \,.\, (s-i+2) \,.\, a^{s-2} . U^{(i)} + (s'+i+3) \,.\, (s'-i+2) \,.\, a^{s'-2} . U'^{(i)} + \&c.$$
$$= \frac{6\,n \,.\, \gamma \,.\, a^n}{(n+3) . \beta} . \{ (s+1) \,.\, a^{s-2} . U^{(i)} + (s'+1) \,.\, a^{s'-2} . U'^{(i)} + \&c. \}. \qquad (e) \qquad [1713]$$

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as in [1710]; hence,

$$i.(i+1) - \frac{6a^3.\rho}{\int \rho.d.a^3} = i.(i+1) - 6 + \frac{6n}{n+3} . \frac{\gamma}{\beta} . a^n + \&c. = \left\{ (i-2) . (i+3) + \frac{6n . \gamma . a^n}{(n+3) . \beta} + \&c. \right\}.$$

Dividing this by a^2, we get the coefficient of $Y^{(i)}$ [1708]; substituting this and [1710] in [1708], it becomes as in [1711], which may be put under the following form,

$$\left(\frac{ddY^{(i)}}{d\,a^2}\right) - (i-2) . (i+3) . \frac{Y^{(i)}}{a^2} + \frac{6}{a} . \left(\frac{dY^{(i)}}{d\,a}\right) = \frac{6n.\gamma.a^n}{(n+3).\beta} . \left\{ \frac{Y^{(i)}}{a^2} + \frac{1}{a} . \left(\frac{dY^{(i)}}{d\,a}\right) \right\} + \&c. \qquad [1711a]$$

* (1212) The first term of $Y^{(i)}$ [1712], namely $a^s . U^{(i)}$, produces, in the first member of [1711a], the following expression, observing that $U^{(i)}$, $U'^{(i)}$, &c., are supposed to be independent of a [1711'], [1712a]

$$\{ s . (s-1) - (i-2) . (i+3) + 6\,s \} . a^{s-2} . U^{(i)} = \{ s\,s + 5\,s - i^2 - i + 6 \} . a^{s-2} . U^{(i)}$$
$$= (s+i+3) . (s-i+2) . a^{s-2} . U^{(i)}, \qquad [1713a]$$

which is the same as the first term of [1713]. In like manner, the second term of [1712], $a^{s'} . U'^{(i)}$, produces the second term of [1713]; which is similar to the first term, changing s into s', and $U^{(i)}$ into $U'^{(i)}$, &c. In this way, the first member of [1713] will be found to correspond to the first member of [1711a]. Proceeding in the same manner with the second member of [1711a], we shall find that the first term of [1712] $a^s . U^{(i)}$, produces the first term of the second member of [1713]; the second term, $a^{s'} . U'^{(i)}$,

Comparing the similar powers of a, we shall have, in the first place,
[1713'] $(s+i+3).(s-i+2)=0$, which gives $s=i-2$, and $s=-i-3$.

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produces the second of [1713]; and so on for the other terms. To satisfy this equation, it is necessary that the coefficients of the same power of a should be equal, in both members.
[1713b] Now the first term of the first member is multiplied by a^{s-2}, and the first term of the second is multiplied by a^{s+n-2}, n being positive [1709vi]; these cannot therefore be of the same power of a, and we must put the coefficient of a^{s-2} equal to nothing, or $(s+i+3).(s-i+2)=0$. This may be satisfied by putting either $s+i+3=0$,
[1713c] or $s-i+2=0$. The first of these expressions gives $s=-i-3$; substituting this in [1712], it becomes $Y^{(i)}=a^{-i-3}.U^{(i)}+a^{s'}.U'^{(i)}+\&c.$ This series being ascending relative to the powers of a [1711'], the greatest negative exponent of a, will be found in the first term $a^{-i-3}.U^{(i)}$, which will therefore exceed the other terms, and become
[1713d] infinite, when a is infinitely small, making $\alpha a.Y^{(i)}$ infinite near the centre of the spheroid; which is contrary to the hypothesis on which the equations [1705, 1711, 1713] are founded, namely, that this quantity is of the order αa. We must therefore make use of the other value of s, deduced from $s-i+2=0$ [1713c], whence $s=i-2$; then, as the first term of [1713] vanishes, the whole expression will become,

$$(s'+i+3).(s'-i+2).a^{s'-2}.U'^{(i)}+(s''+i+3).(s''-i+2).a^{s''-2}.U''^{(i)}+\&c.$$

[1713e]
$$=\frac{6n.\gamma}{(n+3).\beta}.\{(s+1).a^{s+n-2}.U^{(i)}+(s'+1).a^{s'+n-2}.U'^{(i)}+\&c.\}.$$

Comparing the exponents of a in each member, term by term, in the order in which they stand, we shall have, $s'-2=s+n-2$, $s''-2=s'+n-2$, $s'''-2=s''+n-2$, &c.;
[1713f] hence $s'=s+n$, $s''=s'+n=s+2n$, $s'''=s''+n=s+3n$, &c. Then making the first, second, &c., terms of each member of [1713e] respectively equal to each other, we shall get,

[1713g]
$$U'^{(i)}=\frac{6n.\gamma.(s+1)}{(n+3).\beta.(s'+i+3).(s'-i+2)}.U^{(i)},\qquad U''^{(i)}=\frac{6n.\gamma.(s'+1)}{(n+3).\beta.(s''+i+3).(s''-i+2)}.U'^{(i)},\qquad \&c.$$

Substituting this value of $U'^{(i)}=h'.U^{(i)}$, in $U''^{(i)}$, we get another value of $U''^{(i)}$, of the form $U''^{(i)}=h''.U^{(i)}$, and this being substituted in $U'''^{(i)}$, it becomes of the form $U'''^{(i)}=h'''.U^{(i)}$, &c.; h', h'', h''', &c., being used for brevity instead of the coefficients produced by these operations. By means of these values, [1712] becomes,

$$Y^{(i)}=U^{(i)}.a^{s}.\{1+a^{n}.h'+a^{2n}.h''+a^{3n}.h'''+\&c.\}$$

[1714a]
$$=U^{(i)}.a^{i-2}.\{1+a^{n}.h'+a^{2n}.h''+a^{3n}.h'''+\&c.\}.$$

This value of $Y^{(i)}$ may be considered as complete, because the similar series, arising from the other value of $s=-i-3$, must be neglected, for the reasons stated in [1713''], though
[1714b] in general their sum would have been considered as the complete value of $Y^{(i)}$, each being multiplied by a *different* arbitrary quantity $U^{(i)}$.

To each of these values of s, there corresponds a particular series, which, being multiplied by an arbitrary constant quantity, will be an integral of the differential equation in $Y^{(i)}$; the sum of these two integrals will be the complete integral. In the present case, *the series corresponding to* $s=-i-3$ [1713′′] *must be rejected*; for the value of $a \,.\, Y^{(i)}$, resulting from it will be infinite, when a is infinitely small; which would render the radii of the strata near the centre infinite, [1713d]. Therefore, of the two particular integrals of the *expression of* $Y^{(i)}$, *that which corresponds to* $s=i-2$, *is* [1713′′′] *the only one which ought to be used.* This expression will then contain but one arbitrary quantity, which can be determined by means of the function $Z^{(i)}$.*

$Z^{(1)}$ being equal to nothing [1632], $Y^{(1)}$ must also be equal to nothing, [1713′′′′] *so that the centre of gravity of each stratum must be at the centre of gravity of the whole spheroid*; for the differential equation in $Y^{(i)}$ of the preceding article gives,†

Centre of gravity of each stratum is at the centre of gravity of the spheroid.

$$\left(\frac{ddY^{(1)}}{da^2}\right)=\left(\frac{2}{a^2}-\frac{6\rho\,.\,a}{\int\rho\,.\,d\,.\,a^3}\right).\,Y^{(1)}-\frac{6\rho\,.\,a^2}{\int\rho\,.\,d\,.\,a^3}\,.\left(\frac{d\,Y^{(1)}}{d\,a}\right). \qquad [1714]$$

This equation may be satisfied by making $Y^{(1)}=\dfrac{U^{(1)}}{a}$, $U^{(1)}$ being [1714′]

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* (1213) If we substitute $Y^{(i)}$ [1714a] in [1705], the term $U^{(i)}$ may be brought from [1714b'] under the sign of integration, as in [1709a], and then $U^{(i)}$ becomes a multiple of $Z^{(i)}$.

† (1214) Making $i=1$ in [1708], it becomes as in [1714]. This is satisfied by putting $Y^{(i)}=U^{(1)}\,.\,a^{-1}$, as is evident. For its differentials, $U^{(1)}$ being independent of a [1712a], are $\left(\frac{d\,Y^{(1)}}{d\,a}\right)=-\,U^{(1)}\,.\,a^{-2}$, $\left(\frac{d\,dY^{(1)}}{da^2}\right)=2\,U^{(1)}\,.\,a^{-3}$. Substituting [1714$c$] these in [1714], it becomes

$$2\,U^{(1)}\,.\,a^{-3}=\left(\frac{2}{a^2}-\frac{6\rho\,.\,a}{\int\rho\,.\,d\,.\,a^3}\right).\,U^{(1)}\,.\,a^{-1}+\frac{6\rho\,.\,a^2}{\int\rho\,.\,d\,.\,a^3}\,.\,U^{(1)}\,.\,a^{-2}; \qquad [1714c']$$

in which the terms mutually destroy each other. That the expression $Y^{(1)}=U^{(1)}\,.\,a^{-1}$ corresponds to $s=i-2$, may be proved from the values [1713g]. For $i=1$ makes $s=i-2=-1$ [1713′′′], or $s+1=0$; hence $U'^{(i)}$ [1713g] becomes $U'^{(i)}=0$. Substituting this in $U''^{(i)}$ [1713g], we get $U''^{(i)}=0$; and in like manner, $U'''^{(i)}=0$, $U''''^{(i)}=0$, &c.; hence the expression [1712] is reduced to its first term $a^s\,.\,U^{(1)}$, or $Y^{(1)}=U^{(1)}\,.\,a^{-1}$, as above. This is the only part of $Y^{(1)}$ which is to [1714d] be used, as was observed in [1714b].

independent of a. This value of $Y^{(1)}$ is that which corresponds to the equation $s=i-2$ [1714d]; consequently it is the only one which ought to be used. Substituting it in the equation [1705] of the preceding article,
[1714''] supposing $Z^{(1)}=0$, the function $U^{(1)}$ will disappear,* and it may therefore be taken of any value at pleasure; but the condition, that the origin of the radius r is at the centre of gravity of the terrestrial spheroid, makes it vanish.† For we shall see, in the following article [1745], that $Y^{(1)}$ is
[1714'''] then nothing, at the surface of every spheroid, covered with a stratum of fluid, in equilibrium; we shall therefore have, in the present case, $U^{(1)}=0$;
[1714''''] hence $Y^{(1)}$ *is nothing for all the fluid strata of which the spheroid is composed.*

We shall now consider the general equation [1712],

[1715] $$Y^{(i)}=a^{s}\,.\,U^{(i)}+a^{s'}\,.\,U'^{(i)}+\&c.$$

s being, as we have seen [1713'''], equal to $i-2$, s is nothing or positive,
[1715'] when i is equal to, or exceeds, 2; moreover, the functions $U'^{(i)}$, $U''^{(i)}$, &c., are given in terms of $U^{(i)}$, by the equation [1713], so that we shall have,

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* (1215) Putting $i=1$, $Z^{(1)}=0$, $Y^{(1)}=U^{(1)}\,.\,a^{-1}$, in [1705], it becomes,

[1714e] $$0=\frac{4\pi.a}{3}\,.\int_{a}^{1}\rho\,.\,dU^{(1)}-\frac{4\pi}{3a}\,.\,\frac{U^{(1)}}{a}\,.\int_{0}^{a}\rho\,.\,d\,.\,a^{3}+\frac{4\pi}{3a^{2}}\,.\int_{0}^{a}\rho\,.\,d\,.\,(a^{3}\,.\,U^{(1)}).$$

If we bring $U^{(1)}$ from under the sign $\int$, as in [1714b'], the last integral of [1714e] will become $\frac{4\pi\,.\,U^{(1)}}{3a^{2}}\,.\int_{0}^{1}\rho\,.\,d\,.\,a^{3}$, which being equal, and of an opposite sign, to the second integral, destroys it. Lastly, as $U^{(1)}$ is independent of a [1712a], we shall have $dU^{(1)}=0$; therefore the first integral, $\int_{a}^{1}\rho\,.\,dU^{(1)}$, becomes nothing, and the whole expression [1714e] vanishes, leaving $U^{(1)}$ indeterminate; consequently, also, $Y^{(1)}=U^{(1)}\,.\,a^{-1}$ remains indeterminate.

† (1216) It will be shown, in [1745], that if the origin of the radius be at the centre of gravity of the spheroid, we shall have, at its surface, $Y^{(1)}=0$; therefore, at this *surface*,
[1714f] $Y^{(1)}=U^{(1)}\,.\,a^{-1}=0$, or $U^{(1)}=0$; but $U^{(1)}$ [1712a] is independent of a, and must therefore be the same *within* the spheroid, as at the *surface;* hence we shall have generally $U^{(1)}=0$. Substituting this in $Y^{(i)}=\frac{U^{(1)}}{a}$ [1714d], which corresponds to
[1714g] any part whatever of the spheroid, we get generally $Y^{(1)}=0$, when the origin of the radius r is taken at the centre of gravity of the spheroid.

$Y^{(i)}$.

$$Y^{(i)} = h \,.\, U^{(i)} \;;^{*}$$ [1716]

in which h is a function of a, and $U^{(i)}$ is independent of a [1712a]. If we substitute this value of $Y^{(i)}$ in the differential equation [1714], we shall get,

$$\frac{ddh}{da^2} = \left\{ i\,.\,(i+1) - \frac{6\rho\,.\,a^3}{\int\rho\,.\,d\,.\,a^3} \right\} . \frac{h}{a^2} - \frac{6\rho\,.\,a^2}{\int\rho\,.\,d\,.\,a^3} . \frac{dh}{da};$$ [1717]

the product $i\,.\,(i+1)$ is greater than $\frac{6\rho\,.\,a^3}{\int\rho\,.\,d\,.\,a^3}$, when i is equal to, or greater than, 2; because the fraction $\frac{\rho\,.\,a^3}{\int\rho\,.\,d\,.\,a^3}$ is less than unity. For its denominator, $\int\rho\,.\,d\,.\,a^3$, is equal to $\rho\,.\,a^3 - \int a^3\,.\,d\rho$;† and the quantity $-\int a^3\,.\,d\rho$ is positive, when ρ decreases from the centre to the surface. [1717′]

Hence it follows, that h and $\frac{dh}{da}$ *are always positive, from the centre to the surface.* To prove this, suppose that these two quantities are positive, in proceeding from the centre; dh ought then to become negative before h.‡ [1717″] [1717‴]

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* (1217) This general value of $Y^{(i)}$ is the same as in [1714a], putting for brevity $a^{i-2}\,.\,\{1 + a^n\,.\,h' + a^{2n}\,.\,h'' + \&c.\} = h$; so that h will be a function of a, $U^{(i)}$ being independent of a, as in [1712a]. Substituting in [1708] the value of $Y^{(i)}$ [1716], and its differentials $\left(\frac{dY^{(i)}}{da}\right) = \frac{dh}{da} . U^{(i)}$, $\left(\frac{ddY^{(i)}}{da^2}\right) = \frac{ddh}{da^2} . U^{(i)}$; then dividing by $U^{(i)}$, [1714h] we get [1717]. Moreover, from $s = i - 2$ [1713‴], we find that s is nothing when $i = 2$, and s is positive when i exceeds 2, agreeably to the remarks in [1715′].

† (1218) This is a result of the usual formula for the integrating by parts,

$$\int A\,.\,dB = A\,.\,B - \int B\,.\,dA,$$ [1716a]

which is easily proved by taking the differential, and agrees with [1717′], by putting $A = \rho$, $B = a^3$.

‡ (1219) Since h and dh are supposed to be positive at the commencement at the centre of the earth, the positive values of h must increase, in proceeding from the centre to the circumference, as long as dh remains positive; and before h can decrease, dh must become negative. Now it is evident, that before the sign of dh can change from positive to negative, it must pass through the state of $dh = 0$, whilst h still remains positive. At this point, the expression [1717] would become, $\frac{ddh}{da^2} = \left\{ i\,.\,(i+1) - \frac{6\rho\,.\,a^3}{\int\rho\,.\,d\,.\,a^3} \right\} . \frac{h}{a^2}$; now [1717a]

Now it is evident, that for this to take place, dh ought first to become nothing; but the moment it becomes nothing, ddh becomes positive, as
[1717''''] appears by the preceding equation [1717], consequently dh must begin to increase; it cannot therefore become negative; hence it follows that h and dh must always preserve the same sign, from the centre to the surface. Now these two quantities are positive, in commencing at the centre; for we
[1717v] have, by means of the equation [1713], $s'-2=s+n-2$ [1713f]; therefore $s'=i+n-2$ [1713''']; and then we get,*

[1718] $$(s'+i+3).(s'-i+2).U'^{(i)}=\frac{6n.(s+1).\gamma.U^{(i)}}{(n+3).\beta};$$

hence we deduce,

[1719] $$U'^{(i)}=\frac{6.(i-1).\gamma.U^{(i)}}{(n+3).(2i+n+1).\beta}.$$

Therefore we shall have,†

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if $i>2$, we shall have $i.(i+1)>6$, also $\frac{6\rho.a^3}{\int\rho.d.a^3}<6$ [1717']; hence the factor $i.(i+1)-\frac{6\rho.a^3}{\int\rho.d.a^3}$ of the preceding expression will be positive; therefore the second member of [1717a], or the value of $\frac{ddh}{da^2}$, will then be positive; consequently dh will increase, and will retain its positive sign, as in [1717'''']. Therefore, if h and dh
[1717b] are positive at the centre, they will both continue positive, in proceeding from the centre to the surface of the spheroid.

* (1220) The equation [1718] is easily deduced from the value of $U'^{(i)}$ [1713g]; and if we substitute in this $s=i-2$ [1713'''], $s'=s+n=i+n-2$ [1713f], which
[1718a] give $s+1=i-1$, $s'+i+3=2i+n+1$, $s'-i+2=n$, it becomes as in [1719].

† (1221) Substituting $s=i-2$, $s'=i+n-2$ [1718a], and $U'^{(i)}$ [1719], in
[1720a] [1715], it becomes, $Y^{(i)}=a^{i-2}.U^{(i)}+a^{i+n-2}.\frac{6.(i-1).\gamma.U^{(i)}}{(n+3).(2i+n+1).\beta}+\&c.=h.U^{(i)}$, [1716]; being in fact the same as in [1714a]. Dividing by $U^{(i)}$, we get h [1720]. The differential of this value of h, divided by da, gives $\frac{dh}{da}$ [1720].

$$h = a^{i-2} + \frac{6\,.\,(i-1)\,.\,\gamma\,.\,a^{i+n-2}}{(n+3)\,.\,(2i+n+1)\,.\,\beta} + \&c.\,;$$

$$\frac{dh}{da} = (i-2)\,.\,a^{i-3} + \frac{6\,.\,(i-1)\,.\,(i+n-2)\,.\,\gamma\,.\,a^{i+n-3}}{(n+3)\,.\,(2i+n+1)\,.\,\beta} + \&c. \qquad [1720]$$

γ, β and n being positive [1709^{vi}], we perceive that at the centre, h and dh are positive, when i is equal to 2, or exceeds 2;* they are therefore always positive from the centre to the surface. [1720′]

Relative to the Earth, the Moon, Jupiter, &c., $Z^{(i)}$ is nothing, or insensible, when i is equal to, or exceeds, 3;† the equation [1705] of the preceding article then becomes, [1720″]

$$0 = \left\{3a^{2i+1}\,.\int_a^1 \rho\,.\,d\,.\left(\frac{h}{a^{i-2}}\right) - (2i+1)\,.\,a^i h\,.\int_0^a \rho.d.a^3 + 3\,.\int_0^a \rho\,.\,d\,.\,(a^{i+3}h)\right\}.\,U^{(i)}\,; \qquad [1721]$$

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* (1222) If we put $i=2+e$, e will be nothing or positive, when $i=2$, or $i>2$, and the expressions [1720] will become,

$$h = a^e + \frac{6\,.\,(1+e)\,.\,\gamma\,.\,a^{n+e}}{(n+3).(n+5+2e).\beta} + \&c.\,; \qquad \frac{dh}{da} = e\,a^{-1+e} + \frac{6.(1+e).(n+e).\gamma.a^{n-1+e}}{(n+3).(n+5+2e).\beta} + \&c.\,; \qquad [1720b]$$

e, γ, β, n, being positive; and it is evident, that when a is infinitely small, these values of h and $\frac{dh}{da}$ will be reduced to their first terms, which in the present case are always positive; therefore the values of h and $\frac{dh}{da}$ are positive, at the centre of the spheroid, as was assumed in [1717‴]; consequently they will preserve the same signs from the centre to the surface, from what is proved in [1717b]. [1720c]

† (1223) It appears by the formulas [1632], that $\alpha\,.\,Z^{(3)}$, $\alpha\,.\,Z^{(4)}$, &c., are of the order S, S', &c., or of the order of the disturbing forces of the planets; which may be neglected, in computing the figures of the heavenly bodies, because one of the greatest of these forces, namely the disturbing force of the moon upon the earth, has no greater effect on the figure of the earth, than to produce a small ebb and flow of the sea. Hence, in computing the figures of the heavenly bodies, we may neglect all the terms $Z^{(3)}$, $Z^{(4)}$, $Z^{(5)}$, &c. . . . . . $Z^{(i)}$, in which i is equal to 3, or exceeds 3. Therefore, if we neglect $Z^{(i)}$ in [1705], and multiply this formula by $\frac{3\,a^{i+1}.(2i+1)}{4\pi}$, we shall get, [1721a] [1721b]

$$0 = \left\{3\,a^{2i+1}.\int_a^1 \rho.d.\left(\frac{Y^{(i)}}{a^{i-2}}\right) - (2i+1).a^i.Y^{(i)}.\int_0^a \rho.d.a^3 + .\,3\int_0^a \rho\,.\,d\,.\,(a^{i+3}.Y^{(i)})\right\}. \qquad [1721c]$$

If in this we substitute $Y^{(i)} = h\,.\,U^{(i)}$ [1716], and bring $U^{(i)}$ from under the sign $\int$, because it is independent of a [1712a], it becomes as in [1721]. At the surface, the limits

[1721'] the first integral being taken from $a = a$ to $a = 1$, and the other two from $a = 0$ to $a = a$. At the surface, where $a = 1$, this becomes,

[1722] $$0 = \left\{ -(2i+1) \cdot h \cdot \int_0^1 \rho \cdot d \cdot a^3 + 3 \cdot \int_0^1 \rho \cdot d \cdot (a^{i+3} h) \right\} \cdot U^{(i)};$$

which equation may be put under the following form,*

[1723] $$0 = \left\{ -(2i-2) \cdot \rho \cdot h + (2i+1) \cdot h \cdot \int_0^1 a^3 \cdot d\rho - 3 \cdot \int_0^1 a^{i+3} h \cdot d\rho \right\} \cdot U^{(i)}.$$

$d\rho$ is negative from the centre to the surface [1709''''], and h increases in the same interval [1720']; therefore the function

[1723'] $$(2i+1) \cdot h \cdot \int_0^1 a^3 \cdot d\rho - 3 \cdot \int_0^1 a^{i+3} h \cdot d\rho$$

is negative in the same interval.† Hence it follows, that in the preceding

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[1721c'] of the first integral, $a = a$, $a = 1$, change into $a = 1$, $a = 1$; therefore that integral must vanish at the surface, and the two remaining terms will then become as in [1722].
[1721d] The letter h, without the sign of integration [1722], is that corresponding to the surface of the spheroid, where $a = 1$, which may be represented by h_1.

* (1224) Integrating by parts, as in [1716a], we get

$$\int \rho \cdot d \cdot (a^{i+3} h) = \rho \cdot (a^{i+3} h) - \int (a^{i+3} h) \cdot d\rho;$$

also, as in [1717'], $\int \rho \cdot d \cdot a^3 = \rho \cdot a^3 - \int a^3 \cdot d\rho$. The terms without the sign $\int$ vanish, at the first limit of these integrals, $a = 0$; and at the second limit, $a = 1$, these expressions become

[1723a] $$\int_0^1 \rho \cdot d \cdot (a^{i+3} h) = \rho \cdot h - \int_0^1 a^{i+3} h \cdot d\rho, \qquad \text{and} \qquad \int_0^1 \rho \cdot d \cdot a^3 = \rho - \int_0^1 a^3 \cdot d\rho.$$

Substituting these in [1722], we get [1723].

† (1225) The increment dh being positive from the centre to the surface [1720c],
[1723b] h must increase from the centre to the surface, where we shall suppose it becomes equal to h_1 [1721d], so that we shall have $h_1 > h$; moreover, i is positive, and by hypothesis
[1724a] [1720'], greater than 2, in the cases under consideration, and a never exceeds unity; therefore the quantity $(2i+1) \cdot h_1 - 3 a^i h$, must evidently be positive from the centre to the surface. Multiplying this by $a^3 \cdot d\rho$, which is negative, because ρ decreases in proceeding from the centre to the surface, we shall obtain, for the product, the negative quantity $(2i+1) \cdot h_1 \cdot a^3 \cdot d\rho - 3 a^{i+3} h \cdot d\rho$. Now as this element is negative, its integral, taken from $a=0$ to $a=1$, and represented by $(2i+1) \cdot h_1 \cdot \int_0^1 a^3 \cdot d\rho - 3 \cdot \int_0^1 a^{i+3} h \cdot d\rho$, will also be negative. If to this we add the negative quantity $-(2i-2) \cdot \rho \cdot h_1$, the sum

equation, the coefficient of $U^{(i)}$ will be negative, and cannot vanish at the surface; therefore $U^{(i)}$ must be nothing; consequently $Y^{(i)}=0$ [1716], and the expression of the radius of the spheroid becomes,

$$a+\alpha a.\{Y^{(0)}+Y^{(2)}\};$$ [1724]

therefore the surface of each level stratum of the spheroid is elliptical, consequently its external surface is elliptical.

As it regards the earth, $Z^{(2)}$ is, by [1632], equal to $-\frac{g}{2\alpha}.(\mu^2-\frac{1}{3})$,* [1724′]
and the equation [1705] of the preceding article gives,†

$$0=\left\{\tfrac{4}{5}\pi.a^5.\int_a^1\rho.dh-\tfrac{4}{3}\pi.a^2h.\int_0^a\rho.d.a^3+\tfrac{4}{5}\pi.\int_0^a\rho.d.(a^5h)\right\}.U^{(2)}-\frac{g}{2\alpha}.a^5.(\mu^2-\tfrac{1}{3}).$$ [1725]

At the surface, the first integral, $\int_a^1\rho.dh$ is nothing;‡ we shall therefore [1725′]
have, at that surface, where $a=1$ [1702″],

$$U^{(2)}=\frac{-\frac{1}{2\alpha}.g.(\mu^2-\frac{1}{3})}{\frac{4}{3}\pi.h.\int_0^1\rho.d.a^3-\frac{4}{5}\pi.\int_0^1\rho.d.(a^5h)}.$$ [1726]

If $\alpha\varphi$ *represent the ratio of the centrifugal force to the gravity at the* [1726′]
equator, the expression of gravity, neglecting quantities of the order α, being [1727]

$-(2i-2).\rho.h_1+(2i+1).h_1.\int_0^1 a^3.d\rho-3.\int_0^1 a^{i+3}h.d\rho$, which is the factor of [1724b]
$U^{(i)}$ in [1723], will be negative. Therefore the equation [1723] cannot be satisfied in any other way than by putting $U^{(i)}=0$, at the surface of the spheroid. Now $U^{(i)}$ [1712a] is independent of a, and must be the same, within the spheroid, as at the surface; consequently the value of $U^{(i)}$, when $i>2$, is $U^{(i)}=0$. Hence we have, when $i>2$, $Y^{(i)}=0$ [1716]; therefore the general value of the radius of a level stratum [1503″],
will be $a+\alpha a.\{Y^{(0)}+Y^{(1)}+Y^{(2)}\}$, and by putting $Y^{(1)}=0$ [1714g], it becomes [1724c]
as in [1724], which is the equation of an ellipsoid [1503a].

* (1226) This follows from the third equation [1632], neglecting S, S', S'', &c., as in [1721a].

† (1227) Putting, in [1705], $i=2$, and $Z^{(2)}$ as in [1724′], also $Y^{(2)}=h.U^{(2)}$ [1716]; then multiplying by a^3, we obtain the equation [1725], observing that $U^{(2)}$, being independent of a [1712a], may be brought from under the sign of integration.

‡ (1228) The limits of the integral $\int_a^1\rho.dh$, at the surface, become $a=1$ and
$a=1$; therefore $\int_a^1\rho.dh=0$, as in [1721c′]. Substituting this in [1725], putting [1726a]
$a=1$ in terms without the sign $\int$, and dividing by the coefficient of $U^{(i)}$, we get [1726].

[1727′] equal to $\frac{4}{3}\pi . \int \rho . d . a^3$,* we shall have $g = \frac{4}{3}\pi . \alpha\varphi . \int \rho . d . a^3$. Hence

[1728]
$$U^{(2)} = \frac{-\varphi . (\mu^2 - \frac{1}{3})}{2h - \frac{2 . \int_0^1 \rho . d . (a^5 h)}{5 . \int_0^1 \rho . a^2 da}}.$$

[1728′] Therefore, if we include, in the arbitrary constant quantity a, which we have taken for unity, the function

[1729]
$$\alpha Y^{(0)} - \frac{\alpha h \varphi}{3h - \frac{3 . \int_0^1 \rho . d . (a^5 h)}{5 . \int_0^1 \rho . a^2 da}},$$

the radius of the terrestrial spheroid, at its surface, will be,†

Radius of the ellipsoid.

[1730]
$$1 + \frac{\alpha h \varphi . (1 - \mu^2)}{2h - \frac{2 . \int_0^1 \rho . d . (a^5 h)}{5 . \int_0^1 \rho . a^2 da}}.$$

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* (1229) If we neglect terms of the order α, the attraction of the spheroid upon a point of

[1728a] its surface, will be the same as that of a sphere [1506a, b]; which, by putting $r = 1$, becomes $\frac{4}{3}\pi . \int_0^1 \rho . d . a^3$. Multiplying this by $\alpha\varphi$, we shall obtain, by the definition [1726′],

[1728b] the centrifugal force, represented by g [1616xvi]; hence $g = \frac{4}{3}\pi . \alpha\varphi . \int_0^1 \rho . d . a^3$, as in

[1728c] [1727′]. Substituting this value of g in [1726], then dividing the numerator and denominator by $\frac{2}{3}\pi . \int \rho . d . a^3$, or $2\pi . \int \rho . a^2 da$, we get [1728]; in which the first term of the denominator ought to be $2h_1$ [1721d], instead of $2h$.

† (1230) Putting $i = 2$ in [1716], we get $Y^{(2)} = h . U^{(2)}$; substituting this and $U^{(2)}$ [1728] in the expression of the radius [1724], it becomes,

$$a + \alpha a . Y^{(0)} - \frac{\alpha a . h\varphi . (\mu^2 - \frac{1}{3})}{2h_1 - \frac{2 . \int \rho . d . (a^5 h)}{5 . \int \rho . a^2 da}}$$

[1730a]
$$= a + \alpha a . Y^{(0)} - \frac{\frac{2}{3}\alpha a . h\varphi}{2h_1 - \frac{2 . \int \rho . d . (a^5 h)}{5 . \int \rho . a^2 da}} + \frac{\alpha a . h\varphi . (1 - \mu^2)}{2h_1 - \frac{2 . \int \rho . d . (a^5 h)}{5 . \int \rho . a^2 da}}.$$

As a is arbitrary, we may denote the three first terms of this expression by 1 [1725′], which will not alter the last term, always neglecting quantities of the order α^2. Hence this radius will be, at the surface, as in [1730],

[1730b]
$$1 + \frac{\alpha . h_1 \varphi . (1 - \mu^2)}{2h_1 - \frac{2 . \int \rho . d . (a^5 h)}{5 . \int \rho . a^2 da}}.$$

[1730b′] The expression [1730b] does not contain ϖ, and the radius is the same, for the same value of μ,

This radius is that of an ellipsoid of revolution, in which the least semi-axis is unity, and the greatest semi-axis is [1730']

Greatest semi-axis.

$$1+\frac{\alpha h \varphi}{2h-\frac{2.\int_0^1 \rho.d.(a^5 h)}{5.\int_0^1 \rho.a^2 da}}.$$ [1731]

[1731']

Figure of the earth, supposing it to be fluid, is an ellipsoid of revolution.

The figure of the earth, supposing it to be a fluid, must therefore be an ellipsoid of revolution [1730b'], in which every stratum of the same density, is elliptical and of revolution. The ellipticities increase, and the densities*

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whatever be the value of ϖ; therefore the spheroid must be of revolution, and by [1724c], it will be an ellipsoid of revolution. The least semi-axis, at the surface of the spheroid, is found by putting $\mu=1$ in [1730], which makes it equal to unity; and the greatest [1730c]
semi-axis, corresponding to $\mu=0$, is given in [1731].

* (1231) It was shown, in [1724], that the radius of each *level* stratum of the ellipsoid is $a+\alpha a.Y^{(0)}+\alpha a.Y^{(2)}$. [1732a]

If we put $i=2$, in [1716], we shall get $Y^{(2)}=h.U^{(2)}$, h being a function of a, [1732a']
which we shall denote by $h=\psi(a)$, and $U^{(2)}$ being independent of a [1712a]; so that [1732b]
$U^{(2)}$ is the same at the surface of the ellipsoid as upon any of the level strata; therefore it

is equal to the value given in [1728]. Hence $Y^{(2)}=\frac{-\varphi.(\mu^2-\frac{1}{3}).\psi(a)}{2h_1-\frac{2.\int_0^1 \rho.d.(a^5h)}{5.\int_0^1 \rho.a^2 da}}$, and the [1732c]

radius of the level stratum [1732a] becomes $a+\alpha a.Y^{(0)}-\frac{\alpha a.\varphi.(\mu^2-\frac{1}{3}).\psi(a)}{2h_1-\frac{2.\int_0^1 \rho.d.(a^5h)}{5.\int_0^1 \rho.a^2 da}}$;

if we include $\alpha a.Y^{(0)}-\frac{\frac{2}{3}\alpha a.\varphi.\psi(a)}{2h_1-\frac{2.\int_0^1 \rho.d.(a^5 h)}{5.\int_0^1 \rho.a^2 da}}$ in the arbitrary constant quantity a,

as is done in [1728', &c.], it becomes,

Radius of any stratum.

$$a+\frac{\alpha a.\psi(a).(1-\mu^2)}{2h_1-\frac{2.\int_0^1 \rho.d.(a^5 h)}{5.\int_0^1 \rho.a^2 da}};$$ [1732d]

which is similar to [1730], and for the same reason must correspond to an ellipsoid of revolution, in which the least axis is a, found by putting $\mu=1$; and the greatest axis, corresponding to

$\mu=0$, is $$a+\frac{\alpha a.\psi(a)}{2h_1-\frac{2.\int_0^1 \rho.d.(a^5 h)}{5.\int_0^1 \rho.a^2 da}}.$$ [1732e]

*decrease, from the centre to the surface.** *The relation between the ellipticities*
[1731''] *and the densities*, is given by the following differential equation of the second order,†

[1732] $$\frac{ddh}{da^2}=\frac{6h}{a^2}\cdot\left(1-\frac{\rho\,.\,a^3}{3\,.\int\rho\,.\,a^2da}\right)-\frac{2\rho\,.\,a^2}{\int\rho\,.\,a^2da}\cdot\frac{dh}{da}.$$

This equation is not integrable by any known method, except in some particular hypotheses, relative to the density ρ; but if the law of the
[1732'] ellipticity were given, we could easily obtain that of the corresponding density.‡ We have seen that the expression of h, given by the integral of

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* (1232) The two semi-axes of a stratum of revolution, or level surface, are, as in
[1732f] [1732e], a and $a+\dfrac{\alpha a\,.\,\psi(a)}{2h_1-\dfrac{2\,.\int_0^1\rho\,.\,d\,.\,(a^5h)}{5\,.\int_0^1\rho\,.\,a^2da}}$. The ellipticity ε is measured by the ratio of the difference of these axes, divided by the least axis, and is therefore represented by
[1732g] $\varepsilon=\dfrac{\alpha\,.\,\psi(a)}{2h_1-\dfrac{2\,.\int_0^1\rho\,.\,d\,.\,(a^5h)}{5\,.\int_0^1\rho\,.\,a^2da}}$. The quantity α, and the denominator of this expression, are
[1732h] the same for all the level strata. This *ellipticity* must therefore be proportional to $\alpha\,.\,\psi(a)$, or $\alpha\,.\,h$ [1732b], which increases from the centre to the surface [1723b], while the density decreases [1709''''].

† (1233) Putting $i=2$, in [1717], and for $d\,.\,a^3$ its value $3a^2da$, we shall obtain the equation [1732].

‡ (1234) To show, by some examples, the method of computing ρ from h, by means of the formula [1732], we shall suppose $h=a^4$, whence $\frac{dh}{da}=4a^3$, $\frac{ddh}{da^2}=12a^2$. Substituting these in [1732], after transposing $\frac{ddh}{da^2}$ to the second member, it becomes,

[1732i] $$0=-12a^2+\frac{6a^4}{a^2}\cdot\left(1-\frac{\rho\,.\,a^3}{3\,.\int\rho\,.\,a^2da}\right)-\frac{2\rho\,.\,a^2}{\int\rho\,.\,a^2da}\cdot4a^3=-6a^2-\frac{10\rho\,.\,a^5}{\int\rho\,.\,a^2da}.$$

Multiplying this by $-\frac{\int\rho\,.\,a^2da}{2a^2}$, we get $0=3\,.\int\rho\,.\,a^2da+5\rho\,.\,a^3$, whose differential is $0=3\rho\,.\,a^2da+5d\rho\,.\,a^3+15\rho\,.\,a^2da=18\rho\,.\,a^2da+5d\rho\,.\,a^3$. Dividing this by $5a^3\,.\,\rho$, we find $\frac{d\rho}{\rho}=-\frac{18}{5}\cdot\frac{da}{a}$, whose integral [59] Int., is

[1732i'] $$\log.\,\rho=\tfrac{18}{5}\,.\,\log.\,\frac{1}{a}+\log.\,\text{constant}\,.\,c=\log.\,c\,.\,a^{-\frac{18}{5}};\qquad\text{hence}\qquad\rho=c\,a^{-\frac{18}{5}}.$$

this equation contains, in the present case, but one arbitrary constant quantity, which disappears from the preceding value of the radius of the spheroid;* there is therefore but one possible figure of equilibrium, in a spheroid differing but little from a sphere; and it is easy to show, that the [1732″]

If we suppose h to be constant, we shall have $dh=0$, $ddh=0$; and [1732] will become $0=\frac{6h}{a^2}\cdot\left(1-\frac{\rho\,.\,a^3}{3.\int\rho\,.\,a^2da}\right)$; multiplying by $\frac{a^2}{2h}\cdot\int\rho\,.\,a^2\,da$, we get $0=3.\int\rho\,.\,a^2da-\rho\,.\,a^3$, whose differential is $0=3\rho\,.\,a^2da-3\rho\,.\,a^2da+a^3.d\rho=a^3.d\rho$; [1732i′] hence $d\rho=0$, and $\rho=$ constant. Therefore, in general, if the ellipticity be constant, the fluid will be homogeneous.

* (1235) This is similar to what takes place relative to the two values of $Y^{(i)}$, [1708′, 1713″, 1714″, &c.]. For from the differential equation in $Y^{(i)}$ [1708], we have deduced, in [1716], the expression $Y^{(i)}=h.U^{(i)}$; in which h is a function of the quantity a [1714a], supposed to be variable, as in [1708]; and $U^{(i)}$ is an arbitrary quantity [1708′], considered as one of the constant quantities, introduced by the integration of [1708], depending upon $s=i-2$ [1713‴], the other term depending on $s=-i-3$ being rejected, [1713″]. Now if we change $Y^{(i)}$ into h, and put $i=2$, the equation [1708] will become like [1732]. Making the same changes in $Y^{(2)}=U^{(2)}.\psi(a)$ [1732a', b], which was deduced [1732k] from [1708], we shall get, for the value of h, depending on [1732], an expression of the form $h=b.\Omega(a)$, b being an arbitrary quantity, independent of a; and $\Omega(a)$ a function [1732k′] of a. This value of h is also called $\psi(a)$ [1732b]. Putting it for $\psi(a)$, in the expression of the radius [1732d], observing also, that at the surface, where h becomes h_{1}, a becomes 1, and $h_{1}=b\,.\,\Omega(1)$; we shall get, for the expression of the radius,

$$a+\frac{\alpha a\,.\,b\,.\,\Omega(a)\,.\,(1-\mu^2)}{2b\,.\,\Omega(1)-\frac{2\,.\int_0^1\rho\,.\,d\,.\{a^5b\,.\,\Omega(a)\}}{5\,.\int_0^1\rho\,.\,a^2\,da}}\,. \qquad [1732l]$$

Now b being independent of a [1732k'], that term may be brought from under the sign of integration; consequently the numerator and denominator become divisible by the arbitrary constant quantity b. The expression of the radius then becomes,

$$a+\frac{\alpha a\,.\,\Omega(a)\,.\,(1-\mu^2)}{2\,\Omega(1)-\frac{2\,.\int_0^1\rho\,.\,d\,.\{a^5.\Omega(a)\}}{5\,.\int_0^1\rho\,.\,a^2da}}\,;$$

which is free from that quantity b; and this radius can only have one form, when α is small. But if we were to retain the part of $Y^{(i)}$ arising from the series produced by $s=-i-3$, [1732m] [1713″], that value of $Y^{(i)}$ would be composed of two terms of the form [1716], whose sum would be the complete integral [1713″]. In like manner, h, deduced from [1732], would have two terms, $h=b\,.\,\Omega(a)+B\,.\,\Pi(a)$; and by changing the arbitrary constant

[1732''] limits of the oblateness of this figure, are $\frac{1}{2}\alpha\varphi$, and $\frac{5}{4}\alpha\varphi$;* of which the first corresponds to the case where all the mass is collected in the centre, and the second to the case where the mass is homogeneous.

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quantity B into $b.b'$, it would become of the form $h=\psi(a)=b.\{\Omega(a)+b'.\Pi(a)\}$, whence $h_1=b.\{\Omega(1)+b'.\Pi(1)\}$. Substituting these in [1732d], we find that the numerator and denominator of the factor of $(1-\mu^2)$ become divisible by b, therefore b
[1732m] vanishes from the expression; but the other arbitrary quantity b' remains, and renders the form of the ellipsoid, dependent upon it, arbitrary.

* (1236) The greatest semi-axis, corresponding to the surface of the ellipsoid, is given in [1731], and by subtracting from it the smaller semi-axis 1, we obtain the ellipticity

[1732m']
$$\varepsilon=\frac{\alpha h_1\varphi}{2h_1-\frac{2.\int_0^1\rho.d.(a^5h)}{5.\int_0^1\rho.a^2da}}.$$

Dividing the numerator and denominator by h_1, and putting $\int\rho.a^2da=\frac{1}{3}.\int\rho.d.a^3$,

[1732m'''] also, for brevity,
$$H=\frac{\int_0^1\rho.d.\left(a^5.\frac{h}{h_1}\right)}{\int_0^1\rho.d.a^3},$$
we get,

Ellipticity of the earth.

[1732n]
$$\varepsilon=\frac{\alpha\varphi}{2-\frac{6.\int_0^1\rho.d.\left(a^5.\frac{h}{h_1}\right)}{5.\int_0^1\rho.d.a^3}}=\frac{\alpha\varphi}{2-\frac{6}{5}H}.$$

[1732o] Now ρ decreases, and h increases, from the centre to the surface [1732h]; so that $\frac{h}{h_1}<1$; also $a^2.\frac{h}{h_1}<1$. This, being multiplied by a^3, gives $a^5.\frac{h}{h_1}<a^3$. Its differential, multiplied by ρ, gives $\rho.d.\left(a^5.\frac{h}{h_1}\right)<\rho.d.a^3$, both elements being positive. Integrating, we get, $\int_0^1\rho.d.\left(a^5.\frac{h}{h_1}\right)<\int_0^1\rho.d.a^3$, both expressions being also *positive*; substituting H [1732m'''], we get $H.\int_0^1\rho.d.a^3<\int_0^1\rho.d.a^3$; hence $H<1$; therefore H must be a positive quantity, and less than unity, so that its limits must be $H=0$, $H=1$. Substituting these values of H in [1732n], we evidently
[1732p] obtain the limits of the ellipticity; namely, at the *first* limit $H=0$, it becomes $\varepsilon=\frac{1}{2}\alpha\varphi$;
[1732q] and at the *last* limit $H=1$, we get $\varepsilon=\frac{5}{4}\alpha\varphi$, as in [1732'']. The *first* value corresponds to the case where the whole mass is collected, in a spherical point, of an infinite density, at the centre, and surrounded by an infinitely rare fluid; the *second* value corresponds to the case of uniform density. For if the mass be all collected in the centre, the integrals

The direction of gravity, from any point of the surface to the centre, does not form a right line, but a curve, whose elements are perpendicular to the level strata, which it passes through. This curve is the trajectory, intersecting at right angles, all the ellipses, which by their revolution form these level strata. To determine its nature, we shall take for the axis, the radius drawn from the centre to the proposed point of the surface, θ being the angle which this radius makes with the axis of revolution.* We have [1732'''']

[1732^{v}]

$\int \rho \,.\, d \,.\, a^3$, and $\int \rho \,.\, d \,.\left(a^5 \,.\, \frac{h}{h_1}\right)$, will be reduced to the central element, whose radius is $d\,a$, the density of this element being infinite and represented by ρ'. In the other parts of the spheroid, $\rho = 0$. In this case,

$$\int_0^1 \rho \,.\, d \,.\, a^3 = \rho' \,.\, d\,a^3, \qquad \text{and} \qquad \int_0^1 \rho \,.\, d \,.\left(a^5 \,.\, \frac{h}{h_1}\right) = \rho' \,.\, d a^5 \,.\, \frac{h}{h_1}, \qquad [1732r]$$

and the last expression vanishes in comparison with the first; so that we have, from [1732m'''], $H = 0$, and the second formula [1732n] becomes $\varepsilon = \frac{1}{2}\,\alpha\,\varphi$.

If ρ be constant and equal to unity, we shall have $\int_0^1 \rho \,.\, d \,.\, a^3 = \int_0^1 d \,.\, a^3 = a^3 = 1$, because at the surface $a = 1$; and $\int \rho \,.\, d \,.\left(a^5 \,.\, \frac{h}{h_1}\right) = \int d \,.\left(a^5 \,.\, \frac{h}{h_1}\right) = a^5 \,.\, \frac{h}{h_1} = 1$, [1732$s$] because at the surface $a = 1$, and $h = h_1$; hence $H = 1$ [1732m'''], and then, from [1732n], $\varepsilon = \frac{5}{4}\,\alpha\,\varphi$, as in [1732$q$].

* (1237) Thus, in fig. 38, AC is the semi-axis of revolution, AL the greater semi-axis of the ellipsoid, whose surface is CFL; and two of the level strata, infinitely near each other, are $MGEb$, $NHIKB$. $FHGA$ is the proposed *trajectory*, which intersects at right angles all the level strata, so that the element GH is perpendicular to the curves $MGEb$, NHB, at the points G, H. The line AF is taken for the axis of this curve, making the angle $CAF = \theta$, whose cosine is μ. Join A, G, and continue it to A', draw Ge perpendicular to AF, also GI parallel to AF; then if [1733a] [1733b]

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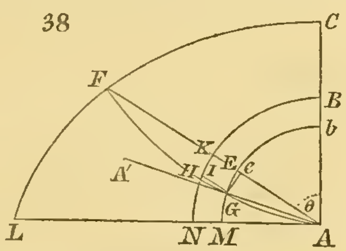

we put $2\,h_1 - \frac{2 \,.\int_0^1 \rho \,.\, d \,.\,(a^5 h)}{5 \,.\int_0^1 \rho \,.\, a^2 d\,a} = \frac{1}{k}$, k must be independent of a, and the general [1733c] expression of the radius AE [1732d] will become

$$AE = a + \alpha k \,.\, a \,.\, \psi(a) \,.\,(1 - \mu^2) = a + \alpha k \,.\, a h \,.\,(1 - \mu^2) \qquad [1732b],$$

as in [1733]. When $\mu = 0$, it becomes $AM = a + \alpha k \,.\, a\,h$, and when $\mu = 1$, it becomes $Ab = a$. In the case of $a = 1$, they will become $AL = 1 + \alpha k \,.\, h$, [1733$d$] $AC = 1$. Lastly, the ordinate $Ge = \alpha y'$ [1733']. [1733e]

just seen, [1732d], that the general expression of the radius of any stratum of the spheroid is

Radius of any stratum.

[1733] $$a + \alpha k \,.\, a h \,.\, (1 - \mu^2),$$

k being independent of a. Hence it is easy to prove, that if we put $\alpha y'$ for

[1733'] the ordinate let fall from any point whatever of the curve upon its axis, we shall have,*

Equation of the vertical curve.

[1734] $$\alpha y' = \alpha \,.\, a k \,.\, \sin. 2\theta \,.\, \left\{ c - \int_0^a \frac{h \,.\, da}{a} \right\},$$

[1734'] c being the complete value of the integral $\int_0^1 \frac{h \,.\, da}{a}$, taken from the centre to the surface.

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* (1238) We have computed, in [1579s], the sine of the angle formed by the radius

[1734a] AGA', and by the arc GH, drawn perpendicular to the curve $MGEb$, at the point G, fig. 38, page 311. If we neglect terms of the order α^2, we may take the angle itself, instead of its sine, and then the formula [1579s] will become, for the ellipsis $MGEb$,

[1734b] $$\text{angle } A'GH = \tfrac{1}{2} \,.\, \frac{AM^2 - Ab^2}{Ab^2} \,.\, \sin. 2 \,.\, CAG.$$

Substituting the values of $AM = a + \alpha k \,.\, a h$, $Ab = a$, [1733d], it becomes

[1734c] $$A'GH = \alpha k \,.\, h \,.\, \sin. 2 \,.\, CAG,$$

and as this is of the order α, we may, by neglecting terms of the order α^2, write $CAF = \theta$,

[1734d] for CAG, and we shall get $A'GH = \alpha k \,.\, h \,.\, \sin. 2\theta$. The angle $GAF = A'GI$ being very small, we may put it equal to its sine $\frac{Ge}{AG}$, and as the ordinate $Gc = \alpha y'$,

[1734e] [1733'], and AG is nearly equal to Ab, or a, we shall have the angle $A'GI = \frac{\alpha y'}{a}$.

[1734f] Subtracting from this, $A'GH$ [1734d], we get the angle $HGI = \frac{\alpha y'}{a} - \alpha k \,.\, h \,.\, \sin. 2\theta$. In the differential triangle HGI, we have $HI = GH \,.\, \sin. HGI$; and as this is of the order α, we may put HI equal to the differential of Gc, or $\alpha dy'$, also $GH = Bb = da$; moreover, the preceding value of HGI may be taken for its sine. In this manner we shall get, from [1734f],

[1734g] $$\alpha dy' = da \,.\, \left\{ \frac{\alpha y'}{a} - \alpha k \,.\, h \,.\, \sin. 2\theta \right\}, \qquad \text{or} \qquad \alpha \,.\, \frac{(a\,dy' - y'da)}{a^2} = -\alpha k \,.\, \sin. 2\theta \,.\, \frac{da}{a} \,.\, h.$$

Integrating this relative to a, considering α, k, θ, as constant, and adding the arbitrary constant quantity $\alpha k \,.\, \sin. 2\theta \,.\, c$ to the second member, we shall get,

[1734h] $$\alpha \,.\, \frac{y'}{a} = \alpha k \,.\, \sin. 2\theta \,.\, c - \alpha k \,.\, \sin. 2\theta \,.\, \int \frac{da}{a} \,.\, h = \alpha k \,.\, \sin. 2\theta \,.\, \left\{ c - \int \frac{da}{a} \,.\, h \right\}.$$

31. *We shall now consider the general case, in which the spheroid is always fluid at its surface, but may contain within it a solid nucleus of any figure whatever, differing but little from a sphere.* The radius, drawn from the centre of gravity of the spheroid to its surface, and the law of gravity at [1734''] Spheroid, composed of a solid nucleus, covered by a fluid.

Multiplying this by a, it becomes as in [1734]. This vanishes at the point A, where $a=0$, because the factor $\alpha a.k$ becomes nothing. It ought also to vanish at the point F, where $a=1$; therefore the factor $c-\int_0^1 \frac{h.da}{a}$ must vanish. Hence $c=\int_0^1 \frac{h.da}{a}$; [1734i] consequently the arbitrary constant quantity c is equal to this integral, taken from $a=0$ to $a=1$, as in [1734'].

If h be constant, the fluid will be homogeneous [1732i'], and

$$-\int\frac{h.da}{a}=-h.\int\frac{da}{a}=h.\log.\frac{1}{a} \qquad [59] \text{ Int.}$$

Hence [1734] becomes $\alpha y'=\alpha a.k.\sin.2\theta.\left\{c+h.\log.\frac{1}{a}\right\}$; in which c must be taken so as to make this vanish at the point F, where $a=1$, and $\log.\frac{1}{a}=\log.1=0$; therefore we must put $c=0$, and the general value of $\alpha y'$ will become

$$\alpha y'=\alpha a.k.\sin.2\theta.h.\log.\frac{1}{a}. \qquad [1734k]$$

This also vanishes at the point A, where $a=0$, because the factor $a.\log.\frac{1}{a}$ becomes nothing when $a=0$. For if we use the tabular logarithms, and put $a=\frac{1}{10^n}$, we [1734l] shall have $\log.\frac{1}{a}=\log.10^n=n$, and $a.\log.\frac{1}{a}=\frac{n}{10^n}$; which decreases rapidly as n increases. For by putting successively $n=1, 2, 3, 4$, &c., it will become $\frac{1}{10}$, $\frac{2}{100}$, $\frac{3}{1000}$, $\frac{4}{10000}$, &c., which evidently becomes nothing when n is infinite. The same result would have been obtained, if we had used hyperbolic logarithms; as is evident from the [1734m] consideration that the hyp. log. of any number is to the tab. log. of the same number, in a constant ratio, represented by the hyperbolic logarithm of $10=2,30258,50929$. [1734n]

If $h=ca$, we shall have $\int_0^a \frac{h.da}{a}=\int_0^a c.da=ca$. Hence [1734] becomes $\alpha y'=\alpha a.k.\sin.2\theta.(c-ca)=\alpha c.k.\sin.2\theta.(a-a^2)$, which is the equation of a parabola. For if we put $a=\frac{1}{2}+y$, and $y'=ck.\sin.2\theta.(\frac{1}{4}-2px)$, it becomes $\alpha c.k.\sin.2\theta.(\frac{1}{4}-2px)=\alpha c.k.\sin.2\theta.(\frac{1}{4}-y^2)$; hence $\frac{1}{4}-2px=\frac{1}{4}-y^2$, [1734o] and $y^2=2px$, the equation of a parabola [379d], whose vertex corresponds to $y=0$, or $a=\frac{1}{2}$.

that surface, have some general properties, which are the more important to be examined, because they are independent of every hypothesis.

Centre of gravity of the surface coincides with that of the spheroid.

The first of these properties is, that in a state of equilibrium, *the fluid part of the spheroid will assume such a form, that the function* $Y^{(1)}$ *will*
[1734'''] *disappear from the expression of the radius drawn from the centre of gravity of the whole spheroid to its surface; so that the centre of gravity of this surface will coincide with that of the spheroid*, [1745b—e].

Equations resulting from the condition that the origin of the co-ordinates is in the centre of gravity.

To prove this, we shall observe, that R being supposed to represent the
[1734''''] radius drawn from the centre of gravity of the spheroid, to any one of its particles, the expression of this particle will be $\rho . R^2 dR . d\mu . d\varpi$; and we shall have,* by means of the properties of the centre of gravity § 12,

[1735]
$$0 = \int \rho . R^3 dR . d\mu . d\varpi . \mu;$$
$$0 = \int \rho . R^3 dR . d\mu . d\varpi . \sqrt{1-\mu^2} . \sin. \varpi;$$
$$0 = \int \rho . R^3 dR . d\mu . d\varpi . \sqrt{1-\mu^2} . \cos. \varpi.$$

[1735'] Supposing the integral $\int \rho . R^3 dR$ to be taken, relative to R, from the origin of R, to the surface of the spheroid, and then developed in a series of the form

[1736]
$$\int_0^R \rho . R^3 dR = N^{(0)} + N^{(1)} + N^{(2)} + N^{(3)} + \&c.,$$

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* (1239) We have shown, in [1480d], that if the spheroid be homogeneous, a particle of
[1735a] its mass will be represented by $dM = R^2 dR . d\mu . d\varpi$; and if this be multiplied
[1735b] respectively by the three co-ordinates [1480b], $x = R . \mu$, $y = R . \sqrt{(1-\mu^2)} . \sin. \varpi$, $z = R . \sqrt{(1-\mu^2)} . \cos. \varpi$, it will produce three products,

[1735c]
$$R^3 dR . d\mu . d\varpi . \mu, \quad R^3 dR . d\mu . d\varpi . \sqrt{(1-\mu^2)} . \sin. \varpi, \quad R^3 dR . d\mu . d\varpi . \sqrt{(1-\mu^2)} . \cos. \varpi;$$

whose integrals [126, 127] represent the mass M of the spheroid, multiplied by the distances
[1735d] of its centre of gravity, from the planes, from which the co-ordinates x, y, z, are measured; and if this centre of gravity be taken for the origin of these co-ordinates, or the origin of R,
[1735e] these integrals must be nothing, by the nature of the centre of gravity [124''']. The same
[1735f] would take place, if the spheroid were not homogeneous. For if the density of the
[1735g] particle dM be ρ instead of 1, its mass will become $dM = \rho . R^2 dR . d\mu . d\varpi$; this being multiplied respectively by the three co-ordinates [1735b], will produce the elements

[1735h]
$$\rho . R^3 dR . d\mu . d\varpi . \mu, \quad \rho . R^3 dR . d\mu . d\varpi . \sqrt{(1-\mu^2)} . \sin. \varpi, \quad \rho . R^3 dR . d\mu . d\varpi . \sqrt{(1-\mu^2)} . \cos. \varpi;$$

whose integrals, as above [1735c], must be put equal to nothing, by the property of the centre of gravity [124''']; hence we get the three equations [1735].

$N^{(i)}$ being,* for all values of i, subjected to the following equation of partial differentials,

$$0=\left\{\frac{d\,.\left\{(1-\mu^2).\left(\frac{d\,N^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}+\frac{\left(\frac{d\,d\,N^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,N^{(i)}.$$ [1737]

we shall have, by [1476], when i differs from unity,

$$0=\int N^{(i)}\,.\,\mu\,d\,\mu\,.\,d\,\varpi\,; \qquad 0=\int N^{(i)}\,.\,d\,\mu\,.\,d\,\varpi\,.\sqrt{1-\mu^2}\,.\,\sin.\,\varpi\,;$$ [1738]
$$0=\int N^{(i)}\,.\,d\,\mu\,.\,d\,\varpi\,.\sqrt{1-\mu^2}\,.\,\cos.\,\varpi.$$

The three preceding equations [1735], depending on the centre of gravity, become,

$$0=\int N^{(1)}\,.\,\mu\,d\,\mu\,.\,d\,\varpi\,; \qquad 0=\int N^{(1)}\,.\,d\,\mu\,.\,d\,\varpi\,.\sqrt{1-\mu^2}\,.\,\sin.\,\varpi.$$ [1739]
$$0=\int N^{(1)}\,.\,d\,\mu\,.\,d\,\varpi\,.\sqrt{1-\mu^2}\,.\,\cos.\,\varpi.$$

$N^{(1)}$ is of the form†

$$N^{(1)}=H\,.\,\mu+H'\,.\sqrt{1-\mu^2}\,.\,\sin.\,\varpi+H''\,.\sqrt{1-\mu^2}\,.\,\cos.\,\varpi.$$ [1740]

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* (1240) The density ρ being a function of R, μ, ϖ, we may find the integral $\int\rho\,.R^3dR$, relative to R; and if we suppose it equal to the expression $N^{(0)}+N^{(1)}+N^{(2)}+\&c.$ [1736], the formulas [1735] will become

$$0=\int d\,\mu\,.\,d\,\varpi\,.\,\mu\,.\,\{N^{(0)}+N^{(1)}+\&c.\}\,;$$
$$0=\int d\,\mu\,.\,d\,\varpi\,.\sqrt{(1-\mu^2)}\,.\,\sin.\,\varpi\,.\,\{N^{(0)}+N^{(1)}+\&c.\}\,;$$ [1738a]
$$0=\int d\,\mu\,.\,d\,\varpi\,.\sqrt{(1-\mu^2)}\,.\,\cos.\,\varpi\,.\,\{N^{(0)}+N^{(1)}+\&c.\}.$$

Now it is shown, in [1480i], that the quantities μ, $\sqrt{(1-\mu^2)}\,.\,\sin.\,\varpi$, $\sqrt{(1-\mu^2)}\,.\,\cos.\,\varpi$, are each of the form $Z^{(1)}$, satisfying the equation [1481]; therefore, by [1476], we may put equal to nothing, in the preceding integrals, all the terms where the index of N differs from unity, as $N^{(0)}$, $N^{(2)}$, $N^{(3)}$, &c.; hence we get the formulas [1738]. Substituting these in [1738a], all the terms will vanish, except those connected with $N^{(1)}$, and these will become as in [1739]; the whole of this calculation being like that in [1482a, 1483a—h]. [1738b]

† (1241) The expression of N [1740] agrees with that in [1483′], putting $A=H$, $B=H'$, $C=H''$. This value of $N^{(1)}$, being substituted in the three expressions [1483], produce, as in [1483″], the following equations $\frac{1}{4}\,.\int N^{(1)}\,.\,d\,\mu\,.\,d\,\varpi\,.\,\mu=\frac{1}{3}\,\pi\,.\,A$, $\frac{1}{4}\,.\int N^{(1)}\,.\,d\mu\,.\,d\varpi\,.\sqrt{(1-\mu^2)}\,.\,\sin.\,\varpi=\frac{1}{3}\pi\,.\,B$, $\frac{1}{4}\,.\int N^{(1)}\,.\,d\mu\,.\,d\varpi\,.\sqrt{(1-\mu^2)}\,.\,\cos.\,\varpi=\frac{1}{3}\pi\,.\,C$. [1740a] The first members of these equations vanish, by means of formulas [1739]; therefore $0=\frac{1}{3}\pi\,.\,A$, $0=\frac{1}{3}\pi\,.B$, $0=\frac{1}{3}\pi\,.C$; hence $A=H=0$, $B=H'=0$, $C=H''=0$, as in [1741]. These values of H, H', H'', being substituted in $N^{(1)}$ [1740], it becomes $N^{(1)}=0$, as in [1742].

Substituting this value in these three equations, we shall find,

[1741] $$H=0, \qquad H'=0, \qquad H''=0;$$

[1742] therefore $N^{(1)}=0$; *which is the condition required by the supposition that the origin of R is at the centre of gravity of the spheroid.*

$N^{(1)}=0$ is the condition that the origin is at the centre of gravity of the spheroid.

We shall now compute the value of $N^{(1)}$, for a spheroid, differing but little from a sphere, and covered by a fluid in equilibrium. In this case, we
[1742'] have $R=a.(1+\alpha y)$ [1503''''], and the integral $\int\rho.R^3dR$ becomes*
[1742''] $\frac{1}{4}.\int\rho.d.\{a^4.(1+4\alpha y)\}$; in which the differential and the integral both refer to the variable quantity a; ρ being a function of a. Substituting for y its value $Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.$ [1464], we shall get,

[1743] $$N^{(1)}=\alpha.\int\rho.d.(a^4.Y^{(1)}).$$

The equation [1705] gives, at the surface, where $a=1$, observing that $Z^{(1)}$ [1632] is nothing,†

[1744] $$\int\rho.d.(a^4.Y^{(1)})=Y^{(1)}.\int\rho.d.a^3;$$

[1741a] * (1242) Since $R^3dR=\frac{1}{4}d.R^4$, and $R=a.(1+\alpha y)$, $R^4=a^4.(1+4\alpha y)$, neglecting α^2, we have $R^3dR=\frac{1}{4}d.\{a^4.(1+4\alpha y)\}$; therefore

[1742a] $$\rho.R^3dR=\tfrac{1}{4}\rho.d.\{a^4.(1+4\alpha y)\}, \quad\text{and}\quad \int\rho.R^3dR=\tfrac{1}{4}.\int\rho.d.\{a^4.(1+4\alpha y)\},$$

as in [1742'']. Substituting in the first member its value [1736], and putting in the second member the value of y [1464], we get,

[1742b] $$N^{(0)}+N^{(1)}+N^{(2)}+\&c.=\{\tfrac{1}{4}.\int\rho.d.a^4.(1+4\alpha.Y^{(0)})\}+\alpha.\int\rho.d.(a^4.Y^{(1)})+\alpha.\int\rho.d.(a^4.Y^{(2)})+\&c.$$

The integrals in this second member affect a only; so that the quantities $\int\rho.d.(a^4.Y^{(1)})$, $\int\rho.d.(a^4.Y^{(2)})$, &c., being substituted for $N^{(1)}$, $N^{(2)}$, &c., would satisfy the equation
[1742c] [1737], because $Y^{(1)}$, $Y^{(2)}$, &c., satisfy the similar equation [1465]. Now from [1479'], the function $\int\rho.R^3.dR$ can be developed but in one way, in a function of the form $N^{(0)}+N^{(1)}+N^{(2)}+\&c.$; it necessarily follows, that the similar terms of the first and second members of the preceding equation, must be equal to each other; hence

[1742d] $$N^{(0)}=\tfrac{1}{4}.\int\rho.d.a^4.(1+4\alpha.Y^{(0)}), \quad N^{(1)}=\alpha.\int\rho.d.(a^4.Y^{(1)}), \quad N^{(2)}=\alpha.\int\rho.d.(a^4.Y^{(2)}), \quad \&c.$$

The second of these equations is the same as [1743].

† (1243) Substituting $Z^{(1)}=0$, and $i=1$, in [1705], and taking the integrals to correspond to the surface of the spheroid, by which means the first integral of [1705], whose limits were $a=a$ and $a=1$, will vanish; we shall get,

[1745a] $$0=-\frac{4\pi}{3a}.Y^{(1)}.\int_0^1\rho.d.a^3.+\frac{4\pi}{3a^2}.\int_0^1\rho.d.(a^4.Y^{(1)});$$

the value of $Y^{(1)}$, in the second member of this equation, being its value at the surface; therefore, $N^{(1)}$ [1742] being nothing, when the origin of R is at the centre of gravity of the spheroid, we shall have also $Y^{(1)}=0$. In this case, $Y^{(1)}=0$. [1745]

32. The permanent state of equilibrium of the heavenly bodies makes known to us some of the properties of their radii. If the planets did not revolve about one of their three principal axes, or very near to one of them, [1745′]
there would be produced, in the position of the axes of rotation, some variations which would become sensible, particularly in the earth. Now by the most accurate observations, no such variations are perceived. Therefore [1745″]

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in which *the term* $Y^{(1)}$, *without the sign* $\int$, *corresponds to the surface of the fluid;* and we [1745a′]
must put the quantity a, which falls without the sign of integration, equal to unity. If we then divide by $\frac{4}{3}\pi$, and transpose the first term, we shall obtain [1744]. Substituting this value of $\int\rho.d.(a^4.Y^{(1)})$ in [1743], we get $N^{(1)}=\alpha.Y^{(1)}.\int\rho.d.a^3$, or $\alpha.Y^{(1)}=\frac{N^{(1)}}{\int\rho.d.a^3}$; and since, by [1742], $N^{(1)}=0$, when the origin of R is at the [1745b]
centre of gravity of the spheroid, we shall have, in the same hypothesis, $Y^{(1)}=0$ *at the surface of the spheroid* [1745a, a']. We shall now compute the centre of gravity of a fluid stratum, at the surface of the spheroid, the thickness of this stratum, measured in the direction of the radius, being supposed constant and equal to dR'; also the density ρ constant and equal to ρ'. Then the elements [1735h] will be

$$\rho'.dR'.R^3.d\mu.d\varpi.\mu,\quad \rho'.dR'.R^3.d\mu.d\varpi.\sqrt{(1-\mu^2)}.\sin.\varpi,\quad \rho'.dR'.R^3.d\mu.d\varpi.\sqrt{(1-\mu^2)}.\cos.\varpi;$$

whose integrals are $\rho'.dR'.\int R^3.d\mu.d\varpi.\mu$, $\rho'.dR'.\int R^3.d\mu.d\varpi.\sqrt{(1-\mu^2)}.\sin.\varpi$, [1745c]
$\rho'.dR'.\int R^3.d\mu.d\varpi.\sqrt{(1-\mu^2)}.\cos.\varpi$, representing, as in [1735d], the products, of the mass of this stratum, by the distance of its centre of gravity from the planes, from which the co-ordinates x, y, z, are counted. Now

$$R^3=a^3.(1+\alpha y)^3=a^3.(1+3\alpha y)=a^3.(1+3\alpha.Y^{(0)})+3\alpha a^3.Y^{(1)}+3\alpha a^3.Y^{(2)}+\&c.$$

If we substitute these in the integrals [1745c], all the terms, except those depending on $Y^{(1)}$, will vanish, for the same reason as those depending on $N^{(0)}$, $N^{(2)}$, $N^{(3)}$, &c., vanish, in [1745d]
[1738b]. Hence these integrals will become,

$$\rho'.dR'.\int 3a^3\alpha.Y^{(1)}.d\mu.d\varpi.\mu,\quad \rho'.dR'.\int 3a^3\alpha.Y^{(1)}.d\mu.d\varpi.\sqrt{(1-\mu^2)}.\sin.\varpi,$$
$$\rho'.dR'.\int 3a^3\alpha.Y^{(1)}.d\mu.d\varpi.\sqrt{(1-\mu^2)}.\cos.\varpi;$$

and since at this surface $Y^{(1)}=0$ [1745], all these integrals will vanish. Hence the distances of the centre of gravity of this stratum, from the planes abovementioned, will be nothing; therefore the centre of gravity must be at the origin of R; or in other words, it must be at the centre of gravity of the spheroid, as was observed in [1734‴]. [1745e]

we must infer, that a long period of time has elapsed since all parts of the heavenly bodies, and particularly the fluid particles on their surfaces, have been arranged in such a manner as to render their state of equilibrium permanent; consequently also their axes of rotation. For it is very natural to suppose, that after a great number of oscillations, the bodies must assume
[1745‴] the forms corresponding to the state of equilibrium, on account of the resistances, suffered by the particles of the fluid. We shall now examine into the conditions arising from this supposition, in the expression of the radii of the heavenly bodies.

If we put x, y, z, for the rectangular co-ordinates of a particle dM of the spheroid, referred to the three principal axes, the axis of x being the axis
[1745⁗] of rotation of the spheroid; we shall have, by the properties of these axes, demonstrated in the first book, [228],

[1746] $$0=\int xy\,.\,dM, \qquad 0=\int xz\,.\,dM, \qquad 0=\int yz\,.\,dM.$$

These integrals must include the whole mass of the spheroid. R is the radius drawn from the origin of the co-ordinates to the particle dM; θ is the angle formed by R and the axis of rotation; ϖ is the angle which the
[1746′] plane, formed by this axis and R, makes with the plane formed by this axis and that principal axis, called the axis of y; we shall then have,*

[1747] $$x=R\,.\,\mu\,; \qquad y=R\,.\sqrt{1-\mu^2}\,.\cos.\varpi\,; \qquad z=R\,.\sqrt{1-\mu^2}\,.\sin.\varpi\,;$$
$$dM=\rho\,.\,R^2dR\,.\,d\mu\,.\,d\varpi.$$

The three equations [1746], given by the nature of the principal axes of rotation, will, by this means, become,†

[1748] $$0=\int\rho\,.\,R^4dR\,.\,d\mu\,.\,d\varpi\,.\,\mu\,.\sqrt{1-\mu^2}\,.\cos.\varpi\,;$$
$$0=\int\rho\,.\,R^4dR\,.\,d\mu\,.\,d\varpi\,.\,\mu\,.\sqrt{1-\mu^2}\,.\sin.\varpi\,;$$
$$0=\int\rho\,.\,R^4dR\,.\,d\mu\,.\,d\varpi\,.\,(1-\mu^2)\,.\sin.2\varpi.$$

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* (1244) These values of x, y, z, agree with [1480*b*], and that of dM with [1480*d*], multiplying it by the density ρ.

† (1245) Substituting the values x, y, z, dM, [1747], in [1746]; the two first of these equations produce the two first of [1748], and the third of [1746] becomes

[1748a] $$0=\int\rho\,.\,R^4dR\,.\,d\mu\,.\,d\varpi\,.\,(1-\mu^2)\,.\sin.\varpi\,.\cos.\varpi\,;$$

multiplying this by 2, and substituting $\sin.2\varpi=2\sin.\varpi\,.\cos.\varpi$, we get the third equation [1748].

We shall suppose the integral $\int \rho \, . \, R^4 \, d\, R$ to be taken, from $R=0$ to [1748''] the value of R corresponding to the surface of the spheroid, and then to be developed in a series of the form,

$$\int \rho \, . \, R^4 \, d\, R = U^{(0)} + U^{(1)} + U^{(2)} + U^{(3)} + \&c. \, ; \qquad [1749]$$

$U^{(i)}$ being, for all values of i, subjected to the equation of partial differentials [1437],

$$0 = \left\{ \frac{d \, . \left\{ (1-\mu^2) \, . \left(\frac{dU^{(i)}}{d\mu} \right) \right\}}{d\mu} \right\} + \frac{\left(\frac{ddU^{(i)}}{d\varpi^2} \right)}{1-\mu^2} + i \, . \, (i+1) \, . \, U^{(i)}. \qquad [1750]$$

We shall have, by the theorem [1476], when i is different from 2, observing that the functions $\mu . \sqrt{1-\mu^2} . \cos . \varpi$, $\mu . \sqrt{1-\mu^2} . \sin . \varpi$, and $(1-\mu^2) . \sin . 2\varpi$, [1750'] are comprised in the form $U^{(2)}$,*

$$0 = \int U^{(i)} \, . \, d\mu \, . \, d\varpi \, . \sqrt{1-\mu^2} \, . \, \cos . \, \varpi \, ;$$
$$0 = \int U^{(i)} \, . \, d\mu \, . \, d\varpi \, . \sqrt{1-\mu^2} \, . \, \sin . \, \varpi \, ; \qquad [1751]$$
$$0 = \int U^{(i)} \, . \, d\mu \, . \, d\varpi \, . (1-\mu^2) \, . \, \sin . 2\varpi .$$

The three equations [1748], depending on the nature of the axes of rotation, will, by this means, become,†

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* (1246) If in the *general* value of $Y^{(2)}$ [1528c], we put successively, all the coefficients equal to nothing, except $B_2^{(1)}$, $A_2^{(1)}$, $A_2^{(2)}$; and then put these quantities equal to 1, we shall have the particular values of $Y^{(2)}$, represented by

$$\mu \, . \sqrt{(1-\mu^2)} \, . \, \cos . \, \varpi, \qquad \mu \, . \sqrt{(1-\mu^2)} \, . \, \sin . \, \varpi, \qquad (1-\mu^2) \, . \, \sin . \, 2\,\varpi, \qquad [1751a]$$

as in [1750']. If we represent any one of these quantities by $Z^{(2)}$, the formula [1476] will give the equations [1751], for all values of i differing from 2.

† (1247) The integral of the first of the equations [1748] being taken relative to a, which affects only ρ, R, it becomes, by using [1749],

$$0 = \int \{ U^{(0)} + U^{(1)} + \&c. \} \, . \, d\mu \, . \, d\varpi \, . \, \mu \, . \sqrt{(1-\mu^2)} \, . \, \cos . \, \varpi \, ;$$

or, as it may be written,

$$0 = \int U^{(0)} \, . \, d\mu \, . \, d\varpi \, . \, \mu \, . \sqrt{(1-\mu^2)} \, . \, \cos . \, \varpi + \int U^{(1)} \, . \, d\mu \, . \, d\varpi \, . \, \mu \, . \sqrt{(1-\mu^2)} \, . \, \cos . \, \varpi$$
$$+ \int U^{(2)} \, . \, d\mu \, . \, d\varpi \, . \, \mu \, . \sqrt{(1-\mu^2)} \, . \cos . \varpi + \&c. \, ; \qquad [1752a]$$

and this, by reason of the first of the formulas [1751], becomes like the first equation [1752]. In the same manner, the second and third of the equations [1748] produce the second and third of [1752], respectively.

[1752]
$$0 = \int U^{(2)} . d\mu . d\varpi . \mu . \sqrt{1-\mu^2} . \cos. \varpi ;$$
$$0 = \int U^{(2)} . d\mu . d\varpi . \mu . \sqrt{1-\mu^2} . \sin. \varpi ;$$
$$0 = \int U^{(2)} . d\mu . d\varpi . (1-\mu^2) . \sin. 2\varpi .$$

Therefore these equations will depend wholly upon the value of $U^{(2)}$. This value is of the form [1528c],

[1753]
$$U^{(2)} = H . (\mu^2 - \tfrac{1}{3}) + H' . \mu . \sqrt{1-\mu^2} . \sin. \varpi + H'' . \mu . \sqrt{1-\mu^2} . \cos. \varpi$$
$$+ H''' . (1-\mu^2) . \sin. 2\varpi + H'''' . (1-\mu^2) . \cos. 2\varpi .$$

Substituting it in the three preceding equations [1752], we shall find,*

[1754]
$$H' = 0, \qquad H'' = 0, \qquad H''' = 0.$$

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* (1249) When we substitute the value of $U^{(2)}$ [1753], in the formulas [1752], we must reduce the products of the sines and cosines of ϖ and its multiples, by the formulas [17—20] Int.; so that an expression of the form $\int d\varpi . \sin. n\varpi . \cos. m\varpi$, will become $\frac{1}{2} . \int d\varpi . \sin. (n+m) . \varpi + \frac{1}{2} . \int d\varpi . \sin. (n-m) . \varpi$. If we suppose n and m to be integral numbers, differing from each other, these integrals will be,

[1754a]
$$\frac{1-\cos. (n+m) . \varpi}{2 . (n+m)} + \frac{1-\cos. (n-m) . \varpi}{2 . (n-m)} ;$$

which vanish at both the limits $\varpi = 0$, $\varpi = 2\pi$, [1470']; hence we have,

[1754b]
$$\int_0^{2\pi} d\varpi . \sin. n\varpi . \cos. m\varpi = 0.$$

Again, when $n = m$, we shall have, as in [1544a],

[1754c]
$$\int_0^{2\pi} d\varpi . \sin.^2 n\varpi = \pi ; \quad \int_0^{2\pi} d\varpi . \cos.^2 n\varpi = \pi ; \quad \int_0^{2\pi} d\varpi . \sin. n\varpi . \cos. n\varpi = 0 ;$$

therefore, in the *first* equation [1752], we need only notice the term H'' of [1753]; in the *second* of these equations, we need only notice the term H'; and in the *third*, we need only notice H'''. Hence these equations become, by using the integrals [1754c],

[1754d]
$$0 = \int \mu d\mu . \sqrt{(1-\mu^2)} . \{H'' . \mu . \sqrt{(1-\mu^2)}\} . \pi ;$$
$$0 = \int \mu d\mu . \sqrt{(1-\mu^2)} . \{H' . \mu . \sqrt{(1-\mu^2)}\} . \pi ;$$
$$0 = \int d\mu . (1-\mu^2) . \{H''' . (1-\mu^2)\} . \pi ;$$

or, by reduction,

[1754e]
$$0 = H'' . \pi . \int d\mu . (\mu^2 - \mu^4) \qquad = H'' . \pi . (\tfrac{1}{3}\mu^3 - \tfrac{1}{5}\mu^5 + \tfrac{2}{15}) ;$$
$$0 = H' . \pi . \int d\mu . (\mu^2 - \mu^4) \qquad = H' . \pi . (\tfrac{1}{3}\mu^3 - \tfrac{1}{5}\mu^5 + \tfrac{2}{15}) ;$$
$$0 = H''' . \pi . \int d\mu . (1 - 2\mu^2 + \mu^4) = H''' . \pi . (\mu - \tfrac{2}{3}\mu^3 + \tfrac{1}{5}\mu^5 + \tfrac{8}{15}) ;$$

these integrals being taken so as to vanish when $\mu = -1$. If we put $\mu = 1$, which is

Conditions required to make $x, y, z,$ principal axes of rotation.

The conditions required, in order that the three axes $x, y, z,$ should be the principal axes of rotation, are therefore reduced to the three preceding equations; and then $U^{(2)}$ [1753] will be of the form, [1754′]

$$U^{(2)} = H.(\mu^2 - \tfrac{1}{3}) + H''''.(1-\mu^2).\cos.2\varpi. \qquad [1755]$$

When the spheroid is a solid, differing but little from a sphere, and covered by a fluid in equilibrium, we shall have $R = a.(1+\alpha y)$ [1742′]; therefore,* [1755′]

$$\int\rho.R^4 dR = \tfrac{1}{5}.\int\rho.d.\{a^5.(1+5\alpha y)\}. \qquad [1756]$$

If we substitute the value of $y = Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$ [1698], we shall get, [1756′]

$$U^{(2)} = \alpha.\int\rho.d.(a^5.Y^{(2)}). \qquad [1757]$$

The equation [1705] gives, at the surface of the spheroid,†

$$\tfrac{4}{5}\pi.\int_0^1\rho.d.(a^5.Y^{(2)}) = \tfrac{4}{3}\pi.Y^{(2)}.\int_0^1\rho.d.a^3 - Z^{(2)}; \qquad [1758]$$

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the other limit of μ [1470′], they become $0 = \tfrac{4}{15}H''.\pi$, $0 = \tfrac{4}{15}H'.\pi$, $0 = \tfrac{16}{15}H'''.\pi$; [1754f] hence $H' = 0$, $H'' = 0$, $H''' = 0$, as in [1754]. Substituting these in [1753], we get $U^{(2)}$ [1755].

* (1250) In like manner as, in [1741a], we have

$$R^4 dR = \tfrac{1}{5}d.R^5 = \tfrac{1}{5}d.\{a.(1+\alpha y)\}^5 = \tfrac{1}{5}d.\{a^5.(1+5\alpha y)\}.$$

Multiplying this by ρ, and integrating, we get [1756]. Substituting y [1698], in the second member of [1756], we find, [1756a]

$$\int\rho.R^4dR = \int\{\tfrac{1}{5}\rho.d.a^5 + \alpha.\rho.d.(a^5.Y^{(0)})\} + \alpha.\int\rho.d.(a^5.Y^{(1)}) + \alpha.\int\rho.d.(a^5.Y^{(2)}) + \&c.;$$

then, by means of [1749], we get,

$$U^{(0)} + U^{(1)} + U^{(2)} + \&c.$$
$$= \int\{\tfrac{1}{5}\rho.d.a^5 + \alpha.\rho.d.(a^5.Y^{(0)})\} + \alpha.\int\rho.d.(a^5.Y^{(1)}) + \alpha.\int\rho.d.(a^5.Y^{(2)}) + \&c.;$$

which is similar to [1742b]. Hence we deduce, as in [1742d],

$$U^{(0)} = \int\{\tfrac{1}{5}\rho.d.a^5 + \alpha.\rho.d(a^5.Y^{(0)})\}; \qquad U^{(1)} = \alpha.\int\rho.d.(a^5.Y^{(1)});$$
$$U^{(2)} = \alpha.\int\rho.d.(a^5.Y^{(2)}); \qquad \&c. \qquad [1756b]$$

This value of $U^{(2)}$ agrees with that in [1757].

† (1251) Putting $i = 2$ in [1705], and supposing the stratum to correspond to the surface of the spheroid; the first of the integrals of this formula will vanish, because the limits of the integral become $a = 1$, $a = 1$; and the whole expression will become, as

[1758] $Y^{(2)}$ and $Z^{(2)}$, in the second member of this equation, are the values at the surface; we then have,

[1759] $$U^{(2)} = \tfrac{5}{3}\alpha \,.\, Y^{(2)} \,.\, \textstyle\int_0^1 \rho \,.\, d \,.\, a^3 - \frac{5\,\alpha \,.\, Z^{(2)}}{4\pi}.$$

The value $Z^{(2)}$ is of the form [1528c],*

[1760] $$Z^{(2)} = -\tfrac{1}{2} g \,.\, (\mu^2 - \tfrac{1}{3}) + g' \,.\, \mu \,.\, \sqrt{1-\mu^2} \,.\, \sin.\varpi + g'' \,.\, \mu \,.\, \sqrt{1-\mu^2} \,.\, \cos.\varpi + g''' \,.\, (1-\mu^2) \,.\, \sin.2\varpi + g'''' \,.\, (1-\mu^2) \,.\, \cos.2\varpi.$$

and that of $Y^{(2)}$ is of the form [1528c],

[1761] $$Y^{(2)} = -h \,.\, (\mu^2 - \tfrac{1}{3}) + h' \,.\, \mu \,.\, \sqrt{1-\mu^2} \,.\, \sin.\varpi + h'' \,.\, \mu \,.\, \sqrt{1-\mu^2} \,.\, \cos.\varpi + h''' \,.\, (1-\mu^2) \,.\, \sin.2\varpi + h'''' \,.\, (1-\mu^2) \,.\, \cos.2\varpi.$$

Conditions arising from the revolution being about a principal axis.

Substituting these values in the preceding equation, and also that of

$$U^{(2)} = H \,.\, (\mu^2 - \tfrac{1}{3}) + H'''' \,.\, (1-\mu^2) \,.\, \cos.2\varpi, \qquad [1755],$$

we shall get,†

[1758a] in [1758], $0 = -\frac{4}{3}\pi \,.\, Y^{(2)} \,.\, \int_0^1 \rho \,.\, d \,.\, a^3 + \frac{4}{5}\pi \,.\, \int_0^1 \rho \,.\, d \,.\, (a^5 \,.\, Y^{(2)}) + Z^{(2)}$; observing that, in the terms which fall without the sign $\int$, we must put $a = 1$. Moreover, the terms $Y^{(2)}$ and $Z^{(2)}$, which fall without the sign $\int$, correspond to the surface of the spheroid, where $a = 1$. Multiplying [1758] by $\frac{5\alpha}{4\pi}$, we get

[1758b] $$\alpha \,.\, \textstyle\int \rho \,.\, d \,.\, (a^5 \,.\, Y^{(2)}) = \frac{5}{3}\alpha \,.\, Y^{(2)} \,.\, \int \rho \,.\, d \,.\, a^3 - \frac{5\,\alpha \,.\, Z^{(2)}}{4\pi};$$

substituting this in [1757], we get [1759].

* (1252) The general values of $Z^{(2)}$, $Y^{(2)}$, assumed in [1760, 1761], are precisely of the same form as in [1528c], the constant coefficients being changed. The coefficients

[1760a] of $(\mu^2 - \frac{1}{3})$, in [1760, 1761], are put *negative*, for the convenience of making them conform to the actual signs in $\alpha \,.\, Z^{(2)}$ [1632], and in $a \,.\, (1 + \alpha y)$ [1648].

† (1253) Substituting in [1759] the values $U^{(2)}$ [1755], $Z^{(2)}$ [1760], $Y^{(2)}$ [1761], it becomes, by connecting together terms of the same form,

$$H \,.\, (\mu^2 - \tfrac{1}{3}) + H'''' \,.\, (1-\mu^2) \,.\, \cos.2\varpi$$

[1761a] $$= \left\{ -\tfrac{5}{3}\alpha h \,.\, \textstyle\int \rho \,.\, d \,.\, a^3 + \frac{5\alpha}{4\pi} \,.\, \frac{1}{2} g \right\} \,.\, (\mu^2 - \tfrac{1}{3}) + \left\{ \tfrac{5}{3}\alpha h' \,.\, \textstyle\int \rho \,.\, d \,.\, a^3 - \frac{5\alpha}{4\pi} \,.\, g' \right\} \,.\, \mu \,.\, \sqrt{(1-\mu^2)} \,.\, \sin.\varpi$$
$$+ \left\{ \tfrac{5}{3}\alpha h'' \,.\, \textstyle\int \rho \,.\, d \,.\, a^3 - \frac{5\alpha}{4\pi} \,.\, g'' \right\} \,.\, \mu \,.\, \sqrt{(1-\mu^2)} \,.\, \cos.\varpi + \left\{ \tfrac{5}{3}\alpha h''' \,.\, \textstyle\int \rho \,.\, d \,.\, a^3 - \frac{5\alpha}{4\pi} \,.\, g''' \right\} \,.\, (1-\mu^2) \,.\, \sin.2\varpi$$
$$+ \left\{ \tfrac{5}{3}\alpha h'''' \,.\, \textstyle\int \rho \,.\, d \,.\, a^3 - \frac{5\alpha}{4\pi} \,.\, g'''' \right\} \,.\, (1-\mu^2) \,.\, \cos.2\varpi.$$

$$h' = \frac{g'}{4\pi . \int \rho . a^2 da}; \qquad h'' = \frac{g''}{4\pi . \int \rho . a^2 da}; \qquad h''' = \frac{g'''}{4\pi . \int \rho . a^2 da}. \qquad [1762]$$

These are the conditions arising from the supposition that the spheroid revolves about one of its principal axes of rotation. This supposition determines the constant quantities h', h'', h''', by means of the values of g', g'', g'''; but it leaves the quantities h and h'''' indeterminate, as well as the functions $Y^{(3)}$, $Y^{(4)}$, &c. [1762′]

If the forces, foreign from the attraction of the particles of the spheroid, be reduced to the centrifugal force, arising from its rotatory motion, we shall have $g' = 0$, $g'' = 0$, $g''' = 0$;* therefore $h' = 0$, $h'' = 0$, $h''' = 0$, [1762″] and the expression of $Y^{(2)}$ will be of the form,

$$Y^{(2)} = -h . (\mu^2 - \tfrac{1}{3}) + h'''' . (1 - \mu^2) . \cos. 2\varpi. \qquad [1763]$$

This equation ought to be satisfied for all values of μ, ϖ; hence the coefficients of the terms of the second member ought to be equal to those of the first; and as the coefficients of the terms $\mu . \sqrt{(1-\mu^2)} . \sin. \varpi$, $\mu . \sqrt{(1-\mu^2)} . \cos. \varpi$, $(1-\mu^2) . \sin. 2\varpi$, vanish in the *first* member, they ought also to vanish in the *second* member; therefore we must have,

$$\tfrac{5}{3}\alpha h' . \int\rho . d . a^3 - \frac{5\alpha}{4\pi} . g' = 0; \quad \tfrac{5}{3}\alpha h'' . \int\rho . d . a^3 - \frac{5\alpha}{4\pi} . g'' = 0; \quad \tfrac{5}{3}\alpha h''' . \int\rho . d . a^3 - \frac{5\alpha}{4\pi} . g''' = 0. \qquad [1761b]$$

Dividing these by $\frac{5}{3}\alpha . \int\rho . d . a^3$, or its equal $5\alpha . \int\rho . a^2 da$, we get, by transposition, the values of h', h'', h''', [1762]. In like manner, by putting the coefficients of $\mu^2 - \frac{1}{3}$, $(1-\mu^2) . \cos. 2\varpi$, in each member of [1761a], equal to each other, respectively, we find,

$$H = -\tfrac{5}{3}\alpha h . \int\rho . d . a^3 + \frac{5\alpha}{4\pi} . \tfrac{1}{2} g; \qquad H'''' = \tfrac{5}{3}\alpha h'''' . \int\rho . d . a^3 - \frac{5\alpha}{4\pi} . g''''. \qquad [1761c]$$

Now the forces acting on the spheroid being known, we shall have the expression of $\alpha . Z^{(2)}$, [1632]; consequently the values g, g', g'', g''', g'''', will be given, and from these we may determine h', h'', h''', [1762]; but h, h'''', depend on the preceding arbitrary quantities H, H'''', [1761c]; therefore h, h'''', are indeterminate, as well as the quantities $Y^{(3)}$, $Y^{(4)}$, [1761d] &c., which are not affected by the equation [1748], or [1752].

* (1254) If the forces S, S', &c., vanish, the value $\alpha . Z^{(2)}$ [1632] will become $\alpha . Z^{(2)} = -\frac{1}{2} g . (\mu^2 - \frac{1}{3})$, or $Z^{(2)} = -\frac{g}{2\alpha} . (\mu^2 - \frac{1}{3})$. Comparing this with [1760], we find that g is changed into $\frac{g}{\alpha}$, and that we must put $g' = 0$, $g'' = 0$, [1762a] $g''' = 0$, $g'''' = 0$. The values g', g'', g''', being substituted in [1762], give $h' = 0$, $h'' = 0$, $h''' = 0$; hence $Y^{(2)}$ [1761] becomes as in [1763].

33. *We shall now consider the expression of gravity at the surface of the*
[1763'] *spheroid*, putting this force equal to p. It is evident, from § 25, [1645''', 1646], that we shall obtain the value of p, by taking the differential of the second member of the equation [1702], relative to r, and dividing this
[1763''] differential by $-dr$, which will give, at the surface,*

[1764]
$$p=\frac{4\pi}{3r^2}\cdot\int_0^1\rho.d.a^3+\frac{4\alpha\pi}{r^2}\cdot\int_0^1\rho.d.\left\{a^3.Y^{(0)}+\frac{2a^4}{3r}.Y^{(1)}+\frac{3a^5}{5r^2}.Y^{(2)}+\frac{4a^6}{7r^3}.Y^{(3)}+\&c.\right\}$$
$$-\alpha r.\{2.Z^{(0)}+2.Z^{(2)}+3r.Z^{(3)}+4r^2.Z^{(4)}+\&c.\};$$

[1764'] these integrals being taken, from $a=0$ to $a=1$. The radius r, at the surface, is equal to $1+\alpha y$, or†

Radius of the earth's surface.

[1765]
$$r=1+\alpha y=1+\alpha.\{Y^{(0)}+Y^{(1)}+Y^{(2)}+Y^{(3)}+\&c.\};$$

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* (1255) It is shown, in [1645''', &c.], that the differential of the *second* member of [1635], taken relative to r, and divided by $-dr$, gives the value of p; and the second member of [1635] is the same as the second member of [1702], because the
[1764a] first members of both these equations are $\int\frac{d\Pi}{\rho}$. In finding this differential, we may neglect the variations of the limits of the integrals, as in [1447n, 1562g]. Moreover, the two first integrals of the second member of [1702] vanish, when the attracting point is at the surface, because the two limits of the integrals are $a=1$, $a=1$; and as the sign $\int$ affects a only, we may introduce r, under that sign, so that the second member of [1702] will become,

[1764b]
$$\frac{4\pi}{3}\cdot\int_0^1\frac{\rho.d.a^3}{r}+4\alpha.\pi\int_0^1\rho.d.\left\{\frac{a^3}{r}.Y^{(0)}+\frac{a^4}{3r^2}.Y^{(1)}\ldots\ldots+\frac{a^{i+3}}{(2i+1).r^{i+1}}.Y^{(i)}+\&c.\right\}$$
$$+\alpha.\{r^2.Z^{(0)}+r^2.Z^{(2)}+r^3.Z^{(3)}\ldots\ldots+r^i.Z^{(i)}+\&c.\};$$

and its differential relative to r, being divided by $-dr$, gives

[1764c]
$$p=\tfrac{4}{3}\pi.\int_0^1\frac{1}{r^2}.\rho.d.a^3+4\alpha\pi.\int_0^1\rho.d.\left\{\frac{a^3}{r^2}.Y^{(0)}+\frac{2a^4}{3r^3}.Y^{(1)}+\frac{3a^5}{5r^4}.Y^{(2)}\ldots\ldots+\frac{(i+1).a^{i+3}}{(2i+1).r^{i+2}}.Y^{(i)}+\&c.\right\}$$
$$-\alpha.\{2r.Z^{(0)}+2r.Z^{(2)}+3r^2.Z^{(3)}+4r^3.Z^{(4)}\ldots\ldots+i.r^{i-1}Z^{(i)}+\&c.\}.$$

which, by bringing r^2 from under the sign of integration, becomes as in [1764].

† (1256) At the surface of the fluid, where $a=1$, we have $r=1+\alpha y$ [1676^x]; and this, by means of the value of y [1756'], becomes as in [1765]. If we neglect α^2, we may put $r=1$, in the terms of [1764] multiplied by α, by which means we shall have,

[1766a]
$$p=\frac{4\pi}{3.(1+\alpha y)^2}\cdot\int_0^1\rho.d.a^3+4\alpha\pi.\int_0^1\rho.d.\{a^3.Y^{(0)}+\tfrac{2}{3}a^4.Y^{(1)}+\tfrac{3}{5}a^5.Y^{(2)}+\&c.\}$$
$$-\alpha.\{2.Z^{(0)}+2.Z^{(2)}+3.Z^{(3)}+\&c.\}.$$

therefore we shall get,

$$p=\tfrac{4}{3}\pi.\int_0^1\rho.d.a^3-\tfrac{8}{3}\alpha\pi.\{Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.\}.\int_0^1\rho.d.a^3$$
$$+4\alpha\pi.\int_0^1\rho.d.\{a^3.Y^{(0)}+\tfrac{2}{3}a^4.Y^{(1)}+\tfrac{3}{5}a^5.Y^{(2)}+\tfrac{4}{7}a^6.Y^{(3)}+\&c.\}$$ [1766]
$$-\alpha.\{2.Z^{(0)}+2.Z^{(2)}+3.Z^{(3)}+4.Z^{(4)}+\&c.\}.$$

We may eliminate the integrals from this expression, by means of the equation [1705], which becomes, at the surface,*

$$\frac{4\pi}{2i+1}.\int_0^1\rho.d.(a^{i+3}.Y^{(i)})=\tfrac{4}{3}\pi.Y^{(i)}.\int_0^1\rho.d.a^3-Z^{(i)};$$ [1767]

therefore, by supposing

$$P=\tfrac{4}{3}\pi.\int_0^1\rho.d.a^3-\tfrac{8}{3}\alpha\pi.Y^{(0)}+4\alpha\pi.\int_0^1\rho.d.(a^3.Y^{(0)})-2\alpha.Z^{(0)},$$ [1768]

we shall find,†

Force of gravity at the surface of the earth.

$$p=P+\alpha.P.\{Y^{(2)}+2.Y^{(3)}+3.Y^{(4)}\,.....+(i-1).Y^{(i)}+\&c.\}$$
$$-\alpha.\{5.Z^{(2)}+7.Z^{(3)}+9.Z^{(4)}\,.....+(2i+1).Z^{(i)}+\&c.\}.$$ [1769]

Substituting in the first term the value of

$$\frac{1}{(1+\alpha y)^2}=1-2\alpha y=1-2\alpha.\{Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.\},$$ [1766b]

it becomes as in [1766].

* (1257) At the surface of the spheroid, the first of the integrals [1705] vanishes, because its limits become $a=1$ and $a=1$; and by putting $a=1$, in the terms without the sign of integration, we get, as in [1767].

$$0=-\tfrac{4}{3}\pi.Y^{(i)}.\int_0^1\rho.d.a^3+\frac{4\pi}{2i+1}.\int_0^1\rho.d.(a^{i+3}Y^{(i)})+Z^{(i)}.$$ [1767a]

† (1258) Substituting the value of P [1768], instead of the corresponding terms of the second member of [1766], this formula becomes,

$$p=P-\tfrac{8}{3}\alpha\pi.\{Y^{(1)}+Y^{(2)}+Y^{(3)}\,.....+Y^{(i)}+\&c.\}.\int_0^1\rho.d.a^3$$
$$+4\alpha\pi.\int_0^1\rho.d.\left\{\tfrac{2}{3}a^4.Y^{(1)}+\tfrac{3}{5}a^5.Y^{(2)}\,.....+\frac{(i+1).a^{i+3}}{2i+1}.Y^{(i)}+\&c.\right\}$$ [1768a]
$$-\alpha.\{2.Z^{(2)}+3.Z^{(3)}\,.....+i.Z^{(i)}+\&c.\};$$

in which the term of the second member, of the order i, being put equal to $\alpha P^{(i)}$, we shall have,

$$\alpha P^{(i)}=-\tfrac{8}{3}\alpha\pi.Y^{(i)}.\int_0^1\rho.d.a^3+4\alpha\pi.\int_0^1\rho.d.\left(\frac{(i+1).a^{i+3}}{2i+1}.Y^{(i)}\right)-\alpha i.Z^{(i)}.$$ [1768b]

[1769'] The variation of gravity at the surface of the earth, was first discovered by observing the lengths of a pendulum vibrating in a second of time. We have seen, in the first book, that these lengths are proportional to gravity.*
[1769''] Therefore if we put l, L, for the lengths of a pendulum, vibrating in a second, corresponding to the gravities p and P, the preceding equation will give,

[1770]
$$\begin{aligned} l = L + \alpha L . \{ Y^{(2)} + 2 . Y^{(3)} + 3 . Y^{(4)} \ldots\ldots + (i-1) . Y^{(i)} + \&c. \} \\ - \frac{\alpha L}{P} . \{ 5 . Z^{(2)} + 7 . Z^{(3)} + 9 . Z^{(4)} \ldots\ldots + (2i+1) . Z^{(i)} + \&c. \}. \end{aligned}$$

[1770'] Relative to the earth, $\alpha Z^{(2)}$ is reduced to $-\frac{1}{2} g . (\mu^2 - \frac{1}{3})$ [1632]; or, as it may be expressed,†

[1771]
$$\alpha Z^{(2)} = -\tfrac{1}{2} \alpha \varphi . P . (\mu^2 - \tfrac{1}{3}),$$

Multiplying [1767] by $(i+1) . \alpha$, and transposing the first term, we get,

[1768c]
$$0 = -4 \alpha \pi . \int_0^1 \rho . d . \left(\frac{(i+1) . a^{i+3} . Y^{(i)}}{2i+1} \right) + \tfrac{4}{3} . (i+1) . \alpha\pi . Y^{(i)} . \int_0^1 \rho . d . a^3 - (i+1) . \alpha . Z^{(i)}.$$

Adding the expressions [1768b, c], we obtain, by putting $\frac{4}{3} \pi . \int_0^1 \rho . d . a^3 = P$ [1768], in the terms multiplied by α,

$$\begin{aligned} \alpha P^{(i)} &= -\tfrac{8}{3} \alpha \pi . Y^{(i)} . \int_0^1 \rho . d . a^3 + \tfrac{4}{3} . (i+1) . \alpha\pi . Y^{(i)} . \int_0^1 \rho . d . a^3 - (2i+1) . \alpha . Z^{(i)} \\ &= \alpha . Y^{(i)} . (i-1) . \tfrac{4}{3} \pi . \int_0^1 \rho . d . a^3 - (2i+1) . \alpha . Z^{(i)} \\ &= \alpha . Y^{(i)} . (i-1) . P - (2i+1) . \alpha . Z^{(i)}. \end{aligned}$$ [1768d]

which is the same as the general term of the second member of [1769]; observing that
[1768e] when $i = 1$, the expression [1768d] becomes $\alpha P^{(1)} = -3 \alpha . Z^{(1)} = 0$ [1632]. Hence the value of p [1768a], becomes as in [1769].

[1768f] * (1259) This follows from $r = \frac{T^2}{\pi^2} . g$ [86]. For if the time of vibration T be given, we shall have the length of the pendulum r proportional to the gravity g. Therefore
[1770a] $L : l :: P : p = \frac{Pl}{L}$. Substituting this value of p in [1769], and multiplying by $\frac{L}{P}$, we obtain [1770].

† (1260) The forces S, S', &c. [1632], being neglected, we have $Z^{(3)} = 0$, $Z^{(4)} = 0$, &c. [1720'']; also $\alpha Z^{(2)} = -\frac{1}{2} g . (\mu^2 - \frac{1}{3})$. Now g [1616xvi] is the centrifugal force at the distance 1 from the axis, or at the surface of the equator nearly; and, by [1769], the
[1770b] gravity at the equator is nearly equal to P; therefore $\alpha \varphi$ is nearly equal to $\frac{g}{P}$, or $g = P . \alpha \varphi$; hence $\alpha Z^{(2)} = -\frac{1}{2} \alpha \varphi . P . (\mu^2 - \frac{1}{3})$, as in [1771]. Substituting these in [1770], we get [1772].

$\alpha\varphi$ being the ratio of the centrifugal force to the gravity at the equator, [1726′]; moreover, $Z^{(3)}$, $Z^{(4)}$, &c., are nothing, [1720″]; therefore we shall have, [1771′]

Length of a pendulum on the earth's surface.

$$l = L + \alpha L.\{Y^{(2)} + 2.Y^{(3)} + 3.Y^{(4)} \ldots\ldots + (i-1).Y^{(i)} + \&c.\} + \tfrac{5}{2}\alpha\varphi.L.(\mu^2 - \tfrac{1}{3}). \qquad [1772]$$

The radius of curvature of a meridian of a spheroid, whose radius is $1+\alpha y$, is,*

Radius of curvature of a meridian on the earth's surface.

$$1 + \alpha.\left(\frac{d.\mu y}{d\mu}\right) + \alpha.\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dy}{d\mu}\right)\right\}}{d\mu}\right\}. \qquad [1773]$$

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* (1261) Referring to fig. 28, page 217, in which CD is the semi-axis of revolution, CA the semi-axis of the equator, $DPpA$ the meridian, arc $DP = s$, $CH = a$, $HP = b'$, $CP = r = 1 + \alpha y$, angle $DCP = \theta$, $\cos.\theta = \mu$, $d\theta = \frac{-d\mu}{\sqrt{(1-\mu^2)}}$ [1773a] [50] Int., $Pp = ds = \sqrt{(da^2 + db'^2)}$ [1579e]. Then, from the first of the equations [372], changing x, y, v, into a, b', θ, respectively, we get, by neglecting terms of the order α^2,

$$ds = \sqrt{(r^2.d\theta^2 + dr^2)} = \sqrt{\{(1+\alpha y)^2.d\theta^2 + (\alpha.dy)^2\}} = (1+\alpha y).d\theta = \frac{-d\mu.(1+\alpha y)}{\sqrt{(1-\mu^2)}}. \qquad [1773b]$$

The radius of curvature, at the point P, being represented by r', we shall have $r' = \frac{ds.db'}{dda}$ [1773c] [53c], ds being constant; and as the sign of r' is arbitrary, it may be written $r' = \frac{-db'}{d.\left(\frac{da}{ds}\right)}$, without considering any quantity as constant. In the triangle CHP, we [1773d] have $CH = CP.\cos.HCP$, $PH = CP.\sin.HCP$; or, in symbols

$$a = (1+\alpha y).\cos.\theta = (1+\alpha y).\mu, \qquad b' = (1+\alpha y).\sqrt{(1-\mu^2)}; \qquad [1773e]$$

whose differentials, relative to μ, are

$$da = (1+\alpha y).d\mu + \alpha\mu.dy;$$
$$db' = \alpha.dy.\sqrt{(1-\mu^2)} - (1+\alpha y).\frac{\mu\,d\mu}{\sqrt{(1-\mu^2)}} = \frac{-\mu\,d\mu}{\sqrt{(1-\mu^2)}}.\left\{1 + \alpha.\left(y - \frac{(1-\mu^2).dy}{\mu\,d\mu}\right)\right\}. \qquad [1773f]$$

Dividing da [1773f] by ds [1773b], and neglecting α^2, we get,

$$\frac{da}{ds} = -\sqrt{(1-\mu^2)}.\left\{1 + \alpha\mu.\frac{dy}{d\mu}\right\}. \qquad [1773g]$$

Its differential is to be taken relative to μ only, because ϖ does not vary in the meridian arc DPA; hence, [1773g′]

Putting c for the length of a degree of a circle, whose radius is equal to that
[1773'] we have put equal to unity, the expression of the degree of the meridian of the spheroid will be,

Length of a degree of the meridian on the earth's surface.

[1774] $$c\cdot\left\{1+\alpha\cdot\left(\frac{d\cdot\mu y}{d\mu}\right)+\alpha\cdot\left(\frac{d\cdot\left\{(1-\mu^2)\cdot\left(\frac{dy}{d\mu}\right)\right\}}{d\mu}\right)\right\}.$$

[1774'] y is equal to $Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.$ [1756']; we may make $Y^{(0)}$ vanish from this expression, by including it in the arbitrary constant quantity, which we have taken for unity; and also $Y^{(1)}$, by fixing the origin of the radius at the centre of gravity of the whole spheroid, [1745]. This radius will, by this means, become,*

[1775] $$1+\alpha\cdot\{Y^{(2)}+Y^{(3)}+Y^{(4)}\ldots\ldots+Y^{(i)}+\&c.\}.$$

$$d\cdot\left(\frac{da}{ds}\right)=\frac{\mu\,d\mu}{\sqrt{(1-\mu^2)}}\cdot\left\{1+\alpha\mu\cdot\frac{dy}{d\mu}\right\}-\sqrt{(1-\mu^2)}\cdot\alpha\cdot dy-\alpha\mu\cdot\frac{ddy}{d\mu}\cdot\sqrt{(1-\mu^2)}$$

[1773h] $$=\frac{\mu\,d\mu}{\sqrt{(1-\mu^2)}}\cdot\left\{1-\alpha\cdot\left(-2\cdot\frac{\mu\cdot dy}{d\mu}+\frac{dy}{\mu\,d\mu}+(1-\mu^2)\cdot\frac{ddy}{d\mu^2}\right)\right\}$$

Substituting this and $d\,b'$ [1773f] in r' [1773d], and neglecting α^2, we get,

$$r'=-\frac{d\,b'}{d\cdot\left(\frac{da}{ds}\right)}=1+\alpha\cdot\left\{y-\frac{(1-\mu^2)\cdot dy}{\mu\,d\mu}\right\}+\alpha\cdot\left\{-2\cdot\frac{\mu\cdot dy}{d\mu}+\frac{dy}{\mu\,d\mu}+(1-\mu^2)\cdot\frac{ddy}{d\mu^2}\right\}$$

[1773i] $$=1+\alpha\cdot\left\{y+\mu\cdot\left(\frac{dy}{d\mu}\right)\right\}+\alpha\cdot\left\{-2\mu\cdot\frac{dy}{d\mu}+(1-\mu^2)\cdot\left(\frac{ddy}{d\mu^2}\right)\right\}$$

[1773k] $$=1+\alpha\cdot\left(\frac{d\cdot(\mu y)}{d\mu}\right)+\alpha\cdot\left\{\frac{d\cdot\left\{(1-\mu^2)\cdot\left(\frac{dy}{d\mu}\right)\right\}}{d\mu}\right\};$$

as is easily perceived by reduction, and developing the two last terms; observing that the differentials of y are taken relative to μ only [1773g']. This value of r' is the same as
[1773l] in [1773]; and if we multiply it by c [1773'], the length of a degree of a circle whose radius is 1, we shall get the length of a degree of the meridian [1774], corresponding to the radius r'.

* (1262) The value of y [1774'] changes r [1702'''] into

[1775a] $$r=a\cdot\{1+\alpha\cdot(Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.)\};$$

and if we put $a+a\alpha\cdot Y^{(0)}=1$, we may, in the terms of the order α, put $a=1$; by which means this value of r will become $r=1+\alpha\cdot(Y^{(1)}+Y^{(2)}+\&c.)$. Supposing the origin of r to be placed at the centre of gravity of the spheroid, we shall have $Y^{(1)}=0$ [1745],
[1775b] and the preceding value of r will be of the form [1775]. Comparing this with its value $1+\alpha y$ [1764'], we get, $y=Y^{(2)}+Y^{(3)}+Y^{(4)}+\&c.$

Now from [1620] we have,

$$\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}=-i\,.\,(i+1)\,.\,Y^{(i)}-\frac{\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}\,; \qquad [1776]$$

therefore the expression of the degree of the meridian [1774] will become,*

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* (1263) If we, for brevity, put y [1775*b*] under the form $y=\Sigma\,Y^{(i)}$ the sign Σ of finite integrals including all positive integral numbers exceeding unity, the expression of the radius of curvature [1773] will become,

$$1+\alpha\,.\,d\,.\left(\frac{\mu\,.\,\Sigma\,Y^{(i)}}{d\,\mu}\right)+\alpha\,.\left\{\frac{d\,.\left\{(1-\mu^2)\,.\,\Sigma\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}$$

$$=1+\alpha\,.\,\Sigma\,.\left\{d\,.\left(\frac{\mu\,.\,Y^{(i)}}{d\,\mu}\right)+\left(\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right)\right\}; \qquad [1776a]$$

but $d\,.\left(\frac{\mu\,.\,Y^{(i)}}{d\,\mu}\right)=Y^{(i)}+\mu\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)$. Adding this to [1776], we get,

$$d\,.\left(\frac{\mu\,.\,Y^{(i)}}{d\,\mu}\right)+\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}$$

$$=-(i^2+i-1)\,.\,Y^{(i)}+\mu\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)-\frac{\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}. \qquad [1776b]$$

This being integrated, relative to Σ, and multiplied by α, we get,

$$\alpha\,.\,\Sigma\,.\left\{d\,.\left(\frac{\mu\,.\,Y^{(i)}}{d\,\mu}\right)+\left(\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right)\right\}$$

$$=-\alpha\,.\,\Sigma\,.\,(i^2+i-1)\,.\,Y^{(i)}+\alpha\mu\,.\,\Sigma\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)-\alpha\,.\,\Sigma\,.\,\frac{\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}. \qquad [1776c]$$

Substituting this in the radius of curvature r' [1776*a*], it becomes,

$$1-\alpha\,.\,\Sigma\,.\,(i^2+i-1)\,.\,Y^{(i)}+\alpha\,\mu\,.\,\Sigma\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)-\alpha\,.\,\Sigma\,.\,\frac{\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}.$$

Multiplying this by c, we get, from [1774] the length of a degree of the meridian

$$=c-\alpha\,c\,.\,\Sigma\,.\,(i^2+i-1)\,.\,Y^{(i)}+\alpha\,c\,\mu\,.\,\Sigma\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)-\alpha\,c\,.\,\Sigma\,.\,\frac{\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}, \qquad [1776d]$$

as in [1777]; putting for i all positive integral numbers, 2, 3, 4, ... ∞.

Length of a degree of the meridian on the surface of the earth.

[1777]
$$c - \alpha c . \{5 . Y^{(2)} + . 11\, Y^{(3)} \ldots\ldots + (i^2+i-1) . Y^{(i)} + \&c.\}$$
$$+ \alpha c \mu . \left\{ \left(\frac{dY^{(2)}}{d\mu}\right) + \left(\frac{dY^{(3)}}{d\mu}\right) + \&c. \right\}$$
$$- \alpha c . \frac{\left\{ \left(\frac{ddY^{(2)}}{d\varpi^2}\right) + \left(\frac{ddY^{(3)}}{d\varpi^2}\right) + \&c. \right\}}{1-\mu^2} .$$

If we compare with each other, the expression of the radius of the earth
[1777'] [1775], the length of the pendulum [1770], and the length of a degree of the meridian [1777]; we shall find that the term $\alpha . Y^{(i)}$, of the expression of the radius, is multiplied by $i-1$, in the length of the pendulum; and
Remarks on the
by i^2+i-1, in the degree of the meridian. Hence it follows, that
[1777''] however small $i-1$ may be,* *this term will be more sensible in the lengths*
effects of the terms $Y^{(3)}$, $Y^{(4)}$, &c.,
of the pendulum, than in the horizontal parallax of the moon, which is proportional to the radius of the earth; and it will be still more sensible in the measures of the degrees, than in the lengths of the pendulum. The reason of
[1777'''] this is, that the terms of the expression of the radius of the earth, suffer
on the radius of the earth, &c.
two differentials in the expression of the degree of the meridian;† and in

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* (1263a) Observing that the least value of i in [1775] is $i=2$, so that $i-1$ must be equal to, or exceed, unity.

† (1264) The second differential of $r = 1 + \alpha y$ [1765], relative to μ, gives $\left(\frac{ddr}{d\mu^2}\right) = \alpha . \left(\frac{ddy}{d\mu^2}\right)$; hence the last term of r' [1773i], becomes

$$\alpha . (1-\mu^2) . \left(\frac{ddy}{d\mu^2}\right) = \alpha . (1-\mu^2) . \left(\frac{ddr}{d\mu^2}\right);$$

multiplying this by c [1773l], we get the corresponding term of the degree of the meridian
[1777a] [1777], $c\alpha . (1-\mu^2) . \left(\frac{ddy}{d\mu^2}\right) = c\alpha . (1-\mu^2) . \left(\frac{ddr}{d\mu^2}\right)$; therefore the degree of the meridian contains a term depending on the second differential of r, as is observed in [1777''']. Now if we substitute the value of y [1775b] in the preceding equation, we shall obtain a series, whose general term is $c\alpha . (1-\mu^2) . \left(\frac{ddY^{(i)}}{d\mu^2}\right)$, and as the value of $Y^{(i)}$ [1510], or [1528d, e, &c.], is composed of terms of the form $A . \mu$, whose second differential, relative to μ, is multiplied by $i.(i-1)$; this will produce, in $c\alpha . (1-\mu^2) . \left(\frac{ddY^{(i)}}{d\mu^2}\right)$, or
[1777b] in the expression [1774], the term $i . (i-1) . c\alpha . (1-\mu^2) . A . \mu^{i-2}$. Hence we perceive the manner in which the factors of the order i^2 are produced in the formula [1774].

taking each differential, these terms are multiplied by the corresponding exponent of μ, in consequence of which they become more important. *In the expression of the variation of two consecutive degrees of the meridian, the* [1777''''] *terms of the radius of the earth suffer three successive differentials*;* *those terms which cause the figure of the earth to vary from an ellipsoid, may by this means become very sensible; and the ellipticity, deduced from this variation, may be very different from that deduced from the observed lengths of a pendulum. These three expressions have the important advantage of being independent of the internal constitution of the earth*;† *that is, they are independent of the* [1777v] *figure and density of the strata; so that if we can determine the functions* $Y^{(2)}$, $Y^{(3)}$, &c., *by the measures of the degrees of the meridian and of the parallaxes, we shall obtain directly the length of the pendulum.* We may, by this means, ascertain whether the law of universal gravitation agrees with [1777vi] the figure of the earth, and with the observed variations of gravity at its surface. These remarkable relations between the expressions of the degrees of the meridian, and the lengths of the pendulum, may also serve to verify any hypothesis, assumed to represent the measures of the degrees of the

These terms are very sensible in the differences of the measures of two consecutive degrees.

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* (1265) Putting C for the length of a degree of the meridian [1774], corresponding to a latitude, whose cosine, neglecting terms of the order α, is μ; we may suppose C to be a function of μ, represented by $C=\varphi(\mu)$; and δC, μ', to be increments of C, μ, respectively, [1777c] arising from an increment of one degree in the latitude. Hence we shall have

$$C+\delta C=\varphi(\mu+\mu'). \qquad [1777d]$$

Developing the second member according to the powers of μ', by Taylor's theorem [617], it will become $C+\delta C=\varphi(\mu)+\mu'.\left(\frac{d.\varphi(\mu)}{d\mu}\right)+\&c$; from which subtracting $C=\varphi(\mu)$, and neglecting the second and higher powers of μ', we shall find,

$$\delta C=\mu'.\left(\frac{d.\varphi(\mu)}{d\mu}\right)=\mu'.\left(\frac{dC}{d\mu}\right). \qquad [1777e]$$

Now from [1777a, b], C contains a term depending on the second differential of y or $Y^{(i)}$, of the order $i^2.c\alpha.A.\mu^{i-2}$; therefore the value of δC will contain a term depending on the third differential of y, or $Y^{(i)}$, which will have a factor of the order i^3.

† (1266) The expressions [1770, 1774, 1775], contain the values of $Y^{(2)}$, $Y^{(3)}$, &c., $Z^{(3)}$, $Z^{(2)}$, &c., corresponding to the surface of the spheroid; but do not contain any of these quantities for the interior of the earth, nor any expressions relative to the density of the internal [1777f] mass, so that those formulas are independent of the internal constitution of the earth.

[1777vii] meridian; as will appear by the application we shall now make of this method, to the hypothesis proposed by Bouguer, to represent the lengths of the degrees, measured in the North, in France, and at the equator.

We shall suppose the expression of the radius of the earth [1775] to be,

[1778] $$1+\alpha\,.\,Y^{(2)}+\alpha\,.\,Y^{(4)},$$

and that

[1779] $$Y^{(2)}=-A\,.\,(\mu^2-\tfrac{1}{3})\,; \qquad Y^{(4)}=-B\,.\,\{\mu^4-\tfrac{6}{7}\mu^2+\tfrac{3}{35}\}.$$

It is evident that these functions of μ satisfy the equations of partial differentials to which $Y^{(2)}$ and $Y^{(4)}$ are subjected.* The variation of a degree of the meridian will be, by what has been said,

[1780] $$\alpha c\,.\,\{3A-\tfrac{102}{7}B\}\,.\,\mu^2+15\,\alpha c\,.\,B\,.\,\mu^4.$$

Bouguer's hypothesis.

Bouguer supposed this variation to be proportional to the fourth power of

[1780′] the sine of the latitude, which is nearly equal to μ^4. Therefore by making the coefficient of μ^2 vanish from the preceding equation, we shall have,†

[1781] $$B=\tfrac{7}{34}A\,;$$

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* (1267) The values of $Y^{(2)}$, $Y^{(4)}$, [1779], are deduced from those in [1528c, e], putting all the coefficients, except $B_2^{(0)}$, $B_4^{(0)}$, equal to nothing, and making $B_2^{(0)}=-A$, $B_4^{(0)}=-B$. The values [1779] give

[1778a] $$\left(\frac{d\,Y^{(2)}}{d\mu}\right)=-2A.\mu;\quad \left(\frac{d\,Y^{(3)}}{d\mu}\right)=0;\quad \left(\frac{d\,Y^{(4)}}{d\mu}\right)=-B.\{4\mu^3-\tfrac{12}{7}\mu\};\quad \left(\frac{dd\,Y^{(2)}}{d\varpi^2}\right)=0;\ \ \&c.$$

Substituting these in the length of a degree [1777], which we shall call C, it becomes

$$C=c+\alpha c.\{5A.(\mu^2-\tfrac{1}{3})+19B.(\mu^4-\tfrac{6}{7}\mu^2+\tfrac{3}{35})\}-\alpha c\mu.\{2A.\mu+B\,.\,(4\,\mu^3-\tfrac{12}{7}\mu)\}$$

[1778b] $$=c-\tfrac{5}{3}\alpha c\,.\,A+\tfrac{57}{35}\alpha c\,.\,B+\alpha c\,.\,(3A-\tfrac{102}{7}B)\,\mu^2+15\,\alpha c\,.\,B\,.\,\mu^4.$$

At the equator, where $\mu=0$, if we denote the length of the degree by C', we shall have $C'=c-\tfrac{5}{3}\alpha c\,.\,A+\tfrac{57}{35}\alpha c\,.\,B$; hence, as in [1780],

[1778c] $$C-C'=\alpha c\,.\,(3A-\tfrac{102}{7}B)\,.\,\mu^2+15\,\alpha c\,.\,B\,.\,\mu^4.$$

† (1267a) If we suppose with Bouguer, that $C-C'$ [1778c] is proportional to μ^4, we must put the coefficient of μ^2 equal to nothing, or $3A-\tfrac{102}{7}B=0$; hence

[1779a] $B=\tfrac{7}{34}A$, as in [1781]; and $Y^{(4)}$ [1779] changes into $Y^{(4)}=-\tfrac{7}{34}A.(\mu^4-\tfrac{6}{7}\mu^2+\tfrac{3}{35})$. Adding this to $Y^{(2)}$ [1779], we get,

[1779a′] $$Y^{(2)}+Y^{(4)}=-A\,.\,(\mu^2-\tfrac{1}{3})-\tfrac{7}{34}A.(\mu^4-\tfrac{6}{7}\mu^2+\tfrac{3}{35})=-\tfrac{7}{34}A.(\mu^4+4\mu^2)+\tfrac{161}{510}A.$$

Hence the radius [1778] drawn from the centre of gravity of the earth to its surface, will be, by putting the radius of the equator equal to unity, [1781']

$$1 - \frac{7\alpha . A}{34} . (4\mu^2 + \mu^4). \qquad [1782]$$

The expression of the length of the pendulum l [1772], will become, by putting L for its value at the equator,* [1782']

$$L + \tfrac{5}{2}\alpha\varphi . L . \mu^2 - \frac{\alpha . A L}{34} . (16\mu^2 + 21\mu^4). \qquad [1783]$$

Lastly, the expression of the degree of the meridian will be, by putting c for its length at the equator,† [1783']

$$c + \tfrac{105}{34}\alpha A c . \mu^4. \qquad [1784]$$

We shall here remark, that from what has been said, the term multiplied by μ^4 is three times more sensible in the expression of the length of the [1784']

Multiplying this by α, and adding the product to unity, we get the radius [1778],

$$r = 1 + \alpha . Y^{(2)} + \alpha . Y^{(4)} = 1 + \tfrac{161}{510}\alpha A - \tfrac{7}{34}\alpha A . (\mu^4 + 4\mu^2). \qquad [1781a]$$

At the equator, where $\mu = 0$, this becomes $1 + \tfrac{161}{510}\alpha A$; and if we suppose the radius of the equator equal to unity, the preceding general expression of the radius r will be, as in [1782], $r = 1 - \tfrac{7}{34}\alpha A . (\mu^4 + 4\mu^2)$, neglecting terms of the order α^2. [1781b]

* (1268) If we substitute the values of $Y^{(2)}$ [1779], and $Y^{(4)}$ [1779a], in [1772], we shall get,

$$l = L - \alpha . A L . \{(\mu^2 - \tfrac{1}{3}) + \tfrac{21}{34} . (\mu^4 - \tfrac{6}{7}\mu^2 + \tfrac{3}{35})\} + \tfrac{5}{2}\alpha\varphi . L . (\mu^2 - \tfrac{1}{3})$$

$$= (L + \tfrac{1}{3}\alpha . A L - \tfrac{9}{170}\alpha . A L - \tfrac{5}{6}\alpha\varphi . L) + \tfrac{5}{2}\alpha\varphi . L . \mu^2 - \frac{\alpha . A L}{34} . (16\mu^2 + 21\mu^4); \qquad [1783a]$$

and by writing L for $L + \alpha . (\tfrac{1}{3} A L - \tfrac{9}{170} . A L - \tfrac{5}{6}\varphi . L)$, neglecting α^2, we find $l = L + \tfrac{5}{2}\alpha\varphi . L . \mu^2 - \frac{\alpha . A L}{34} . (16\mu^2 + 21\mu^4)$, as in [1783]. At the equator, where $\mu = 0$, this length of the pendulum becomes L, as in [1782']. [1783b]

† (1269) Putting $B = \tfrac{7}{34} A$ [1781], in [1778b], we get

$$C = c - \tfrac{679}{510}\alpha c . A + \tfrac{105}{34}\alpha c . A . \mu^4;$$

which, by writing c for $c - \tfrac{679}{510}\alpha c . A$, and neglecting α^2, becomes

$$C = c + \tfrac{105}{34}\alpha c . A . \mu^4, \qquad [1784a]$$

as in [1784]. This gives c for the length of a degree at the equator, as in [1783'].

pendulum, than in that of the radius of the earth;* and five times more sensible in the expression of the length of a degree, than in that of the length of the pendulum; lastly, on the mean parallel of latitude, it will be four times more sensible in the expression of the variation of the consecutive
[1784″] degrees, than in that of the degree itself.† According to Bouguer, the difference of the degrees of the pole and of the equator divided by the
[1784‴] degree of the equator, is $\frac{959}{56753}$; which is the ratio required in this hypothesis, by comparing the degrees measured in Pello, Paris, and at the equator.‡ This ratio is equal to $\frac{105}{34}\,\alpha A$; therefore we shall find,

[1785] $$\alpha A = 0{,}0054717.$$

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* (1270) The coefficient of μ^4, in the expression of the radius of the earth [1782], is $-\frac{7}{34}\alpha.A$; that in the length of the pendulum [1783], divided by the value L, is $-\frac{21}{34}\alpha.A$, which is three times the former. Again, the preceding term, relative to the
[1784b] pendulum, is to the corresponding term of the degree of the meridian [1784] $\frac{105}{34}\alpha.A$, as one to five, as in [1784′].

† (1271) The variation of two consecutive degrees of latitude is $\delta C = \mu'.\left(\frac{dC}{d\mu}\right)$ [1777e]; which, by substituting the value of C [1784], becomes $\frac{420}{34}\mu'.\alpha Ac.\mu^3$, μ' being
[1784c] the increment of μ, arising from an increase of one degree in the latitude. Now if an arc, whose sine is μ, be increased 1°, the sine will be increased by $1^\circ.\sqrt{(1-\mu^2)} = \mu'$ nearly, [52] Int.; hence $\delta C = \frac{420}{34}.1^\circ.\sqrt{(1-\mu^2)}.\alpha Ac.\mu^3$, and $\frac{\delta C}{1^\circ} = \frac{420}{34}\alpha Ac.\sqrt{(1-\mu^2)}.\mu^3$. Dividing this by the variable part of [1784] $\frac{105}{34}\alpha Ac.\mu^4$, we get the ratio $4.\frac{\sqrt{(1-\mu^2)}}{\mu}$, which, in the mean latitude, where $\mu = \sqrt{(1-\mu^2)} = \sqrt{\tfrac{1}{2}}$, becomes simply 4, as in [1784″].

‡ (1272) The terms of the ratio $\frac{959}{56753}$ represent, in toises, according to Bouguer, the length of a sexagesimal degree at the equator, and the excess of the polar degree above the equatorial. In decimals, this ratio is 0,0168978. Now by putting successively $\mu=0$, and $\mu=1$, in [1784],
[1785a] we obtain the equatorial degree $=c$, and polar degree $= c + \frac{105}{34}\alpha c.A$; the difference of these, divided by the former, is $\frac{105}{34}\alpha.A = 0{,}0168978$; hence $\alpha.A = 0{,}0054717$, as in [1785]. It may be observed, that the ratio, computed from the table of measures in [2010], agrees nearly with the above. For, by that table, the lengths of the degrees at the equator, Paris, and Lapland, are respectively 25538,85, 25658,28, 25832,25; subtracting the first from the two last, we get 119,43, 293,40; which being divided by
[1785b] the fourth power of the sines of the latitude of Paris $51^\circ{,}3327$, and Lapland $73^\circ{,}7037$, become respectively 440,1 and 416,9. The mean of these 428,5, divided by the degree of the equator 25538,85, gives the ratio 0,0168, nearly as in [1785a].

Taking the length of the pendulum at the equator for unity; the variation of this length, in any place whatever, will be,* [1785‴]

$$-\frac{0{,}0054717}{34}\cdot\{16\,\mu^2+21\,\mu^4\}+\tfrac{5}{2}\,\alpha\,\varphi\,.\,\mu^2. \qquad [1786]$$

We have, by § 19,† $\alpha\,\varphi=0{,}00345113$; hence $\tfrac{5}{2}\,\alpha\,\varphi=0{,}0086278$, and the preceding formula becomes, [1786′]

$$0{,}0060529\,.\,\mu^2-0{,}0033796\,.\,\mu^4. \qquad [1787]$$

At Pello, where $\mu=\sin.\,74^\circ\,22'$, this formula gives 0,0027016, for the variation of the length of the pendulum. According to observations, the variation is 0,0044625,‡ which is much greater than by the theory; therefore, [1787′] [1787″]

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* (1273) Substituting $L=1$, and αA [1785] in [1783], we obtain, for the length of the pendulum, the expression $1-\frac{0{,}0054717}{34}\,.\,(16\,\mu^2+21\,\mu^4)+\frac{5}{2}\alpha\varphi\,.\,\mu^2$; from which subtracting the length corresponding to the equator, we shall get the variation from the equatorial length, as in [1786]. [1785c]

† (1274) In [1584], we have $q=\frac{g}{\frac{4}{3}\pi\rho}$, and by putting $\psi=0$, in [1583], we shall get the whole gravity at the equator $p=\frac{4\pi\rho\,.\,k\,.\,(1+\lambda^2)\,.\,(\lambda-\text{ang. tang. }\lambda)}{\lambda^3\,.\,\sqrt{(1+\lambda^2)}}$. Now the centrifugal force, at the distance 1, is g [1583″]; therefore at the surface of the equator, where the distance is $k.\sqrt{(1+\lambda^2)}$ [1565c], the centrifugal force will be $k.\sqrt{(1+\lambda^2)}.g$. Dividing this by the preceding expression of the gravity p, at the equator, we get the quantity named $\alpha\,\varphi$ [1771′]; hence, by means of [1589a], we find, [1787a]

$$\alpha\varphi=\frac{k\,.\,(1+\lambda^2)\,.\,\lambda^3\,.\,g}{4\,\pi\,\rho\,.\,k\,.\,(1+\lambda^2)\,.\,(\lambda-\text{ang. tang. }\lambda)}=\frac{\lambda^3\,.\,g}{4\,\pi\,\rho\,.\,(\lambda-\text{ang. tang. }\lambda)}$$

$$=\frac{\lambda^3\,.\,g}{4\,\pi\,\rho\,.\,(\frac{1}{3}\lambda^3-\frac{1}{5}\lambda^5+\&c.)}=\frac{g}{\frac{4}{3}\pi\rho}\cdot\frac{1}{1-\frac{3}{5}\lambda^2+\&c.}=\frac{g}{\frac{4}{3}\pi\rho}\cdot(1+\tfrac{3}{5}\lambda^2+\&c.)$$

$$=q\,.\,(1+\tfrac{3}{5}\lambda^2+\&c.). \qquad [1787b]$$

Substituting q, λ^2, [1592], we get nearly $\alpha\,\varphi=0{,}0034675$; and [1786] becomes $0{,}006093\,.\,\mu^2-0{,}0033796\,.\,\mu^4$, differing a little from [1786′, 1787]. [1787c]

‡ (1275) In the table [2038] the length of the pendulum at the equator is 0,99669, that at the latitude of $74^\circ\,22'$ is 1,00137, and $\frac{1{,}00137}{{,}99669}=1{,}00469$; so that if the length of the pendulum at the equator be taken for unity, that in the latitude $74^\circ\,22'$ will exceed it by ,00469, which differs a little from 0,0044625 [1787″]. [1787d]

as the hypothesis of Bouguer does not agree with the observed lengths of the pendulum, it must be rejected.

34. *We shall apply the general results we have here obtained, to the case in which the spheroid is not acted upon by any foreign attractions, supposing it to*
[1787‴] *be formed of elliptical strata, having their centre in the centre of gravity of the spheroid.* We have seen [1731′], that this is the case when the earth is
[1787⁗] supposed to have been originally fluid; it is also the case when the figures of the strata are similar. For the equation [1705] becomes at the surface, where $a=1$,*

Equation to find $Y^{(i)}$ at the surface of the spheroid;

[1788]
$$0=Y^{(i)}.\int_0^1\rho.a^2\,da-\frac{1}{2i+1}.\int_0^1\rho.d.(a^{i+3}.Y^{(i)})-\frac{Z^{(i)}}{4\pi}.$$

[1788′] *The strata being supposed similar*, the value of $Y^{(i)}$ is, for each of them, the the same as at the surface; therefore $Y^{(i)}$ is independent of a, and we shall have,†

application to the case where the strata are similar.

[1789]
$$Y^{(i)}.\int_0^1\rho.a^2\,da.\left\{1-\left(\frac{i+3}{2i+1}\right).a^i\right\}=\frac{Z^{(i)}}{4\pi}.$$

If i be equal to 3, or greater than 3, $Z^{(i)}$ will be nothing relative to the
[1789′] earth;‡ and the factor $1-\left(\frac{i+3}{2i+1}\right).a^i$ will be always positive; therefore

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* (1276) The first integral of [1705] vanishes at the surface, because the limits are
[1788a] $a=1$, $a=1$; and if we put $\int\rho.d.a^3=3.\int\rho.a^2da$, in the second of these integrals; also $a=1$, in the terms without the sign $\int$; the formula [1705] becomes,

[1788b]
$$0=-4\pi.Y^{(i)}.\int_0^1\rho.a^2da+\frac{4\pi}{2i+1}.\int_0^1\rho.d.(a^{i+3}Y^{(i)})+Z^{(i)};$$

Dividing this by -4π, we get [1788].

† (1277) As $Y^{(i)}$ is independent of a, and the sign $\int$ refers to a only, we may bring $Y^{(i)}$ from under that sign; then

$$\frac{1}{2i+1}.\int\rho.d.(a^{i+3}.Y^{(i)})=Y^{(i)}.\int\frac{1}{2i+1}.\rho.d.a^{i+3}=Y^{(i)}.\int\frac{i+3}{2i+1}.\rho.a^{i+2}da;$$

substituting this in [1788], we get [1789].

‡ (1278) The forces S, S', &c., being insensible [1647vii] when computing the permanent figure of the earth, the values $Z^{(1)}$, $Z^{(3)}$, $Z^{(4)}$, &c., [1632], will be nothing,
[1788c] or generally $Z^{(i)}=0$, if i be equal to 3, or greater than 3. Now in the formula [1789],

we shall have $Y^{(i)}=0$, when $i=3$, or $i>3$. $Y^{(1)}$ is nothing when we fix the origin of the radii at the centre of gravity of the spheroid, [1745]. Lastly we have, by § 33,*

$$Z^{(2)}=-\tfrac{1}{2}\varphi\,.\,(\mu^2-\tfrac{1}{3})\,.\,4\pi\,.\int_0^1\rho\,.\,a^2\,d\,a\,;$$ [1790]

hence we find,

$$Y^{(2)}=-\frac{\tfrac{1}{2}\varphi\,.\,(\mu^2-\tfrac{1}{3})\,.\int_0^1\rho\,.\,a^2 d\,a}{\int_0^1\rho\,.\,a^2\,d\,a\,.\,(1-a^2)}\,;$$ [1791]

and the earth will be an ellipsoid of revolution. Therefore *we shall consider, in a general manner, the case in which the figure of the earth is formed by the revolution of an ellipsis about its axis.* [1791′]

Case of the earth being supposed an ellipsoid of revolution.

We have, in this case, by fixing the origin of the radii of the earth, at the centre of gravity of the earth,† [1791″]

$$Y^{(1)}=0,\qquad Y^{(3)}=0,\qquad Y^{(4)}=0,\qquad \&c.\,;$$
$$Y^{(2)}=-h\;(\mu^2-\tfrac{1}{3})\,;$$ [1792]

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the factor $1-\left(\frac{i+3}{2i+1}\right).a^i$, is always positive within the limits of the integral; because a never exceeds 1, and $\frac{i+3}{2i+1}<1$, when $i>2$, as in [1789′]. Hence the integral $\int_0^1\rho.a^2 d\,a.\left\{1-\left(\frac{i+3}{2i+1}\right).a^i\right\}$ must be a positive quantity, which we shall denote by A. Substituting this, and $Z^{(i)}=0$ [1788c], in [1789], we get $Y^{(i)}.A=0$. Hence $Y^{(i)}=0$, when $i>2$. [1788d]

* (1279) In [1771] we have $\alpha\,.\,Z^{(2)}=-\tfrac{1}{2}\alpha\varphi\,.\,P\,.\,(\mu^2-\tfrac{1}{3})$; and if we neglect α^2, we may, from [1768], put $P=\tfrac{4}{3}\pi\,.\int_0^1\rho\,.\,d\,.\,a^3=4\pi\,.\int_0^1\rho\,.\,a^2\,d\,a$; hence $Z^{(2)}$ becomes as in [1790]. Substituting this, and $i=2$, in [1789], we get [1788e]

$$Y^{(2)}\,.\int_0^1\rho\,.\,a^2\,d\,a\,.\,(1-a^2)=-\tfrac{1}{2}\varphi\,.\,(\mu^2-\tfrac{1}{3})\,.\int_0^1\rho\,.\,a^2\,d\,a.$$ [1791a]

Dividing this by the factor of $Y^{(2)}$, we get [1791]; which corresponds with an ellipsoid of revolution [1503a, 1730′]. This value of $Y^{(2)}$, computed for the surface, is also, by hypothesis [1788′], the same for all the strata of the spheroid.

† (1280) In this computation it is supposed, that the earth is composed of elliptical strata of revolution [1791′], having the same centre, and their axes in the same situations, but the ellipticities and densities variable. Hence it will follow, that the above origin of the radii [1792a]

h being a function of a. We also have, [1788c, e],

[1793]
$$Z^{(1)}=0, \qquad Z^{(3)}=0, \qquad Z^{(4)}=0, \qquad \&c.;$$
$$\alpha \,.\, Z^{(2)} = -\tfrac{1}{2}\alpha\varphi \,.\, (\mu^2-\tfrac{1}{3}) \,.\, \tfrac{4}{3}\pi \,.\, \textstyle\int_0^1 \rho \,.\, d \,.\, a^3;$$

therefore the equation [1705] will give, at the surface,*

[1794]
$$0 = 6 \,.\, \textstyle\int_0^1 \rho \,.\, d \,.\, (a^5 h) + 5 \,.\, (\varphi - 2h) \,.\, \int_0^1 \rho \,.\, d \,.\, a^3. \qquad (1)$$

This equation shows the relation which must exist, between the densities of the strata of the spheroid and their ellipticities, in order that the equilibrium may take place. For the radius of a stratum being,† [1792d],

[1794′]
$$a \,.\, \{1 + \alpha \,.\, Y^{(0)} - \alpha h \,.\, (\mu^2 - \tfrac{1}{3})\}$$

[1791″], is also the centre of gravity of the ellipsoid included by any stratum, as that corresponding to a. The general expression of the radius of this included ellipsoid is, by
[1792b] [1503a, 1724c], $R = a \,.\, (1 + \alpha . Y^{(0)} + \alpha . Y^{(1)} + \alpha . Y^{(2)})$; hence $Y^{(3)}=0$, $Y^{(4)}=0$, &c., as in [1792]. Then, by reasoning as in [1745], we shall find, that at the surface of this enclosed ellipsoid $Y^{(1)}=0$; so that in general, for all the strata, $Y^{(1)}=0$, as in [1792]; and the preceding radius of this included spheroid will be

[1792c]
$$R = a + \alpha a \,.\, (Y^{(0)} + Y^{(2)}).$$

Lastly, as the spheroid is of revolution [1792a], R must be independent of ϖ; hence ϖ must vanish from $Y^{(2)}$ [1528c], and we shall have, $Y^{(2)} = B^{(0)} \,.\, (\mu^2 - \tfrac{1}{3})$. Putting
[1792d] $B^{(0)} = -h$, it becomes as in [1792]; consequently $R = a.\{1 + \alpha . Y^{(0)} - \alpha h . (\mu^2 - \tfrac{1}{3})\}$, as in [1794′].

* (1281) Putting $i=2$ in [1788], and substituting the value of $Z^{(2)}$ [1790], we
[1794a] shall get, $0 = Y^{(2)} . \int_0^1 \rho \,.\, a^2 da - \tfrac{1}{5} . \int_0^1 \rho \,.\, d \,.\, (a^5 \,.\, Y^{(2)}) + \tfrac{1}{2}\varphi \,.\, (\mu^2 - \tfrac{1}{3}) \,.\, \int_0^1 \rho \,.\, a^2 da$;
which, by using $Y^{(2)}$ [1792], becomes,

$$0 = -h \,.\, (\mu^2-\tfrac{1}{3}) \,.\, \textstyle\int_0^1 \rho \,.\, a^2 da + \tfrac{1}{5} \,.\, (\mu^2-\tfrac{1}{3}) \,.\, \int_0^1 \rho \,.\, d\,(a^5 h) + \tfrac{1}{2}\varphi \,.\, (\mu^2-\tfrac{1}{3}) \,.\, \int_0^1 \rho \,.\, a^2 da;$$

the quantity $\mu^2 - \tfrac{1}{3}$ being brought from under the sign $\int$, because the integrals refer to a only. Dividing this by $\tfrac{1}{30} \,.\, (\mu^2 - \tfrac{1}{3})$, we obtain,

[1794b]
$$0 = 6 \,.\, \textstyle\int_0^1 \rho \,.\, d \,.\, (a^5 h) + (15\varphi - 30h) \,.\, \int_0^1 \rho \,.\, a^2 da;$$

and by putting $\int \rho \,.\, a^2 da = \tfrac{1}{3} \,.\, \int \rho \,.\, d \,.\, a^3$, it becomes as in [1794].

† (1282) $Y^{(0)}$ being any arbitrary constant quantity, we may take for it the value $-\tfrac{1}{3}h$, in order to render the expression of R [1794′] more simple; and then we shall get,
[1795a] $R = a \,.\, \{1 - \tfrac{1}{3}\alpha h - \alpha h \,.\, (\mu - \tfrac{1}{3})\} = a \,.\, (1 - \alpha h \,.\, \mu^2)$, as in [1795].

if we suppose $Y^{(0)} = -\frac{1}{3}h$, which may be done, this radius will become,

$$a \,.\, \{1 - \alpha h \,.\, \mu^2\}\,; \qquad [1795]$$

and then αh *will be the ellipticity of the stratum.** [1795′]

At the surface of the spheroid, the radius is $1 - \alpha h \,.\, \mu^2$; hence we see [1795″]
that *the diminutions of the radii, in proceeding from the equator to the poles, are proportional to* μ^2; *therefore they are proportional to the square of the* [1795‴]
sine of the latitude.†

The radius of the earth, and the length of a degree of the meridian, vary as the square of the sine of the latitude.

The increment of the degrees of the meridian, in proceeding from the equator to the poles, is, by the preceding article, equal to $3c.\alpha h.\mu^2$;‡ *c being* [1796]
the degree of the equator. This increment is therefore proportional to the square of the sine of the latitude.

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* (1283) Putting successively $\mu = 1$, and $\mu = 0$, in [1795], we obtain the polar and equatorial semi-axes of the stratum $a - a \,.\, \alpha h$ and a, whose difference is $a \,.\, \alpha h$; [1795*b*]
dividing this by the semi-axis, we get the ellipticity αh [1795′].

† (1284) At the surface of the spheroid $a = 1$, the general expression of the radius [1795] becomes $1 - \alpha h \,.\, \mu^2$; and the semi-axes [1795*b*], $1 - \alpha h$, 1. The difference between the equatorial radius 1, and the general expression of the radius $1 - \alpha h \,.\, \mu^2$, is $\alpha h \,.\, \mu^2$, which is proportional to μ^2, or to the square of the sine of the latitude, nearly [1648‴]. [1795*c*]

‡ (1285) Substituting the values [1792] in [1777], it makes all the terms depending on $Y^{(i)}$ vanish, except $Y^{(2)}$: also $\left(\frac{d\,dY^{(2)}}{d\varpi^2}\right) = 0$. Hence the general expression of a degree of the meridian, represented by C [1777*c*], becomes,

$$C = c - \alpha c \,.\, 5 \,.\, Y^{(2)} + \alpha c \mu \,.\, \left(\frac{dY^{(2)}}{d\mu}\right) = c + 5c \,.\, \alpha h \,.\, (\mu^2 - \tfrac{1}{3}) - 2c \,.\, \alpha h \,.\, \mu^2$$
$$= c - \tfrac{5}{3} c \,.\, \alpha h + 3c \,.\, \alpha h \,.\, \mu^2. \qquad [1795d]$$

At the equator $\mu = 0$, and at the poles $\mu = 1$; hence the increment of the degree, in proceeding from the equator to the poles, is $3c \,.\, \alpha h$, as in [1796]. If we put $c - \frac{5}{3} c \,.\, \alpha h = z$, $3c \,.\, \alpha h = y$, we shall have $C = z + y \,.\, \mu^2$; and the equatorial [1795*e*]
degree will be z, the polar degree $z + y$. From this value of z we get $c = \frac{z}{1 - \frac{5}{3}\alpha h}$, and then $y = 3\alpha h \,.\, c = 3\alpha h \,.\, \frac{z}{1 - \frac{5}{3}\alpha h}$. From this last equation, we find $\alpha h = \frac{y}{3z + \frac{5}{3}y} =$ the ellipticity [1795*b*]. [1795*f*]

The equation [1794] shows, that if the densities are supposed to decrease,
[1796'] from the centre to the surface, the ellipticity of the spheroid will be less than in the case of homogenity, unless the ellipticities increase, in proceeding from the surface to the centre, in a greater ratio than in the inverse ratio of
[1797] the square of the distances from the centre. For if we suppose $h=\frac{u}{a^2}$, we shall have,*

[1798] $$\int \rho\,.\,d\,.\,(a^5 h)=\int \rho\,.\,d\,.\,(a^3 u)=u\,.\int \rho\,.\,d\,.\,a^3+\int(d u\,.\int a^3\,.\,d\rho).$$

If the ellipticities increase in a less ratio than $\frac{1}{a^2}$, u will increase from
[1798'] the centre to the surface, consequently du will be positive;† moreover, $d\rho$ is negative, because the densities are supposed to decrease from the centre to the surface [1709'''']; therefore $\int(du\,.\int a^3\,.\,d\rho)$ is a negative quantity,‡ and by putting, at the surface,

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* (1286) Substituting, in the first member of [1798], the value of h [1797], it becomes $\int\rho\,.\,d\,.\,(a^3 u)$, as in the second expression [1798]. Now we have

$$a^3\,.\,\rho=\int a^3\,.\,d\rho+\int\rho\,.\,d\,.\,a^3,$$

as is easily perceived by taking the differential of both members. Multiplying this by du, and adding $\rho u\,.\,d\,.\,a^3$, we get,

[1796a] $$a^3\,.\,\rho\,.\,du+\rho u\,.\,d\,.\,a^3=du\,.\int a^3\,.\,d\rho+du\,.\int\rho\,.\,d\,.\,a^3+\rho u\,.\,d\,.\,a^3,$$

the first member of which is evidently equal to $\rho\,.\,d\,(a^3 u)$, and the two terms of the second member $du\,.\int\rho\,.\,d\,.\,a^3+\rho u\,.\,d\,.\,a^3$ are equal to $d\,.\,(u\,.\int\rho\,.\,d\,.\,a^3)$, as is easily proved by development; hence [1796a] becomes

[1796b] $$\rho\,.\,d\,.\,(a^3 u)=d\,.\,(u\,.\int\rho\,.\,d\,.\,a^3)+du\,.\int a^3\,.\,d\rho\,;$$

the integral of which is $\int\rho\,.\,d\,.\,(a^3 u)=u\,.\int\rho\,.\,d\,.\,a^3+\int(du\,.\int a^3.d\rho)$, as in [1798].

† (1287) The ellipticity αh is proportional to $\frac{u}{a^2}$ [1797], and if we suppose u to be constant, the ellipticity will increase, exactly in proportion to $\frac{1}{a^2}$ while a decreases, in proceeding from the surface of the spheroid to the centre. If the ellipticity increase in a less ratio than $\frac{1}{a^2}$, as is supposed in [1798'], it must be because *u decreases*, while *a decreases;* and on the same hypothesis, u must increase with a, so that when a is increased by the positive element da, u will also be increased by the positive element du, as in [1798'].

[1796c] ‡ (1287a) If we represent this negative quantity by $\int(du\,.\int a^3\,.\,d\rho)=-f\,.\int_0^1\rho\,.\,d\,.\,a^3$, f being positive, the integral being taken from $a=0$, at the centre, to $a=1$, at the

$$\int_0^1 \rho \,.\, d \,.\, (a^5 h) = (h - f) \,.\, \int_0^1 \rho \,.\, d \,.\, a^3, \qquad [1799]$$

f will be a positive quantity. This being supposed, the equation [1794] will give,*

$$h = \frac{5\varphi - 6f}{4}; \qquad [1800]$$

consequently αh will be less than $\frac{5}{4}\alpha\varphi$; therefore it will be less than in the case of homogeneity, where, $d\rho$ being nothing, f is also equal to nothing [1800b]. [1800′]

Hence it follows, that in the most probable hypotheses, the oblateness of the spheroid is less than $\frac{5}{4}\alpha\varphi$. For it is natural to suppose that the strata of the spheroid become denser in approaching towards the centre; and that the ellipticities increase, from the surface to the centre, in a less ratio than $\frac{1}{a^2}$, because this ratio gives an infinite radius to a stratum infinitely near to the centre,† which is absurd. These suppositions are so much the more probable, as they become necessary, in the case where the spheroid was originally fluid; then the denser strata are, as we have seen [1732h], nearest to the centre, and the ellipticities are so far from increasing, in [1800″] [1800‴] [1800⁗]

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surface; the expression [1798] will become,

$$\int_0^1 \rho \,.\, d \,.\, (a^5 h) = u \,.\, \int_0^1 \rho \,.\, d \,.\, a^3 - f \,.\, \int_0^1 \rho \,.\, d \,.\, a^3 = (u - f) \,.\, \int_0^1 \rho \,.\, d \,.\, a^3.$$

Now at the surface, where $a = 1$, we have $u = h$ [1797]. Substituting this in the preceding expression, it becomes as in [1799].

* (1288) Substituting the value of $\int_0^1 \rho \,.\, d \,.\, (a^5 h)$ [1799], in [1794], it becomes

$$0 = 6 \,.\, (h - f) \,.\, \int_0^1 \rho \,.\, d \,.\, a^3 + 5 \,.\, (\varphi - 2h) \,.\, \int_0^1 \rho \,.\, d \,.\, a^3; \qquad [1800a]$$

dividing this by $\int_0^1 \rho \,.\, d \,.\, a^3$, we get, $0 = 6 \,.\, (h - f) + 5 \,.\, (\varphi - 2h)$; whence we easily obtain h, as in [1800]. If the density be constant, $d\rho = 0$, $\int a^3 \,.\, d\rho = 0$, $du \,.\, \int a^3 \,.\, d\rho = 0$, $\int (du \,.\, \int a^3 \,.\, d\rho) = 0$; this last quantity is equal to $-f \,.\, \int \rho \,.\, d \,.\, a^3$, [1796c], and $\int \rho \,.\, d \,.\, a^3$ is finite; therefore we shall have $f = 0$, and then $h = \frac{5}{4}\varphi$, [1800]. [1800b]

† (1289) Substituting $\alpha h = \alpha u \,.\, a^{-2}$ [1797] in the difference between the polar and equatorial axes of any stratum, [1795b], it becomes, $a \,.\, \alpha h = \alpha u \,.\, a^{-1}$; which is infinite when $a = 0$ and u finite. [1800c]

proceeding from the surface to the centre, that, on the contrary, they decrease [1732h].

Supposing the body to be an ellipsoid of revolution, covered by a
[1800v] homogeneous fluid of any depth, and putting a' for the least semi-axis of
[1800vi] the solid ellipsoid, also $\alpha h'$ for its ellipticity, we shall have, at the surface of the fluid,*

[1801] $$\int_0^1 \rho \,.\, d \,.\, (a^5 h) = h - a'^5 h' + \int_0^{a'} \rho \,.\, d \,.\, (a^5 h) \,;$$

the integral in the second member of this equation being taken relative to
[1801′] the inner solid ellipsoid, from its centre to its surface, and the density of the fluid which covers it being put equal to unity. The equation [1794] will give, for the expression of the ellipticity αh of the terrestrial spheroid,†

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* (1290) The integral of the first member of [1801], whose limits are $a=0$, $a=1$, may be divided into two parts; the one between the limits $a=0$, $a=a'$, and represented by $\int_0^{a'} \rho \,.\, d \,.\, (a^5 h)$; and the other between the limits $a=a'$, and $a=1$; the density of this last part being $\rho=1$, so that it will be expressed by

$$\int_{a'}^1 \rho \,.\, d \,.\, (a^5 h) = \int_{a'}^1 d \,.\, (a^5 h).$$

The general integral of this expression is $a^5 h - a'^5 h'$; the constant quantity $-a'^5 h'$ being added, so as to make it vanish at the first limit, where $a=a'$, and $h=h'$. At the second limit, where $a=1$, it becomes, $\int_{a'}^1 \rho \,.\, d \,.\, (a^5 h) = h - a'^5 h'$; adding this to the other
[1801a] part $\int_0^{a'} \rho \,.\, d \,.\, (a^5 h)$, we get the whole value of $\int_0^1 \rho \,.\, d \,.\, (a^5 h)$, as in [1801]; which is a general formula, corresponding to any value of h whatever. We may, in the same manner, find the value of $\int_0^1 \rho \,.\, d \,.\, a^3$, which is used in the next note; but it is easier to derive it from [1801], by supposing the arbitrary value of h to be represented by $h=a^{-2}$, whence $h'=a'^{-2}$, $a^5 h = a^3$; and as the value of h, in the first term of the second member of [1801], corresponds to the surface, where $a=1$, that term will become $h=1$, and
[1801b] the whole expression [1801] will give $\int_0^1 \rho \,.\, d \,.\, a^3 = 1 - a'^3 + \int_0^{a'} \rho \,.\, d \,.\, a^3$.

† (1291) Substituting in [1794] the value of $\int_0^1 \rho \,.\, d\,(a^5 h)$, [1801], also that of $\int_0^1 \rho \,.\, d \,.\, a^3$ [1801b], we shall get,

$$0 = 6 \,.\, \{h - a'^5 h' + \int_0^{a'} \rho \,.\, d \,.\, (a^5 h)\} + 5 \,.\, (\varphi - 2h) \,.\, \{1 - a'^3 + \int_0^{a'} \rho \,.\, d \,.\, a^3\},$$

or by reduction,

[1802a] $$\begin{aligned} &h \,.\, (4 - 10\, a'^3 + 10 \,.\, \int_0^{a'} \rho \,.\, d \,.\, a^3) \\ &= 5\,\varphi \,.\, (1 - a'^3 + \int_0^{a'} \rho \,.\, d \,.\, a^3) - 6\, h' \,.\, a'^5 + 6 \,.\, \int_0^{a'} \rho \,.\, d \,.\, (a^5 h) \,. \end{aligned}$$

Dividing this by the coefficient of h, we obtain h, and then αh, as in [1802].

Ellipticity.

$$\alpha h = \frac{5\,\alpha\varphi\,.\,\{1 - a'^3 + \int_0^{a'} \rho\,.\,d\,.\,a^3\} - 6\,\alpha h'\,.\,a'^5 + 6\,\alpha\,.\int_0^{a'} \rho\,.\,d\,.\,(a^5 h)}{4 - 10\,a'^3 + 10\,.\int_0^{a'} \rho\,.\,d\,.\,a^3}\,; \qquad [1802]$$

the integrals being taken from $a = 0$ to $a = a'$. [1802']

We shall now consider the law of gravity; or what is the same thing, the length of a pendulum, at the surface of the spheroid, supposing it to be elliptical and in equilibrium. The value of l, found in the preceding article [1802''] [1772], becomes in this case,*

$$l = L + \alpha L\,.\,\{\tfrac{5}{2}\varphi - h\}\,.\,(\mu^2 - \tfrac{1}{3}). \qquad [1803]$$

Putting therefore $L' = L - \frac{1}{3}\alpha L\,.\,(\frac{5}{2}\varphi - h)$, we shall have, by neglecting [1803'] quantities of the order α^2,

Length of a pendulum;

$$l = L' + \alpha L'\,.\,(\tfrac{5}{2}\varphi - h)\,.\,\mu^2. \qquad [1804]$$

From this equation it follows, that L' is the length of a pendulum, vibrating [1804'] in a second, at the equator, and that *this length increases, from the equator to the poles, in proportion of the square of the sine of the latitude* [1648'''].

it increases as the square of the sine of the latitude.

If we put $\alpha\varepsilon$ *for the excess of the length of the pendulum at the pole, above* [1804''] *its length at the equator, divided by this last length, we shall have,*†

$$\alpha\varepsilon = \alpha\,.\,(\tfrac{5}{2}\varphi - h)\,; \qquad [1805]$$

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* (1292) Substituting the values [1792] in [1772], and reducing, using L' [1803'], and neglecting α^2, we get successively,

$$\begin{aligned} l &= L - \alpha h\,.\,L\,.\,(\mu^2 - \tfrac{1}{3}) + \tfrac{5}{2}\alpha\varphi\,.\,L\,.\,(\mu^2 - \tfrac{1}{3}) = L + \alpha L\,.\,(\tfrac{5}{2}\varphi - h)\,.\,(\mu^2 - \tfrac{1}{3}) \\ &= L - \tfrac{1}{3}\alpha L\,.\,(\tfrac{5}{2}\varphi - h) + \alpha L\,.\,(\tfrac{5}{2}\varphi - h)\,.\,\mu^2 = L' + \alpha L\,.\,(\tfrac{5}{2}\varphi - h)\,.\,\mu^2 \\ &= L' + \alpha L'\,.\,(\tfrac{5}{2}\varphi - h)\,.\,\mu^2, \qquad \text{as in [1803, 1804].} \end{aligned} \qquad [1803a]$$

At the equator, where $\mu = 0$, the value of [1804] becomes L'; therefore the general value l [1804] exceeds that at the equator L', by the quantity $\alpha L'\,.\,(\frac{5}{2}\varphi - h)\,.\,\mu^2$, which [1803b] is proportional to the square of the sine of the latitude [1648'''].

† (1293) The excess of the general length of the pendulum l, above that at the equator, found in [1803b], becomes at the pole, where $\mu = 1$, $\alpha L'\,.\,(\frac{5}{2}\varphi - h)$. Dividing this by the length of the pendulum at the equator L', we get the quantity $\alpha\varepsilon$ [1804'']; hence $\alpha\varepsilon = \alpha\,.\,(\frac{5}{2}\varphi - h)$, as in [1805]. Transposing αh, we obtain Clairaut's beautiful theorem [1806a] [1806], in which terms of the order α^2 are neglected, and this is sufficiently accurate in the present state of practical astronomy. Mr. Airy has shown how to obtain a similar formula, in which such terms are noticed, in the Transactions of the Royal Society of London for 1826.

Clairaut's remarkable equation between the ellipticity of the earth αh, and the variation in the lengths of a pendulum on its surface $\alpha \varepsilon$.

consequently,

[1806] $$\alpha \varepsilon + \alpha h = \tfrac{5}{2} \alpha \varphi,$$

which is a remarkable equation between the ellipticity of the earth and the variation of the length of the pendulum, from the equator to the poles. In the

[1806′] case of homogeneity, $\alpha h = \frac{5}{4} \alpha \varphi$;* therefore in this case we have,

[1807] $$\alpha \varepsilon = \alpha h;$$

but if *the spheroid be heterogeneous, the two quantities* $\alpha \varepsilon$, αh, *will vary from* $\frac{5}{4} \alpha \varphi$, *so that the one of them will exceed* $\frac{5}{4} \alpha \varphi$, *as much as the other*

[1808] *falls short of the same quantity.*

35. The planets being supposed to be covered by a fluid in equilibrium, it becomes necessary, in the calculation of their attractions, to compute the attraction of a spheroid, whose surface is a fluid in equilibrium. This may

[1808′] be done, by a very simple process, in the following manner. Resuming the equation [1505], we may eliminate the signs of integration, by means of the equation [1705], which gives at the surface of the spheroid,†

[1809] $$\frac{4\pi}{2i+1} \cdot \int_0^1 \rho \cdot d \cdot (a^{i+3} \cdot Y^{(i)}) = \tfrac{4}{3} \pi \cdot Y^{(i)} \cdot \int_0^1 \rho \cdot d \cdot a^3 - Z^{(i)}.$$

If we fix the origin of r at the centre of gravity of the spheroid, it will make

[1809′] $Y^{(1)} = 0$ [1745]; moreover $Z^{(1)} = 0$ [1632]; and $Y^{(0)}$ being arbitrary,

[1810] [1704‴], we may suppose it *to vanish from* V.‡ The equation [1505] will then become,

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* (1294) In the case of homogeneity, we have by [1800*b*], $h = \frac{5}{4} \varphi$, or $\alpha h = \frac{5}{4} \alpha \varphi$; substituting this in [1805], we get $\alpha \varepsilon = \alpha \cdot (\frac{5}{2} \varphi - \frac{5}{4} \varphi) = \frac{5}{4} \alpha \varphi$, or $\alpha \varepsilon = \alpha h$, as in

[1807*a*] [1807]. If we suppose $\alpha h = \frac{5}{4} \alpha \varphi + g'$, the value of $\alpha \varepsilon$, deduced from [1805], will be $\alpha \varepsilon = \frac{5}{4} \alpha \varphi - g'$, as in [1808].

† (1295) The equation [1809] may be deduced from [1705], observing, as in [1788*a*], that the first integral vanishes at the surface; and that $a = 1$, in all the terms without the

[1809*a*] sign of integration. This equation takes place for all integral values of i, which are equal to, or greater than, unity, as is observed in [1704‴].

‡ (1296) In the term depending on $Y^{(i)}$, [1505], the sign $\int$ affects a, which is also supposed to be contained implicitly in ρ, $Y^{(i)}$; but r, and $(2i+1)$ may be brought from

[1809*b*] under that sign so that this term becomes $\frac{4\alpha\pi}{(2i+1) \cdot r^{i+1}} \cdot \int \rho \cdot d \cdot (a^{i+3} \cdot Y^{(i)})$, which is

Value of V for a spheroid covered by a fluid in equilibrium.

$$V=\frac{4\pi}{3r}\cdot\int_0^1\rho\cdot d\cdot a^3+\frac{4\alpha\pi}{3r^3}\cdot\left\{Y^{(2)}+\frac{Y^{(3)}}{r}+\frac{Y^{(4)}}{r^2}+\&c.\right\}\cdot\int_0^1\rho\cdot d\cdot a^3$$
$$-\frac{\alpha}{r^3}\cdot\left\{Z^{(2)}+\frac{Z^{(3)}}{r}+\frac{Z^{(4)}}{r^2}+\&c.\right\}. \quad [1811]$$

We may observe, that in this formula, $\frac{4}{3}\pi\cdot\int_0^1\rho\cdot d\cdot a^3$ denotes the mass of the spheroid; since when r is infinite, the value of V is equal to the mass [1811′] M of the spheroid, divided by r.* This being supposed, the attraction of the spheroid, parallel to r, will be $-\left(\frac{dV}{dr}\right)$; the attraction perpendicular [1811″]

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the same as the first member of [1809], multiplied by $\alpha\cdot r^{-i-1}$; and the second member of [1809], multiplied in the same manner, becomes $\frac{4\alpha\pi}{3r^{i+1}}\cdot Y^{(i)}\cdot\int_0^1\rho\cdot d\cdot a^3-\frac{\alpha\cdot Z^{(i)}}{r^{i+1}}$. Therefore this represents the value of the term of [1505], depending on $Y^{(i)}$, i being any integral positive number, excluding $i=0$ [1809a]. If in this expression, we put $i=1$, [1809c] $i=2$, &c., and substitute the corresponding terms in [1505], it will become,

$$V=\frac{4\pi}{3r}\cdot\int_0^1\rho.d.a^3+\frac{4\alpha\pi}{r}\cdot\int_0^1\rho.d.(a^3.Y^{(0)})+\left\{\frac{4\alpha\pi}{3r^2}\cdot Y^{(1)}\cdot\int_0^1\rho.d.a^3-\frac{\alpha.Z^{(1)}}{r^2}\right\}$$
$$+\left\{\frac{4\alpha\pi}{3r^3}\cdot Y^{(2)}\cdot\int_0^1\rho.d.a^3-\frac{\alpha.Z^{(2)}}{r^3}\right\}+\&c. \quad [1810a]$$

Substituting in this $Y^{(1)}=0$, $Z^{(1)}=0$, [1809′], and supposing $Y^{(0)}$ to be nothing, or that $\alpha\cdot\int_0^1\rho\cdot d\cdot(a^3\cdot Y^{(0)})$ is included as the arbitrary constant quantity of the integral $\int_0^1\rho\cdot d\cdot a^3$, we shall get,

$$V=\frac{4\pi}{3r}\cdot\int_0^1\rho.d.a^3+\left\{\frac{4\alpha\pi}{3r^3}\cdot Y^{(2)}\cdot\int_0^1\rho.d.a^3-\frac{\alpha.Z^{(2)}}{r^3}\right\}$$
$$+\left\{\frac{4\alpha\pi}{3r^4}\cdot Y^{(3)}\cdot\int_0^1\rho.d.a^3-\frac{\alpha.Z^{(3)}}{r^4}\right\}+\&c.; \quad [1810b]$$

and by a different arrangement of the terms, it becomes as in [1811]. We may observe, that the words "*it to vanish from V*," were inserted in [1810], instead of the equation $\frac{4}{3}\pi\cdot Y^{(0)}-Z^{(0)}=0$, given in the original.

* (1297) V [1385‴] represents the sum of all the particles of the spheroid, divided by [1810c] their distances from the attracted point. If we suppose the attracted point to be at a very great distance r from the centre of the spheroid, and any particle dM of the spheroid to be at the distance $r-r'$ from the attracted point, $\frac{r'}{r}$ being very small, we shall have, $\int\frac{dM}{r-r'}=\int\frac{dM}{r}\cdot\left(1+\frac{r'}{r}+\frac{r'^2}{r^2}+\&c.\right)=\frac{1}{r}\cdot\int dM+\frac{1}{r^2}\cdot\int r'\cdot dM+\&c.$; which will [1810$d$] become $V=\frac{M}{r}$, if r' be infinitely small in comparison with r.

[1811'''] to this radius, in the plane of the meridian, will be $-\frac{\sqrt{1-\mu^2}}{r}\cdot\left(\frac{dV}{d\mu}\right)$;

Expressions of the attractions.

lastly, the attraction perpendicular to the same radius, in the direction of

[1811''''] the parallel of latitude, will be, $-\frac{\left(\frac{dV}{d\varpi}\right)}{r\cdot\sqrt{1-\mu^2}}$.* The expression of V,

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* (1298) It follows from [455'], that if the three rectangular co-ordinates of the attracted point be represented by r, μ', ϖ', the attraction of the spheroid, resolved in directions parallel

[1811a] to these co-ordinates, and tending towards the origin, will be represented by

[1811b] $$-\left(\frac{dV}{dr}\right); \qquad -\left(\frac{dV}{d\mu'}\right); \qquad -\left(\frac{dV}{d\varpi'}\right);$$

which may, for brevity, be represented by R', R'', R''', respectively, as in [1447t].

In the annexed figure, which is similar to that in Vol. I, page 181, C is the centre of the spheroid; A the attracted point; $PEQAP$ a spherical surface described about the centre C, with the radius

[1811c] $CA=CP=CE=CQ=Cq=r$; CP is the polar semi-axis; EQq an arc of the equator; $PaAQ$, $PbBQ$, two meridians, infinitely near to each other;

[1811d] AB, ab, parallels of latitude, infinitely near to each other; PE the meridian from which the angle ϖ is counted, so

[1811e] that $ECQ=\varpi$, $PCA=\theta$. Then, as in [275a, &c.], we shall find the arc $Qq=r.d\varpi$, arc $AB=r.d\varpi.\sin.\theta$

[1811f] $=r.d\varpi.\sqrt{(1-\mu^2)}$, arc $Aa=-r.d\theta=\frac{r.d\mu}{\sqrt{(1-\mu^2)}}$;

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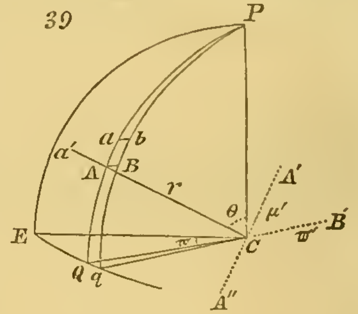

[1811g] observing that $\cos.\theta=\mu$, and $d\theta=\frac{-d\mu}{\sqrt{(1-\mu^2)}}$ [50] Int. If we now draw through

[1811h] the centre C, the lines CA', CB', parallel to the arcs Aa, AB, respectively; we may suppose the rectangular axes of the co-ordinates r, μ', ϖ', to be represented by CA, CA', CB', respectively; and we shall have

$$Aa'=dr; \qquad \text{arc } Aa=d\mu'=\frac{r.d\mu}{\sqrt{(1-\mu^2)}}; \qquad \text{arc } AB=d\varpi'=r.d\varpi.\sqrt{(1-\mu^2)}.$$

[1811i] Moreover, it is evident, from what has been said in [28a, b], that if we substitute these values of dr, $d\mu'$, $d\varpi'$, in [1811b], we shall obtain the attraction of the spheroid, resolved in directions parallel to these lines, which will therefore be,

Expressions of the attraction.

[1811k] $$R'=-\left(\frac{dV}{dr}\right); \qquad R''=-\frac{\sqrt{(1-\mu^2)}}{r}\cdot\left(\frac{dV}{d\mu}\right); \qquad R'''=-\frac{\left(\frac{dV}{d\varpi}\right)}{r.\sqrt{(1-\mu^2)}}.$$

relative to the earth, supposed to be elliptical,* becomes,

$$V=\frac{M}{r}+\frac{(\frac{1}{2}\,\alpha\,\varphi-\alpha\,h)}{r^3}\,.\,M\,.\,(\mu^2-\tfrac{1}{3})\;;$$ [1812]

M being the mass of the earth. [1812′]

36. Although the law of attraction in the inverse ratio of the square of the distance, is the only one which particularly interests us; yet the [1812″] equation [1456] affords so simple an expression of the force of gravity at the surface of a homogeneous spheroid, in equilibrium, whatever be the exponent† of the power of the distance to which the attraction is proportional, [1812‴] that we have thought proper here to give it. The attraction being as the power n of the distance; if we denote by dm the mass of a particle of

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If we put $\mu=\cos.\theta$, we shall have,

$$\left(\frac{dV}{d\theta}\right)=\left(\frac{dV}{d\mu}\right).\left(\frac{d\mu}{d\theta}\right)=-\left(\frac{dV}{d\mu}\right).\sin.\theta=-\left(\frac{dV}{d\mu}\right).\sqrt{(1-\mu^2)},$$

and $R''=\frac{1}{r}.\left(\frac{dV}{d\theta}\right)$; or, as it may be written, $-\frac{1}{r}.\left(\frac{dV}{d\theta}\right)$, by taking the axis CA'' on the continuation of the line AC', instead of CA'; so that the direction of the force R'' may correspond to the positive values of $d\theta$. Then the preceding forces will become,

$$R'=-\left(\frac{dV}{dr}\right);\qquad R''=-\frac{1}{r}.\left(\frac{dV}{d\theta}\right);\qquad R'''=-\frac{1}{r.\sin.\theta}.\left(\frac{dV}{d\varpi}\right).$$ [1811*l*]

If either of these forces be positive, it will tend to decrease the corresponding ordinate; and the contrary if the sign be negative, [1811*a*]. [1811*m*]

* (1299) It is also supposed, in formula [1812], that the ellipsoid is of revolution; in which case we shall have, as in [1792*b*, *d*], $Y^{(3)}=0$, $Y^{(4)}=0$, &c.; and $Y^{(2)}=-h.(\mu^2-\frac{1}{3})$, [1792]. Substituting these, and the values of $Z^{(1)}$, $Z^{(2)}$, &c., [1793], in [1811], we get, by using $\frac{4}{3}\pi.\int_0^1\rho.d.a^3=M$ [1811′], [1812*a*]

$$V=\frac{4\pi}{3r}.\int_0^1\rho.d.a^3-\frac{4\alpha\pi.h}{3r^3}.(\mu^2-\tfrac{1}{3}).\int_0^1\rho.d.a^3+\frac{\alpha\varphi}{2r^3}.(\mu^2-\tfrac{1}{3}).\tfrac{4}{3}\pi.\int_0^1\rho.d.a^3$$

$$=\frac{M}{r}-\frac{\alpha h}{r^3}.(\mu^2-\tfrac{1}{3}).M+\frac{\alpha\varphi}{2r^3}.(\mu^2-\tfrac{1}{3}).M=\frac{M}{r}+\frac{(\frac{1}{2}\alpha\varphi-\alpha h)}{r^3}.M.(\mu^2-\tfrac{1}{3}),$$ [1812*b*]

as in [1812].

† (1299*a*) Excluding however, as in [1455′], all negative values of the exponent n exceeding 2. [1812*c*]

the spheroid, and by f its distance from the attracted point; the action of dm upon that point, multiplied by the element of its direction $-df$, will be $-dm\,.\,f^n df$. The integral of this quantity, taken relative to f,

[1812''''] is* $-\frac{dm\,.\,f^{n+1}}{n+1}$; and the sum of these integrals, extended to the whole

[1813] spheroid, is $-\frac{V}{n+1}$, supposing, as in [1448], $V=\int f^{n+1}\,.\,dm$.

[1813'] If the spheroid be a homogeneous fluid, endowed with a rotatory motion, and not attracted by any foreign body; we shall have, at its surface, in the case of equilibrium, by § 23,†

[1814] $$\text{constant} = -\frac{V}{n+1} + \tfrac{1}{2}g\,.\,r^2\,.\,(1-\mu^2);$$

[1814'] r being the radius, drawn from the centre of gravity of the spheroid to its surface, and g [1616^{xvi}, &c.] the centrifugal force, at the distance 1 from the axis of rotation.

The gravity p, at the surface of the spheroid, is equal to the differential of

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* (1300) This computation is made in a similar manner to that in [1616^{x}, &c.], and the

[1813a] integral of the expression $-\frac{dm\,.\,f^{n+1}}{n+1}$ is $-\frac{1}{n+1}\,.\int dm\,.\,f^{n+1}$; but by [1448], changing dM into dm, we have $\int dm\,.\,f^{n+1}=V$, and the preceding expression becomes $-\frac{V}{n+1}$, as in [1813].

† (1301) The integral of the product of the centrifugal force, multiplied by the element of its direction, is $\tfrac{1}{2}g\,.\,r^2\,.\,(1-\mu^2)$ [1616^{xviii}, 1618]; to this add the similar integral

[1814a] $-\frac{V}{n+1}$ [1813], arising from the attraction of the particles; the sum

$$-\frac{V}{n+1} + \tfrac{1}{2}g\,.\,r^2\,.\,(1-\mu^2),$$

represents the expression $\int(F\,.\,df + F'\,.\,df' + \&c.)$, treated of [1616^{ix}, &c.]; and

[1814b] this, by [1615], is equal to $\int\frac{d\Pi}{\rho}$. Hence $\int\frac{d\Pi}{\rho} = -\frac{V}{n+1} + \tfrac{1}{2}g\,.\,r^2\,.\,(1-\mu^2)$,

which is similar to the equation [1635]; and since $\int\frac{d\Pi}{\rho}$ is constant at the surface of the fluid [1635b'], this equation will become as in [1814].

the second member of this equation, taken relative to r, and divided by [1814'']
$-dr$;* hence we get,

$$p=\frac{1}{n+1}\cdot\left(\frac{dV}{dr}\right)-g\cdot r\cdot(1-\mu^2). \qquad [1815]$$

We shall now resume the equation [1456], which corresponds to the surface of the spheroid, [supposing n to be any positive number from 0 to [1815']
∞, or negative from 0 to -2, inclusive],†

$$\left(\frac{dV}{dr}\right)=A'-\frac{(n+1).A}{2a}+\frac{(n+1).V}{2a}. \qquad [1816]$$

This equation, combined with the preceding, gives, [for the same values [1816']
of n [1815']],‡

$$p=\text{constant}+\left\{\frac{(n+1).r}{4a}-1\right\}.g.r.(1-\mu^2). \qquad [1817]$$

At the surface of the spheroid, r is nearly equal to a; putting therefore

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* (1302) This method of finding p is precisely like that in [1763''], where it is proved that the differential of $\int\frac{d\Pi}{\rho}$, taken relative to r, and divided by $-dr$, gives nearly the value of p; and it is shown, in [1814b], that $\int\frac{d\Pi}{\rho}$ is equal to the second member [1815a]
of [1814]; therefore the differential, taken in this manner, and divided by $-dr$, gives p, as in [1815].

† (1303) The limits of n [1815'] are the same as in [1455']; they are not in the original [1816a]
work. Similar restrictions are introduced in [1816', 1817', 1818'], so as to include all the formulas [1816—1819].

‡ (1304) Substituting $\left(\frac{dV}{dr}\right)$ [1816] in [1815], we get,

$$p=\frac{A'}{n+1}-\frac{A}{2a}+\frac{V}{2a}-g.r.(1-\mu^2).$$

Multiplying [1814] by $\frac{n+1}{2a}$, we get, $\frac{V}{2a}=\text{constant}+\frac{(n+1).g.r^2.(1-\mu^2)}{4a}$;
substituting this in the preceding value of p, and including the terms $\frac{A'}{n+1}-\frac{A}{2a}$, in [1817a]
the arbitrary constant quantity of $\frac{V}{2a}$, it becomes as in [1817]. If we put $r=a=1$, in the factor $\left(\frac{(n+1).r}{4a}-1\right).r$, it becomes $\frac{n+1}{4}-1=\frac{n-3}{4}$; hence [1817] becomes as in [1818].

[1817′] $a=1$, in order to simplify the calculation, we shall find, [n being as above],

[1818] $$p=\text{constant}+\left(\frac{n-3}{4}\right).g.(1-\mu^2).$$

[1818′] If P be the gravity at the equator of the spheroid, and $\alpha\varphi$ the ratio of the centrifugal force to the gravity at the equator [1647^ix]; we shall have, [n being as before],*

Expression of gravity, varying as the power n.

[1819] $$p=P.\left\{1+\left(\frac{3-n}{4}\right).\alpha\varphi.\mu^2\right\}.$$

[1819′] *Hence it follows, that from the equator to the poles, gravity varies in proportion to the square of the sine of the latitude* [1648‴]. In the case of nature, where $n=-2$, we shall have,

[1820] $$p=P.\{1+\tfrac{5}{4}\alpha\varphi.\mu^2\};$$

which agrees with what we have found before.† But it is very remarkable,

[1820′] that if $n=3$, we shall have $p=P$; that is, *if the attraction be proportional to the cube of the distance, the gravity at the surface of a homogeneous spheroid will be everywhere the same, whatever be the rotatory motion.*‡

Remarkable result, when the attraction is as the cube of the distance.

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* (1305) The value of p [1818] becomes, at the equator, where $\mu=0$,

$$P=\text{constant}+\frac{(n-3)}{4}.g;$$

[1818a] and if we substitute, in [1818], P instead of these terms, we shall get,

$$p=P-\frac{(n-3)}{4}.g.\mu^2;$$

but by hypothesis [1818′], $\alpha\varphi=\frac{g}{P}$, or $g=\alpha\varphi.P$; hence

[1818b] $$p=P-\frac{(n-3)}{4}.\alpha\varphi.P.\mu^2,$$

as in [1819]; which becomes as in [1820], when $n=-2$.

[1818c] † (1306) Putting, in [1648′], $P=\frac{4}{3}\pi\alpha.\{1-\frac{2}{3}\alpha\varphi-\frac{4}{4}\alpha\varphi.\frac{1}{3}\}$, it becomes $p=P+\frac{4}{3}\pi\alpha.\frac{5}{4}\alpha\varphi.\mu^2=P+P.\frac{5}{4}\alpha\varphi.\mu^2$, neglecting α^2, as in [1820].

[1820a] ‡ (1307) It may be proper to remark, that terms of the order α^2 are neglected in this computation.

37. In the investigation of the figures of the heavenly bodies, we have only noticed quantities of the order α; but it is easy, by the preceding analysis, to extend the approximations to quantities of the order α^2, or to higher orders.* For this purpose, we shall consider the figure of a homogeneous fluid mass, in equilibrium, and covering a spheroid which differs but little from a sphere; the whole having a rotatory motion, as is the case with the earth and planets. The condition of the equilibrium at the surface gives the equation [1635],† [1820''] Manner of noticing terms of the order α^2, α^3, &c.; [1820'''] in a homogeneous spheroid.

$$\text{constant} = V - \tfrac{1}{2} g \cdot r^2 \cdot (\mu^2 - \tfrac{1}{3}) + [\tfrac{1}{3} g \cdot r^2]. \qquad [1821]$$

The value of V is composed, *first*, of the attraction of the solid spheroid, covered by the fluid, upon *a particle of the surface, whose co-ordinates are* r, θ, ϖ; *second*, of the attraction of the fluid mass upon the same particle. [1821'] Now the sum of these two attractions is the same as the sum of the following attractions: *first*, that of the solid spheroid, supposing the density of each stratum to be diminished by the density of the fluid; *second*, that of a spheroid, having the same external surface as the fluid, and the same density as the fluid. [1821''] [1821'''] We shall suppose V' to be the first of these attractions, and V'' the second, so that $V = V' + V''$. Then we shall have, by supposing g to be of the order α, and equal to $\alpha g'$, [1821'''']

$$\text{constant} = V' + V'' - \tfrac{1}{2} \alpha g' \cdot r^2 \cdot (\mu^2 - \tfrac{1}{3}) + [\tfrac{1}{3} \alpha g' \cdot r^2]. \qquad [1822]$$

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* (1307*a*) We have already given, in [1560*a*, &c.], Poisson's method of noticing the higher powers of α, in the computation of V and its differentials.

† (1308) The external forces S, S', &c., being by hypothesis nothing, $Z^{(3)}$, $Z^{(4)}$, &c. [1632] must vanish; $\alpha . Z^{(2)}$ will become $= -\tfrac{1}{2} g . (\mu^2 - \tfrac{1}{3})$, and $\alpha . Z^{(0)} = \tfrac{1}{3} g$; substituting these in [1635], we get $\int \frac{d\Pi}{\rho} = V + \tfrac{1}{3} g \cdot r^2 - \tfrac{1}{2} g \cdot r^2 . (\mu^2 - \tfrac{1}{3})$. But we have, at the surface of the fluid, $\int \frac{d\Pi}{\rho} = \text{constant}$, [1635*b*']; hence [1821*a*]

$$\text{constant} = V + \tfrac{1}{3} g \cdot r^2 - \tfrac{1}{2} g \cdot r^2 . (\mu^2 - \tfrac{1}{3})\,;$$

which is the same as in [1821], except in the term $+\tfrac{1}{3} g \cdot r^2$, neglected by the author. This produces a similar term, which I have inserted between the brackets, at the end of each of the equations [1821, 1822, 1828, 1834, 1836], in addition to those given by the author, in the original work. Putting $V = V' + V''$ [1821''''] in [1821], it becomes as in [1822]. [1821*b*]

We have seen, in [1436], that V' may be developed in a series of the form,

[1823] $$V'=\frac{U^{(0)}}{r}+\frac{U^{(1)}}{r^2}+\frac{U^{(2)}}{r^3}+\frac{U^{(3)}}{r^4}+\&c.,$$

$U^{(i)}$ being subjected to the equation of partial differentials [1437],

[1824] $$0=\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dU^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddU^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i.(i+1).U^{(i)};$$

and we may determine $U^{(i)}$, by the analysis of § 17 [1533, 1541, &c.], with the requisite degree of accuracy, when the figure of the spheroid is known.

Also V'' may be developed in a series of the form,

[1825] $$V''=\frac{U_{,}^{(0)}}{r}+\frac{U_{,}^{(1)}}{r^2}+\frac{U_{,}^{(2)}}{r^3}+\&c.;$$

$U_{,}^{(i)}$ being subjected to the same equation of partial differentials as $U^{(i)}$,
[1825′] [1824]. If we take the density of the fluid for the unity of density, we shall have [1541],*

[1826] $$U_{,}^{(i)}=\frac{4\pi}{(i+3).(2i+1)}.Z^{(i)};$$

r^{i+3} being supposed to be developed in the series [1533],

[1827] $$r^{i+3}=Z^{(0)}+Z^{(1)}+Z^{(2)}+\&c.;$$

in which $Z^{(i)}$ is subjected to the same equation of partial differentials as $U^{(i)}$ [1824]. Therefore the equation of equilibrium [1822], will become,†

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* (1309) Putting $\rho=1$, and $U_{,}^{(i)}$ for $U^{(i)}$ in [1541], we get,

$$U_{,}^{(i)}=\frac{4\pi}{(i+3).(2i+1)}.\int\left(\frac{dZ^{(i)}}{da}\right).da=\frac{4\pi}{(i+3).(2i+1)}.Z^{(i)},$$

as in [1824]. Now from [1533] we have $R^{i+3}=Z'^{(0)}+Z'^{(1)}+Z'^{(2)}+\&c.$; and if we change μ', ϖ', $Z'^{(i)}$, into μ, ϖ, $Z^{(i)}$, respectively, as in [1540'''''], it will become,
[1825a] $r^{i+3}=Z^{(0)}+Z^{(1)}+Z^{(2)}+\&c.$; r being the radius of the attracted point of the surface [1821′]. $Z^{(i)}$ satisfies an equation similar to [1534].

[1827a] † (1310) If we suppose the sign Σ of finite integrals, to include all values of i except $i=0$, the expressions of V', V'', [1823, 1825], will become, by using [1826],

[1828a] $$V'=\frac{U^{(0)}}{r}+\Sigma.\frac{U^{(i)}}{r^{i+1}};\qquad V''=\frac{U_{,}^{(0)}}{r}+\Sigma.\frac{U_{,}^{(i)}}{r^{i+1}}=\frac{U_{,}^{(0)}}{r}+\Sigma.\frac{4\pi}{(i+3).(2i+1).r^{i+1}}.Z^{(i)}.$$

$$\text{constant} = \frac{U^{(0)}}{r} + \frac{U_{,}^{(0)}}{r} + \Sigma \cdot \frac{1}{r^{i+1}} \cdot \left\{ U^{(i)} + \frac{4\pi}{(i+3)\cdot(2i+1)} \cdot Z^{(i)} \right\}$$ [1828]

$$-\tfrac{1}{2}\alpha g' \cdot r^2 \cdot (\mu^2 - \tfrac{1}{3}) + [\tfrac{1}{3}\alpha g' \cdot r^2];$$

i being equal to unity, or greater than unity. [1828′]

If the distance r, of the attracted particle from the centre of the spheroid, be infinite, V will be equal to the sum of the masses of the spheroid and [1828″] fluid, divided by r; and if we put this sum equal to m, we shall have, $U^{(0)} + U_{,}^{(0)} = m$, [1828$b$]. We shall carry on the approximation so [1828‴] as to include quantities of the order α^2, supposing

$$r = 1 + \alpha y + \alpha^2 y';$$ [1829]

which gives,*

$$r^{i+3} = 1 + (i+3)\cdot\alpha y + \frac{(i+2)\cdot(i+3)}{1\cdot 2}\cdot\alpha^2 y^2 + (i+3)\cdot\alpha^2 y'.$$ [1830]

We shall suppose,

$$y = Y^{(1)} + Y^{(2)} + Y^{(3)} + \&c.;$$

$$y' = Y'^{(1)} + Y'^{(2)} + Y'^{(3)} + \&c.;$$ [1831]

$$y^2 = M^{(0)} + M^{(1)} + M^{(2)} + M^{(3)} + \&c.;$$

Substituting these in [1822], we get [1828]. When r is infinite, the values of V', V'', [1823, 1825], are reduced to their first terms $\frac{U^{(0)}}{r}$, $\frac{U_{,}}{r}$, and then

$$V = V' + V'' = \frac{U^{(0)} + U_{,}^{(0)}}{r};$$

but, by [1385‴], or [1810d], V then becomes $\frac{M}{r}$, or, in the present notation, $\frac{m}{r}$; hence $\frac{U^{(0)} + U_{,}^{(0)}}{r} = \frac{m}{r}$, and $U^{(0)} + U_{,}^{(0)} = m$, as in [1828‴]. [1828$b$]

* (1311) The value of r [1829], raised to the power $i+3$, by the binomial theorem, becomes as in [1830], neglecting α^3. Substituting in [1830] the values of r^{i+3} [1827], y, y', y^2, [1831], we shall get,

$$Z^{(0)} + Z^{(1)} + Z^{(2)} + \&c. = 1 + (i+3)\cdot\alpha\cdot\{Y^{(1)} + Y^{(2)} + \&c.\}$$ [1832a]

$$+ \frac{(i+2)\cdot(i+3)}{1\cdot 2}\cdot\alpha^2\cdot\{M^{(0)} + M^{(1)} + \&c.\} + (i+3)\cdot\{Y'^{(1)} + Y'^{(2)} + \&c.\};$$

and as there can be but one way of developing a function in the forms $Z^{(i)}$, $Y^{(i)}$, [1479′], the terms of the form $Z^{(i)}$, $Y^{(i)}$, must be equal to each other in [1832a]; hence we obtain the equation [1832], i being any integral positive number, *excluding* $i=0$.

$Y^{(i)}$, $Y'^{(i)}$, and $M^{(i)}$, being subjected to the same equation of partial differentials as $U^{(i)}$ [1824], and we shall have,

[1832] $$Z^{(i)} = (i+3) \,.\, \alpha \,.\, Y^{(i)} + \frac{(i+2) \,.\, (i+3)}{1 \,.\, 2} \,.\, \alpha^2 \,.\, M^{(i)} + (i+3) \,.\, \alpha^2 \,.\, Y'^{(i)}.$$

[1832′] We shall then observe, that $U^{(i)}$ is a quantity of the order α, since it would be nothing, if the spheroid were a sphere.* Hence, if we carry on the approximation to quantities of the order α^2, $U^{(i)}$ will be of the form,

[1833] $$U^{(i)} = \alpha \,.\, U'^{(i)} + \alpha^2 \,.\, U''^{(i)}.$$

Therefore, if we substitute these values in the preceding equation of equilibrium [1828], and change r into $1 + \alpha y + \alpha^2 y'$ [1829], we shall get, by neglecting quantities of the order α^3,†

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* (1312) Adding the equations [1823, 1825], and using $V' + V'' = V$ [1821′′′′],

[1832b] $U^{(0)} + U_{,}^{(0)} = m$ [1828′′′], we get $V = \frac{m}{r} + \frac{U^{(1)}}{r^2} + \frac{U^{(2)}}{r^3} + \&c. + \frac{U_{,}^{(1)}}{r^2} + \frac{U_{,}^{(2)}}{r^3} + \&c.$

If the spheroid be a sphere, whose mass is m, we shall have [1506b] $V = \frac{m}{r}$; which is the first term of the preceding value; therefore the remaining terms $\frac{U^{(1)}}{r}$, $\frac{U^{(2)}}{r^2}$, &c., must be of the same order as the excess of the spheroid above the sphere; that is, of the order α, as in [1832′].

† (1313) Putting in [1828] $U^{(0)} + U_{,}^{(0)} = m$ [1828′′′], and substituting the value of $Z^{(i)}$ [1832], and $U^{(i)}$ [1833], we get,

[1834a] $$\text{constant} = \frac{m}{r} + \Sigma \,.\, \frac{1}{r^{i+1}} \,.\, \left\{ \begin{array}{l} \alpha \,.\, U'^{(i)} + \alpha^2 \,.\, U''^{(i)} + \frac{4\alpha\pi}{2i+1} \,.\, Y^{(i)} \\ + \frac{4\alpha^2\pi \,.\, (i+2)}{2 \,.\, (2i+1)} \,.\, M^{(i)} + \frac{4\alpha^2\pi}{2i+1} \,.\, Y'^{(i)} \end{array} \right\}$$
$$- \tfrac{1}{2}\alpha g' \,.\, r^2 \,.\, (\mu^2 - \tfrac{1}{3}) + \tfrac{1}{3}\alpha g' \,.\, r^2.$$

Now from [1829] we obtain, by neglecting α^3,

[1834b] $$\frac{m}{r} = \frac{m}{1 + \alpha y + \alpha^2 y'} = m \,.\, (1 - \alpha y + \alpha^2 y^2 - \alpha^2 y');$$

therefore the first term of the expression [1834a] produces the terms multiplied by m in [1834]. The same value of r gives $\frac{1}{r^{i+1}} = 1 - (i+1) \,.\, \alpha y - \&c.$; hence

$$\frac{\alpha \,.\, U'^{(i)}}{r^{i+1}} = \alpha \,.\, U'^{(i)} - (i+1) \,.\, \alpha^2 y \,.\, U'^{(i)};$$

[1834c] $$\frac{4\alpha\pi \,.\, Y^{(i)}}{(2i+1) \,.\, r^{i+1}} = \frac{4\alpha\pi \,.\, Y^{(i)}}{2i+1} - \frac{4\alpha^2\pi \,.\, (i+1) \,.\, y \,.\, Y^{(i)}}{2i+1}.$$

$$\text{constant} = m\,.\{1 - \alpha y + \alpha^2 y^2 - \alpha^2 y'\}$$

$$+\Sigma\,.\left\{\begin{array}{l} \alpha\,.\,U'^{(i)} + \alpha^2\,.\,U''^{(i)} - (i+1)\,.\,\alpha^2 y\,.\,U'^{(i)} \\ + \dfrac{4\,\alpha\,\pi}{2i+1}\,.\,Y^{(i)} - \dfrac{4\,\alpha^2\pi\,.\,(i+1)}{2\,i+1}\,.\,y\,.\,Y^{(i)} + \dfrac{4\,\alpha^2\pi}{2i+1}\,.\,Y'^{(i)} \\ + \dfrac{4\,\alpha^2\pi\,.\,(i+2)}{2\,.\,(2\,i+1)}\,.\,M^{(i)} \end{array}\right\} \qquad [1834]$$

$$- \frac{\alpha g'\,.\,(1+2\,\alpha y)}{2}\,.\,(\mu^2 - \tfrac{1}{3}) + [\tfrac{1}{3}\,\alpha g'\,.\,(1+2\,\alpha y)].$$

Putting the terms of the order α, and the terms of the order α^2, separately equal to nothing, we shall obtain the two following equations,*

$$\Sigma\,.\left(m - \frac{4\,\pi}{2i+1}\right).Y^{(i)} = \Sigma\,.\,U'^{(i)} - \frac{g'}{2}\,.\,(\mu^2 - \tfrac{1}{3})\,; \qquad [1835]$$

$$\Sigma\,.\left(m - \frac{4\,\pi}{2i+1}\right).\,Y'^{(i)} = C' + \Sigma\,.\left\{\begin{array}{l} U''^{(i)} - (i+1)\,.\,y\,.\,U'^{(i)} - \dfrac{4\pi\,.\,(i+1)}{2i+1}\,.\,y\,.\,Y^{(i)} \\ \qquad + \left\{m + \dfrac{4\pi\,.\,(i+2)}{2\,.\,(2\,i+1)}\right\}.\,M^{(i)} \end{array}\right\} \qquad [1836]$$

$$- g'\,y\,.\,(\mu^2 - \tfrac{1}{3}) + [\tfrac{2}{3}\,g'\,y]\,;$$

The other terms under the sign Σ, being of the order α^2, we may in them put $r=1$, and by substituting these values, the terms under that sign become as in [1834]. Lastly, from [1829] we get $r^2 = 1 + 2\alpha y$, to be substituted in the two terms multiplied by g' [1834a], [1834d] to obtain the corresponding terms in [1834].

* (1314) The equation [1834] must be satisfied for all values of μ, ϖ, α; the coefficients of the different powers of α must therefore be put equal to nothing. Now the constant term of the second member independent of α, is m; that of the order α, independent of $(\mu^2 - \frac{1}{3})$, is equal to $\frac{1}{3}\,\alpha g'$; and if we put that of the order α^2 equal to $-\alpha^2\,.\,C'$, the sum will be $m + \frac{1}{3}\,\alpha g' - \alpha^2\,.\,C'$, which is to be put equal to the constant term of the [1836a] first member. Then the terms independent of α are $m = m$. Those of the order α give,

$$\tfrac{1}{3}\,\alpha g' = - m\,.\,\alpha y + \alpha\,.\,\Sigma\,.\left\{U'^{(i)} + \frac{4\,\pi}{2i+1}\,.\,Y^{(i)}\right\} - \frac{\alpha g'}{2}\,.\,(\mu^2 - \tfrac{1}{3}) + \tfrac{1}{3}\,\alpha g'. \qquad [1836b]$$

Dividing by α, and substituting the value of y [1831], we get,

$$0 = - m\,.\,\Sigma\,.\,Y^{(i)} + \Sigma\,.\left\{U'^{(i)} + \frac{4\,\pi}{2i+1}\,.\,Y^{(i)}\right\} - \frac{g'}{2}\,.\,(\mu^2 - \tfrac{1}{3})\,;$$

transposing the first and third terms, it becomes as in [1835]. The terms of [1834, 1836a],

C' being an arbitrary constant quantity. The first of these equations
[1836′] determines $Y^{(i)}$, consequently also the value of y.* Substituting these values in the second member of the second equation, we may develop it by the method of [1530v], in a series of the form,

[1837] $$N^{(0)}+N^{(1)}+N^{(2)}+\&c.\,;$$

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depending on α^2, being divided by α^2, give

[1836c] $$-C'=m\,.\,(y^2-y')+\Sigma\,.\left\{\begin{array}{l} U''^{(i)}-(i+1)\,.\,y\,.\,U'^{(i)}-\dfrac{4\pi.(i+1)}{(2i+1)}\,.\,y\,.\,Y^{(i)} \\ \qquad+\dfrac{4\pi}{2i+1}\,.\,Y'^{(i)}+\dfrac{4\pi.(i+2)}{2.(2i+1)}\,.\,M^{(i)} \end{array}\right\}$$
$$-g'y\,.\,(\mu^2-\tfrac{1}{3})+\tfrac{2}{3}g'y.$$

Using the values [1831], the term $m\,.\,(y^2-y')$ becomes $\Sigma\,.\,(m\,.\,M^{(i)}-m\,.\,Y'^{(i)})$; substituting this in the preceding equation, and transposing the terms depending on $Y'^{(i)}$ to the first member, we get [1836].

* (1315) The form of the internal solid spheroid being known, its attraction, or rather the value of V' [1823], will be given; consequently the general value of $U^{(i)}$ [1823] will be given, and then $U'^{(i)}$, $U''^{(i)}$, [1833]; also g' will be found by means of the known rotatory motion. Therefore the second member of [1835] will be given, and as it can be developed in only one manner, in functions of the form $Y^{(i)}$, $U^{(i)}$, [1479′], it follows that the functions of the first and second members of [1835], must be separately equal to each other, for any value of i; therefore when $i=2$, we shall have,

[1839a] $$(m-\tfrac{4}{5}\pi)\,.\,Y^{(2)}=U'^{(2)}-\tfrac{1}{2}g'\,.\,(\mu^2-\tfrac{1}{3})\,;$$

[1839b] and for all other values of i exceeding $i=0$ [1828′], we have $\left(m-\dfrac{4\pi}{2i+1}\right).Y^{(i)}=U'^{(i)}$. Hence we may find $Y^{(1)}$, $Y^{(2)}$, $Y^{(3)}$, &c.; and by substitution, in the first equation [1831], we shall get the value of y, in terms of the known quantities $U'^{(1)}$, $U'^{(2)}$, &c., and g'. These values of y, $U'^{(i)}$, $U''^{(i)}$, being substituted in the second member of [1836], it becomes known, and may be developed, in a given function of the form [1837], or

[1839c] $\Sigma\,.\,N^{(i)}$; so that [1836] will become $\Sigma\,.\left(m-\dfrac{4\pi}{2i+1}\right).\,Y'^{(i)}=\Sigma\,.\,N^{(i)}$; and as this second member can be developed in only one manner, in the proposed form [1479′], this equation must exist separately for each value of i; therefore $\left(m-\dfrac{4\pi}{2i+1}\right).\,Y'^{(i)}=N^{(i)}$; hence we get $Y'^{(i)}$ [1838]; and by putting $i=1$, $i=2$, &c., we obtain,

[1839d] $Y'^{(1)}=\dfrac{N^{(1)}}{m-\frac{4}{3}\pi}$; $Y'^{(2)}=\dfrac{N^{(2)}}{m-\frac{4}{5}\pi}$; &c.; whose sum is equal to y' [1831], as in [1839]; hence, by substituting the values of y and y', thus found, in [1829], we obtain r.

$N^{(i)}$ being subjected to the same equation of partial differentials as $U^{(i)}$, [1837′]
[1824]; and we may determine the constant quantity C', so that $N^{(0)}$ may be nothing, and we shall have,

$$Y'^{(i)} = \frac{N^{(i)}}{m - \frac{4\pi}{2i+1}}; \qquad [1838]$$

consequently,

$$y' = \frac{N^{(1)}}{m - \frac{4}{3}\pi} + \frac{N^{(2)}}{m - \frac{4}{5}\pi} + \frac{N^{(3)}}{m - \frac{4}{7}\pi} + \&c. \qquad [1839]$$

Hence the expression of the radius r of the surface of the fluid will be determined, neglecting quantities of the order α^3; and we may, in the same manner, carry on the approximation as far as we wish. We shall say no [1839′]
more upon this subject, since there is no other difficulty than that arising from the length of the calculations; but we may deduce from the preceding analysis this important conclusion, namely, that we can affirm that the equilibrium is rigorously possible, though we cannot state the precise figure which will satisfy it. For we can always find a series of figures, which, by [1839″]
substitution in the equation of equilibrium, will leave remainders that become successively smaller, and will finally be less than any assignable quantity.

CHAPTER V.

COMPARISON OF THE PRECEDING THEORY WITH OBSERVATIONS.

38. To compare the preceding theory with observations, we must know the figure of the terrestrial meridian, and the curve described by a series of geodetical operations. *We shall suppose a plane to be drawn, through the axis of rotation of the earth, and the zenith of any place upon its surface. This plane, continued infinitely, will describe in the heavens the circumference*
[1839‴] *of a great circle, which will be the meridian of that place. All the points of*
[1839⁗] *the surface of the earth, which have their zenith in this circumference, are*
Celestial and terrestrial meridians. *under the same celestial meridian. These points form, upon the surface of the earth, a curve, which is the corresponding terrestrial meridian.*

Equation of the earth's surface. To determine this curve, we shall represent the equation of the surface of the earth by*

[1840] $$u=0;$$

[1840′] u being a function of three rectangular co-ordinates x, y, z. If x', y', z', *be the three rectangular co-ordinates of the vertical line, which passes through*
[1840″] *the point of the surface of the earth, determined by the co-ordinates* x, y, z, we shall have, by the theory of curve surfaces, the following equations,†

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* (1316) The nature of the equation of a surface $u=0$ is explained in [19b''', c, d, e], for a plane and a sphere. The equation of an ellipsoid [1363], after transposing all the terms to the second member, and dividing by k^2, becomes of the form $u=0$; as in the
Equation of an ellipsoid. following expression, which will be used hereafter,

[1840a] $$u=1-\frac{x^2}{k^2}-\frac{m\,y^2}{k^2}-\frac{n\,z^2}{k^2}=0.$$

† (1317) The situation of a right line, drawn in any direction whatever, may be
[1840b] determined, when we know the places of the two lines, formed by projecting it upon any

$$0=\left(\frac{du}{dx}\right).dy'-\left(\frac{du}{dy}\right).dx';$$
Equations of the vertical to the surface.
$$0=\left(\frac{du}{dx}\right).dz'-\left(\frac{du}{dz}\right).dx'.$$ [1841]

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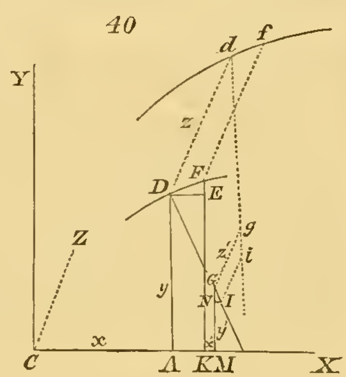
two of the planes in which the axes of the co-ordinates are situated; as for example, those of xy and yz. For if we erect, upon these two lines, planes perpendicular to the planes of xy, yz, they will intersect each other in the proposed line. We shall suppose that d, f, are two points of the surface of the earth, infinitely near to each other, and having the co-ordinate z equal in both points; so that for the point d, the co-ordinates are



$$CA=x, \qquad AD=y, \qquad Dd=z;$$

and for the point f,

$$CK=x+dx, \qquad KF=y+dy, \qquad Ff=z;$$ [1840c]

DF being the projection of df upon the plane of xy. Then drawing DE parallel and equal to AK, we shall have $DE=dx$, $EF=dy$. If dgi be the vertical line drawn through the point d, perpendicular to the surface of the earth, and DGI its projection upon the plane of xy, we shall have, for the co-ordinates of any point g of the line dgi, $CM=x'$, $MG=y'$, $Gg=z'$, [1840′]; and the co-ordinates of the point i, infinitely near to g, will be $x'+dx'$, $y'-dy'$, $z'+dz'$; the negative sign being given to dy', because in the present figure, dy' decreases as dx' increases. Hence if I be the point of projection of i, and IN be let fall, perpendicularly upon GM, we shall have, $GN=-dy'$, $NI=dx'$. Now it is evident from the construction, that the equal and parallel lines df, DF, are perpendicular to the plane $IDdi$; therefore the line DGI, in that plane, must be perpendicular to DF; and as GN, NI, are respectively perpendicular to DE, FE, the triangles DEF, GNI, must be similar. Hence $DE:EF::GN:NI$; or in symbols, $dx:dy::-dy':dx'$, or $dx=\frac{-dy.dy'}{dx'}$. At the point d, the equation of the [1840d] surface of the earth is $u=0$, corresponding to the co-ordinates x, y, z. At the point f, the co-ordinates x, y, become $x+dx$, $y+dy$, respectively, z remaining unaltered; and then the equation $u=0$ becomes, by development as in [610], and retaining only the first power of dx, dy, $u+\left(\frac{du}{dx}\right).dx+\left(\frac{du}{dy}\right).dy=0$; in which u and its differentials correspond to the point d, where $dx=0$, $dy=0$. The difference of these two equations is $\left(\frac{du}{dx}\right).dx+\left(\frac{du}{dy}\right).dy=0$; and by substituting the preceding value of dx, it becomes $-\left(\frac{du}{dx}\right).\frac{dy.dy'}{dx'}+\left(\frac{du}{dy}\right).dy=0$. Multiplying this by $-\frac{dx'}{dy}$, we [1840e]

Multiplying the first of these equations by the indeterminate quantity λ, and adding the product to the second equation, we shall obtain,

Differential equation of a plane parallel to the vertical line.

[1842]
$$dz' = \left\{ \frac{\left(\frac{du}{dz}\right) + \lambda . \left(\frac{du}{dy}\right)}{\left(\frac{du}{dx}\right)} \right\} . dx' - \lambda . dy'.$$

[1842'] This equation is that of any plane parallel to the vertical line just mentioned.*

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get the first of the equations [1841], depending on the projection of the vertical line dg upon the plane of xy. It is evident, that the projection of the same line, upon the plane of xz, must produce a similar equation, which may be deduced from the preceding, by changing y into z, and y' into z', as in the second of the equations [1841]. If for any point d of the surface, the co-ordinates x, y, z, be *given*, also the equation of the surface $u=0$, we
[1840f] shall have the values of $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$, $\left(\frac{du}{dz}\right)$. Substituting these in [1841], we shall get *two* differential equations, containing dx', dy', dz', and constant quantities, which represent the differential equation of the vertical line dgi.

* (1318) The equation of a plane, passing through the origin of the co-ordinates x', y', z', which, for distinction we shall call the *first plane*, is $z' = Ax' + By'$, [19c]. If a
[1840g] *second plane* be drawn parallel to the first plane, and at the distance C from it, measured in the direction of the co-ordinate z', the preceding value of z' will be increased by C, and the
[1841a] equation of this *second* plane will be $z' = Ax' + By' + C$. Now the differentials of both
[1842a] these expressions are identical, namely $dz' = A.dx' + B.dy'$. This may therefore be considered as the general differential equation of *parallel planes; the values of* A, B, *being the same in all these parallel planes.* Again, if any line be drawn, at pleasure, in the

Differential equation of parallel planes.

[1842a'] *first* plane, the *second* plane will be parallel to that line; therefore the equation

[1842b]
$$dz' = A.dx' + B.dy'$$

Plane drawn parellel to any line, in another plane.

may also be considered as the equation of any *second* plane, drawn parallel to *any line*, described in the *first* plane. We shall now proceed to apply these principles, supposing the *first* plane to be drawn through the line dgi fig. 40, page 359, and the *second* plane to be drawn parallel to this *first* plane.

It was proved, in the preceding note, that the two equations [1841] comprise the conditions necessary for the line dgi to be a *vertical* line. These *two* equations may be included in one *more general* expression, by multiplying the first by a constant but indeterminate quantity λ, and adding the product to the second; the sum is

[1842c]
$$\lambda . \left(\frac{du}{dx}\right) . dy' + \left(\frac{du}{dx}\right) . dz' - \left\{ \lambda . \left(\frac{du}{dy}\right) + \left(\frac{du}{dz}\right) \right\} . dx' = 0.$$

The vertical line, when infinitely produced, coincides with the celestial meridian, whilst its base is distant from the plane of the meridian, by a finite quantity; it may therefore be considered as parallel to this plane.* The differential equation of the plane of the celestial meridian may therefore [1842″] coincide with the preceding equation, by assigning a proper value to the quantity λ. We shall now suppose

$$dz' = a \,.\, dx' + b \,.\, dy', \qquad [1843]$$

to be the equation of the plane of the celestial meridian.† Comparing it with the equation [1842], we shall get,

Equation of the celestial meridian.

$$\left(\frac{du}{dz}\right) - a \,.\, \left(\frac{du}{dx}\right) - b \,.\, \left(\frac{du}{dy}\right) = 0. \qquad (a) \qquad [1844]$$

This expression is evidently equivalent to the two equations [1841]; since the equation [1842c] cannot exist, for all values of the indeterminate quantity λ, unless the equations [1841] are separately satisfied. If we divide [1842c] by the factor of dz', and transpose the first and last terms, we shall obtain [1842]; and by putting

$$\left(\frac{du}{dz}\right) + \lambda \,.\, \left(\frac{du}{dy}\right) = A \,.\, \left(\frac{du}{dx}\right), \qquad \text{and} \qquad -\lambda = B, \qquad [1842c']$$

it will become $dz' = A \,.\, dx' + B \,.\, dy'$. This is the same as the differential equation of *parallel planes* [1842a]; and as it includes both the equations [1841], or the differential equations of the vertical line dgi, this *vertical* line must be situated in one of these *parallel* planes; therefore it follows, from [1842b], that the equation $dz' = A.dx' + B.dy'$, [1842d] or [1842], is the differential equation of *any* plane, drawn *parallel* to that vertical line.

* (1319) It appears from the definition [1839⁗], that if the vertical line igd fig. 40, page 359, be continued infinitely to the heavens, it will coincide with the celestial meridian, [1842e] though the point of that *line*, upon the surface of the earth, might be distant from that meridian by a finite quantity, considered as the base of a triangle. Now this base being finite and the sides infinite, the angle included by these sides must be infinitely small, or nothing; so that the two sides may be considered as parallel.

† (1320) The general equation of any plane is $z' = Ax' + By' + C$ [1841a]. Its differential is $dz' = A \,.\, dx' + B \,.\, dy'$; and by putting $A = a$, $B = b$, it becomes of the same form as [1843]. Comparing the equations [1842, 1843], we find that the coefficients of dx', dy', become equal, by putting

$$a = \frac{\left(\frac{du}{dz}\right) + \lambda \,.\, \left(\frac{du}{dy}\right)}{\left(\frac{du}{dx}\right)}, \qquad b = -\lambda. \qquad [1844a]$$

To obtain the constant quantities a and b, we shall suppose that we have, as given quantities, the co-ordinates of the base of the vertical line, drawn
[1844'] parallel to the axis of rotation of the earth, and those of another vertical line, drawn at any other given place upon its surface. Substituting these co-ordinates successively in the preceding equation, we shall obtain two
[1844''] equations, by means of which we may determine a and b. *The preceding*
Terrestrial meridian. *equation* [1844], *combined with that of the surface* $u=0$, *will give the curve of the terrestrial meridian, passing through the given place.**

[1844'''] If the earth be an ellipsoid, u will be a rational and integral function of
The terrestrial meridian is generally a curve of double curvature. the second degree in x, y, z; the equation [1844] will then be that of a plane,† whose intersection with the surface of the earth, will form the terrestrial meridian. In general, *if the surface be of a more complicated form,*
[1845] *this meridian will be a curve of double curvature.*

In this case, the line, determined by geodetical measures, is not that of the terrestrial meridian. To describe this, we must form the first horizontal
[1845'] triangle, so that one of the angles may have its summit at the origin of the

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Substituting this value of $\lambda=-b$, in the expression of a; then multiplying by $\left(\frac{du}{dx}\right)$, we get $a\cdot\left(\frac{du}{dx}\right)=\left(\frac{du}{dz}\right)-b\cdot\left(\frac{du}{dy}\right)$, as in [1844].

* (1321) From [1844], we get the value of z, in terms of x, y. Substituting this in the
[1844b] given equation of the surface [1840], it becomes $0=\text{function of }(x,y)$. Hence, for any proposed value of x, we may find the corresponding value of y; and these, being substituted in z, will give z in terms of x; so that the curve of the meridian may be determined, by points, for all values of x.

† (1322) The partial differentials of the equation of an ellipsoid [1840a], give

[1844c] $$\left(\frac{du}{dz}\right)=-\frac{2nz}{k^2},\qquad \left(\frac{du}{dx}\right)=-\frac{2x}{k^2},\qquad \left(\frac{du}{dy}\right)=-\frac{2my}{k^2}.$$

Substituting these in [1844], we get,

$$\frac{-2nz+2ax+2mby}{k^2}=0,\qquad\text{or}\qquad z=\frac{a}{n}\cdot x+\frac{mb}{n}\cdot y;$$

which is the same as the equation of a plane [19c]. If the equation $u=0$ contained powers of x, y, z, greater than the second power, the equation [1844], would be that of a surface of
[1844d] a higher order than the first degree; and the intersection of that surface with the surface of the earth, would in general form a curve of double curvature [25b].

curve; and the other two angles their summits upon any two visible objects. We must then find the direction of the first side of the curve, with respect [1845′] to the sides of the triangle, and its length, to the point where it meets the side which connects the two objects. We must then form a second horizontal triangle, with these objects, and a third object, situated at a [1845″] greater distance from the origin of the curve. This second triangle will not be in the same plane as the first; it will have nothing in common with it, except the side formed by the two first objects. Therefore the continuation of the first side of the curve, will be elevated above the plane of this second triangle; but we bend it down upon the plane in such a manner, *that it will always form the same angles with the side which is common to both* [1845‴] *triangles;* and it is evident, that to do this, we must bend it towards the plane, in the direction of the vertical. This therefore is the characteristic [1845ᵛ] property of the curve described by geodetical operations: *its first side, which* Curve described by geodetical operations. *may be supposed in any direction, is a tangent to the surface of the earth; its second side is the continuation of this tangent, bent downwards in the direction of the vertical; its third side is the continuation of the second side, bent* [1846] *downwards in the direction of the vertical;* and so on, for the other sides.*

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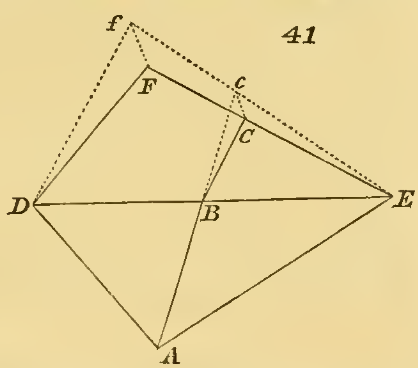
* (1323) For the sake of illustration, we shall observe, that in the same manner as a curve is considered a polygon of an infinite number of sides, any surface may be supposed to be composed of an infinite number of plane triangles. We shall therefore suppose DAE, DFE, fig. 41, to be two such contiguous triangles upon the surface of the earth, having the side DE in common; A being the commencement of the meridian line AB, described by geodetical operations; D, E, the objects placed at the summit of the first triangle DAE; F the object placed at the summit of the second triangle. These two triangles being in different planes; we may suppose the second DFE to revolve about the common side or base DE, till it comes into the situation DfE *in the plane of the first triangle* DAE, continued [1845a] to f; the point F having described the infinitely small arc Ff, *which may be considered as perpendicular to the surface or plane of the triangle* DFE. If the line BC be so situated [1845b] in the second triangle, as to cause it, by this revolution, to come into the situation Bc, on the continuation of the right line AB, this line BC will represent the continuation of the meridian on the plane of the second triangle. In like manner, the continuation of



If in the point where two of these sides meet, we draw, in the plane
[1846′] which is a tangent to the surface of the spheroid, a line perpendicular to one of these sides; it is evident that it will be perpendicular to the other side. Hence it follows, that the sum of these sides is the shortest line that can be described upon this surface between the extreme points.* Therefore *the lines described by geodetical operations, have the property of being the shortest*
[1847] *that can be described, upon the surface of the spheroid, between any two points*

Line described by geodetical operations is a minimum.

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the line BC is to be made to revolve about the side FE, till it meets the plane of the third triangle; and so on for the other sides, as in [1846].

In this trigonometrical process, we can determine the triangle ADE, by measuring one of its sides, as AE, and all its angles, DAE, DEA, ADE; we must also observe the angle BAE, which the meridian line makes with the side AE. Then, by the usual
[1845c] rules of trigonometry, we may compute the sides AB, BE, BD, AD, DE; also the angle $ABD = cBE = CBE$; and having obtained, from observation, the three angles of the triangle DEF, or DEf, we shall also have the three angles of the triangle EBc, or EBC; by means of which we may compute the second side BC of the meridian line. In this way the meridian may be continued through any number of successive triangles.

* (1324) Instead of restricting ourselves to the case here treated of by the author, we shall suppose, more generally, that $ADBE$, $CDBE$, fig. 42, are two plane surfaces, inclined to each other in any given angle, and intersecting each other in the right line DE. Then it is required to connect any two given points A, C, by two right lines AB, BC, drawn in these planes, so that the sum of these lines $AB + BC$, may be a minimum. From the points A, C, let fall
[1846a] the perpendiculars AA', CC', upon the line DE; putting $AA' = a$, $CC' = c$, $A'C' = b$, $A'B = x$, $BC' = b - x$; then $AB = \sqrt{(a^2 + x^2)}$, $BC = \sqrt{\{c^2 + (b - x)^2\}}$, whose sum $\sqrt{(a^2+x^2)} + \sqrt{\{c^2 + (b-x)^2\}}$ is to be a *minimum*. Its differential, relative to x, put equal to nothing, and divided by dx, gives

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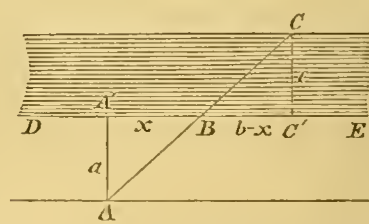

[1846b] $$\frac{x}{\sqrt{(a^2+x^2)}} = \frac{b - x}{\sqrt{\{c^2+(b-x)^2\}}}, \qquad \text{or} \qquad \frac{A'B}{AB} = \frac{BC'}{BC};$$

which is equivalent to $\cos. ABD = \cos. CBE$, or angle $ABD =$ angle CBE.
[1847a] Therefore, to render $AB + BC$ a minimum, we must make the angles ABD, CBE, equal to each other, which is the property of the arc of the meridian, or of any other arc, described upon the surface of the earth by geodetical operations, [1845⁗]. The forms and situations of the infinitely small planes, into which the surface of the spheroid is supposed to

of their surface;* and by what we have seen, in the first book [49″], they would be described by a body moving uniformly, upon that surface.

Let x, y, z, be the rectangular co-ordinates of any point of the curve; [1847′]
$x+dx$, $y+dy$, $z+dz$, will be the co-ordinates of a point infinitely near to it. If ds be the element of the curve, and we suppose this element to be continued by a quantity equal to ds, we shall have [1847″]

$$x+2dx, \qquad y+2dy, \qquad z+2dz,$$

for the co-ordinates of the extremity of this line;† and if we bend it down in the direction of the vertical, the co-ordinates of this extremity will become,

$$x+2dx+ddx, \qquad y+2dy+ddy, \qquad z+2dz+ddz; \qquad [1847'''']$$

therefore $-ddx$, $-ddy$, $-ddz$, will be the co-ordinates of the [1847⁗]

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be divided, being wholly arbitrary, we may suppose them to be so situated, that the line of [1847b]
intersection DBE may be perpendicular to AB, and in that case it must also be perpendicular to BC, because the angle $CBE=ABD$, [1847a]. This agrees with what is stated above, [1846′].

* (1324a) Hence it follows, that on a spherical surface, the geodetical curve is a great [1847c]
circle; because a great circle, connecting two points, is, as is well known, the shortest curve line that can be drawn between them, upon that surface.

† (1325) The co-ordinates of the point A, fig. 41, page 363, being x, y, z, those of the point B, infinitely near to A, will be $x+dx$, $y+dy$, $z+dz$; the increments in proceeding from A to B, being dx, dy, dz; and by construction, the point c is on the continuation of AB; so that if we make, as in [1847″], $Bc=AB$, the increments [1848a]
of the co-ordinates, in proceeding from B to c, will also be dx, dy, dz; making the co-ordinates of the point c equal to $x+2dx$, $y+2dy$, $z+2dz$. Again, the co-ordinates of the point B being $x+dx$, $y+dy$, $z+dz$; their differentials are $dx+ddx$, $dy+ddy$, $dz+ddz$, which represent the increments of those co-ordinates, in proceeding from the second point B to the third point C; and by adding them respectively to the co-ordinates of B, we have the co-ordinates of C,

$$x+2dx+ddx, \qquad y+2dy+ddy, \qquad z+2dz+ddz. \qquad [1848b]$$

as in [1847‴]. This point C is the base of the line cC, which is supposed to be perpendicular to the surface DFE [1845b]. Subtracting the co-ordinates of the point C, from the corresponding ones of the point c, at the summit of the vertical line cC, we get the co-ordinates of that point c, referred to the point C, as the origin; and these co-ordinates will be $-ddx$, $-ddy$, $-ddz$, as in [1847⁗]. [1848b′]

vertical line, counted from its base. Consequently we shall have, by the nature of the vertical line, supposing $u=0$ to be the equation of the surface of the earth,*

Equation of the geodetical curve, or line.

$$0=\left(\frac{du}{dx}\right).ddy-\left(\frac{du}{dy}\right).ddx;$$

$$0=\left(\frac{du}{dx}\right).ddz-\left(\frac{du}{dz}\right).ddx;$$

[1848] $$0=\left(\frac{du}{dy}\right).ddz-\left(\frac{du}{dz}\right).ddy;$$

which equations *are different from those of the terrestrial meridian.* In these
[1848'] equations, ds *ought to be supposed constant;* for it is evident, that if we

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* (1326) If we take the integrals of [1841], considering x', y', z', variable; x, y, z, and the terms depending on u, as constant, we shall get,

$$0=\left(\frac{du}{dx}\right).y'-\left(\frac{du}{dy}\right).x'+\text{constant}; \qquad 0=\left(\frac{du}{dx}\right).z'-\left(\frac{du}{dz}\right).x'+\text{constant}.$$

At the point d, fig. 40, page 359, we have $x'=x$, $y'=y$, $z'=z$; and then these equations become,

[1848c] $$0=\left(\frac{du}{dx}\right).y-\left(\frac{du}{dy}\right).x+\text{constant}; \qquad 0=\left(\frac{du}{dx}\right).z-\left(\frac{du}{dz}\right).x+\text{constant}.$$

Subtracting these respectively from the preceding, we get,

$$0=\left(\frac{du}{dx}\right).(y'-y)-\left(\frac{du}{dy}\right).(x'-x); \qquad 0=\left(\frac{du}{dx}\right).(z'-z)-\left(\frac{du}{dz}\right).(x'-x);$$

and it is evident, that if we take the point of the surface d for the origin of the co-ordinates, we shall have $x=0$, $y=0$, $z=0$; and the preceding equations will become,

[1848c'] $$0=\left(\frac{du}{dx}\right).y'-\left(\frac{du}{dy}\right).x'; \qquad 0=\left(\frac{du}{dx}\right).z'-\left(\frac{du}{dz}\right).x'.$$

[1848d] x', y', z', being counted from the point d of the surface of the earth, fig. 40, corresponding to the point C, fig. 41, page 363. Now we have shown, in [1848b], that the co-ordinates of the point c, of the vertical line Cc, counted from the point C, as the origin, are respectively $-ddx$, $-ddy$, $-ddz$. Substituting these for x', y', z', respectively, in the preceding equations [1848c'], we get,

[1848e] $$0=-\left(\frac{du}{dx}\right).ddy+\left(\frac{du}{dy}\right).ddx; \qquad 0=-\left(\frac{du}{dx}\right).ddz+\left(\frac{du}{dz}\right).ddx;$$

and by changing the signs, they become like the two first equations [1848]. Multiplying the

neglect infinitely small quantities of the third order,* we may suppose that the continuation of the element ds meets the base of the vertical in which it is deflected. [1848'']

We shall now enter into an examination of the results which can be obtained, relative to the figure of the earth, from the geodetical admeasurements, made in the direction of the meridian, or in a direction perpendicular to the meridian. We may always *suppose that there is an ellipsoid, touching each point of the surface of the earth, upon which the geodetical admeasurements, also the latitudes and the longitudes, counted from the point of contact, for a small extent, would be the same as upon that surface.* If the whole surface were that of an ellipsoid, the ellipsoidal tangent would be the same throughout the whole surface; but if the figures of the meridians be not elliptical, as there is reason to believe is the case; then the ellipsoidal tangent will vary in different countries, and it can only be determined by geodetical measures, made in different directions. It would be very interesting to ascertain, in this manner, the osculatory ellipsoids of a great number of places upon the surface of the earth. [1848'''] Osculatory ellipsoid. [1848''''] [1848^v]

Equation of the surface of the earth.

We shall put,†

$$u = x^2 + y^2 + z^2 - 1 - 2\alpha u' = 0, \qquad [1849]$$

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first of the equations [1848] by $-\left(\frac{du}{dz}\right)$, the second by $\left(\frac{du}{dy}\right)$, and dividing the sum by $\left(\frac{du}{dx}\right)$, we obtain the third equation of [1848], which was not given by the author, but is here introduced, because it furnishes the third equation of [1850]. The equations [1848] are evidently different from those of the terrestrial meridian, determined in [1844b, &c.], from the formulas [1840, 1844]. [1848f]

* (1327) Supposing, as in [1845b, 1848b], cC, fig. 41, page 363, to be perpendicular to BC, we shall have, $BC = \sqrt{(Bc^2 - Cc^2)} = Bc - \frac{Cc^2}{2Bc} - \&c.$ Therefore BC differs from Bc by a quantity of the order $\frac{Cc^2}{2Bc}$, which is of the *third* order; because Bc is of the first order ds, and Cc is of the second order. In the original work, this difference is stated to be of the *fourth* order. [1848g]

† (1328) If the earth were an ellipsoid, we should have, as in [1840a],

$$u = 1 - \frac{1}{k^2} \cdot x^2 - \frac{m}{k^2} \cdot y^2 - \frac{n}{k^2} \cdot z^2;$$

for the equation of the surface of the spheroid, which we shall suppose to differ
[1849'] *but very little from a sphere, whose radius is unity; so that* α *may be a very small coefficient, whose square we shall neglect.* u' may always be considered as a function of only two variable quantities x and y; for if we suppose it
[1849''] to be a function of x, y, z, we may eliminate z, by means of the equation*
[1849'''] $z=\sqrt{1-x^2-y^2}$. This being premised, the three equations [1848], corresponding to the shortest line described upon the surface of the earth, become,†

Equation of the geodetical line, neglecting terms of the order α^2. First form.

[1850]
$$\left.\begin{aligned} x\,d\,d\,y-y\,d\,d\,x&=\alpha\,.\left(\frac{d\,u'}{d\,x}\right).\,d\,d\,y-\alpha\,.\left(\frac{d\,u'}{d\,y}\right).\,d\,d\,x\,;\\ x\,d\,d\,z-z\,d\,d\,x&=\alpha\,.\left(\frac{d\,u'}{d\,x}\right).\,d\,d\,z\,;\\ y\,d\,d\,z-z\,d\,d\,y&=\alpha\,.\left(\frac{d\,u'}{d\,y}\right).\,d\,d\,z.\end{aligned}\right\}\qquad(O)$$

[1850'] We shall call this curve by the name of *the geodetical line.*

[1849a] the semi-axes of the ellipsoid [1363''], being k, $\frac{k}{\sqrt{m}}$, $\frac{k}{\sqrt{n}}$; and as these differ from each other only by quantities of the order α; m and n will differ from unity only by quantities of the same order α. If we now put

[1849a'] $$\frac{1}{k^2}=1+\alpha g,\qquad \frac{m}{k^2}=1+\alpha g',\qquad \frac{n}{k^2}=1+\alpha g'';$$

[1849b] the preceding value of u will be $u=1-x^2-y^2-z^2-\alpha\,.\,(g\,x^2+g'\,y^2+g''\,z^2)$;
[1849c] which, by putting $gx^2+g'y^2+g''z^2=-2u'$, becomes of the form [1849]. In a similar way, the equation of any other figure of the spheroid may be reduced to the form [1849].

* (1329) If we neglect α^2, we may reject terms of the order α, in u' [1849]; but if we neglect terms of the order α, we shall have $z=\sqrt{(1-x^2-y^2)}$ [1849]; and this value of z being substituted in u', it will become a function of x, y. In the case of an ellipsoid [1849c] we shall get $-2\,u'=g\,x^2+g'\,y^2+g''\,.\,(1-x^2-y^2)$, or
[1849d] $-2\,u'=g''+(g-g'')\cdot x^2+(g'-g'')\cdot y^2$.

† (1330) The partial differentials of u [1849], relative to x, y, z, are

[1850a] $$\left(\frac{du}{dx}\right)=2\,x-2\,\alpha\,.\left(\frac{d\,u'}{d\,x}\right);\qquad \left(\frac{d\,u}{dy}\right)=2\,y-2\,\alpha\,.\left(\frac{d\,u'}{d\,y}\right);\qquad \left(\frac{du}{dz}\right)=2\,z\,;$$

u' being considered as a function of x, y, only, [1849'']. Substituting these in [1848], dividing by 2, and transposing the terms depending on α, we get [1850].

We shall put r for the radius, drawn from the centre of the earth to its surface; θ the angle which this radius makes with the axis of rotation, supposing this to be the axis of z; φ the angle, formed by the intersection of the plane of xz, with the plane included by the radius r and the axis z; we shall then have,* [1850″] [1850‴]

$$x = r \,.\, \sin.\, \theta \,.\, \cos.\, \varphi\,; \qquad y = r \,.\, \sin.\, \theta \,.\, \sin.\, \varphi\,; \qquad z = r \,.\, \cos.\, \theta. \qquad [1851]$$

Hence we deduce,†

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* (1331) x, y, z, [1851], are found in the same manner as y, z, x, [460], putting $\varpi = \varphi$. The values [1851] give $x^2 + y^2 + z^2 = r^2$ [461]; hence [1849] becomes $r^2 - 1 - 2\,\alpha\, u' = 0$, or $r^2 = 1 + 2\,\alpha\, u'$; whose square root, neglecting α^2, is $r = 1 + \alpha\, u'$, which is used hereafter. [1850b] [1850c]

† (1332) The differentials of the equations [1851] are,

$$\begin{aligned} dx &= dr \,.\, \sin.\,\theta \,.\, \cos.\,\varphi + r\,d\theta \,.\, \cos.\,\theta \,.\, \cos.\,\varphi - r\,d\varphi \,.\, \sin.\,\theta \,.\, \sin.\,\varphi\,; \\ dy &= dr \,.\, \sin.\,\theta \,.\, \sin.\,\varphi + r\,d\theta \,.\, \cos.\,\theta \,.\, \sin.\,\varphi + r\,d\varphi \,.\, \sin.\,\theta \,.\, \cos.\,\varphi\,; \\ dz &= dr \,.\, \cos.\,\theta - r\,d\theta \,.\, \sin.\,\theta. \end{aligned} \qquad [1851a]$$

Multiplying the first of these equations by $-y = -r \,.\, \sin.\,\theta \,.\, \sin.\,\varphi$, and the second by $x = r.\sin.\theta.\cos.\varphi$; putting for brevity $A = \sin.^2\theta \,.\, \sin.\varphi \,.\, \cos.\varphi$, $B = \sin.\theta \,.\, \cos.\theta \,.\, \sin.\varphi \,.\, \cos.\varphi$; we get,

$$\begin{aligned} -y\,dx &= -A \,.\, r\,dr - B \,.\, r^2\,d\theta + r^2\,d\varphi \,.\, \sin.^2\theta \,.\, (\sin.^2\varphi)\,; \\ x\,dy &= A \,.\, r\,dr + B \,.\, r^2\,d\theta + r^2\,d\varphi \,.\, \sin.^2\theta \,.\, (\cos.^2\varphi)\,; \\ x\,dy - y\,dx &= r^2\,d\varphi \,.\, \sin.^2\theta \,.\, (\sin.^2\varphi + \cos.^2\varphi) = r^2\,d\varphi \,.\, \sin.^2\theta\,; \end{aligned} \qquad [1851a']$$

as in [1852]. The same result might also be easily obtained, from the differential of $\frac{y}{x} = \text{tang.}\,\varphi$, deduced from the values of x, y, [1851].

Multiplying the values [1851] by $\cos.\varphi$, or $\sin.\varphi$, we obtain the four following equations,

$$\begin{aligned} x \,.\, \cos.\,\varphi &= r \,.\, \sin.\,\theta \,.\, \cos.^2\varphi, & -z \,.\, \cos.\,\varphi &= -r \,.\, \cos.\,\theta \,.\, \cos.\,\varphi, \\ y \,.\, \sin.\,\varphi &= r \,.\, \sin.\,\theta \,.\, \sin.^2\varphi, & -z \,.\, \sin.\,\varphi &= -r \,.\, \cos.\,\theta \,.\, \sin.\,\varphi\,; \end{aligned} \qquad [1851b]$$

then multiplying these respectively by dz, dx, dz, dy, [1851a], we find,

$$\begin{aligned} x\,dz \,.\, \cos.\,\varphi &= r\,dr \,.\, \{\sin.\,\theta \,.\, \cos.\,\theta \,.\, \cos.^2\varphi\} - r^2\,d\theta \,.\, \{\sin.^2\theta \,.\, \cos.^2\varphi\}\,; \\ -z\,dx \,.\, \cos.\,\varphi &= -r\,dr \,.\, \{\sin.\,\theta \,.\, \cos.\,\theta \,.\, \cos.^2\varphi\} - r^2\,d\theta \,.\, \{\cos.^2\theta \,.\, \cos.^2\varphi\} + B \,.\, r^2 d\varphi\,; \\ y\,dz \,.\, \sin.\,\varphi &= r\,dr \,.\, \{\sin.\,\theta \,.\, \cos.\,\theta \,.\, \sin.^2\varphi\} - r^2\,d\theta \,.\, \{\sin.^2\theta \,.\, \sin.^2\varphi\}\,; \\ -z\,dy \,.\, \sin.\,\varphi &= -r\,dr \,.\, \{\sin.\,\theta \,.\, \cos.\,\theta \,.\, \sin.^2\varphi\} - r^2\,d\theta \,.\, \{\cos.^2\theta \,.\, \sin.^2\varphi\} - B \,.\, r^2\,d\varphi. \end{aligned} \qquad [1851c]$$

Geodetical line.

[1852] $$r^2 . \sin.^2 \theta . d\varphi = x\,dy - y\,dx\,;$$

Second form,

[1852′] $$-r^2 . d\theta = (x\,dz - z\,dx) . \cos.\varphi + (y\,dz - z\,dy) . \sin.\varphi\,;$$

[1852″] $$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 . d\theta^2 + r^2 . d\varphi^2 . \sin.^2 \theta.$$

neglecting α^2.

[1852‴] Considering u' as a function of x and y, and putting ψ for the latitude; we
[1853] may suppose, in this function, $r = 1$, and $\psi = 100° - \theta$, and we shall have,*

[1854] $$x = \cos.\psi . \cos.\varphi\,; \qquad y = \cos.\psi . \sin.\varphi\,;$$

hence we shall obtain,†

If we add these four equations, the first member of the sum will become like the second member of [1852′]. In the second member, the coefficients of $r\,dr$, as well as those of $r^2 d\varphi$, will mutually destroy each other; and the remaining terms will be, by reduction, as in the first member of [1852′],

[1851d] $$-r^2 d\theta . \{\cos.^2\varphi . (\sin.^2\theta + \cos.^2\theta) + \sin.^2\varphi . (\sin.^2\theta + \cos.^2\theta)\}$$
$$= -r^2 d\theta . \{\cos.^2\varphi + \sin.^2\varphi\} = -r^2 d\theta.$$

If we now, for a moment, put C, D, for the coefficients of $\sin.\varphi$, $\cos.\varphi$, in the value of dy [1581a]; that is $C = dr . \sin.\theta + r\,d\theta . \cos.\theta$, $D = r\,d\varphi . \sin.\theta$; we shall
[1851e] have $dx = C . \cos.\varphi - D . \sin.\varphi$, $dy = C . \sin.\varphi + D . \cos.\varphi$; hence ds^2 [44] becomes, by successive reductions,

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$= C^2 . (\cos.^2\varphi + \sin.^2\varphi) + 2CD . (-\sin.\varphi . \cos.\varphi + \sin.\varphi . \cos.\varphi) + D^2 . (\sin.^2\varphi + \cos.^2\varphi) + dz^2$$
[1851f] $$= C^2 + D^2 + dz^2 = (dr . \sin.\theta + r\,d\theta . \cos.\theta)^2 + (dr . \cos.\theta - r\,d\theta . \sin.\theta)^2 + D^2$$
$$= dr^2 . (\sin.^2\theta + \cos.^2\theta) + 2r\,dr . d\theta . (\sin.\theta . \sin.\theta - \sin.\theta . \cos.\theta) + r^2 d\theta^2 . (\cos.^2\theta + \sin.^2\theta) + D^2$$
$$= dr^2 + r^2 d\theta^2 + D^2, \qquad \text{as in [1852″]}.$$

It may be observed, that this result might be obtained very easily by a geometrical method, similar to that used in note 642.

* (1333) In fig. 28, page 217, the angle PGD is equal to the complement of the
[1853a] latitude $= 100° - \psi$, and $PCD = \theta$. The difference of these angles is GPC, which is of the order of the ellipticity α. Now as u' is multiplied by α, in [1849, 1850], we may neglect terms of the order α, in the values of x, y, contained in u'; therefore, from
[1853b] [1850c], we may put $r = 1$, also $\theta = 100° - \psi$, in x, y, [1851], and they will become as in [1854].

† (1334) Supposing u' to be a function of x, y, the complete differential will be as in the
[1855a] first member of [1855]; but if we suppose u' to be a function of ψ, φ, [1854], this

$$\left(\frac{d\,u'}{d\,x}\right).\,d\,x+\left(\frac{d\,u'}{d\,y}\right).\,d\,y=\left(\frac{d\,u'}{d\,\psi}\right).\,d\,\psi+\left(\frac{d\,u'}{d\,\varphi}\right).\,d\,\varphi. \qquad [1855]$$

But we have,

$$x^2+y^2=\cos.^2\psi\,; \qquad \frac{y}{x}=\text{tang.}\,\varphi\,; \qquad [1856]$$

hence we find,*

$$d\,\psi=-\frac{(x\,d\,x+y\,d\,y)}{\sin.\psi\,.\,\cos.\psi}\,; \qquad d\,\varphi=\frac{(x\,d\,y-y\,d\,x)}{x^2}\,.\,\cos.^2\varphi. \qquad [1857]$$

Substituting these values of $d\psi$, $d\varphi$, in the preceding differential of u', and comparing separately the coefficients of $d\,x$ and $d\,y$, we get,†

$$\left(\frac{d\,u'}{d\,x}\right)=-\frac{\cos.\varphi}{\sin.\psi}.\left(\frac{d\,u'}{d\,\psi}\right)-\frac{\sin.\varphi}{\cos.\psi}.\left(\frac{d\,u'}{d\,\varphi}\right)\,; \qquad [1858]$$

$$\left(\frac{d\,u'}{d\,y}\right)=-\frac{\sin.\varphi}{\sin.\psi}.\left(\frac{d\,u'}{d\,\psi}\right)+\frac{\cos.\varphi}{\cos.\psi}.\left(\frac{d\,u'}{d\,\varphi}\right). \qquad [1858']$$

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differential will be as in the second member of [1855]; and it is evident that these two expressions ought to be equal to each other.

* (1335) The sum of the squares of x, y, [1854], putting $\cos.^2\varphi+\sin.^2\varphi=1$, gives the first of the equations [1856]. Dividing the value of y [1854], by that of x, we get the second equation [1856]. The differentials of these two equations are

$$2\,x\,d\,x+2\,y\,d\,y=-2\cos.\psi\,.\,\sin.\psi\,.\,d\psi\,; \qquad \frac{x\,d\,y-y\,d\,x}{x^2}=\frac{d\,\varphi}{\cos.^2\varphi}. \qquad [1856a]$$

Dividing the first by $-2\cos.\psi\,.\,\sin.\psi$, and multiplying the second by $\cos.^2\varphi$, we get [1857].

† (1336) This substitution being made in [1855], it becomes,

$$\left(\frac{d\,u'}{d\,x}\right).\,d\,x+\left(\frac{d\,u'}{d\,y}\right).\,d\,y=\left(\frac{d\,u'}{d\,\psi}\right).\left\{\frac{-x\,d\,x-y\,d\,y}{\sin.\psi\,.\,\cos.\psi}\right\}+\left(\frac{d\,u'}{d\,\varphi}\right).\left\{\frac{x\,d\,y\,.\cos.^2\varphi-y\,d\,x\,.\cos.^2\varphi}{x^2}\right\}\,;$$

which must exist for all values of $d\,x$, $d\,y$. If we put $d\,y=0$, and divide by $d\,x$, we obtain $\left(\frac{d\,u'}{d\,x}\right)=-\left(\frac{d\,u'}{d\,\psi}\right).\frac{x}{\sin.\psi\,.\cos.\psi}-\left(\frac{d\,u'}{d\,\varphi}\right).\frac{y\,.\cos.^2\varphi}{x^2}$; substituting, in the second member, the values of x, y, [1854], we get [1858]. In like manner, by putting $dx=0$, and dividing by dy, we get $\left(\frac{d\,u'}{d\,y}\right)=-\left(\frac{d\,u'}{d\,\psi}\right).\frac{y}{\sin.\psi\,.\cos.\psi}+\left(\frac{d\,u'}{d\,\varphi}\right).\frac{x\,.\cos.^2\varphi}{x^2}$; [1856b] which, by a similar process, is easily reduced to the form [1858'].

From these we obtain,*

$$\left(\frac{d\,u'}{d\,x}\right).\,d\,d\,y-\left(\frac{d\,u'}{d\,y}\right).\,d\,d\,x$$

[1859] $$=-\frac{\left(\frac{d\,u'}{d\,\psi}\right)}{\sin.\psi\,.\,\cos.\psi}.\,(x\,d\,d\,y-y\,d\,d\,x)-\frac{\left(\frac{d\,u'}{d\,\varphi}\right)}{\cos.^2\psi}.\,(x\,d\,d\,x+y\,d\,d\,y).$$

[1860] Now by neglecting quantities of the order α, we have† $x d d y - y d d x = 0$; also the two equations,

[1861] $$x\,d\,d\,z-z\,d\,d\,x=0, \qquad y\,d\,d\,z-z\,d\,d\,y=0.$$

From these two we obtain,‡

[1862] $$z\,d\,d\,z=\frac{z^2.\,(x\,d\,d\,x+y\,d\,d\,y)}{x^2+y^2}.$$

[1863] The equation $x^2+y^2+z^2=1$, gives,§

[1864] $$x\,d\,d\,x+y\,d\,d\,y+z\,d\,d\,z+d\,s^2=0.$$

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* (1337) Multiplying the equation [1858] by $d d y$, and [1858′] by $-d d x$; then adding the products, we get,

$$\left(\frac{d\,u'}{d\,x}\right).\,d\,d\,y-\left(\frac{d\,u'}{d\,y}\right).\,d\,d\,x$$

[1859a] $$=-\left(\frac{d\,u'}{d\,\psi}\right).\left\{\frac{\cos.\varphi\,.\,d\,d\,y-\sin.\varphi\,.\,d\,d\,x}{\sin.\psi}\right\}-\left(\frac{d\,u'}{d\,\varphi}\right).\left\{\frac{\cos.\varphi\,.\,d\,d\,x+\sin.\varphi\,.\,d\,d\,y}{\cos.\psi}\right\}.$$

Substituting $\cos.\varphi=\frac{x}{\cos.\psi}$, $\sin.\varphi=\frac{y}{\cos.\psi}$, [1854], we obtain [1859].

† (1338) This equation, and the two equations [1861], are found by putting $\alpha=0$, in [1850].

‡ (1339) Multiplying the first equation [1861] by $x z$, the second by $y z$, and adding
[1862a] the products, we get $(x^2+y^2)\,.\,z\,d\,d\,z-z^2\,.\,(x\,d\,d\,x+y\,d\,d\,y)=0$; dividing this by x^2+y^2, we get [1862].

§ (1340) Neglecting α in [1849], we get $x^2+y^2+z^2=1$, whose second differential is $2\,x\,d\,d\,x+2\,y\,d\,d\,y+2\,z\,d\,d\,z+2\,.\,(d\,x^2+d\,y^2+d\,z^2)=0$; substituting $d\,s^2$, [1852″], and dividing by 2, we obtain [1864]. This, by using $z\,d\,d\,z$ [1862], becomes,

[1863a] $$x\,d\,d\,x+y\,d\,d\,y+\frac{z^2.\,(x\,d\,d\,x+y\,d\,d\,y)}{x^2+y^2}+d\,s^2=0.$$

Substituting the preceding value of $z\,d\,d\,z,$ we shall have,

$$x\,d\,d\,x+y\,d\,d\,y=-(x^2+y^2)\,.\,d\,s^2=-d\,s^2\,.\,\cos.^2\psi\,;$$ [1865]

consequently,*

$$\left(\frac{d\,u'}{d\,x}\right).\,d\,d\,y-\left(\frac{d\,u'}{d\,y}\right).\,d\,d\,x=\left(\frac{d\,u'}{d\,\varphi}\right).\,d\,s^2.$$ [1866]

Therefore the first of the equations [1850] will give, by integration,†

$$r^2\,d\,\varphi\,.\,\sin.^2\theta=c\,d\,s+\alpha\,d\,s\,.\int d\,s\,.\left(\frac{d\,u'}{d\,\varphi}\right);\qquad (p)$$ [1867]

c being an arbitrary constant quantity.

The second of the equations [1850] gives,‡

$$d\,.\,(x\,d\,z-z\,d\,x)=\alpha\,.\left(\frac{d\,u'}{d\,x}\right).\,d\,d\,z\,;$$ [1868]

Multiplying by x^2+y^2, we get $(x^2+y^2+z^2).(xddx+yddy)+(x^2+y^2).ds^2=0$; transposing the last term, and using [1863], we obtain

$$x\,d\,d\,x+y\,d\,d\,y=-(x^2+y^2)\,.\,d\,s^2=-\cos.^2\psi\,.\,d\,s^2,$$ [1856], as in [1865].

* (1341) Substituting [1860, 1865] in [1859], we get [1866].

† (1342) Multiplying [1866] by α, and substituting the product in the first of the equations [1850], we get $x\,d\,d\,y-y\,d\,d\,x=\alpha\,.\left(\frac{d\,u'}{d\varphi}\right).\,d\,s^2$; its integral is [1866a]

$$x\,d\,y-y\,d\,x=c\,d\,s+\alpha\,d\,s\,.\int\left(\frac{d\,u'}{d\varphi}\right).\,d\,s,$$

as is easily proved by taking the differential; observing that $d\,s$ is constant [1848′]. Hence, by using [1852], we obtain [1867].

‡ (1343) The equation [1868] is evidently the same as the second of [1850]. Now if we substitute in [1862], for $x\,d\,d\,x+y\,d\,d\,y,$ its value $-d\,s^2\,.\,\cos.^2\psi$ [1865], we get $z\,d\,d\,z=-\frac{z^2\,d\,s^2\,.\,\cos.^2\psi}{x^2+y^2}$. Dividing this by z, and using $x^2+y^2=\cos.^2\psi$ [1856], [1868a] we obtain $d\,d\,z=-z\,d\,s^2$. But from [1863, 1856], we have

$$z^2=1-(x^2+y^2)=1-\cos.^2\psi=\sin.^2\psi\,;$$

hence $z=\sin.\psi$, and the preceding value of $d\,d\,z$ becomes, $d\,d\,z=-d\,s^2\,.\,\sin.\psi$, [1868b] as in [1869]; substituting this in [1868], we get [1870].

but it is evident from what precedes, that we have

[1869] $$d\,d\,z = -\,d\,s^2 \,.\, \sin.\,\psi\,;$$

hence we get,

[1870] $$d\,.\,(x\,d\,z - z\,d\,x) = -\,\alpha\,d\,s^2 \,.\, \left(\frac{d\,u'}{d\,x}\right) . \sin.\,\psi.$$

In like manner,*

[1871] $$d\,.\,(y\,d\,z - z\,d\,y) = -\,\alpha\,d\,s^2 \,.\, \left(\frac{d\,u'}{d\,y}\right) . \sin.\,\psi\,;$$

therefore we shall have,†

[1872] $$\begin{aligned} r^2\,d\,\theta = {} & c'\,d\,s \,.\, \sin.\,\varphi + c''\,d\,s \,.\, \cos.\,\varphi \\ & -\alpha\,d\,s \,.\, \cos.\,\varphi \,.\, \textstyle\int d\,s \,.\, \left\{ \left(\frac{d\,u'}{d\,\psi}\right) . \cos.\,\varphi + \left(\frac{d\,u'}{d\,\varphi}\right) . \sin.\,\varphi \,.\, \mathrm{tang.}\,\psi \right\} \qquad (q) \\ & -\alpha\,d\,s \,.\, \sin.\,\varphi \,.\, \textstyle\int d\,s \,.\, \left\{ \left(\frac{d\,u'}{d\,\psi}\right) . \sin.\,\varphi - \left(\frac{d\,u'}{d\,\varphi}\right) . \cos.\,\varphi \,.\, \mathrm{tang.}\,\psi \right\}. \end{aligned}$$

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* (1344) The last of the equations [1850] is easily put under the form,

[1871a] $$d\,.\,(y\,d\,z - z\,d\,y) = \alpha \,.\, \left(\frac{d\,u'}{d\,y}\right) . \,d\,d\,z\,;$$

substituting $d\,d\,z = -\,d\,s^2 \,.\, \sin.\,\psi$ [1869], we get [1871].

† (1345) Taking the integrals of [1870, 1871], and adding the constant quantities $-\,c''\,d\,s$, $-\,c'\,d\,s$, [1848'], we get,

[1872a] $$\begin{aligned} x\,d\,z - z\,d\,x &= -\,c''\,d\,s - \alpha\,d\,s \,.\, \textstyle\int d\,s \,.\, \left(\frac{d\,u'}{d\,x}\right) . \sin.\,\psi\,; \\ y\,d\,z - z\,d\,y &= -\,c'\,d\,s - \alpha\,d\,s \,.\, \textstyle\int d\,s \,.\, \left(\frac{d\,u'}{d\,y}\right) . \sin.\,\psi. \end{aligned}$$

Multiplying the first by $-\cos.\,\varphi$, the second by $-\sin.\,\varphi$, and adding the products, we get,

[1872b] $$\begin{aligned} & -(x\,d\,z - z\,d\,x) \,.\, \cos.\,\varphi - (y\,d\,z - z\,d\,y) \,.\, \sin.\,\varphi \\ & = c'\,d\,s \,.\, \sin.\,\varphi + c''\,d\,s \,.\, \cos.\,\varphi + \alpha\,d\,s \,.\, \cos.\,\varphi \,.\, \textstyle\int d\,s \,.\, \left(\frac{d\,u'}{d\,x}\right) . \sin.\,\psi \\ & \qquad + \alpha\,d\,s \,.\, \sin.\,\varphi \,.\, \textstyle\int d\,s \,.\, \left(\frac{d\,u'}{d\,y}\right) . \sin.\,\psi. \end{aligned}$$

The first member of this equation is equal to $r^2 \,.\, d\,\theta$ [1852']; and if we substitute, in the second member, the values of $\left(\frac{d\,u'}{d\,x}\right)$, $\left(\frac{d\,u'}{d\,y}\right)$, [1858, 1858'], we shall get [1872].

Geodetical line in the direction of the meridian.

We shall, in the first place, consider the case, in which the first side of the geodetical line is parallel to the corresponding plane of the celestial meridian. [1872′]
In this case, $d\varphi$, dr, are both of the order α;* and by neglecting quantities of the order α^2, we shall find, [1872″]

$$ds = -r\,d\theta\,; \qquad [1873]$$

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* (1346) To illustrate what is said relative to the curve of the meridian, we have given the annexed figure; in which CX, CY, CZ, are the radii of the earth, drawn parallel to the axes of x, y, z, respectively; CZ being the axis of rotation, and ZCE the plane of the *celestial* meridian, corresponding to the commencement H of the *geodetical* line $HhIiL$. $ZmIF$, Zif, are two curves formed upon the surface of the earth, by two planes ZCF, ZCf, passing through the axis CZ, and intersecting the plane of xy, in the infinitely near points F, f. Drawing im perpendicular to Im, also the radii CF, Cf, CI, &c., we shall have, [1850″, &c.], [1872c]

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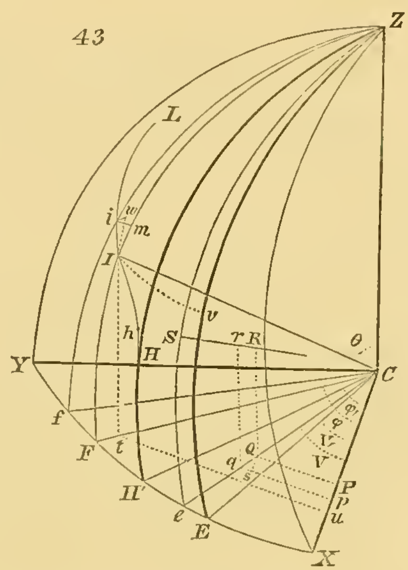

$$ZCI = \theta, \qquad XCF = \varphi, \qquad FCf = d\varphi\,; \qquad [1872d]$$

$$CI = r, \qquad HI = s, \qquad Ii = ds. \qquad [1872e]$$

At the first point H of the curve, the angles θ, φ, become respectively,

$$\theta_{,} = ZCH, \qquad \varphi_{,} = XCH'\,; \qquad [1872e']$$

CH' being the line of intersection of the plane $ZCH'H$ with the equator XCY; the line CH not being drawn in the figure, to prevent confusion. If the earth were spherical, the curve $HIiL$ would coincide with the plane of the celestial meridian CZE, and the arc im would be nothing in comparison with Ii; or, in other words, the angle mIi would be nothing. Moreover it is evident, that if the ellipticity of the osculatory ellipsoid, at the point I, be of the order α, the angle mIi will generally be of the order α; consequently $\frac{mi}{Ii}$, will be of the order α; and [1872f]
as Ff is generally of the same order as mi, the ratio $\frac{Ff}{Ii}$ will also be of the order α. [1872g]
Now Ff is nearly equal to $rd\varphi$, or φ, because r is nearly equal to unity [1850e]; moreover $Ii = ds$, therefore $\frac{d\varphi}{ds}$ is of the order α. Hence if we divide the equation [1867] [1872h]
by ds, the first member will be of the order α; consequently its second member will also be of the order α, and as the last term of this second member is multiplied by α, the other term c will also be of the order α; and we may incidentally remark, that the value of c is

[1873'] *the arc s being supposed to increase from the equator to the poles.* As the latitude is expressed by ψ, it is evident that we shall have,

[1874] $$\theta = 100^{\circ} - \psi - \left(\frac{dr}{d\psi}\right);$$

[1872i] computed in [1888]. Again, if we neglect, in [1852''], the terms dr^2, $r^2 d\varphi^2 . \sin.^2\theta$, of the order α^2, in comparison with ds^2, $d\theta^2$, or $d\psi^2$, [1872h, 1850c], we get $ds^2 = r^2 d\theta^2$,
[1872k] whose square root is $ds = -r\,d\theta$, as in [1873]; the negative sign being used, because θ decreases when s increases, [1850'', 1873'].

* (1347) If the point i of the geodetical line HIL, be projected upon the plane of $ZCFI$, as in the point i' of the annexed figure, which is drawn separately for distinctness; the perpendicular $i\,i'$ will be of the order α, in comparison with ds [1872'']; and if we neglect
[1874a] terms of the order α^2, we shall have $Ci = Ci'$. Continuing the line CI towards I', also the line Ci' towards k, so as to make $Ck = CI$, and drawing IZ' parallel to CZ, we shall have very nearly the angle $Z'Ii' = \psi$, because this line IZ', continued infinitely towards the heavens, marks the place of the pole; and the line Ii', continued in like manner, marks nearly the plane of the true horizon; so that the angle $Z'Ii'$, represents nearly the elevation of the pole above the horizon, or the latitude of the place ψ. Moreover $Z'II' = \theta$, and

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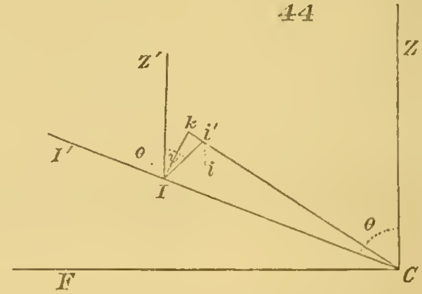

the angle $kIi' = \frac{ki'}{Ik}$; therefore $Z'Ii' + Z'II' - kIi' = I'Ik = 100^{\circ}$ becomes,

in symbols, $\psi + \theta - \frac{ki'}{Ik} = 100^{\circ}$, or $\theta = 100^{\circ} - \psi + \frac{ki'}{Ik}$. But $Ik = -r\,d\theta$,
[1874b]
$ki' = -dr$; the negative signs being prefixed, because the spheroid being oblate, the

angle θ decreases with r; hence $\theta = 100^{\circ} - \psi + \frac{dr}{r\,d\theta}$. In the last term of this

[1874c] expression $\frac{dr}{r\,d\theta}$, which is of the order α [1872i, k], we may put $r = 1$, and

$d\theta = -d\psi$, and it will become $\theta = 100^{\circ} - \psi - \frac{dr}{d\psi}$, as in [1874]; observing that

we may change $\frac{dr}{d\psi}$ into $\left(\frac{dr}{d\psi}\right)$. For the radius r being considered as a function of

ψ, φ, its complete differential is $dr = \left(\frac{dr}{d\psi}\right).d\psi + \left(\frac{dr}{d\varphi}\right).d\varphi$. Dividing this by $d\psi$,

[1874d] we get $\frac{dr}{d\psi} = \left(\frac{dr}{d\psi}\right) + \left(\frac{dr}{d\varphi}\right).\frac{d\varphi}{d\psi}$. Now $\left(\frac{dr}{d\varphi}\right)$ is nothing in a sphere whose centre

which gives,*

$$d\theta = -d\psi - \alpha\, d\psi \,.\left(\frac{d\,d\,u'}{d\psi^2}\right); \qquad [1874']$$

hence we find,

$$d\,s = d\psi \,.\left\{1 + \alpha u' + \alpha \,.\left(\frac{d\,d\,u'}{d\psi^2}\right)\right\}. \qquad [1875]$$

Therefore by putting ε for the difference of latitude of the two extreme points of the arc s, we shall have,† [1875′]

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is the origin of r, and in this spheroid it is of the order α; moreover this is multiplied by $\frac{d\varphi}{d\psi}$, which is nearly equal to $\frac{d\varphi}{ds}$, and is also of the order α [1872h]; therefore the product is of the order α^2, and may be neglected, making $\frac{dr}{d\psi} = \left(\frac{dr}{d\psi}\right)$. [1874$e$]

* (1348) The differential of $r = 1 + \alpha u'$ [1850c] is $\left(\frac{dr}{d\psi}\right) = \alpha \,.\left(\frac{du'}{d\psi}\right)$; hence [1874$f$]
[1874] becomes $\theta = 100^\circ - \psi - \alpha \,.\left(\frac{du'}{d\psi}\right)$, and the differential of this is

$$d\theta = -d\psi - \alpha \,.\, d \,.\left(\frac{du'}{d\psi}\right). \qquad [1874f']$$

Now u' is a function of r, ψ, φ; and dr, $d\varphi$, [1872i, &c.], are of the order α, in comparison with $d\psi$; therefore, in the term $\alpha \,.\, d \,.\left(\frac{du'}{d\psi}\right)$, we may, by neglecting α^2, reject the variations of r, φ, considering ψ only as variable; and we shall have,

$$\alpha \,.\, d \,.\left(\frac{du'}{d\psi}\right) = \alpha \,.\left(\frac{d\,d\,u'}{d\psi^2}\right) . \,d\psi.$$

Substituting this in [1874f'], we get the equation [1874′]. Multiplying this expression by $-r = -(1 + \alpha u')$ [1850c], we obtain $-r\,d\theta = d\psi \,.\left\{1 + \alpha u' + \alpha \,.\left(\frac{d\,d\,u'}{d\psi^2}\right)\right\} = ds$, [1874$g$]
[1873], as in [1875].

† (1349) If $\psi_{,}$ be the value of ψ, at the commencement of the geodetical line, and we put $\psi - \psi_{,} = \varepsilon'$, $d\psi = d\varepsilon'$, and $u' + \left(\frac{d\,d\,u'}{d\psi^2}\right) = U$, the expression [1875] will become $d\,s = d\,\varepsilon' + \alpha\, d\,\varepsilon' . U$; U being a function of ε, φ, r, which varies in proceeding along the geodetical line s; but the variations of φ, r, [1872i], are of the order α in comparison with those of ψ or ε'; so that if we neglect terms of the order α^2, in ds [1875a], we may consider ε' as the only variable quantity in U; and by formula [617], it will become, [1875a] [1875b]

Arc of the geodetical curve.

[1876] $$s = \varepsilon + \alpha\varepsilon . \left\{ u'_{,} + \left(\frac{d\,d\,u'_{,}}{d\psi^2}\right) \right\} + \frac{\alpha\,\varepsilon^2}{1.2} . \left\{ \left(\frac{d\,u'_{,}}{d\psi}\right) + \left(\frac{d^3 u'_{,}}{d\psi^3}\right) \right\} + \&c. ;$$

$u'_{,}$ being the value of u', at the origin of s.*

[1876′] If the earth be a solid of revolution, the geodetical line will always be in the plane of the same meridian; it will vary from it, if the parallels of latitude be not circles; the observations made upon this variation may therefore serve to elucidate this important point of the theory of the earth.
[1876″] We shall therefore resume the equation [1867]; observing that $d\varphi$,† and the

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$U = U_{,} + \varepsilon' . \left(\frac{dU_{,}}{d\psi}\right) + \&c.$, $U_{,}$ being the value of U when $\varepsilon' = 0$. Substituting this in [1875a], we get $ds = d\varepsilon' + \alpha\,d\varepsilon' . U_{,} + \alpha . \varepsilon'\,d\varepsilon' . \left(\frac{dU_{,}}{d\psi}\right) + \&c.$ Its integral gives,
[1875c] $s = \varepsilon' + \alpha\varepsilon' . U_{,} + \tfrac{1}{2}\alpha\varepsilon'^2 . \left(\frac{dU_{,}}{d\psi}\right) + \&c.$ At the extremity of the geodetical line, where $\varepsilon' = \varepsilon$, it becomes $s = \varepsilon + \alpha\varepsilon . U_{,} + \tfrac{1}{2}\alpha\varepsilon^2 . \left(\frac{dU_{,}}{d\psi}\right) + \&c.$ Substituting in this the value
[1875d] of $U_{,} = u'_{,} + \left(\frac{d\,d\,u'_{,}}{d\psi^2}\right)$ [1875a], corresponding to the first point of the curve, it becomes as in [1876].

* (1349a) The expression of s [1876] may be considerably simplified, and the terms of the order ε^2, and other even powers of ε avoided; by taking $u'_{,}$, $U_{,}$, to correspond to the *mean* latitude, between the two extremities of the arc s, instead of that at its origin. In this
[1875e] case we shall put, as above, $\psi - \psi_{,} = \varepsilon'$, $U = u' + \left(\frac{d\,d\,u'}{d\psi^2}\right)$; $\psi_{,}$ being this mean latitude, and ψ the latitude at any other point of the arc. We shall then have, as in
[1875f] [1875a—c], $d\psi = d\varepsilon'$; $ds = d\varepsilon' + \alpha . d\varepsilon' . U = d\varepsilon' + \alpha . d\varepsilon' . U_{,} + \alpha . \varepsilon\,d\varepsilon' . \left(\frac{dU_{,}}{d\psi}\right) + \&c.$ Integrating this, and adding a constant quantity, so as to make s vanish at the origin, where
[1875g] $\varepsilon' = -\tfrac{1}{2}\varepsilon$; we get $s = (\varepsilon' + \tfrac{1}{2}\varepsilon) + \alpha . (\varepsilon' + \tfrac{1}{2}\varepsilon) . U_{,} + \tfrac{1}{2}\alpha . (\varepsilon'^2 - \tfrac{1}{4}\varepsilon^2) . \left(\frac{dU_{,}}{d\psi}\right)$, neglecting ε'^3. Now putting $\varepsilon' = \tfrac{1}{2}\varepsilon$, corresponding to the other extremity of the curve, the terms of the order ε^2 will vanish, and we shall get,

[1875h] $$s = \varepsilon + \alpha\varepsilon . U_{,} = \varepsilon + \alpha\varepsilon . \left\{ u'_{,} + \left(\frac{d\,d\,u'_{,}}{d\psi^2}\right) \right\} ;$$

in which $u'_{,}$ corresponds to the mean latitude abovementioned. Terms of the order ε^3 are neglected in this formula, but it is easily continued to any degree of accuracy.

† (1350) c and $d\varphi$ [1872h—i] are of the order α; therefore we may neglect terms of
[1876a] the order α, in the factors of these quantities in [1867], putting $r = 1$ [1850c],

constant quantity c, are of the order α [1872h—i]. We may suppose, in that equation, $r=1$, $ds=d\psi$, and $\theta=100^\circ-\psi$; hence we shall have, [1876‴]

$$d\varphi \,.\, \cos.^2\psi = c\,d\psi + \alpha\, d\psi \,.\, f d\psi \,.\, \left(\frac{du'}{d\varphi}\right). \qquad [1877]$$

Now if we put V for the angle, formed by the plane of the celestial meridian, [1877′] and the plane of xz, from which the angle φ is counted, as its origin,* we shall have $dx'\,.\,\text{tang.}\,V=dy'$; x', y', z', being the co-ordinates of this [1878] meridian, whose differential equation we have found, in the preceding article [1843], to be

$$dz' = a\,dx' + b\,dy'. \qquad [1879]$$

Comparing this with [1878], we find a and b to be infinite, and [1879′]

$$-\frac{a}{b} = \text{tang.}\,V. \qquad [1880]$$

Then from [1844], we obtain,†

$$0 = \left(\frac{du}{dx}\right) . \,\text{tang.}\,V - \left(\frac{du}{dy}\right); \qquad [1881]$$

$ds=d\psi$ [1875], $\theta=100^\circ-\psi$. Substituting these in [1867], we get [1877], in [1876b] which terms of the order α^2 are neglected.

* (1351) ZCX, fig. 43, page 375, is the plane of zx; ZCE, ZCe, the planes of the celestial meridians, corresponding respectively to the points H, I, of the geodetical line HIL. We shall suppose R, r, to be two points of the plane ZCe, infinitely near to each other, the co-ordinates of the first point R, being $CP=x'$, $PQ=y'$, $QR=z'$; [1877a] those of the second point r, $Cp=x'+dx'$, $pq=y'+dy'$, $qr=z'+dz'$. Drawing Qs parallel and equal to Pp, we shall have $Qs=dx'$, $sq=dy'$, the angle $sQq=XCe=V$. In the rectangular triangle Qsq, we have $Qs\,.\,\text{tang.}\,sQq=sq$, or in symbols $dx'.\text{tang.}V=dy'$, as in [1878]. This is the equation of the plane of the [1877b] celestial meridian ZCe, and is equivalent to that assumed in [1843, 1879], which may be put under the form $-\frac{a}{b}\,.\,dx'+\frac{1}{b}\,.\,dz'=dy'$. Comparing this with [1877b], we get, $-\frac{a}{b}=\text{tang.}\,V$, $\frac{1}{b}=0$. This last gives $b=\infty$, and the first makes a of the same [1877c] order as b, as in [1879′, 1880].

† (1352) Dividing [1844] by b, we get $\frac{1}{b}\,.\left(\frac{du}{dz}\right)-\frac{a}{b}\,.\left(\frac{du}{dx}\right)-\left(\frac{du}{dy}\right)=0$, and by [1881$a$] using the values [1877$c$], it becomes as in [1881].

hence we deduce,*

[1882] $$0 = x \cdot \text{tang.}\, V - y - \alpha \cdot \left(\frac{d\,u'}{d\,x}\right) \cdot \text{tang.}\, V + \alpha \cdot \left(\frac{d\,u'}{d\,y}\right).$$

[1883] We may suppose† $V = \varphi$, in the terms multiplied by α; moreover, $\frac{y}{x} = \text{tang.}\, \varphi$; hence we have,‡

[1882a] * (1353) Substituting in [1881] the two first expressions [1850a], and dividing by 2, it becomes as in [1882].

† (1354) The distance of the point I of the geodetical line IIL, fig. 43, page 375,
[1883a] from the plane of the corresponding celestial meridian, ZCe, is of the order α; and by neglecting such quantities, we may put the angle $XCe = XCF$, or $V = \varphi$. Again, if
[1883b] from the point I we let fall upon CF the perpendicular It; and from t let fall upon CX the perpendicular tu, we shall have $Cu = x$, $ut = y$, [1840″], angle $uCt = \varphi$ [1850‴]; and
[1883c] in the rectangular triangle Cut, we have $\frac{ut}{Cu} = \text{tang.}\, uCt$, or $\frac{y}{x} = \text{tang.}\, \varphi$, [1883].

‡ (1355) Substituting the values of $\left(\frac{d\,u'}{d\,x}\right)$, $\left(\frac{d\,u'}{d\,y}\right)$, [1858], in [1882], after transposing the terms depending on α, we get,

[1884a] $$x \cdot \text{tang.}\, V - y = \alpha \cdot \text{tang.}\, V \cdot \left\{ -\frac{\cos.\varphi}{\sin.\psi} \cdot \left(\frac{d\,u'}{d\,\psi}\right) - \frac{\sin.\varphi}{\cos.\psi} \cdot \left(\frac{d\,u'}{d\,\varphi}\right) \right\}$$
$$- \alpha \cdot \left\{ -\frac{\sin.\varphi}{\sin.\psi} \cdot \left(\frac{d\,u'}{d\,\psi}\right) + \frac{\cos.\varphi}{\cos.\psi} \cdot \left(\frac{d\,u'}{d\,\varphi}\right) \right\}.$$

Putting in the second member $V = \varphi$ [1883], also $\text{tang.}\, \varphi \cdot \cos.\varphi = \sin.\varphi$, the terms connected with $\alpha \cdot \left(\frac{d\,u'}{d\,\psi}\right)$ mutually destroy each other, and the coefficient of $-\frac{\alpha}{\cos.\psi} \cdot \left(\frac{d\,u'}{d\,\varphi}\right)$ is

$$\text{tang.}\, \varphi \cdot \sin.\varphi + \cos.\varphi = \frac{\sin.\varphi}{\cos.\varphi} \cdot \sin.\varphi + \cos.\varphi = \frac{1}{\cos.\varphi} \cdot \{\sin.^2\varphi + \cos.^2\varphi\} = \frac{1}{\cos.\varphi};$$

[1884b] consequently the equation [1884a] becomes $x \cdot \text{tang.}\, V - y = -\frac{\alpha \cdot \left(\frac{d\,u'}{d\,\varphi}\right)}{\cos.\psi \cdot \cos.\varphi}$. Multiplying this by $-\frac{1}{x} \cdot \cos.\psi \cdot \cos.\varphi$, we get,

$$\cos.\psi \cdot \cos.\varphi \cdot \left\{ \frac{y}{x} - \text{tang.}\, V \right\} = \frac{1}{x} \cdot \alpha \cdot \left(\frac{d\,u'}{d\,\varphi}\right);$$

which, by means of [1883], and x [1854], is easily reduced to the form [1884].

$$\cos.\psi . \cos.\varphi . \{\text{tang.}\,\varphi - \text{tang.}\,V\} = \frac{\alpha . \left(\frac{du'}{d\varphi}\right)}{\cos.\psi . \cos.\varphi}; \qquad [1884].$$

which gives,*

$$\varphi - V = \frac{\alpha . \left(\frac{du'}{d\varphi}\right)}{\cos.^2\psi}. \qquad [1885]$$

The first side of the geodetical line being supposed parallel to the plane of the celestial meridian [1872′], the differentials of the angle V, and of the distance $(\varphi - V) . \cos.\psi$, from the origin of the curve, to the plane of the celestial meridian, must be nothing at this origin.† Hence we shall have, at this point, [1886]

$$\frac{d\varphi}{d\psi} = (\varphi - V) . \text{tang.}\,\psi = \frac{\alpha . \left(\frac{du'}{d\varphi}\right) . \text{tang.}\,\psi}{\cos.^2\psi}; \qquad [1887]$$

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* (1356) From [34′, 22] Int., we have

$$\text{tang.}\,\varphi - \text{tang.}\,V = \frac{\sin.\varphi}{\cos.\varphi} - \frac{\sin.V}{\cos.V} = \frac{\sin.\varphi . \cos.V - \cos.\varphi . \sin.V}{\cos.\varphi . \cos.V} = \frac{\sin.(\varphi - V)}{\cos.\varphi . \cos.V}. \qquad [1885a]$$

Substituting this in [1884], and multiplying by $\frac{\cos.V}{\cos.\psi}$, we get,

$$\sin.(\varphi - V) = \frac{\alpha . \left(\frac{du'}{d\varphi}\right)}{\cos.^2\psi} . \frac{\cos.V}{\cos.\varphi};$$

in which the second member is of the order α; therefore the first member, or $\sin.(\varphi - V)$, [1885b] must be of the same order; and by neglecting terms of the order α^3, we may put it equal to $\varphi - V$ [43] Int. Lastly, as V differs from φ by quantities of the order α [1885b], we may put $\frac{\cos.V}{\cos.\varphi} = 1$, [61] Int.; and then the preceding equation will become as in [1885]. [1885c]

† (1357) Supposing ZCE, fig. 43, page 375, to be the plane of the celestial meridian, corresponding to the *first* point H of the geodetical curve, we shall have [1887a] $XCE = V_{,}$; and for *any* other point I of that curve, we have $XCF = \varphi$; hence the angle $ECF = \varphi - V_{,}$, and as this angle would be nothing if the earth were spherical, it must be of the order α in the case under consideration. Therefore if we neglect quantities of the order α^2, in finding the distance Iv of the point I from the plane ZCE, we may suppose [1887b] the angle $FCI = \psi$, also $CE = CF = CI = 1$. Then the angle $ECF = \varphi - V_{,}$, the arc $EF = \varphi - V_{,}$, and the distance Iv may be found very nearly, as if it were a

therefore the equation [1877] will give,*

[1888] $$c = \alpha \cdot \left(\frac{d u'_{,}}{d \varphi}\right) \cdot \text{tang.}\, \psi_{,} \,;$$

in which $u'_{,}$ and $\psi_{,}$ correspond to the origin of the arc s.

At the end of the measured arc, the side of the curve makes, with the plane of the corresponding celestial meridian, an angle which is nearly equal
[1888'] to the differential of $(\varphi - V) \cdot \cos. \psi$, divided by $d\psi$, V being supposed
[1888''] constant, in taking the differentials;† therefore if we denote that angle by ϖ, we shall have,

[1889] $$\varpi = \frac{d\varphi}{d\psi} \cdot \cos. \psi - (\varphi - V) \cdot \sin. \psi.$$

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parallel of latitude, included between two meridians ZE, ZF, of a spherical surface;
[1887c] consequently it will be represented nearly by $Iv = EF \cdot \cos. FCI = (\varphi - V_{,}) \cdot \cos. \psi$, as in [1886]. Now the first infinitely small element Hh of the geodetical curve [1872'], is parallel to the plane of the celestial meridian ZCE, corresponding to the first point H; therefore the differential of the general expression of the distance Iv [1887c], which is
[1887d] $d\varphi \cdot \cos. \psi - (\varphi - V_{,}) \cdot \sin. \psi \cdot d\psi$, becomes nothing at that first point; and by using the accented letters corresponding to that point, as in [1892], we have

$$0 = d\varphi_{,} \cdot \cos. \psi_{,} - (\varphi_{,} - V_{,}) \cdot \sin. \psi_{,} \cdot d\psi_{,}.$$

Dividing by $d\psi_{,} \cdot \cos. \psi_{,}$, we get $\frac{d\varphi_{,}}{d\psi_{,}} = (\varphi_{,} - V_{,}) \cdot \text{tang.}\, \psi_{,}$; substituting [1885], it

[1887e] becomes $\frac{d\varphi_{,}}{d\psi_{,}} = \frac{\alpha \cdot \left(\frac{du'_{,}}{d\varphi}\right) \cdot \text{tang.}\, \psi_{,}}{\cos.^2 \psi_{,}}$, as in [1887].

[1888a] * (1358) As the integral $\int d\psi \cdot \left(\frac{d u'}{d\varphi}\right)$ [1877], commences with the first point of the geodetical curve, it is nothing at that point; hence if we accent the letters, as in [1892], we shall have, at the origin of the curve,

[1888b] $$d\varphi_{,} \cdot \cos.^2 \psi_{,} = c\, d\psi_{,}, \qquad \text{or} \qquad c = \frac{d\varphi_{,}}{d\psi_{,}} \cdot \cos.^2 \psi_{,} \,;$$

substituting [1887e], we obtain [1888].

† (1359) Supposing a plane Iw, fig. 43, to be drawn through I, parallel to the plane of the corresponding celestial meridian ZCc, and letting fall upon it the perpendicular iw, it will be equal to the differential of the distance of the point I from the plane of this meridian.
[1889a] This distance is easily deduced from [1887c], changing $V_{,}$ into V, which corresponds to the

If we substitute the value of $\left(\frac{d\varphi}{d\psi}\right)$, deduced from the equation [1867], and also the preceding value of $\varphi - V$ [1885], we shall obtain,*

$$\varpi = \frac{\alpha}{\cos.\psi} \cdot \left\{ \left(\frac{d u_{\prime}'}{d\varphi}\right) . \text{tang.}\,\psi_{\prime} - \left(\frac{d u'}{d\varphi}\right) . \text{tang.}\,\psi + \int d\psi \, . \left(\frac{d u'}{d\varphi}\right) \right\}; \qquad [1890]$$

the integral being taken from the beginning to the end of the measured arc. We shall put ε for the difference of the latitude of these two extreme points; supposing ε to be so small that we may neglect its square; we shall then have,† [1890′]

$$\varpi = -\frac{\alpha\,\varepsilon\,.\,\text{tang.}\,\psi}{\cos.\psi} \cdot \left\{ \left(\frac{d u'}{d\varphi}\right) . \text{tang.}\,\psi + \left(\frac{d\,d u'}{d\varphi\, d\psi}\right) \right\}. \qquad [1891]$$

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given plane ZCe; hence it becomes $(\varphi - V)\,.\cos.\psi$, and its differential is $iw = d\varphi\,.\cos.\psi - d\psi\,.(\varphi - V)\,.\sin.\psi$; V being considered constant, because this distance is measured from the *given* plane corresponding to V. Dividing iw by $Ii = ds$, we obtain the sine of the angle iIw, which, on account of its smallness, may be taken for the angle itself, or ϖ [1888″]; and since by [1875] ds is nearly equal to $d\psi$, this angle will become $\frac{iw}{Ii} = \varpi = \frac{d\varphi}{d\psi}\,.\cos.\psi - (\varphi - V)\,.\sin.\psi$, as in [1889]. [1889b]

* (1360) Substituting c [1888], in [1877], and dividing by $d\psi.\cos.^2\psi$, we get,

$$\frac{d\varphi}{d\psi} = \alpha \, . \left(\frac{d u_{\prime}'}{d\varphi}\right) . \frac{\text{tang.}\,\psi_{\prime}}{\cos.^2\psi} + \frac{\alpha}{\cos.^2\psi} . \int d\psi \, . \left(\frac{d u'}{d\varphi}\right). \qquad [1889c]$$

Using this and [1885], we obtain, from [1889],

$$\varpi = \alpha \, . \left(\frac{d u_{\prime}'}{d\varphi}\right) . \frac{\text{tang.}\,\psi_{\prime}}{\cos.\psi} + \frac{\alpha}{\cos.\psi} . \int d\psi \, . \left(\frac{d u'}{d\varphi}\right) - \frac{\alpha\,.\left(\frac{d u'}{d\varphi}\right)}{\cos.^2\psi} . \sin.\psi\,; \qquad [1889d]$$

which is easily reduced to the form [1890].

† (1361) Putting for brevity U equal to the second member of [1890], also $\psi = \psi_{\prime} + \varepsilon'$, [1890a] and developing it as in [1875a, b], according to the powers of ε', we shall get, neglecting ε'^2, and higher powers of ε', $\varpi = U_{\prime} + \varepsilon' \, . \left(\frac{dU_{\prime}}{d\psi}\right)$. Supposing now, for a moment, W to [1890b] represent the terms of the second member of [1890] between the braces, and $W_{\prime}$ its value at the origin of the curve, where $\psi = \psi_{\prime}$; we shall have,

$$W = \left(\frac{d u_{\prime}'}{d\varphi}\right) . \text{tang.}\,\psi_{\prime} - \left(\frac{d u'}{d\varphi}\right) . \text{tang.}\,\psi + \int d\psi \, . \left(\frac{d u'}{d\varphi}\right); \qquad U = \frac{\alpha}{\cos.\psi} . W. \qquad [1890c]$$

[1891] The values of ψ, $\left(\frac{d\,u'}{d\,\varphi}\right)$, and $\left(\frac{d\,d\,u'}{d\,\varphi\,d\,\psi}\right)$, ought, for greater accuracy, *to be made to correspond to the middle of the measured arc.* The angle ϖ is positive when it falls on the same side of the meridian as the increments of φ.*

[1892] To obtain the difference of longitude of the two meridians corresponding to the extremities of the arc, we shall observe that $u'_{,}$, $V_{,}$, $\psi_{,}$, $\varphi_{,}$, being the values of u', V, ψ, φ, at the origin, we have,†

The differential of this value of U, relative to ψ, is $\left(\frac{d\,U}{d\,\psi}\right)=\frac{\alpha.\sin.\psi}{\cos.^2\psi}\cdot W+\frac{\alpha}{\cos.\psi}\cdot\left(\frac{d\,W}{d\,\psi}\right)$;

[1890d] hence, at the origin, we have $\left(\frac{dU_{,}}{d\,\psi}\right)=\frac{\alpha\,.\sin.\psi_{,}}{\cos.^2\psi_{,}}\cdot W_{,}+\frac{\alpha}{\cos.\psi_{,}}\cdot\left(\frac{d\,W_{,}}{d\,\psi}\right)$. At the origin of the curve, where $\psi=\psi_{,}$, $u'=u'_{,}$, [1892], the integral $\int d\,\psi\,.\left(\frac{d\,u'}{d\,\varphi}\right)$ [1888a] vanishes, and the value of W [1890c] becomes $W_{,}=0$; whence $U_{,}=\frac{\alpha}{\cos.\psi_{,}}\cdot W_{,}=0$,

[1890e] [1890c]. Substituting these in [1890b, d], we get $\varpi=\varepsilon'.\left(\frac{dU_{,}}{d\,\psi}\right)=\frac{\alpha\,\varepsilon'}{\cos.\psi_{,}}\cdot\left(\frac{d\,W_{,}}{d\,\psi}\right)$. The differential of W [1890c], relative to ψ, becomes, by observing that the first term $\left(\frac{d\,u'_{,}}{d\,\varphi}\right).\,\text{tang}.\,\psi_{,}$ is constant, relative to this differentiation,

[1890f]
$$\left(\frac{dW}{d\,\psi}\right)=-\left(\frac{d\,d\,u'}{d\,\varphi\,d\,\psi}\right).\,\text{tang}.\,\psi-\left(\frac{d\,u'}{d\,\varphi}\right)\cdot\frac{1}{\cos.^2\psi}+\left(\frac{d\,u'}{d\,\varphi}\right);$$

in which the coefficient of $\left(\frac{d\,u'}{d\,\varphi}\right)$ is $-\frac{1}{\cos.^2\psi}+1=-\,\text{tang}.^2\psi$ [34'''] Int.; hence $\left(\frac{d\,W_{,}}{d\,\psi}\right)=-\left(\frac{d\,d\,u'_{,}}{d\,\varphi\,d\,\psi_{,}}\right).\,\text{tang}.\,\psi_{,}-\left(\frac{du'_{,}}{d\,\varphi}\right).\,\text{tang}.^2\,\psi_{,}$. Substituting this in [1890e],

[1891a] we get $\varpi=-\frac{\alpha\,\varepsilon'.\,\text{tang}.\,\psi_{,}}{\cos.\psi_{,}}\cdot\left\{\left(\frac{d\,d\,u'}{d\,\varphi\,d\,\psi_{,}}\right)+\left(\frac{du'_{,}}{d\,\varphi}\right).\text{tang}.\psi_{,}\right\}$; which, by changing ε' into ε, to correspond to the whole arc, becomes as in [1891], ψ being used for $\psi_{,}$; observing that, in all these calculations, we have neglected terms of the order α^2, and ε^2.

[1892a] * (1362) This is conformable to the calculation in [1889a, b], where the positive values of ϖ correspond to the case in which the line $i\,w$, fig. 43, page 375, falls on the same side of the meridian as the increments of φ.

[1893a] † (1363) The second formula [1893] is the same as [1885], and the first formula is deduced from this, by changing φ, V, u', ψ, into $\varphi_{,}$, $V_{,}$, $u'_{,}$, $\psi_{,}$, respectively, to make them correspond to the first point of the curve.

$$\varphi_{\prime} - V_{\prime} = \frac{\alpha \cdot \left(\frac{d u_{\prime}'}{d \varphi}\right)}{\cos.^2 \psi_{\prime}} ; \qquad \varphi - V = \frac{\alpha \cdot \left(\frac{d u'}{d \varphi}\right)}{\cos.^2 \psi} ;$$ [1893]

but we have nearly, by neglecting the square of ε.* [1893′]

$$\varphi - \varphi_{\prime} = \frac{c\, \varepsilon}{\cos.^2 \psi_{\prime}} ; \qquad c = \alpha \cdot \left(\frac{d u_{\prime}'}{d \varphi}\right) . \text{tang.} \psi_{\prime} ;$$ [1894]

therefore we shall have,†

$$V - V_{\prime} = - \frac{\alpha\, \varepsilon}{\cos.^2 \psi_{\prime}} \cdot \left\{ \left(\frac{d u_{\prime}'}{d \varphi}\right) . \text{tang.} \psi_{\prime} + \left(\frac{d d u_{\prime}'}{d \varphi\, d \psi}\right) \right\} ;$$ [1895]

hence we obtain this very simple equation,

Difference of longitude of the extremities of the geodetical curve.

$$(V - V_{\prime}) . \sin. \psi_{\prime} = \varpi ;$$ [1896]

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* (1364) If we put, as in [1875a], $\psi = \psi_{\prime} + \varepsilon'$, considering ε' as the variable quantity, instead of ψ, and $\psi_{\prime}$ as constant; we shall have $d \psi = d \varepsilon'$, and [1877] will become, $d \varphi . \cos.^2 . (\psi_{\prime} + \varepsilon') = c\, d \varepsilon' + \alpha\, d \varepsilon' . \int d \varepsilon' . \left(\frac{d u'}{d \varphi}\right)$. If we neglect terms of the order ε'^2, as in [1893′], we may omit the integral of the second member, and retain only the first term $c\, d \varepsilon'$. Now this term being of the order ε', the first term must be of the same order; therefore we may neglect the term ε', in $\cos.^2 (\psi_{\prime} + \varepsilon')$, and we shall get $d \varphi . \cos.^2 \psi_{\prime} = c\, d \varepsilon'$. The integral of this, taken so as to vanish when $\varphi = \varphi_{\prime}$, is $(\varphi - \varphi_{\prime}) . \cos.^2 \psi_{\prime} = c \varepsilon'$; whence we easily deduce the first of the equations [1894], ε being put for ε'. The value of c [1894] is the same as in [1888]. [1894a] [1894b]

† (1365) Putting for brevity $U = \alpha . \left(\frac{d u'}{d \varphi}\right) . (\cos. \psi)^{-2}$, we shall have, from the second of the equations [1893], $\varphi - V = U$. If we put, as in [1894a], $\psi = \psi_{\prime} + \varepsilon'$, and develop U, according to the powers of ε', retaining only the first power, we shall obtain $\varphi - V = U_{\prime} + \varepsilon' . \left(\frac{d U_{\prime}}{d \psi}\right)$; $U_{\prime}$ being the value of U when $\varepsilon' = 0$. In the case of $\varepsilon' = 0$, this equation becomes $\varphi_{\prime} - V_{\prime} = U_{\prime}$. Subtracting the first of these equations from the second, we get, $V - V_{\prime} = (\varphi - \varphi_{\prime}) - \varepsilon' . \left(\frac{d U_{\prime}}{d \psi}\right)$; and by using the values of $\varphi - \varphi_{\prime}$, c, [1894], we obtain $V - V_{\prime} = \frac{\alpha\, \varepsilon'}{\cos.^2 \psi_{\prime}} . \left(\frac{d u_{\prime}'}{d \varphi}\right) . \text{tang.} \psi_{\prime} - \varepsilon' . \left(\frac{d U_{\prime}}{d \psi}\right)$. Now the differential of U [1895a], relative to ψ, gives, [1895a] [1895b] [1895c]

$$\left(\frac{d U}{d \psi}\right) = \alpha . \left(\frac{d d u'}{d \varphi\, d \psi}\right) . (\cos. \psi)^{-2} + 2\, \alpha . \left(\frac{d u'}{d \varphi}\right) . (\cos. \psi)^{-3} . \sin. \psi$$

$$= \alpha . (\cos. \psi)^{-2} . \left\{ \left(\frac{d d u'}{d \varphi\, d \psi}\right) + 2 . \left(\frac{d u'}{d \varphi}\right) . \text{tang.} \psi \right\} ;$$ [1895c']

Thus we may determine, by observation alone, independently of any knowledge of the figure of the earth, the difference of longitude corresponding to the
[1896'] *extremities of the measured arc;* and if the value of ϖ be so great, that it cannot be imputed to the errors of the observations, we shall be sure that the earth is not a spheroid of revolution.

We shall now consider the case where the first side of the geodetical line is
[1896''] *perpendicular to the corresponding plane of the celestial meridian.* If we take this plane for the plane of $z\,x$; the cosine of the angle, formed by the side

Perpendicular to the meridian.

[1897] of the geodetical line and the plane, will be $\frac{\sqrt{dx^2+dz^2}}{ds}$.* Now this

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and by changing u', $\cos.\psi$, $\tang.\psi$, into $u_{\prime}'$, $\cos.\psi_{\prime}$, $\tang.\psi_{\prime}$, respectively, we get the value of $\left(\frac{dU_{\prime}}{d\psi}\right)$. Substituting this in [1895c], and changing ε' into ε, we obtain,

$$V-V_{\prime}=\frac{\alpha\varepsilon}{\cos.^2\psi_{\prime}}\cdot\left(\frac{du_{\prime}'}{d\varphi}\right).\tang.\psi_{\prime}-\frac{\alpha\varepsilon}{\cos.^2\psi_{\prime}}\cdot\left\{\left(\frac{ddu_{\prime}'}{d\varphi\,d\psi}\right)+2\cdot\left(\frac{du_{\prime}'}{d\varphi}\right).\tang.\psi_{\prime}\right\}.$$

This may be easily reduced to the form [1895]. Multiplying this last expression by $\sin.\psi_{\prime}$,

[1895d] we get $(V-V_{\prime}).\sin.\psi_{\prime}=-\frac{\alpha\varepsilon.\tang.\psi_{\prime}}{\cos.\psi_{\prime}}\cdot\left\{\left(\frac{du_{\prime}'}{d\varphi}\right).\tang.\psi+\left(\frac{ddu_{\prime}'}{d\varphi\,d\psi}\right)\right\}$; in which the second member is the same as the expression of ϖ [1891]; hence we get [1896].

* (1366) In the annexed figure, which is similar to fig. 43, page 375, except that the points X, E, coincide, CZ is the axis of z; CX the axis of x; CY the axis of y;

[1897a] $HhIL$ the geodetical line, whose first element Hh is perpendicular to the plane XCZ; this plane being that of the celestial meridian, corresponding to the point H [1896'']. Now the arc ds of the geodetical line s, is inclined to the axis of y, by an angle whose cosine is

[1897b] $\frac{dy}{ds}$ [40b]; therefore the sine of the same angle is

[1897c] $$\left(1-\frac{dy^2}{ds^2}\right)^{\frac{1}{2}}=\frac{(ds^2-dy^2)^{\frac{1}{2}}}{ds};$$

and by substituting the value of $ds^2=dx^2+dy^2+dz^2$

[1897d] [1852''], it becomes as above, $\frac{(dx^2+dz^2)^{\frac{1}{2}}}{ds}$. But by hypothesis, the first element Hh is perpendicular to the plane XCZ, or parallel to CY; therefore the

[1897e] preceding angle of inclination, and its sine $\frac{(dx^2+dz^2)^{\frac{1}{2}}}{ds}$,

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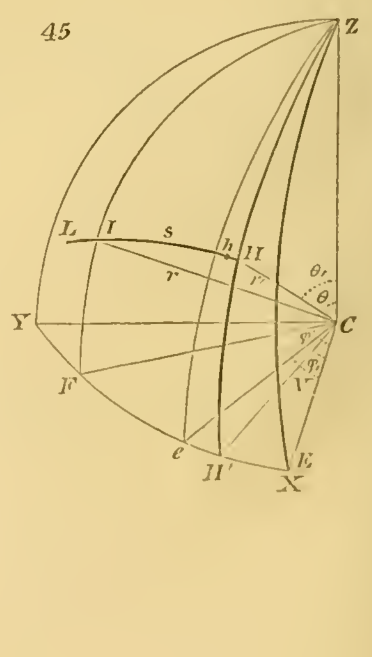

vanishes, at the origin of the curve; hence we have $dx=0$, $dz=0$; [1897′]
consequently,

$$d\,.\,(r\,.\,\sin.\theta\,.\,\cos.\varphi)=0\,; \qquad d\,.\,(r\,.\,\cos.\theta)=0\,; \qquad [1898]$$

therefore,*

$$r\,d\theta=r\,d\varphi\,.\,\sin.\theta\,.\,\cos.\theta\,.\,\text{tang.}\varphi\,; \qquad [1899]$$

but by neglecting quantities of the order α^2, we have,†

$$ds=r\,d\varphi\,.\,\sin.\theta\,; \qquad [1900]$$

therefore we have, at the origin of the curve,

$$\frac{d\theta}{ds}=\frac{\text{tang.}\varphi\,.\,\cos.\theta}{r}\,. \qquad [1901]$$

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are nothing at that point; hence $(dx^2+dz^2)^{\frac{1}{2}}=0$. This equation cannot exist, unless we have separately $dx=0$, $dz=0$; which, by substituting the values of x, z, [1897f] [1851], give the two formulas [1898].

* (1367) Developing the equations [1898], we get,

$$dr.\sin.\theta.\cos.\varphi+r\,d\theta\,.\cos.\theta.\cos.\varphi-r\,d\varphi.\sin.\theta.\sin.\varphi=0\,; \quad dr.\cos.\theta-r\,d\theta.\sin.\theta=0. \qquad [1898a]$$

Multiplying the first by $\cos.\theta$, the second by $-\sin.\theta\,.\,\cos.\varphi$, and adding the products, we obtain,

$$0=r\,d\theta\,.\,\cos.\varphi\,.\,\{\cos.^2\theta+\sin.^2\theta\}-r\,d\varphi\,.\,\sin.\theta\,.\,\cos.\theta\,.\,\sin.\varphi$$

$$=r\,d\theta\,.\,\cos.\varphi-r\,d\varphi\,.\,\sin.\theta\,.\,\cos.\theta\,.\,\sin.\varphi. \qquad [1898b]$$

Dividing this by $\cos.\varphi$, we get the expression [1899], which corresponds to the commencement of the geodetical line.

† (1368) From [1852″] we have $ds=\sqrt{(r^2d\varphi^2\,.\,\sin.^2\theta+dr^2+r^2d\theta^2)}$; and dr [1899a] [1850c] is of the order α. Moreover, Hh is perpendicular to the plane ZCX; and as this plane is inclined to the plane $ZCH'H$, by a quantity of the order α, it follows, that the angle ZCH cannot differ from the angle ZCh but by quantities of the order α. Now these angles represent the values of θ, corresponding to the points H, h; their difference $d\theta$ must therefore be of the order α, at the commencement of the curve. Hence, by neglecting quantities of the order α^2, we have $ds=\sqrt{(r^2d\varphi^2.\sin.^2\theta)}=r\,d\varphi.\sin.\theta$, [1900a] as in [1900]. Substituting this, in the second member of [1899], we get,

$$r\,d\theta=ds\,.\,\cos.\theta\,.\,\text{tang.}\varphi.$$

Dividing this by $r\,ds$, we obtain [1901].

The constant quantity c'', of the equation [1872], is equal to the value of
[1901′] $x\,d\,z - z\,d\,x$, at the origin of the curve; therefore it is nothing,* and the equation [1872] gives, at that point,†

[1902] $$\frac{d\,\theta}{d\,s} = \frac{c'}{r^2}\,.\,\sin.\,\varphi.$$

[1902′] Now by observing that φ is of the order α, and neglecting quantities of the order α^2, we get $\sin.\,\varphi = \text{tang.}\,\varphi$;‡ hence we have,§

[1903] $$c' = r_{,}\,.\,\cos.\,\theta_{,}.$$

[1903′] *The quantities* $r_{,}$ *and* $\theta_{,}$ *correspond to the beginning of the curve;* therefore,

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* (1369) From [1872a], we have,

[1900a] $$x\,d\,z - z\,d\,x = -c''\,d\,s - \alpha\,d\,s\,.\,\textstyle\int d\,s\,.\left(\frac{d\,u'}{d\,x}\right).\,\sin.\,\psi\,;$$

and as the integral of the second member vanishes, at the beginning of the curve, where all
[1900b] the integrals in [1872] are supposed to commence, we shall have $x\,d\,z - z\,dx = -c''ds$; so that $x\,d\,z - z\,d\,x$, at that point, is equal to $-c''\,d\,s$, instead of c'', as stated by the author. Now at the beginning of the curve, $d\,x = 0$, $d\,z = 0$, [1897f]; therefore, at
[1901a] that point, $x\,d\,z - z\,d\,x = 0$, consequently also $-c''\,d\,s = 0$; and as ds [1848′] is constant, we have $c'' = 0$.

† (1370) Putting $c'' = 0$, in [1872], and then observing that, at the *first* point of the curve, the integrals of the second member are nothing [1900b]; we shall have, at that point,
[1902a] $r^2\,d\,\theta = c'\,d\,s\,.\,\sin.\,\varphi$; dividing this by $r^2\,d\,s$, we get [1902].

[1902b] ‡ (1371) From [34′, 44] Int., we have $\frac{\sin.\,\varphi}{\text{tang.}\,\varphi} = \cos.\,\varphi = 1 - \frac{1}{2}\varphi^2 + \&c.$ Now at the first point of the curve, φ, or $\varphi_{,} = XCH'$, fig. 45, page 386, is of the order α;
[1902c] therefore, by neglecting α^2, we shall have, as above, $\frac{\sin.\,\varphi_{,}}{\text{tang.}\,\varphi_{,}} = 1$, or $\sin.\,\varphi_{,} = \text{tang.}\,\varphi_{,}$.

§ (1372) The two values of $\frac{d\,\theta}{d\,s}$ [1901, 1902], being put equal to each other, give $\frac{\text{tang.}\,\varphi\,.\,\cos.\,\theta}{r} = \frac{c'}{r^2}\,.\,\sin.\,\varphi.$ Dividing by the coefficient of c', we get,

$$c' = r\,.\,\cos.\,\theta\,.\,\frac{\text{tang.}\,\varphi}{\sin.\,\varphi}.$$

[1903a] Changing r, θ, into $r_{,}$, $\theta_{,}$, to correspond to the first point of the curve, and putting $\frac{\text{tang.}\,\varphi_{,}}{\sin.\,\varphi_{,}} = 1$, [1902c], we get c' [1903], in which terms of the order α^2 are neglected.

by observing that, at this point, the angle φ is the same which we have [1903″]
before called $\varphi_{,}-V_{,}$, [1893], and found to be equal to $\frac{\alpha\cdot\left(\frac{du'_{,}}{d\varphi}\right)}{\cos.^2\psi_{,}}$;* we [1904]
shall have, at this origin of the curve,†

$$\frac{d\theta_{,}}{ds}=\alpha\cdot\left(\frac{du'_{,}}{d\varphi}\right)\cdot\frac{\sin.\psi_{,}}{\cos.^2\psi_{,}}. \qquad [1905]$$

The equation [1872] then gives,‡

$$\frac{dd\theta_{,}}{ds^2}=\frac{\cos.\theta_{,}}{r_{,}}\cdot\frac{d\varphi_{,}}{ds}-\alpha\cdot\left(\frac{du'_{,}}{d\psi}\right). \qquad [1906]$$

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* (1373) In [1896″], the plane of zx is taken for the plane of the celestial meridian, corresponding to the first point of the geodetical line; therefore $V_{,}$ [1877′, 1892] is nothing, and the expression of $\varphi_{,}-V_{,}$ [1893] becomes, as in [1904], $\varphi_{,}=\frac{\alpha\cdot\left(\frac{du'_{,}}{d\varphi}\right)}{\cos.^2\psi_{,}}$. [1904a]

† (1374) Changing r, θ, into $r_{,}$, $\theta_{,}$, in [1902], and using c' [1903], we get, at the origin of the curve, $\frac{d\theta_{,}}{ds}=\frac{\sin.\varphi_{,}}{r_{,}^2}\cdot c'=\frac{\sin.\varphi_{,}}{r_{,}^2}\cdot r_{,}\cdot\cos.\theta_{,}=\frac{\cos.\theta_{,}}{r_{,}}\cdot\sin.\varphi_{,}$. Now $\varphi_{,}$ [1904a] being of the order α, we may change $\sin.\varphi_{,}$ into $\varphi_{,}$, and then using its value [1904a], we get $\frac{d\theta_{,}}{ds}=\alpha\cdot\left(\frac{du'_{,}}{d\varphi}\right)\cdot\frac{\cos.\theta_{,}}{r_{,}\cdot\cos.^2\psi_{,}}$. At the point H, fig. 45, $\theta_{,}$ differs from the complement [1905a] of the latitude $\psi_{,}$ by quantities of the order α [1876‴]; therefore if we neglect α^2, we may put $\cos.\theta_{,}=\sin.\psi_{,}$, also $r_{,}=1$, [1850c], in the preceding expression, and then it becomes as in [1905].

‡ (1375) We must substitute $c''=0$ [1901a], in [1872], then take the differential, observing that ds is constant [1848′]; we must also neglect terms of the order α^2. This calculation may be much simplified, by observing that at the commencement H of the curve, φ, or $\varphi_{,}$, [1902b], is of the order α; so that in terms multiplied by α, after taking the differential, we may put $\cos.\varphi=1$, and neglect the terms multiplied by $\alpha\cdot\sin.\varphi$. Now [1906a] if we represent by $\int Uds$, $\int U'ds$, the terms under the sign $\int$ in the second and third lines of [1872], this formula will become,

$$r^2d\theta=c'ds\cdot\sin.\varphi-\alpha ds\cdot\cos.\varphi\cdot\int Uds-\alpha ds\cdot\sin.\varphi\cdot\int U'ds.$$

Its differential is,

$$2rdr\cdot d\theta+r^2\cdot dd\theta$$

$$=c'ds\cdot d\varphi\cdot\cos.\varphi+\alpha ds.d\varphi.\sin.\varphi.\int Uds-\alpha ds^2.U.\cos.\varphi-\alpha ds.d\varphi.\cos.\varphi.\int U'ds-\alpha ds^2.U'.\sin.\varphi \qquad [1906b]$$

Reducing this, by means of the values of φ [1906a], and observing that $\int Uds=0$, [1906c]

but we have,*

[1907] $$\frac{d\varphi_{,}}{ds}=\frac{1}{r_{,}\,.\,\sin.\theta_{,}}\,;\qquad r_{,}=1+\alpha u_{,}'\,;\qquad \theta_{,}=100^{\circ}-\psi_{,}-\alpha\,.\left(\frac{du_{,}'}{d\psi}\right);$$

therefore we shall get,†

[1908] $$\frac{dd\theta_{,}}{ds^{2}}=(1-2\alpha u_{,}')\,.\,\text{tang.}\,\psi_{,}+\alpha\,.\left(\frac{du_{,}'}{d\psi}\right).\,\text{tang.}^{2}\,\psi_{,}.$$

Now by observing, that at the beginning of the curve, we have

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$\int U'ds=0$, at the commencement of the integral [1900b], we get, for the second member, $c'ds\,.\,d\varphi-\alpha ds^{2}\,.\,U$. If we neglect in U the terms multiplied by $\sin.\varphi$, and put in the other term $\cos.\varphi=1$, we shall find $U=\left(\frac{du'}{d\psi}\right)$; and the equation [1906b] will become, at the first point of the curve,

[1906d] $$2r_{,}dr_{,}\,.\,d\theta_{,}+r_{,}^{2}\,.\,dd\theta_{,}=c'ds\,.\,d\varphi_{,}-\alpha ds^{2}\,.\left(\frac{du_{,}'}{d\psi}\right).$$

The terms $dr_{,}$, $d\theta_{,}$, [1850c, 1905], are each of the order α, in comparison with ds; therefore we may neglect the first term of [1906d]. Then dividing by $r_{,}^{2}ds^{2}$, and using

[1906e] c' [1903], we get $\frac{dd\theta_{,}}{ds^{2}}=\frac{\cos.\theta_{,}}{r_{,}}\,.\,\frac{d\varphi_{,}}{ds}-\frac{\alpha}{r_{,}^{2}}\,.\left(\frac{du_{,}'}{d\psi}\right)$. Putting $r_{,}^{2}=1$, in this last term, we finally obtain [1906].

* (1376) The first of the equations [1907] is easily deduced from [1900], the second

[1906f] from [1850c], the third from [1874], substituting $\left(\frac{dr_{,}}{d\psi}\right)=\alpha\,.\left(\frac{du_{,}'}{d\psi}\right)$, deduced from [1850c], for the first point of the curve.

† (1377) Substituting, in [1906] the values of $\frac{d\varphi_{,}}{ds}$, $r_{,}$, [1907], we get successively,

$$\frac{dd\theta_{,}}{ds^{2}}=\frac{\cos.\theta_{,}}{r_{,}^{2}\,.\,\sin.\theta_{,}}-\alpha\,.\left(\frac{du_{,}'}{d\psi}\right)=\frac{\text{cotang.}\,\theta_{,}}{(1+\alpha u_{,}')^{2}}-\alpha\,.\left(\frac{du_{,}'}{d\psi}\right)$$

[1907a] $$=(1-2\alpha u_{,}')\,.\,\text{cotang.}\,\theta_{,}-\alpha\,.\left(\frac{du_{,}'}{d\psi}\right).$$

But from $\theta_{,}$ [1907] we get, by development [617], and using [54] Int.,

[1907b] $$\text{cotang.}\,\theta_{,}=\text{tang.}\left\{\psi_{,}+\alpha\,.\left(\frac{du_{,}'}{d\psi}\right)\right\}=\text{tang.}\,\psi_{,}+\alpha\,.\,(1+\text{tang.}^{2}\,\psi_{,})\,.\left(\frac{du_{,}'}{d\psi}\right).$$

Substituting this in [1907a], and reducing, we get [1908].

$$\frac{d\varphi_{,}}{ds}=\frac{1}{r_{,}\,.\,\sin.\theta_{,}}=\frac{1}{\cos.\psi_{,}}\,.\left\{1-\alpha u_{,}'+\alpha\,.\left(\frac{du_{,}'}{d\psi}\right).\,\mathrm{tang.}\,\psi_{,}\right\},^{*} \qquad [1909]$$

the equation [1867] will give,†

$$c=r_{,}\,.\,\sin.\theta_{,}. \qquad [1910]$$

Hence we deduce,‡

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* (1378) The value of $\theta_{,}$ [1907] gives, by using [61] Int.,

$$\sin.\theta_{,}=\cos.\left\{\psi_{,}+\alpha\,.\left(\frac{du_{,}'}{d\psi}\right)\right\}=\cos.\psi_{,}-\alpha\,.\left(\frac{du'}{d\psi}\right).\sin.\psi_{,}$$

$$=\cos.\psi_{,}\,.\left\{1-\alpha\,.\left(\frac{du_{,}'}{d\psi}\right).\,\mathrm{tang.}\,\psi_{,}\right\}; \qquad [1909a]$$

hence $\dfrac{1}{\sin.\theta_{,}}=\dfrac{1}{\cos.\psi_{,}}\,.\left\{1+\alpha\,.\left(\dfrac{du_{,}'}{d\psi}\right).\,\mathrm{tang.}\,\psi_{,}\right\}$, neglecting terms of the order α^2.

Substituting this, and $\dfrac{1}{r_{,}}=(1-\alpha u_{,}')$) [1907], in the first equation [1907], it becomes [1909b]
as in [1909].

† (1379) Since the integral, in the second member of [1867], commences with s, [1900b], it will vanish at the first point of the curve; and we shall have, at that point, $r_{,}^2 d\varphi_{,}\,.\sin.^2\theta_{,}=c\,ds$. Dividing this by ds, and substituting $\dfrac{d\varphi_{,}}{ds}$ [1907], we get c [1910]. [1909c]

‡ (1380) Taking the differential of [1867], writing $r_{,}$, $\varphi_{,}$, $\theta_{,}$, $u_{,}'$, for r, φ, θ, u', in order to obtain the values corresponding to the first point of the curve, we get,

$$2r_{,}dr_{,}\,.\,d\varphi_{,}\,.\sin.^2\theta_{,}+r_{,}^2\,.\,dd\varphi_{,}\,.\sin.^2\theta_{,}+2r_{,}^2\,.\,d\varphi_{,}\,.\,d\theta_{,}\,.\sin.\theta_{,}\,.\cos.\theta_{,}=\alpha\,ds^2\,.\left(\frac{du_{,}'}{d\varphi}\right).$$

Dividing this by $r_{,}^2\,.\,ds^2\,.\sin.^2\theta_{,}$, we get, by transposition,

$$\frac{dd\varphi_{,}}{ds^2}=-2\,.\,\frac{dr_{,}}{r_{,}ds}\,.\,\frac{d\varphi_{,}}{ds}-2\,.\,\frac{d\varphi_{,}}{ds}\,.\,\frac{d\theta_{,}}{ds}\,.\,\frac{\cos.\theta_{,}}{\sin.\theta_{,}}+\frac{\alpha\,.\left(\frac{du_{,}'}{d\varphi}\right)}{r_{,}^2\,.\sin.^2\theta_{,}}.$$

Now the differential of r [1850c], divided by ds, gives, at the first point of the curve, $\dfrac{dr_{,}}{ds}=\alpha\,.\,\dfrac{du_{,}'}{ds}$; and by [1907], $\dfrac{d\varphi_{,}}{ds}=\dfrac{1}{r_{,}\,.\sin.\theta_{,}}$. Substituting these, we obtain, [1910a]

$$\frac{dd\varphi_{,}}{ds^2}=-\frac{2\alpha\,.\,\frac{du_{,}'}{ds}}{r_{,}^2\,.\sin.\theta_{,}}-\frac{2\,.\,\frac{d\theta_{,}}{ds}\,.\cos.\theta_{,}}{r_{,}\,.\sin.^2\theta_{,}}+\frac{\alpha\,.\left(\frac{du_{,}'}{d\varphi}\right)}{r_{,}^2\,.\sin.^2\theta_{,}};$$

and as the third term of the second member is of the order α, we may, by neglecting α^2, put $r=1$, and $\sin.\theta_{,}=\cos.\psi_{,}$, [1850c, 1909a]; and then the preceding expression becomes as in [1911].

[1911] $$\frac{d\,d\,\varphi_{,}}{d\,s^2} = -\,\frac{2\,\alpha\,.\,\frac{d\,u_{,}'}{d\,s}}{r_{,}^2\,.\,\sin.\,\theta_{,}} - \frac{2\,.\,\frac{d\,\theta_{,}}{d\,s}\,.\,\cos.\,\theta_{,}}{r_{,}\,.\,\sin.^2\,\theta_{,}} + \frac{\alpha\,.\left(\frac{d\,u_{,}'}{d\,\varphi}\right)}{\cos.^2\,\psi_{,}}\,;$$

consequently,*

[1912] $$\frac{d\,d\,\varphi_{,}}{d\,s^2} = -\,\alpha\,.\left(\frac{d\,u_{,}'}{d\,\varphi}\right).\,\frac{(2-\cos.^2\,\psi_{,})}{\cos.^4\,\psi_{,}}\,.$$

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* (1381) The first equation [1907], multiplied by $\frac{d\,u_{,}'}{d\,\varphi_{,}}$, gives, $\frac{d\,u_{,}'}{d\,s} = \frac{\frac{d\,u_{,}'}{d\,\varphi_{,}}}{r_{,}\,.\,\sin.\,\theta_{,}}$. Substituting this, also $\frac{d\,\theta_{,}}{d\,s}$ [1905], in [1911], it becomes,

$$\frac{d\,d\,\varphi_{,}}{d\,s^2} = -\,\frac{2\,\alpha\,.\,\frac{d\,u_{,}'}{d\,\varphi_{,}}}{r_{,}^3\,.\,\sin.^2\,\theta_{,}} - \frac{2\,\alpha\,.\left(\frac{d\,u_{,}'}{d\,\varphi}\right).\,\sin.\,\psi_{,}\,.\,\cos.\,\theta_{,}}{r_{,}\,.\,\cos.^2\,\psi_{,}\,.\,\sin.^2\,\theta_{,}} + \frac{\alpha\,.\left(\frac{d\,u_{,}'}{d\,\varphi}\right)}{\cos.^2\,\psi_{,}}\,;$$

and as each term of the second member is of the order α, we may, by neglecting α^2, put
[1912a] $r_{,}=1$, $\sin.\,\theta_{,}=\cos.\,\psi_{,}$, $\cos.\,\theta_{,}=\sin.\,\psi_{,}$, [1907]; hence

[1912b] $$\frac{d\,d\,\varphi_{,}}{d\,s^2} = -\,\frac{2\,\alpha\,.\,\frac{d\,u_{,}'}{d\,\varphi_{,}}}{\cos.^2\,\psi_{,}} - \frac{2\,\alpha\,.\left(\frac{d\,u_{,}'}{d\,\varphi}\right).\,\sin.^2\,\psi_{,}}{\cos.^4\,\psi_{,}} + \frac{\alpha\,.\left(\frac{d\,u_{,}'}{d\,\varphi}\right)}{\cos.^2\,\psi_{,}}\,.$$

Now u' being a function of r, θ, φ, its complete differential is

[1912c] $$d\,u' = \left(\frac{d\,u'}{d\,\varphi}\right).\,d\,\varphi + \left(\frac{d\,u'}{d\,r}\right).\,d\,r + \left(\frac{d\,u'}{d\,\theta}\right).\,d\,\theta.$$

Dividing this by $d\,\varphi_{,}$, and then writing $u_{,}'$, for u', &c., we get,

[1912d] $$\frac{d\,u_{,}'}{d\,\varphi_{,}} = \left(\frac{d\,u_{,}'}{d\,\varphi}\right) + \left(\frac{d\,u_{,}'}{d\,r}\right).\,\frac{d\,r_{,}}{d\,\varphi_{,}} + \left(\frac{d\,u_{,}'}{d\,\theta}\right).\,\frac{d\,\theta_{,}}{d\,\varphi_{,}}.$$

But the first equation [1907] shows that $d\,\varphi_{,}$ is of the same order as $d\,s$; therefore $\frac{d\,r_{,}}{d\,\varphi_{,}}$ is of the same order as $\frac{d\,r_{,}}{d\,s}$; and since, by [1850c], $d\,r_{,}$ is of the order α, $\frac{d\,r_{,}}{d\,\varphi_{,}}$ must be of the order α. In like manner, $\frac{d\,\theta_{,}}{d\,\varphi_{,}}$ is of the same order as $\frac{d\,\theta_{,}}{d\,s}$; which, by [1905], is of the order α. Hence, if we multiply [1912d] by α, and neglect α^2, we shall get,

[1912e] $$\alpha\,.\,\frac{d\,u_{,}'}{d\,\varphi} = \alpha\,.\left(\frac{d\,u_{,}'}{d\,\varphi}\right).$$

Substituting this in [1912b], we obtain,

$$\frac{d\,d\,\varphi_{,}}{d\,s^2} = -\,\alpha\,.\left(\frac{d\,u_{,}'}{d\,\varphi}\right).\,\frac{1}{\cos.^4\,\psi_{,}}\,.\,\{2\,.\,\cos.^2\,\psi_{,} + 2\,.\,\sin.^2\,\psi_{,} - \cos.^2\,\psi_{,}\}\,;$$

and since $2\,.\,\cos.^2\,\psi_{,} + 2\,.\,\sin.^2\,\psi_{,} = 2$, it may be reduced to the form [1912].

The equation [1874f—f'],

$$\theta = 100^\circ - \psi - \alpha \cdot \left(\frac{d\,u'}{d\,\psi}\right), \qquad [1913]$$

gives, by retaining, among the terms of the order s^2, only those which are independent of α,* [1913']

$$\psi - \psi_{,} = -s \cdot \frac{d\,\theta_{,}}{d\,s} - \tfrac{1}{2}\,s^2 \cdot \frac{d\,d\,\theta_{,}}{d\,s^2} - \frac{\alpha\,s}{\cos.\psi_{,}} \cdot \left(\frac{d\,d\,u_{,}'}{d\,\varphi\,d\,\psi}\right); \qquad [1914]$$

therefore,†

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* (1382) Putting for brevity $U = \theta + \alpha \cdot \left(\frac{d\,u'}{d\,\psi}\right)$, we shall obtain, from [1913], [1914a] $\psi - 100^\circ = -U$; and at the commencement of the arc, $\psi_{,} - 100^\circ = -U_{,}$; hence $\psi - \psi_{,} = U_{,} - U$. Now U may be considered as a function of the constant quantities $\theta_{,}$, $\varphi_{,}$, &c., and the variable quantity s. This may be developed according to the powers of s [607], in the form $U = U_{,} + s \cdot \frac{d\,U_{,}}{ds} + \tfrac{1}{2} \cdot s^2 \cdot \frac{ddU_{,}}{ds^2}$, neglecting s^3; hence $\psi - \psi_{,} = -s \cdot \frac{dU_{,}}{ds} - \tfrac{1}{2}\,s^2 \cdot \frac{d\,d\,U_{,}}{d\,s^2}$. The first and second differentials of the equation [1914b] $U = \theta + \alpha \cdot \left(\frac{d\,u'}{d\,\psi}\right)$, multiplying them by $-\frac{s}{ds}$, $-\frac{s^2}{2\,ds^2}$, respectively, and then accenting U, θ, u', give,

$$-s \cdot \frac{dU_{,}}{ds} = -s \cdot \frac{d\,\theta_{,}}{d\,s} - \frac{\alpha\,s}{d\,s} \cdot d \cdot \left(\frac{d\,u_{,}'}{d\,\psi}\right), \qquad -\tfrac{1}{2}\,s^2 \cdot \frac{d\,d\,U_{,}}{d\,s^2} = -\tfrac{1}{2}\,s^2 \cdot \frac{d\,d\,\theta_{,}}{d\,s^2},$$

neglecting terms of a less order. Substituting these in [1914b], we get,

$$\psi - \psi_{,} = -s \cdot \frac{d\,\theta_{,}}{ds} - \tfrac{1}{2}\,s^2 \cdot \frac{d\,d\,\theta_{,}}{d\,s^2} - \frac{\alpha\,s}{d\,s} \cdot d \cdot \left(\frac{d\,u_{,}'}{d\,\psi}\right). \qquad [1914c]$$

In the last term of this expression, which is of the order $\alpha\,s$, we may, by using the first of the equations [1907], put $d\,s = r_{,} \cdot \sin.\theta_{,} \cdot d\,\varphi_{,} = \cos.\psi_{,} \cdot d\,\varphi_{,}$ nearly; by this means, it becomes, $-\frac{\alpha\,s}{\cos.\psi_{,} \cdot d\,\varphi_{,}} \cdot d \cdot \left(\frac{d\,u_{,}'}{d\,\psi}\right)$. Now by proceeding as in [1912c—e], we find, that if we change $u_{,}'$ into $\left(\frac{d\,u_{,}'}{d\,\psi}\right)$, we may change $\frac{1}{d\,\varphi_{,}} \cdot d \cdot \left(\frac{d\,u_{,}'}{d\,\psi}\right)$, into $\left(\frac{d\,d\,u_{,}'}{d\,\varphi\,d\,\psi}\right)$, neglecting terms of the order $\alpha^2 s$, &c.; hence the expression [1914c] becomes as in [1914].

† (1383) Substituting in [1914] the value [1905], also $\frac{d\,d\,\theta_{,}}{d\,s^2} = \text{tang.}\,\psi_{,}$ [1908], neglecting α^2, αs^2, &c., and putting $\frac{\sin.\psi_{,}}{\cos.\psi_{,}} = \text{tang.}\,\psi_{,}$, it becomes as in [1915]. In [1915a]

Difference of latitude of the extremities of the geodetical curve.

[1915] $$\psi - \psi_{,} = -\frac{\alpha s}{\cos.\psi_{,}} \cdot \left\{ \left(\frac{d u'_{,}}{d \varphi}\right) \cdot \text{tang.}\,\psi_{,} + \left(\frac{d d u'_{,}}{d \varphi\, d \psi}\right) \right\} - \tfrac{1}{2} s^2 \cdot \text{tang.}\,\psi_{,}.$$

*The difference of the latitudes, at the two extremities of the measured arc, will therefore give the value of the function,**

[1916] $$-\frac{\alpha s}{\cos.\psi_{,}} \cdot \left\{ \left(\frac{d u'_{,}}{d \varphi}\right) \cdot \text{tang.}\,\psi_{,} + \left(\frac{d d u'_{,}}{d \varphi\, d \psi}\right) \right\}.$$

It is remarkable, that for the same arc, measured in the direction of the meridian, this function is, by what precedes, equal to†

[1917] $$\frac{\varpi}{\text{tang.}\,\psi_{,}};$$

it may thus be determined by both methods; and we can judge whether the values found, by the difference of the latitudes, or by the azimuth angle ϖ, *are produced by the errors of the observations, or by the excentricities of the parallels of latitude of the earth.*

If we retain only the first power of s, we have,‡

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the original work, the factor α was attached to $\left(\frac{d d u'_{,}}{d \varphi\, d \psi}\right)$ in [1915, 1916], making it, by mistake, of the order α^2.

* (1384) $\psi_{,}$, ψ, $\psi - \psi_{,}$ s, being known from observation, we may, by means of

[1916a] [1915], obtain the function [1916], which is the first term of the second member of that expression.

† (1385) Dividing [1891a] by $\text{tang.}\,\psi_{,}$, we get,

[1917a] $$\frac{\varpi}{\text{tang.}\,\psi_{,}} = -\frac{\alpha \varepsilon}{\cos.\psi_{,}} \cdot \left\{ \left(\frac{d u'_{,}}{d \varphi}\right) \cdot \text{tang.}\,\psi_{,} + \left(\frac{d d u'_{,}}{d \varphi\, d \psi}\right) \right\};$$

in which the coefficient of ε, in the second member, is the same as that of s in [1916]. Hence this coefficient may be determined, either by the value of ϖ, or by that of $\psi - \psi_{,}$.

‡ (1386) Developing φ, according to the powers of s, in the same manner as we have done with U, in [1914b], we get $\varphi = \varphi_{,} + s \cdot \frac{d\varphi_{,}}{ds} + \frac{1}{2} s^2 \cdot \frac{dd\varphi_{,}}{ds^2} + \&c.$ If we retain

[1918a] only the first power of s, it will become $\varphi - \varphi_{,} = s \cdot \frac{d\varphi_{,}}{ds}$, as in [1918]. Substituting [1909], we get the second expression [1918].

$$\varphi-\varphi_{,}=s\,.\,\frac{d\varphi_{,}}{ds}=\frac{s}{\cos.\psi_{,}}\,.\left\{1-\alpha u_{,}'+\alpha\,.\left(\frac{du_{,}'}{d\psi}\right).\,\text{tang.}\,\psi_{,}\right\}. \qquad [1918]$$

Difference of the angles φ, $\varphi_{,}$, neglecting s^2.

$\varphi-\varphi_{,}$ *is not the difference of longitude of the two extremities of the arc* s; this difference is equal to $V-V_{,}$.* Now we have, by what precedes, [1885],

$$\varphi-V=\frac{\alpha\,.\left(\frac{du'}{d\varphi}\right)}{\cos.^2\psi}; \qquad [1919]$$

which gives,†

$$\varphi-V-(\varphi_{,}-V_{,})=\frac{\alpha s\,.\left(\frac{ddu_{,}'}{d\varphi ds}\right)}{\cos.^2\psi_{,}}=\frac{\alpha s\,.\left(\frac{ddu_{,}'}{d\varphi^2}\right)}{\cos.^3\psi_{,}}; \qquad [1920]$$

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* (1387) This is evident, because the difference of longitude of two places, usually determined by celestial observations, must be equal to the angular inclination of the *celestial* [1919a] meridians $V-V_{,}$.

† (1388) If we put, for brevity, the second member of the expression [1919] equal to φ', we shall have $(\varphi-V)=\varphi'$. Developing this second member, according to the [1921a] powers of s, and retaining only the first power of s, as in [1918a], we shall get,

$$\varphi-V=\varphi_{,}'+s\,.\,\frac{d\varphi_{,}'}{ds}, \qquad \text{and} \qquad \varphi_{,}-V_{,}=\varphi_{,}';$$

hence we obtain $\varphi-V-(\varphi_{,}-V_{,})=s\,.\,\frac{d\varphi_{,}'}{ds}$. Now $\varphi'=\alpha\,.\left(\frac{du'}{d\varphi}\right).(\cos.\psi)^{-2}$ is a [1921b] function of r, θ, φ, of the order α; these three last quantities depending respectively on $r_{,}$, $\theta_{,}$, $\varphi_{,}$, and the variable quantity s; but from [1905, 1907, 1915], the variations of θ, r, ψ, are the order α, or of the order s^2; which will produce only terms of the order α^2, or αs^2, in the second member of [1921b], and they may therefore be neglected. Hence it will only be necessary to consider φ as variable, in the second member of [1921b], and we shall have $d\varphi_{,}'=\left(\frac{d\varphi_{,}'}{d\varphi}\right).d\varphi$; therefore the equation [1921b] will become,

$$\varphi-V-(\varphi_{,}-V_{,})=s\,.\,\frac{d\varphi_{,}'}{ds}=s\,.\left(\frac{d\varphi_{,}'}{d\varphi}\right).\,\frac{d\varphi_{,}}{ds}. \qquad [1921c]$$

Substituting in this the value of $\varphi_{,}'$, corresponding to the second member of [1919]; also the value of $\frac{d\varphi_{,}}{ds}=\frac{1}{\cos.\psi_{,}}$, deduced from the equations [1907], we obtain the two expressions [1920]. Subtracting this from [1918], we get [1921].

therefore,

[1921] $$V-V_{,}=\frac{s}{\cos.\psi_{,}}\cdot\left\{1-\alpha u'_{,}+\alpha\cdot\left(\frac{du'_{,}}{d\psi}\right)\cdot\text{tang.}\psi_{,}-\frac{\alpha\cdot\left(\frac{ddu'_{,}}{d\varpi^{2}}\right)}{\cos.^{2}\psi_{,}}\right\}.$$

Difference of longitude, neglecting αs^2, and s^3.

For greater accuracy, we must add to this value of $V-V_{,}$, the term depending on s^3, which is independent of α, and may be computed upon the hypothesis that the earth is spherical; this term is equal to*

[1922] $$-\tfrac{1}{3}s^{3}\cdot\frac{\text{tang.}^{2}\psi_{,}}{\cos.\psi_{,}};$$

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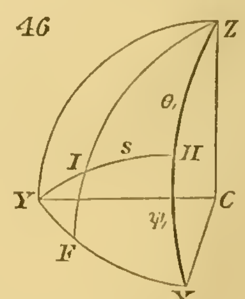
* (1389) In computing the value of $\varphi-\varphi_{,}$ [1918a], we have neglected the terms

[1922a] $$\tfrac{1}{2}s^{2}\cdot\frac{dd\varphi_{,}}{ds^{2}}+\tfrac{1}{6}s^{3}\cdot\frac{d^{3}\varphi_{,}}{ds^{3}}+\&c.;$$

therefore terms of the same orders are neglected in [1921]. Now the term $\tfrac{1}{2}s^{2}\cdot\frac{dd\varphi_{,}}{ds^{2}}$ is of the order αs^2, as is evident from [1912], and this is neglected in [1913'], on account of the smallness of α and s; α being of the same order as the ellipticity of the earth, or about $\frac{1}{300}$; and s, in some of the great surveys, as that in France, may be 6 or 8 degrees, or $\frac{1}{10}$ of the radius of the earth, taken as unity; so that αs^2 may be of the order

[1922b] $\frac{1}{300\,.\,10^{2}}$, but is generally much less; it is therefore neglected. We shall soon see that [1921] contains also a term depending on s^3, which is not multiplied by α; and as s^3 may be of the order $\frac{1}{10^{3}}$, it ought to be retained. To compute this term, we shall suppose the part of $V-V_{,}$ [1923], depending on s^3, to be developed according to the powers of α, in the

[1922c] form $P\,.\,s^{3}+P'\,.\,\alpha s^{3}+P''\,.\,\alpha^{2}s^{3}+\&c.$; P, P', &c., being independent of α. If $\alpha=0$, this becomes simply $P\,.\,s^3$, and the spheroid becomes a sphere; therefore this term, computed on the supposition that the earth is spherical, may be substituted in [1921]. In a spherical surface, fig. 46, the arcs ZHX, ZIF, HIY, become great circles, forming the right angled spherical triangle ZHI, the angle ZHI being a right angle, [1896'']; and from [1345³¹], we get,



[1922d] $$\cot.IZH=\cot.IH\,.\,\sin.ZH,\qquad\text{or}$$
$$\cot.(V-V_{,})=\cot.s\,.\,\sin.\theta_{,}=\cot.s\,.\,\cos.\psi_{,};$$

hence $\text{tang.}(V-V_{,})=\frac{\text{tang.}s}{\cos.\psi_{,}}$. If we, for a moment, put $\frac{\text{tang.}s}{\cos.\psi_{,}}=c$, it becomes $\text{tang.}(V-V_{,})=c$; and by using [48, 45] Int., retaining terms of the order c^3, inclusively, we get,

hence we shall have, Difference of longitude, including a term of the order s^3.

$$V-V_{,}=\frac{s}{\cos.\psi_{,}}\cdot\left\{1-\alpha u_{,}'+\alpha.\left(\frac{du_{,}'}{d\psi}\right).\text{tang.}\psi_{,}-\frac{\alpha.\left(\frac{ddu_{,}'}{d\varphi^2}\right)}{\cos.^2\psi_{,}}-\tfrac{1}{3}s^2.\text{tang.}^2\psi_{,}\right\}. \qquad [1923]$$

There remains yet to be determined, the azimuth angle at the end of the arc s. For this purpose, we shall put x', y', for the values of the co-ordinates x, y, *referred to the celestial meridian corresponding to the end of the arc* s; [1923′]
and it is then evident, that the cosine of the azimuth angle is equal to* $\frac{\sqrt{dx'^2+dz^2}}{ds}$. If we refer the co-ordinates x, y, to the plane of the celestial [1924]
meridian corresponding to the origin of the arc, its first side being supposed perpendicular to the plane of the meridian [1896″], we shall have,†

$$V-V_{,}=e-\tfrac{1}{3}e^3=\frac{\text{tang.}s}{\cos.\psi_{,}}-\tfrac{1}{3}.\frac{\text{tang.}^3s}{\cos.^3\psi_{,}}=\frac{s+\frac{1}{3}s^3}{\cos.\psi_{,}}-\tfrac{1}{3}.\frac{s^3}{\cos.^3\psi_{,}}=\frac{s}{\cos.\psi_{,}}-\frac{\frac{1}{3}s^3}{\cos.^3\psi_{,}}.(-\cos.^2\psi_{,}+1)$$
$$=\frac{s}{\cos.\psi_{,}}-\tfrac{1}{3}s^3.\frac{\sin.^2\psi_{,}}{\cos.^3\psi_{,}}=\frac{s}{\cos.\psi_{,}}-\tfrac{1}{3}s^3.\frac{\text{tang.}^2\psi_{,}}{\cos.\psi_{,}}. \qquad [1922e]$$

The term of this expression depending on s^3, is equal to $P.s^3$ [1922c], to be added to [1921], to obtain [1923], in conformity to the remark [1921—1922]. This expression of $V-V_{,}$ contains no term of the order s^2, which is agreeable to what has been shown in [1922f]
[1922a—b], where the terms of the order s^2 are multiplied by α.

* (1390) The sine of the angle, formed by the first point of the geodetical line, and the line drawn parallel to the axis of y, is $\frac{\sqrt{(dx^2+dz^2)}}{ds}$ [1897d], being the same as the cosine [1926a]
of the angle formed by the geodetical line and the plane of xz, as in [1897]. In like manner, the cosine of the angle formed by the last point I of the same geodetical line, fig. 45, page 386, and the plane of $x'z$, is $\frac{\sqrt{(dx'^2+dz^2)}}{ds}$; this plane being that [1926b]
corresponding to the celestial meridian of this point [1923′]. Now the azimuth angle is that formed by the geodetical line and the celestial meridian; therefore the cosine of this azimuth angle is equal to $\frac{\sqrt{(dx'^2+dz^2)}}{ds}$, as in [1924].

† (1391) The co-ordinates of the first point H of the geodetical line, fig. 45, page 386, being represented by $x_{,}$, $y_{,}$, $z_{,}$, we shall have, by [1897′], $\frac{dx_{,}}{ds}=0$; $\frac{dz_{,}}{ds}=0$. Substituting these in [1897c—d], $\frac{dx_{,}^2}{ds^2}+\frac{dy_{,}^2}{ds^2}+\frac{dz_{,}^2}{ds^2}=1$, we get $\frac{dy_{,}^2}{ds^2}=1$, or $\frac{dy_{,}}{ds}=1$, as in [1925]. [1926c]

[1925] $$\frac{dx_{\prime}}{ds}=0; \qquad \frac{dz_{\prime}}{ds}=0; \qquad \frac{dy_{\prime}}{ds}=1;$$

consequently, by retaining only the first power of s, we obtain,*

[1926] $$\frac{dx}{ds}=s\,.\,\frac{ddx_{\prime}}{ds^2}; \qquad \frac{dz}{ds}=s\,.\,\frac{ddz_{\prime}}{ds^2}.$$

Now we have,†

[1927] $$x'=x\,.\,\cos.(V-V_{\prime})+y\,.\,\sin.(V-V_{\prime});$$

therefore, $V-V_{\prime}$ being of the order s, as appears by what has been said [1921], we shall get,‡

[1928] $$\frac{dx'}{ds}=s\,.\,\frac{ddx_{\prime}}{ds^2}+(V-V_{\prime})\,.\,\frac{dy_{\prime}}{ds}.$$

Now we have, [1851],

[1929] $$x=r\,.\,\sin.\theta\,.\,\cos.\varphi; \qquad z=r\,.\,\cos.\theta;$$

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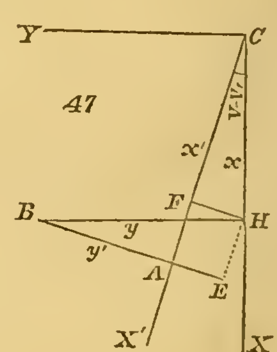
* (1392) The expression of $\frac{dx}{ds}$, corresponding to any point of the geodetical line, may be considered as a function of the value of that quantity, at the commencement of the curve, considered as given, and the variable quantity s. This may be developed according to the powers of s, as in [607, 608]; and if we notice only the first power, we shall have

[1926d] $\frac{dx}{ds}=\frac{dx_{\prime}}{ds}+s\,.\,\frac{ddx_{\prime}}{ds^2}$. This, by means of the first equation [1925], becomes as in the first

[1926e] equation [1926]. In like manner we get $\frac{dz}{ds}$ [1926]; also $\frac{dy}{ds}=\frac{dy_{\prime}}{ds}+s\,.\,\frac{ddy_{\prime}}{ds^2}$.

† (1393) Changing the rectangular co-ordinates x, y, into x', y', as in [586], or in the adjoined figure, which is similar to that in Vol. I, page 351; making also the angle of inclination of

[1926f] the axes x, x', equal to $V-V_{\prime}$ instead of θ; the first of the equations [586] will become as in [1927]. Now $V-V_{\prime}$ being of the order s [1921], if we neglect s^2, we may put, by [43, 44] Int.,



[1926g] $$\cos.(V-V_{\prime})=1; \qquad \sin.(V-V_{\prime})=V-V_{\prime};$$

hence [1927] will become,

[1927a] $$x'=x+(V-V_{\prime})\,.\,y.$$

‡ (1394) In the equation [1927a], the co-ordinates x, y, x', are supposed to

[1928a] correspond to any point whatever of the geodetical line, and are therefore *variable;* while

and if we neglect quantities of the order α^2; observing that $\varphi_{,}$, $\frac{d\,d\,\varphi_{,}}{d\,s^2}$, and $\frac{d\,\theta_{,}}{d\,s}$, are quantities of the order α; we shall find,* [1930]

$$\frac{d\,d\,x_{,}}{d\,s^2}=\alpha\,.\,\frac{d\,d\,u'_{,}}{d\,s^2}\,.\,\sin.\,\theta_{,}+r_{,}\,.\,\frac{d\,d\,\theta_{,}}{d\,s^2}\,.\,\cos.\,\theta_{,}-r_{,}\,.\,\sin.\,\theta_{,}\,.\,\frac{d\,\varphi_{,}^2}{d\,s^2}\,. \qquad [1931]$$

We then have,†

V, $V_{,}$, correspond to the *extremities* of that curve, and are *constant*. Taking the differential of this equation, and dividing by $d\,s$, we get $\frac{d\,x'}{d\,s}=\frac{d\,x}{d\,s}+(V-V_{,})\,.\,\frac{d\,y}{d\,s}$. Substituting $\frac{d\,x}{d\,s}$ [1926], $\frac{d\,y}{d\,s}$ [1926e], and neglecting the term $s\,.\,(V-V_{,})\,.\,\frac{d\,d\,y_{,}}{d\,s^2}$, [1928b] which, by [1921], is of the order s^2, it becomes as in [1928].

* (1395) The differential of x [1929] produces the first of the equations [1851a]. Its differential being again taken, gives,

$$\begin{aligned} d\,d\,x= & \,d\,d\,r\,.\,\sin.\,\theta\,.\,\cos.\,\varphi+r\,.\,d\,d\,\theta\,.\,\cos.\,\theta\,.\,\cos.\,\varphi-r\,.\,\sin.\,\theta\,.\,\cos.\,\varphi\,.\,d\,\varphi^2 \\ & +2\,d\,r\,.\,d\,\theta\,.\,\cos.\,\theta\,.\,\cos.\,\varphi-2\,d\,r\,.\,d\,\varphi\,.\,\sin.\,\theta\,.\,\sin.\,\varphi-2\,r\,.\,d\,\varphi\,.\,d\,\theta\,.\,\cos.\,\theta\,.\,\sin.\,\varphi \\ & -r\,.\,d\,\theta^2\,.\,\sin.\,\theta\,.\,\cos.\,\varphi-r\,.\,d\,d\,\varphi\,.\,\sin.\,\theta\,.\,\sin.\,\varphi. \end{aligned} \qquad [1929a]$$

Dividing this by $d\,s^2$, and putting the accents below r, θ, φ, x, we shall obtain the value of $\frac{d\,d\,x_{,}}{d\,s^2}$; which may be much simplified, by neglecting terms of the order α^2. For by [1904a], $\varphi_{,}$, or $\sin.\,\varphi_{,}$, is of the order α; so that if we neglect α^2, we may put $\cos.\,\varphi_{,}=1$. Moreover, $\frac{d\,r_{,}}{d\,s}$ [1910a], $\frac{d\,\theta_{,}}{d\,s}$ [1905], and $\frac{d\,d\,\varphi_{,}}{d\,s^2}$ [1912], are each of [1929b] the order α; hence the second member of [1929a] is reduced to its three first terms, namely,

$$\frac{d\,d\,x_{,}}{d\,s^2}=\frac{d\,d\,r_{,}}{d\,s^2}\,.\,\sin.\,\theta_{,}+r_{,}\,.\,\frac{d\,d\,\theta_{,}}{d\,s^2}\,.\,\cos.\,\theta_{,}-r_{,}\,.\,\sin.\,\theta_{,}\,.\,\frac{d\,\varphi_{,}^2}{d\,s^2}\,. \qquad [1929c]$$

Substituting in this the second differential of [1850c], divided by $d\,s^2$, which gives $\frac{d\,d\,r_{,}}{d\,s^2}=\alpha\,.\,\frac{d\,d\,u'_{,}}{d\,s^2}$, we obtain the expression [1931]. [1929d]

† (1396) u' is a function of r, φ, ψ, whose complete differential is,

$$d\,u'=\left(\frac{d\,u'}{d\,r}\right).\,d\,r+\left(\frac{d\,u'}{d\,\varphi}\right).\,d\,\varphi+\left(\frac{d\,u'}{d\,\psi}\right).\,d\,\psi. \qquad [1931a]$$

The differential of this, being multiplied by $\frac{\alpha}{d\,s^2}$, gives, putting for brevity $w=\left(\frac{d\,u'}{d\,r}\right)$, $w'=\left(\frac{d\,u'}{d\,\varphi}\right)$, $w''=\left(\frac{d\,u'}{d\,\psi}\right)$, [1931a']

[1932] $$\alpha \cdot \frac{d\,d\,u'_{,}}{d\,s^2} = \alpha \cdot \left(\frac{d\,d\,u'_{,}}{d\,\varphi^2}\right) \cdot \frac{d\,\varphi_{,}^2}{d\,s^2} - \alpha \cdot \left(\frac{d\,u'_{,}}{d\,\psi}\right) \cdot \frac{d\,d\,\theta_{,}}{d\,s^2} = \frac{\alpha \cdot \left(\frac{d\,d\,u'_{,}}{d\,\varphi^2}\right)}{\cos.^2 \psi_{,}} - \alpha \cdot \left(\frac{d\,u'_{,}}{d\,\psi}\right) \cdot \text{tang.}\,\psi_{,}\,;$$

Moreover,*

[1933] $$d\,s = r_{,} \cdot \sin.\,\theta_{,} \cdot d\,\varphi_{,}\,;$$

[1931b] $$\alpha \cdot \frac{d\,d\,u'}{d\,s^2} = \alpha \cdot \left\{ \frac{d\,w}{d\,s} \cdot \frac{d\,r}{d\,s} + \frac{d\,w'}{d\,s} \cdot \frac{d\,\varphi}{d\,s} + \frac{d\,w''}{d\,s} \cdot \frac{d\,\psi}{d\,s} + w \cdot \frac{d\,d\,r}{d\,s^2} + w' \cdot \frac{d\,d\,\varphi}{d\,s^2} + w'' \cdot \frac{d\,d\,\psi}{d\,s^2} \right\}.$$

Changing r, φ, ψ, &c., into $r_{,}$, $\varphi_{,}$, $\psi_{,}$, &c., we shall obtain the value of $\alpha \cdot \frac{d\,d\,u'_{,}}{d\,s^2}$ corresponding to the first point of the curve. Of the six terms of which this expression is composed, four are of the order α^2, and may therefore be neglected; namely, the first and fourth, depending on $\alpha \cdot d\,r$, $\alpha \cdot d\,d\,r$, [1850c]; the fifth, depending on $\alpha \cdot d\,d\,\varphi$, [1912]; and the third, which depends on $\alpha \cdot \frac{d\,\psi_{,}}{d\,s}$. That this third term is of the
[1931c]

[1931d] order α^2, is easily proved from [1874f'], $d\,\psi = -\,d\,\theta - \alpha \cdot d \cdot \left(\frac{d\,u'}{d\,\psi}\right)$; which, by neglecting α^2, gives $\alpha \cdot \frac{d\,\psi_{,}}{d\,s} = -\,\alpha \cdot \frac{d\,\theta_{,}}{d\,s}$, of the order α^2 [1905]. Therefore the expression [1931b] is reduced to the second and sixth terms, depending on $d\,w'$, w''. Now as w' is a function of r, φ, ψ, its complete differential $d\,w'$, is of the same form as that of $d\,u'$ [1931a]; but we may neglect the terms multiplied by $d\,r$, $d\,\psi$, because they are of the order α [1931c], and produce in $\alpha \cdot d\,w'$, only terms of the order α^2; the remaining term is $d\,w' = \left(\frac{d\,w'}{d\,\varphi}\right) \cdot d\,\varphi$; or, by substituting the value of w' [1931a'],

[1931e] $$d\,w' = \left(\frac{d\,d\,u'}{d\,\varphi^2}\right) \cdot d\,\varphi.$$

Substituting this, and $d\,d\,\psi = -\,d\,d\,\theta$ — &c. [1931d], in the second and sixth terms of [1931b], we get $\alpha \cdot \frac{d\,d\,u'}{d\,s^2} = \alpha \cdot \left(\frac{d\,d\,u'}{d\,\varphi^2}\right) \cdot \frac{d\,\varphi^2}{d\,s^2} - \alpha \cdot w'' \cdot \frac{d\,d\,\theta}{d\,s^2}$. Putting the marks below the letters u', φ, θ, &c., so as to correspond to the first point of the curve, and using w'' [1931a'], we shall get the first expression [1932]. As this is of the order α, we may,

[1931f] by neglecting α^2, substitute $\frac{d\,\varphi_{,}}{d\,s} = \frac{1}{\cos.\,\psi_{,}}$ [1909], and $\frac{d\,d\,\theta_{,}}{d\,s^2} = \text{tang.}\,\psi_{,}$ [1908], and we shall get the second expression [1932].

* (1397) This is the same as [1900], marking the letters r, θ, φ, to correspond with the first point of the curve.

therefore, by substituting for $r_{,}$, $\theta_{,}$, $\frac{d\varphi_{,}}{ds}$, $\frac{dd\theta_{,}}{ds^2}$, their preceding values, we shall have,*

$$\frac{ddx_{,}}{ds^2}=(1-\alpha u_{,}')\cdot\frac{\sin.^2\psi_{,}}{\cos.\psi_{,}}+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\text{tang.}^2\psi_{,}\cdot\sin.\psi_{,}$$
$$-\frac{1}{\cos.\psi_{,}}\cdot\left\{1-\alpha u_{,}'+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\text{tang.}\psi_{,}\right\}+\frac{\alpha\cdot\left(\frac{ddu_{,}'}{d\varphi^2}\right)}{\cos.\psi_{,}}. \quad [1934]$$

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* (1398) The first term of the second member of [1931] being of the order α, we may put in it, $\sin.\theta_{,}=\cos.\psi_{,}$ [1907], neglecting α^2; and this term will become, by using [1932], [1934a]

$$\alpha\cdot\frac{ddu_{,}'}{ds^2}\cdot\sin.\theta_{,}=\frac{\alpha}{\cos.\psi_{,}}\cdot\left(\frac{ddu_{,}'}{d\varphi^2}\right)-\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\sin.\psi_{,}. \quad [1934b]$$

The value of θ [1907] gives, by using [60] Int.,

$$\cos.\theta_{,}=\sin.\left\{\psi_{,}+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\right\}=\sin.\psi_{,}+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\cos.\psi_{,}, \quad [1934c]$$

Multiplying this by $r_{,}=1+\alpha u_{,}'$ [1907], we get, by neglecting always α^2,

$$r_{,}\cdot\cos.\theta_{,}=\left\{1+\alpha u_{,}'+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\frac{\cos.\psi_{,}}{\sin.\psi_{,}}\right\}\cdot\sin.\psi_{,};$$

and multiplying this by the expression [1908],

$$\frac{dd\theta_{,}}{ds^2}=\left\{1-2\alpha u_{,}'+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\text{tang.}\psi_{,}\right\}\cdot\text{tang.}\psi_{,}, \quad [1934d]$$

we find,

$$r_{,}\cdot\frac{dd\theta_{,}}{ds^2}\cdot\cos.\theta_{,}=\left\{1-\alpha u_{,}'+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\text{tang.}\psi_{,}+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\frac{\cos.\psi_{,}}{\sin.\psi_{,}}\right\}\cdot\sin.\psi_{,}\cdot\text{tang.}\psi_{,}$$
$$=(1-\alpha u_{,}')\cdot\sin.\psi_{,}\cdot\text{tang.}\psi_{,}+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\text{tang.}^2\psi_{,}\cdot\sin.\psi_{,}+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\sin.\psi_{,}$$
$$=(1-\alpha u_{,}')\cdot\frac{\sin.^2\psi_{,}}{\cos.\psi_{,}}+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\text{tang.}^2\psi_{,}\cdot\sin.\psi_{,}+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\sin.\psi_{,}. \quad [1934e]$$

Using the values of $\frac{d\varphi_{,}}{ds}$ [1907, 1909], we get successively,

$$-r_{,}\cdot\sin.\theta_{,}\cdot\frac{d\varphi_{,}^2}{ds^2}=-\frac{d\varpi_{,}}{ds}=-\frac{1}{\cos.\psi_{,}}\cdot\left\{1-\alpha u_{,}'+\alpha\cdot\left(\frac{du_{,}'}{d\psi}\right)\cdot\text{tang.}\psi_{,}\right\}. \quad [1934f]$$

Adding together the three equations [1934b, e, f], the first member becomes equal to the value of $\frac{ddx_{,}}{ds^2}$ [1931], and the second member becomes as in [1934], observing that the last term of [1934b] is destroyed by the last term of [1934e].

[1934′] Now we have just found, by neglecting the higher powers of s,*

[1935] $$V-V_{,}=\frac{s}{\cos.\psi_{,}}\cdot\left\{1-\alpha u_{,}'+\alpha.\left(\frac{du_{,}'}{d\psi}\right).\,\mathrm{tang}.\psi_{,}-\frac{\alpha.\left(\frac{ddu_{,}'}{d\varphi^2}\right)}{\cos.^2\psi_{,}}\right\};$$

[1936] and $\frac{dy_{,}}{ds}=1$ [1926c]; therefore we have,†

[1937] $$\frac{dx'}{ds}=s.(1-\alpha u_{,}').\frac{\sin.^2\psi_{,}}{\cos.\psi_{,}}+\alpha s.\left(\frac{du_{,}'}{d\psi}\right).\,\mathrm{tang}.^2\psi_{,}.\sin.\psi_{,}-\alpha s.\left(\frac{ddu_{,}'}{d\varphi^2}\right).\frac{\sin.^2\psi_{,}}{\cos.^3\psi_{,}}.$$

We shall in like manner find,‡

[1938] $$\frac{dz}{ds}=-s.(1-\alpha u_{,}').\sin.\psi_{,}-\alpha s.\left(\frac{du_{,}'}{d\psi}\right).\,\mathrm{tang}.^2\psi_{,}.\cos.\psi_{,}+\alpha.\left(\frac{ddu_{,}'}{d\varphi^2}\right).s.\frac{\sin.\psi_{,}}{\cos.^2\psi_{,}};$$

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* (1399) The expression [1935] is the same as [1923], neglecting s^3, α^2.

† (1400) Multiplying [1934] by s, and adding the product to [1935], we get the value of $\frac{dx'}{ds}=s.\frac{ddx_{,}}{ds^2}+(V-V_{,}).\frac{dy_{,}}{ds}$ [1928, 1936], as in [1937]; observing that the

[1935a] terms $\pm\frac{s}{\cos.\psi_{,}}.\left\{1-\alpha u_{,}'+\alpha.\left(\frac{du_{,}'}{d\psi}\right).\,\mathrm{tang}.\psi_{,}\right\}$ destroy each other; and that the coefficient of $-\frac{\alpha s}{\cos.^3\psi_{,}}.\left(\frac{ddu_{,}'}{d\varphi^2}\right)$ is $-\cos.^2\psi_{,}+1=\sin.^2\psi_{,}$.

[1936a] ‡ (1401) The second differential of $z=r.\cos.\theta$ [1851], is

[1936b] $$ddz=ddr.\cos.\theta-2dr.d\theta.\sin.\theta-r.dd\theta.\sin.\theta-r.d\theta^2.\cos.\theta.$$

Changing z, r, θ, into $z_{,}$, $r_{,}$, $\theta_{,}$, respectively, we shall get $ddz_{,}$; multiplying this by $\frac{s}{ds^2}$, we shall obtain the value of $\frac{dz}{ds}$ [1926], neglecting s^2, α^2,

[1936c] $$\frac{dz}{ds}=s.\frac{ddr_{,}}{ds^2}.\cos.\theta_{,}-2s.\frac{dr_{,}}{ds}.\frac{d\theta_{,}}{ds}.\sin.\theta_{,}-sr_{,}.\frac{dd\theta_{,}}{ds^2}.\sin.\theta_{,}-sr_{,}.\frac{d\theta_{,}^2}{ds^2}.\cos.\theta_{,}.$$

We may neglect the second and fourth terms of this expression, because $\frac{d\theta_{,}}{ds}$ [1905], and $\frac{dr_{,}}{ds}$ [1850c], are of the order α; therefore these two terms are of the order α^2. In the

[1936d] first term we have, from [1850c], $ddr_{,}=\alpha.ddu_{,}'$; and as this term is of the order α,

[1936e] we may put $\cos.\theta_{,}=\sin.\psi_{,}$ [1934c], and it will become $\alpha s.\frac{ddu_{,}'}{ds^2}.\sin.\psi_{,}$. The

and the cosine of the azimuth angle, at the end of the arc s, will then become,*

$$s\,.\,\text{tang.}\,\psi_{,}\,.\left\{1-\alpha u_{,}'+\alpha\,.\left(\frac{d u_{,}'}{d\psi}\right).\,\text{tang.}\,\psi_{,}-\frac{\alpha\,.\left(\frac{d d u_{,}'}{d\varphi^2}\right)}{\cos.^2\psi_{,}}\right\}. \qquad [1939]$$

This cosine being very small, it may be taken for the complement of the azimuth angle; consequently this angle is

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only remaining term is the third, $-s r_{,}\,.\,\sin.\theta_{,}\,.\frac{d d\theta_{,}}{d s^2}$; but from [1909] we have,

$$r_{,}\,.\,\sin.\,\theta_{,}=\left\{1+\alpha u_{,}'-\alpha\,.\left(\frac{d u_{,}'}{d\psi}\right).\,\text{tang.}\,\psi_{,}\right\}.\,\cos.\,\psi_{,}\,; \qquad [1936e']$$

multiplying this by $-s\,.\frac{d d\theta_{,}}{d s^2}$ [1934d], we get,

$$-s r_{,}\,.\,\sin.\theta_{,}\,.\frac{d d\theta_{,}}{d s^2}=-s\,.\,(1-\alpha u_{,}')\,.\,\cos.\,\psi_{,}\,.\,\text{tang.}\,\psi_{,}=-s\,.\,(1-\alpha u_{,}')\,.\,\sin.\,\psi_{,}. \qquad [1936f]$$

Connecting this with the other term [1936e], we get the value of [1936c],

$$\frac{dz}{ds}=\alpha s\,.\frac{d d u_{,}'}{d s^2}\,.\,\sin.\,\psi_{,}-s\,.\,(1-\alpha u_{,}')\,.\,\sin.\,\psi_{,}. \qquad [1936g]$$

Substituting in this, the second expression of $\alpha\,.\frac{d d u_{,}'}{d s^2}$ [1932], and putting, in one of the terms, $\sin.\,\psi_{,}=\cos.\,\psi_{,}\,.\,\text{tang.}\,\psi_{,}$, it becomes as in [1938].

* (1402) If we put the second member of [1937] equal to A, we shall have, by comparing it with [1938], $\frac{dx'}{ds}=A$, $\frac{dz}{ds}=-A\,.\frac{\cos.\psi_{,}}{\sin.\psi_{,}}$; hence we get successively, [1939a] by using [1937],

$$\frac{(dx'^2+dz^2)^{\frac{1}{2}}}{ds}=\left(A^2+A^2\,.\frac{\cos.^2\psi_{,}}{\sin.^2\psi_{,}}\right)^{\frac{1}{2}}=\frac{A}{\sin.\psi_{,}}\,.\,(\sin.^2\psi_{,}+\cos.^2\psi_{,})^{\frac{1}{2}}=\frac{A}{\sin.\psi_{,}} \qquad [1939b]$$

$$=s\,.\,(1-\alpha u_{,}')\,.\frac{\sin.\psi_{,}}{\cos.\psi_{,}}+\alpha s\,.\left(\frac{du_{,}'}{d\psi}\right).\,\text{tang.}^2\psi_{,}-\alpha s\,.\left(\frac{d d u_{,}'}{d\varphi^2}\right).\frac{\sin.\psi_{,}}{\cos.^3\psi_{,}}$$

$$=s\,.\,\text{tang.}\,\psi_{,}\,.\left\{1-\alpha u_{,}'+\alpha\,.\left(\frac{d u_{,}'}{d\psi}\right).\,\text{tang.}\,\psi_{,}-\frac{\alpha}{\cos.^2\psi_{,}}\,.\left(\frac{d d u_{,}'}{d\varphi^2}\right)\right\}. \qquad [1939c]$$

The first member of this expression represents, in [1924], the cosine of the azimuth angle, at the end of the geodetical line; and the last member agrees with its value [1939]. This cosine being very small, it is nearly equal to the complement of the corresponding angle; therefore, by subtracting [1939] from 100°, we obtain the azimuth angle nearly, as in [1940].

[1940] $$100^\circ - s\,.\,\text{tang.}\,\psi_{,}\,.\left\{1 - \alpha u_{,}' + \alpha\,.\left(\frac{d u_{,}'}{d\psi}\right).\,\text{tang.}\,\psi_{,} - \frac{\alpha\,.\left(\frac{d d u_{,}'}{d \varsigma^2}\right)}{\cos.^2\psi_{,}}\right\}.$$

For greater accuracy, we must add the part depending on s^3, which is independent of α, and may be obtained upon the supposition that the earth is spherical. This part is equal to*

[1941] $$\tfrac{1}{3}\,s^3\,.\,\{\tfrac{1}{2} + \text{tang.}^2\psi_{,}\}\,.\,\text{tang.}\,\psi_{,}\,;$$

Azimuth angle, at the termination of the geodetical line.

therefore the azimuth angle, at the end of the arc s, is equal to

[1942] $$100^\circ - s\,.\text{tang.}\psi_{,}\,.\left\{1 - \alpha u_{,}' + \alpha\,.\left(\frac{du_{,}'}{d\psi}\right).\,\text{tang.}\psi_{,} - \frac{\alpha\,.\left(\frac{d d u_{,}'}{d\varsigma^2}\right)}{\cos.^2\psi_{,}} - \tfrac{1}{3}s^2\,.\,(\tfrac{1}{2} + \text{tang.}^2\psi_{,})\right\}.$$

General expression of the radius of curvature upon any surface.

The radius of curvature of the geodetical line, drawn at any angle with the plane of the meridian, being put equal to R, *we shall have* [53],

[1943] $$R = \frac{d\,s^2}{\sqrt{(d\,d\,x)^2 + (d\,d\,y)^2 + (d\,d\,z)^2}},$$

[1943′] $d\,s$ being supposed constant. The equation [1849],

[1944] $$x^2 + y^2 + z^2 = 1 + 2\,\alpha\,u',$$

gives,†

[1945] $$x\,d\,d\,x + y\,d\,d\,y + z\,d\,d\,z = -\,d\,s^2 + \alpha\,.\,d\,d\,u'.$$

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* (1403) Using fig. 46, page 396, putting the azimuth angle $ZIH = A$, and $A' = 100^\circ - A$; then supposing the earth to be spherical; we shall have, in the spherical triangle ZHI, $\cot.ZIH = \sin.HI\,.\cot.ZH$ [1345[31]], or $\text{tang.}A' = \sin.s\,.\,\text{tang.}\psi_{,}$. Hence, from [48] Int., $A' = (\sin.s\,.\,\text{tang.}\,\psi_{,}) - \tfrac{1}{3}\,.\,(\sin.s\,.\,\text{tang.}\psi_{,})^3 + \&c.$ Substituting
[1941a] $\sin.s = s - \tfrac{1}{6}s^3 + \&c.$ [43] Int., we get $A' = s\,.\text{tang.}\psi_{,} - \tfrac{1}{3}s^3.\{\tfrac{1}{2}\text{tang.}\psi_{,} + \text{tang.}^3\psi_{,}\} + \&c.$, and $A = 100^\circ - A' = 100^\circ - s\,.\,\text{tang.}\,\psi_{,} + \tfrac{1}{3}s^3\,.\,(\tfrac{1}{2} + \text{tang.}^2\psi_{,})\,.\,\text{tang.}\,\psi_{,} + \&c.$; in which the part depending on s^3 is as in [1941]; adding this to the expression [1940], we obtain the azimuth angle [1942].

† (1405) The first differential of [1944], divided by 2, is

$$\alpha\,.\,d\,u' = x\,d\,x + y\,d\,y + z\,d\,z\,;$$

the second differential, using [1852″], gives

[1945a] $$\alpha\,.\,d\,d\,u' = x\,d\,d\,x + y\,d^2 y + z\,d^2 z + d\,x^2 + d\,y^2 + d\,z^2 = x\,d\,d\,x + y\,d\,d\,y + z\,d\,d\,z + d\,s^2,$$

as in [1945].

If we add the square of this equation to the squares of the equations [1850], we shall get, by neglecting terms of the order α^2,*

$$(x^2+y^2+z^2)\cdot\{(d\,d\,x)^2+(d\,d\,y)^2+(d\,d\,z)^2\}=d\,s^4-2\,\alpha\,d\,s^2\cdot d\,d\,u'\,;$$ [1946]

Radius of curvature of the geodetical line, in any direction.

hence we deduce,

$$R=1+\alpha\,u'+\alpha\,.\,\frac{d\,d\,u'}{d\,s^2}\,.$$ [1947]

In the direction of the meridian, we have,†

$$\alpha\,.\,\frac{d\,d\,u'}{d\,s^2}=\alpha\,.\left(\frac{d\,d\,u'}{d\,\psi^2}\right).$$ [1948]

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* (1406) If for brevity, we put

$$2\,xy\,.\,ddx\,.\,ddy=[xy],\qquad 2\,xz\,.\,ddx\,.\,ddz=[xz],\qquad 2\,yz\,.\,ddy\,.\,ddz=[yz],$$

and then take the squares of the equations [1850, 1945], and place them beneath each other, always neglecting α^2, we shall get,

$$\begin{array}{l} y^2\,.\,(d\,d\,x)^2+x^2\,.\,(d\,d\,y)^2 \qquad\qquad : \quad -[xy]=0, \\ z^2\,.\,(d\,d\,x)^2 \qquad : \quad +x^2\,.\,(d\,d\,z)^2-[xz]=0, \\ \quad : \quad +z^2\,.\,(d\,d\,y)^2+y^2\,.\,(d\,d\,z)^2-[yz]=0, \\ x^2\,.\,(d\,d\,x)^2+y^2\,.\,(d\,d\,y)^2+z^2\,.\,(d\,d\,z)^2+[xy]+[xz]+[yz]=d\,s^4\,.\left(1-2\,\alpha\,.\,\frac{d\,d\,u'}{d\,s^2}\right). \end{array}$$ [1946a]

Adding these four equations, we shall find that each of the quantities $(d\,d\,x)^2$, $(d\,d\,y)^2$, $(d\,d\,z)^2$, in the first member of the sum, has the factor $x^2+y^2+z^2$, and the remaining terms destroy each other; therefore it becomes as in [1946]. Substituting in this, the value of $x^2+y^2+z^2=1+2\,\alpha\,u'$, [1944], and dividing by

$$\left\{1-2\,\alpha\,.\,\frac{d\,d\,u'}{d\,s^2}\right\}\cdot\{(d\,d\,x)^2+(d\,d\,y)^2+(d\,d\,z)^2\},\qquad\text{we get,}$$

$$\frac{d\,s^4}{(d\,d\,x)^2+(d\,d\,y)^2+(d\,d\,z)^2}=1+2\,\alpha\,u'+2\,\alpha\,.\,\frac{d\,d\,u'}{d\,s^2}=R^2\qquad[1943],$$ [1946b]

whose square root gives R [1947].

† (1407) The value of $\alpha\,.\,\frac{d\,d\,u'}{d\,s^2}$ [1931b], contains six terms; of which the first and fourth, depending on $d\,r$, $d\,d\,r$, [1850c]; also the second and fifth, depending on $d\,\varphi$, [1948a] $d\,d\,\varphi$, [1887], are of the order α^2, and may be neglected. The sixth term, $\alpha\,.\,w''\,.\,\frac{d\,d\,\psi}{d\,s^2}$, is also of the order α^2; for if we divide [1875] by the coefficient of $d\,\psi$, neglecting α^2, we

Radius of curvature in the direction of the meridian; therefore,

[1949] $$R = 1 + \alpha u' + \alpha . \left(\frac{d\,d\,u'}{d\,\psi^2}\right).$$

In the direction perpendicular to the meridian, we have, by what precedes, [1932],

[1950] $$\alpha . \frac{d\,d\,u_{,}'}{d\,s^2} = \frac{\alpha . \left(\frac{d\,d\,u_{,}'}{d\,\varphi^2}\right)}{\cos.^2 \psi_{,}} - \alpha . \left(\frac{d\,u_{,}'}{d\,\psi}\right) . \text{tang.}\, \psi_{,}\,;$$

in the direction of the perpendicular to the meridian. consequently,*

[1951] $$R = 1 + \alpha u_{,}' - \alpha . \left(\frac{d\,u_{,}'}{d\,\psi}\right) . \text{tang.}\, \psi_{,} + \frac{\alpha . \left(\frac{d d u_{,}'}{d\,\varphi^2}\right)}{\cos.^2 \psi_{,}}.$$

[1952] *If in the expression of* $V - V_{,}$ [1923], *we put* $\frac{s}{R} = s'$; *it will take this very simple form*, corresponding to a sphere whose radius is R,†

[1948b] shall get $d\,\psi = d\,s . \left\{ 1 - \alpha u' - \alpha . \left(\frac{d\,d\,u'}{d\,\psi^2}\right) \right\}$; whose differential is of the order $\alpha . d\,s$, $d\,s$ being constant; hence $\alpha . \frac{d\,d\,\psi}{d\,s^2}$ is of the order α^2. The only remaining term

[1948c] is the third, which gives $\alpha . \frac{d\,d\,u'}{d\,s^2} = \alpha . \frac{d w''}{d s} . \frac{d\,\psi}{d\,s} = \alpha . \frac{d w''}{d\,s}$ [1948b]. Now w'' being considered as a function of r, φ, ψ, its differential $d\,w''$ may be put under the same form as $d\,u'$ [1931a]; and since $\alpha . d\,r$, $\alpha . d\,\varphi$, [1948a], produce only terms of the order α^2, we need only retain the term depending on $d\,\psi$; therefore,

[1948d] $$d\,w'' = \left(\frac{d\,w''}{d\,\psi}\right) . d\,\psi = \left(\frac{d\,d\,u'}{d\,\psi^2}\right) . d\,\psi \qquad [1931a'].$$

Substituting this in [1948c], we get, $\alpha . \frac{d\,d\,u'}{d\,s^2} = \alpha . \left(\frac{d\,d\,u'}{d\,\psi^2}\right) . \frac{d\,\psi}{d\,s}$; which, by putting as above, $\frac{d\,\psi}{d\,s} = 1$, becomes as in [1948]. Substituting this in [1947], we obtain [1949].

* (1408) Substituting [1950] in [1947], we get [1951] at the origin of the line.

† (1409) If we divide $\frac{s}{\cos.\psi_{,}}$ by the value of R [1951], neglecting α^2, we find,

[1953a] $$\frac{s}{R} . \frac{1}{\cos.\psi_{,}} = \frac{s}{\cos.\psi_{,}} . \left\{ 1 - \alpha u_{,}' + \alpha . \left(\frac{d u_{,}'}{d\,\psi}\right) . \text{tang.}\, \psi_{,} - \frac{\alpha . \left(\frac{d\,d\,u_{,}'}{d\,\varphi^2}\right)}{\cos.^2 \psi_{,}} \right\}.$$

Simple forms of the difference of longitudes and azimuths of the extremities of the geodetical line, drawn perpendicular to the meridian.

$$V-V_{,}=\frac{s'}{\cos.\psi_{,}}\cdot\{1-\tfrac{1}{3}s'^2.\,\mathrm{tang.}^2\psi_{,}\}.$$ [1953]

The expression of the azimuth angle becomes,*

$$100^\circ-s'.\,\mathrm{tang.}\,\psi_{,}\,.\,\{1-\tfrac{1}{3}s'^2.\,(\tfrac{1}{2}+\mathrm{tang.}^2\psi_{,})\}.$$ [1954]

We shall put λ *for the angle which the first side of the geodetical line forms with the corresponding plane of the celestial meridian*, and we shall have,† [1954']

$$\frac{d\,d\,u'}{d\,s^2}=\left(\frac{d\,u'}{d\,\varphi}\right).\frac{d\,d\,\varphi}{d\,s^2}+\left(\frac{d\,u'}{d\,\psi}\right).\frac{d\,d\,\psi}{d\,s^2}+\left(\frac{d\,d\,u'}{d\,\varphi^2}\right).\frac{d\,\varphi^2}{d\,s^2}+2.\left(\frac{d\,d\,u'}{d\,\varphi\,d\,\psi}\right).\frac{d\,\varphi}{d\,s}.\frac{d\,\psi}{d\,s}+\left(\frac{d\,d\,u'}{d\,\psi^2}\right).\frac{d\,\psi^2}{d\,s^2}.$$ [1955]

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the second member of which is equal to the four first terms of the value of $V-V_{,}$ [1923]; substituting these, and $s=s'.R$ [1952], we get successively,

$$V-V_{,}=\frac{s}{R.\cos.\psi_{,}}-\frac{s^3.\,\mathrm{tang.}^2\psi_{,}}{3\cos.\psi_{,}}=\frac{s'}{\cos.\psi_{,}}-\frac{R^3.\,s'^3.\,\mathrm{tang.}^2\psi_{,}}{3\cos.\psi_{,}}.$$ [1953b]

If we neglect terms of the order αs^3, we may put, in the last term, $R^3=1$ [1951], and we shall obtain [1953]. [1953c]

* (1410) Multiplying [1953a] by $-\sin.\psi_{,}$, we shall obtain the value of $-\frac{s}{R}.\mathrm{tang.}\psi_{,}$, or $-s'.\mathrm{tang.}\psi_{,}$ [1952]; and the four terms, of which this expression is composed, will be found the same as the four first terms between the braces in [1942]. This last expression may therefore be represented by $100^\circ-s'.\,\mathrm{tang.}\,\psi_{,}+\frac{1}{3}s^3.\,\mathrm{tang.}\,\psi_{,}\,.\,\{\frac{1}{2}+\mathrm{tang.}^2\psi_{,}\}$. If we neglect αs^3, we may, as in [1953c, 1952], change s^3, into s'^3, and the expression will become as in [1954]. [1954a]

† (1411) If we neglect α^2, in [1947], we may, in the value of $\frac{d\,d\,u'}{d\,s^2}$ [1931b], neglect terms of the order α, such as those depending on $d\,r$, $d\,d\,r$, [1850c]; and we shall have, by arranging the terms in a different order, [1955a]

$$\frac{d\,d\,u'}{d\,s^2}=w'.\frac{d\,d\,\varphi}{d\,s^2}+w''.\frac{d\,d\,\psi}{d\,s^2}+\frac{d\,w'}{d\,s}.\frac{d\,\varphi}{d\,s}+\frac{d\,w''}{d\,s}.\frac{d\,\psi}{d\,s}.$$ [1955b]

In the expressions of $d\,w'$, $d\,w''$, similar to that of $d\,u'$ [1931a], we may, in like manner, neglect $d\,r$, and we shall have, by using [1931a'],

$$d\,w'=\left(\frac{d\,w'}{d\,\varphi}\right).d\,\varphi+\left(\frac{d\,w'}{d\,\psi}\right).d\,\psi=\left(\frac{d\,d\,u'}{d\,\varphi^2}\right).d\,\varphi+\left(\frac{d\,d\,u'}{d\,\varphi\,d\,\psi}\right).d\,\psi;$$ [1955c]

$$d\,w''=\left(\frac{d\,w''}{d\,\varphi}\right).d\,\varphi+\left(\frac{d\,w''}{d\,\psi}\right).d\,\psi=\left(\frac{d\,d\,u'}{d\,\varphi\,d\,\psi}\right).d\,\varphi+\left(\frac{d\,d\,u'}{d\,\psi^2}\right).d\,\psi.$$ [1955d]

Substituting these, and w', w'', [1931a'], in [1955b], we obtain [1955].

But in the hypothesis that the earth is spherical, we have,*

Formulas corresponding to a spherical surface.

$$\frac{d\varphi_{,}}{ds}=\frac{\sin.\lambda}{\cos.\psi_{,}}; \qquad \frac{dd\varphi_{,}}{ds^2}=\frac{2.\sin.\lambda.\cos.\lambda}{\cos.\psi_{,}}.\mathrm{tang.}\psi_{,};$$

[1956] $$\frac{d\psi_{,}}{ds}=\cos.\lambda; \qquad \frac{dd\psi_{,}}{ds^2}=-\sin.^2\lambda.\mathrm{tang.}\psi_{,}.$$

Therefore,

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* (1412) Since we neglect terms of the order α, in $\frac{ddu'}{ds^2}$ [1955b] we may use, for $d\varphi$, $d\psi$, $dd\varphi$, $dd\psi$, their values corresponding to a sphere; in which $\alpha=0$, and the radius $=1=CZ$, as in the annexed figure, which is similar to fig. 43. In this case, the geodetical line HIi, is a great circle, as well as the arcs ZHH', ZIF, Zif; and the infinitely small arc im is perpendicular to Im. Then we have $XH'=\varphi_{,}$, $XF=\varphi$, $Ff=d\varphi$,
[1955e] $ZH=\theta_{,}$, $H'H=\psi_{,}$, $ZI=\theta$, $FI=\psi$, $Im=d\psi$, $HI=s$, $Ii=ds$; angles $ZHI=\lambda$, $ZIi=\lambda'$; and in the rectangular triangle Iim, we have

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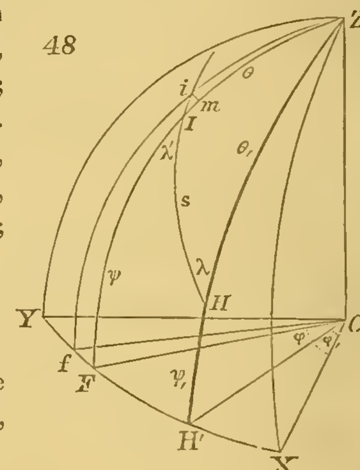

[1955f] $$Im=d\psi=ds.\cos.\lambda', \qquad im=ds.\sin.\lambda'.$$

But $im=Ff.\sin.ZI=d\varphi.\cos.\psi$, hence $d\varphi.\cos.\psi=ds.\sin.\lambda'$. Dividing this by $ds.\cos.\psi$, and the value of $d\psi$ by ds, we obtain

[1956a] $$\frac{d\varphi}{ds}=\frac{\sin.\lambda'}{\cos.\psi}; \qquad \frac{d\psi}{ds}=\cos.\lambda'.$$

[1956b] At the commencement H of the geodetical line, φ, ψ, λ', become respectively $\varphi_{,}$, $\psi_{,}$, λ, [1892, 1954′], and the preceding equations become like the values of $\frac{d\varphi_{,}}{ds}$, $\frac{d\psi_{,}}{ds}$, [1956].

In the spherical triangle ZHI, we have $\sin.ZHI:\sin.ZI::\sin.ZIH:\sin.ZH$;
[1956c] or in symbols, $\sin.\lambda:\cos.\psi::\sin.\lambda':\cos.\psi_{,}$; hence $\sin.\lambda'.\cos.\psi=\sin.\lambda.\cos.\psi_{,}$. The quantities λ, $\psi_{,}$, in the second member, being constant, the differential of this equation,
[1956d] divided by $ds.\cos.\lambda'.\cos.\psi$, gives $\frac{d\lambda'}{ds}=\frac{d\psi}{ds}.\frac{\sin.\psi}{\cos.\psi}.\frac{\sin.\lambda'}{\cos.\lambda'}$. Now the differential of $\frac{d\varphi}{ds}$ [1956a], divided by ds, gives successively, by substituting the preceding values of $\frac{d\lambda'}{ds}$, $\frac{d\psi}{ds}$,

$$\frac{d\,d\,u_{\prime}'}{d\,s^2}=2.\frac{\sin\lambda\,.\cos.\lambda}{\cos.\psi_{\prime}}.\left\{\left(\frac{d\,u_{\prime}'}{d\,\varphi}\right).\mathrm{tang}.\psi_{\prime}+\left(\frac{d\,d\,u_{\prime}'}{d\,\varphi\,d\,\psi}\right)\right\}-\sin.^2\lambda.\,\mathrm{tang}.\psi_{\prime}.\left(\frac{d\,u_{\prime}'}{d\,\psi}\right)$$
$$+\left(\frac{d\,d\,u_{\prime}'}{d\,\varphi^2}\right).\frac{\sin.^2\lambda}{\cos.^2\psi_{\prime}}+\left(\frac{d\,d\,u_{\prime}'}{d\,\psi^2}\right).\cos.^2\lambda\,;^{*} \qquad [1957]$$

consequently *the radius of curvature R, in the direction of the geodetical line, is*

Radius of curvature of the geodetical line, in any direction. First form.

$$R=1+\alpha u_{\prime}'+2\alpha.\frac{\sin.\lambda\,.\cos.\lambda}{\cos.\psi_{\prime}}.\left\{\left(\frac{d\,u_{\prime}'}{d\,\varphi}\right).\mathrm{tang}.\psi_{\prime}+\left(\frac{d\,d\,u_{\prime}'}{d\,\varphi\,d\,\psi}\right)\right\}$$
$$-\alpha.\sin.^2\lambda.\mathrm{tang}.\psi_{\prime}.\left(\frac{d\,u_{\prime}'}{d\,\psi}\right)+\alpha.\left(\frac{d\,d\,u_{\prime}'}{d\,\varphi^2}\right).\frac{\sin.^2\lambda}{\cos.^2\psi_{\prime}}+\alpha.\left(\frac{d\,d\,u_{\prime}'}{d\,\psi^2}\right).\cos.^2\lambda.\dagger \qquad [1958]$$

Putting for brevity,

$$\frac{d\,d\,\varphi}{d\,s^2}=\frac{d\,\lambda'}{d\,s}.\frac{\cos.\lambda'}{\cos.\psi}+\frac{d\,\psi}{d\,s}.\frac{\sin.\psi\,.\sin.\lambda'}{\cos.^2\psi}=\frac{d\,\psi}{d\,s}.\frac{\sin.\psi\,.\sin.\lambda'}{\cos.^2\psi}+\frac{d\,\psi}{d\,s}.\frac{\sin.\psi\,.\sin.\lambda'}{\cos.^2\psi}$$
$$=2.\frac{d\,\psi}{d\,s}.\frac{\sin.\psi\,.\sin.\lambda'}{\cos.^2\psi}=2.\frac{d\,\psi}{d\,s}.\mathrm{tang}.\psi.\frac{\sin.\lambda'}{\cos.\psi}=2.\cos.\lambda'.\mathrm{tang}.\psi.\frac{\sin.\lambda'}{\cos.\psi}$$
$$=2.\frac{\sin.\lambda'.\cos.\lambda'}{\cos.\psi}.\mathrm{tang}.\psi. \qquad [1956e]$$

Making the same changes as in [1956b], we get $\frac{d\,d\,\varphi_{\prime}}{d\,s^2}$, corresponding to the first point of the curve, as in [1956]. Proceeding in the same manner with the differential of the second of the equations [1956a], we get,

$$\frac{d\,d\,\psi}{d\,s^2}=-\frac{d\,\lambda'}{d\,s}.\sin.\lambda'=-\frac{d\,\psi}{d\,s}.\frac{\sin.\psi}{\cos.\psi}.\frac{\sin.^2\lambda'}{\cos.\lambda'}=-\cos.\lambda'.\frac{\sin.\psi}{\cos.\psi}.\frac{\sin.^2\lambda'}{\cos.\lambda'}=-\sin.^2\lambda'.\mathrm{tang}.\psi\,; \qquad [1956f]$$

which, by changing the symbols, as in [1956b], gives the last of the equations [1956].

* (1413) Changing in [1955] u', φ, ψ, into $u_{\prime}'$, $\varphi_{\prime}$, $\psi_{\prime}$, to obtain its value corresponding to the first point H of the curve, and then substituting the values [1956], we get, without any reduction, the formula [1957]. [1957a]

† (1414) Substituting in [1947] the value of $\frac{d\,d\,u_{\prime}'}{d\,s^2}$ [1957], we obtain the expression of R [1958]; then putting $2.\sin.\lambda.\cos.\lambda=\sin.2\lambda$; $\sin.^2\lambda=\frac{1}{2}-\frac{1}{2}.\cos.2\lambda$; $\cos.^2\lambda=\frac{1}{2}+\frac{1}{2}.\cos.2\lambda$; we shall find, that the terms independent of λ, are [1958a] represented by K [1959]; and the coefficients of $\sin.2\lambda$, $\cos.2\lambda$, are respectively A, B, [1959]; so that the whole expression of R becomes as in [1960].

$$K = 1 + \alpha u'_{,} - \tfrac{1}{2}\alpha . \text{tang.}\psi_{,} . \left(\frac{d u'_{,}}{d \psi}\right) + \tfrac{1}{2} . \frac{\alpha . \left(\frac{d d u'_{,}}{d \varphi^2}\right)}{\cos.^2 \psi_{,}} + \tfrac{1}{2}\alpha . \left(\frac{d d u'_{,}}{d \psi^2}\right);$$

[1959]
$$A = \frac{\alpha}{\cos.\psi_{,}} . \left\{ \left(\frac{d u'_{,}}{d \varphi}\right) . \text{tang.}\psi_{,} + \left(\frac{d d u'_{,}}{d \varphi\, d \psi}\right) \right\};$$

$$B = \frac{\alpha}{2} . \text{tang.}\psi_{,} . \left(\frac{d u'_{,}}{d \psi}\right) - \frac{\frac{\alpha}{2} . \left(\frac{d d u'_{,}}{d \varphi^2}\right)}{\cos.^2 \psi_{,}} + \frac{\alpha}{2} . \left(\frac{d d u'_{,}}{d \psi^2}\right);$$

General expression of the radius of curvature of the geodetical line, inclined to the meridian by the angle λ. Second form.

we shall have,

[1960]
$$R = K + A . \sin. 2\lambda + B . \cos. 2\lambda.$$

[1960′] *The observations of the azimuth angles, and the difference of the latitudes of the extremities of two geodetical lines; the one measured in the direction of the meridian, the other in a direction perpendicular to the meridian; will give, by what precedes, the values of* A, B, K. For the radius of curvature will be given, by observation, in both these directions; and if we put R' for the radius in the direction of the meridian, and R'' for that in the direction of the perpendicular to the meridian, we shall have,*

[1961]
$$K = \frac{R' + R''}{2};$$

$$B = \frac{R' - R''}{2};$$

and the value of A will be determined, either by the azimuth of the arc, measured in the direction of the meridian, or by the difference of latitude of
[1961′] the extremities of the arc, measured in the direction perpendicular to the meridian.† We shall thus have the radius of curvature of the geodetical

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* (1415) If the first side of the geodetical line be in the direction of the celestial meridian, we shall have $\lambda = 0$ [1954′]; and R [1960] becomes $R' = K + B$; but if the first side of the geodetical line be perpendicular to the corresponding celestial meridian,
[1960a] we shall have $\lambda = 100^{\circ}$, and R [1960] becomes $R'' = K - B$. The half sum, and the half difference, of these values of R', R'', give K, B, [1961].

† (1416) If we substitute the value of A [1959], in [1891a, 1915], we shall get, by
[1961a] changing ε' into ε, $\varpi = -A\varepsilon . \text{tang.}\psi_{,}$; $\psi - \psi_{,} = -As - \tfrac{1}{2}s^2 . \text{tang.}\psi_{,}$; ϖ being the azimuth, and $\psi - \psi_{,}$ the difference of latitude abovementioned. Either of these being found by observation, and having also ε, s, $\psi_{,}$, we may thence determine the value of A.

line, whose first side makes any angle whatever with the plane of the meridian.

If we take $2E$ such that $A = B \cdot \text{tang.}\, 2E$, we shall have,* [1962]

Third form.

$$R = K + \sqrt{A^2+B^2} \cdot \cos.\,(2\lambda - 2E).$$ [1963]

The greatest radius of curvature corresponds to $\lambda = E$; therefore the corresponding geodetical line forms the angle E, with the plane of the meridian. *The least radius of curvature corresponds to* $\lambda = 100^\circ + E$; *if this least radius be* r, *and the greatest radius* r', *we shall have*,† [1963′] [1963″]

Fourth form.

$$R = r + (r' - r) \cdot \cos.^2 (\lambda - E);$$ [1964]

$\lambda - E$ *being the angle which the geodetical line corresponding to* R, *makes with that corresponding to* r'.

Osculatory ellipsoid.

We have before observed, [1848‴], that at each point of the surface of the earth, we may suppose an osculatory ellipsoid to be formed, upon which the degrees in every direction are sensibly the same as upon the earth, for a [1964′]

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* (1417) The expression [1962] gives

$$(A^2 + B^2)^{\frac{1}{2}} = B \cdot (1 + \text{tang.}^2 2E)^{\frac{1}{2}} = \frac{B}{\cos.\,2E} \qquad [34'''] \text{ Int.}$$ [1962a]

Substituting the same value of A [1962], in the two last terms of [1960], we get successively, by using [34′, 24] Int.,

$$A \cdot \sin.\,2\lambda + B \cdot \cos.\,2\lambda = B \cdot \text{tang.}\,2E \cdot \sin.\,2\lambda + B \cdot \cos.\,2\lambda$$

$$= \frac{B}{\cos.\,2E} \cdot \{\sin.\,2E \cdot \sin.\,2\lambda + \cos.\,2E \cdot \cos.\,2\lambda\} = \frac{B}{\cos.\,2E} \cdot \cos.\,(2\lambda - 2E)$$

$$= (A^2 + B^2)^{\frac{1}{2}} \cdot \cos.\,(2\lambda - 2E).$$ [1962b]

Substituting this in [1960], we get [1963].

† (1418) Putting $\lambda = E$, and $R = r'$, in [1963], we get $r' = K + (A^2 + B^2)^{\frac{1}{2}}$; then putting $\lambda = 100^\circ + E$, and $R = r$, we obtain $r = K - (A^2 + B^2)^{\frac{1}{2}}$. The half sum, and the half difference of these values, give

$$K = \tfrac{1}{2} \cdot (r' + r), \qquad (A^2 + B^2)^{\frac{1}{2}} = \tfrac{1}{2} \cdot (r' - r).$$ [1963a]

Substituting these in [1963], and using also $\tfrac{1}{2} \cdot \cos.\,(2\lambda - 2E) = \cos.^2 (\lambda - E) - \tfrac{1}{2}$, [34] Int., we get successively, as in [1964],

$$R = \tfrac{1}{2} \cdot (r' + r) + \tfrac{1}{2} \cdot (r' - r) \cdot \cos.\,(2\lambda - 2E)$$

$$= \tfrac{1}{2} \cdot (r' + r) + (r' - r) \cdot \cos.^2(\lambda - E) - \tfrac{1}{2} \cdot (r' - r) = r + (r' - r) \cdot \cos.^2 (\lambda - E).$$

Corrected expression of the radius of the spheroid.

small distance about the point of contact. We shall express the radius of this ellipsoid by the function*

[1965] $$1 - \alpha \,.\, \sin.^2 \psi \,.\, \{1 + h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)\} + [\alpha h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)] :$$

[1963b] * (1419) The last term of [1965], $\alpha h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)$, is not in the original work.

Term omitted by the author.

The expression of the radius given by the author being

[1964a] $$1 - \alpha \,.\, \sin.^2 \psi \,.\, \{1 + h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)\} ;$$

which is not sufficiently general, as evidently appears, by putting $\psi = 0$, or $\psi = 100°$. For at the equator, where $\psi = 0$, the radius becomes equal to 1, which corresponds to an ellipsoid of *revolution*, and is not so general as might be assumed; and at the pole,

[1964a'] where $\psi = 100°$, it becomes $1 - \alpha \,.\, \{1 + h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)\}$, which is *not constant*, as it ought to be; since it contains the variable quantity φ. The true value of this radius may be investigated in the following manner; being nearly the same as in a paper on this subject I published in the fourth volume of the Memoirs of the American Academy of Arts and Sciences.

The equation of the ellipsoid [1363] may be put under the form

[1964b] $$x^2 + y^2 + z^2 = k^2 + (1 - m) \,.\, y^2 + (1 - n) \,.\, z^2 ;$$

and by using r^2, [1850b], in the first member; also y, z, [1851], in the second member, it becomes $r^2 = k^2 + (1 - m) \,.\, r^2 \,.\, \sin.^2 \theta \,.\, \sin.^2 \varphi + (1 - n) \,.\, r^2 \,.\, \cos.^2 \theta$. Now $1 - m$, $1 - n$, are of the order α [1849a—a']; hence r differs from k by quantities of the same order, so that if we neglect α^2, we may put, in the second member, $r = k$, $\theta = 100° - \psi$,

[1964c] [1913], $\cos.^2 \psi = 1 - \sin.^2 \psi$; and we shall get successively,

$$r = k \,.\, \{1 + \tfrac{1}{2} \,.\, (1 - m) \,.\, \sin.^2 \theta \,.\, \sin.^2 \varphi + \tfrac{1}{2} \,.\, (1 - n) \,.\, \cos.^2 \theta\}$$

$$= k \,.\, \{1 + \tfrac{1}{2} \,.\, (1 - m) \,.\, (1 - \sin.^2 \psi) \,.\, \sin.^2 \varphi + \tfrac{1}{2} \,.\, (1 - n) \,.\, \sin.^2 \psi\}$$

[1964d] $$= k \,.\, \{1 + \tfrac{1}{2} \,.\, (1 - m) \,.\, \sin.^2 \varphi - \sin.^2 \psi \,.\, [\tfrac{1}{2} \,.\, (1 - m) \,.\, \sin.^2 \varphi - \tfrac{1}{2} \,.\, (1 - n)]\}.$$

Substituting $\sin.^2 \varphi = \tfrac{1}{2} - \tfrac{1}{2} \,.\, \cos.\, 2 \varphi$, and putting for brevity $\tfrac{1}{4} \,.\, (1 - m) = - \alpha h$, $\tfrac{1}{4} \,.\, (1 - m) - \tfrac{1}{2} \,.\, (1 - n) = \alpha$, we get,

$$r = k \,.\, \{1 - \alpha h + \alpha h \,.\, \cos.\, 2 \varphi - \sin.^2 \psi \,.\, (\alpha + \alpha h \,.\, \cos.\, 2 \varphi)\} ;$$

[1964e] and if we put $k = 1 + \alpha h$, it becomes $r = 1 + \alpha h \,.\, \cos.\, 2 \varphi - \alpha \,.\, \sin.^2 \psi \,.\, (1 + h \,.\, \cos.\, 2 \varphi)$. If we suppose the meridian ZCX, fig. 43, page 375, from which the angle φ is counted, to be moved backwards by an angle β, and the meridian ZCY to be moved also. by the same quantity; it will change φ into $\varphi + \beta$, and the value of r will become,

[1965a] $$r = 1 - \alpha \,.\, \sin.^2 \psi \,.\, \{1 + h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)\} + \alpha h \,.\, \cos.\, 2 \,.\, (\varphi + \beta),$$

which agrees with the corrected expression we have given in [1965]. This, when $\psi = 0$, becomes $1 + \alpha h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)$, which varies in different parts of the equator; and when $\psi = 100°$, it becomes $1 - \alpha$, which is constant; therefore this corrected expression is not liable to the objection mentioned in [1964a'].

the longitudes φ being counted from a given meridian. The expression of the terrestrial arc, measured in the direction of the meridian, will be, by what precedes,* [1965]

Corrected arc in the direction of the meridian.

$$s=\varepsilon-\frac{\alpha\varepsilon}{2}.\{1+h.\cos.2.(\varphi+\beta)\}.\{1+3.\cos.^2\psi-3\varepsilon.\sin.^2\psi\}$$
$$+[\alpha\varepsilon.h.\cos.(2\varphi+2\beta)]. \qquad [1966]$$

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* (1420) Putting the expressions of r [1850c, 1965a] equal to each other, then rejecting 1, which occurs in both members, dividing by α, and writing for brevity $q=1+h.\cos.2.(\varphi+\beta)$, we get, [1965$b$]

$$u'=-\sin.^2\psi.\{1+h.\cos.2.(\varphi+\beta)\}+h.\cos.2.(\varphi+\beta) \qquad [1965c]$$
$$=-q.\sin.^2\psi+h.\cos.2.(\varphi+\beta). \qquad [1965d]$$

The partial differentials of this last expression, relative to ψ, are

$$\left(\frac{du'}{d\psi}\right)=-2q.\sin.\psi.\cos.\psi=-q.\sin.2\psi;$$
$$\left(\frac{d^2u'}{d\psi^2}\right)=-2q.\cos.2\psi; \qquad \left(\frac{d^3u'}{d\psi^3}\right)=4q.\sin.2\psi. \qquad [1965e]$$

The differential of this value of $\left(\frac{du'}{d\psi}\right)$ relative to φ gives, by using q [1965b],

$$\left(\frac{ddu'}{d\varphi d\psi}\right)=-\left(\frac{dq}{d\varphi}\right).\sin.2\psi=2h.\sin.2\psi.\sin.2.(\varphi+\beta). \qquad [1965f]$$

In like manner the partial differentials of u' [1965c], relative to φ, give

$$\left(\frac{du'}{d\varphi}\right)=2h.\sin.^2\psi.\sin.2.(\varphi+\beta)-2h.\sin.2.(\varphi+\beta)$$
$$=-2h.\cos.^2\psi.\sin.2.(\varphi+\beta); \qquad [1965g]$$
$$\left(\frac{ddu'}{d\varphi^2}\right)=-4h.\cos.^2\psi.\cos.2.(\varphi+\beta). \qquad [1965h]$$

Changing in these expressions the quantities φ, ψ, u', q, into $\varphi_{,}$, $\psi_{,}$, $u'_{,}$, $q_{,}$, respectively, [1965i] we shall get the values to be substituted in [1876], to obtain the following expression of s; using for reduction $\sin.^2\psi_{,}=\frac{1}{2}-\frac{1}{2}.\cos.2\psi_{,}$:

$$s=\varepsilon+\alpha\varepsilon.\{-q_{,}.\sin.^2\psi_{,}+h.\cos.2.(\varphi_{,}+\beta)-2q_{,}.\cos.2\psi_{,}\}$$
$$+\tfrac{1}{2}\alpha\varepsilon^2.\{-q_{,}.\sin.2\psi_{,}+4q_{,}.\sin.2\psi_{,}\}$$
$$=\varepsilon+\alpha\varepsilon.\{h.\cos.2.(\varphi_{,}+\beta)-\tfrac{1}{2}q_{,}-\tfrac{3}{2}q_{,}.\cos.2\psi_{,}\}+\tfrac{3}{2}\alpha\varepsilon^2.q_{,}.\sin.2\psi_{,}$$
$$=\varepsilon-\tfrac{1}{2}\alpha\varepsilon.q_{,}.\{1+3.\cos.2\psi_{,}-3\varepsilon.\sin.2\psi_{,}\}+\alpha\varepsilon.h.\cos.2.(\varphi_{,}+\beta). \qquad [1965k]$$

Substituting $q_{,}$ [1965b], it becomes as in [1966], the accents of $q_{,}$, $\varphi_{,}$, &c., being omitted [1965l] by the author.

If the measured arc be of considerable length, and the latitudes of some intermediate points between the extremes have been observed, as in France;
[1966′] we shall have, by means of these measures, both the length of the radius taken for unity, and the value of $\alpha \,.\, \{1 + h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)\}$.* We shall then have by what precedes,

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This function becomes more simple, by using the values of ψ, φ, corresponding to the
[1965m] middle latitude between the extreme points of the arc s, as in [1875e—h], where the even powers of ε are avoided; and the same method may be used with advantage, in several of the formulas of this article. For if we suppose generally the arc s to be a function of ε, which can be developed in a series of the form,

[1965n] $$s = A_0 + A_1 \,.\, \varepsilon + A_2 \,.\, \varepsilon^2 + A_3 \,.\, \varepsilon^3 + \&c.;$$

in which A_0, A_1, &c., are independent of ε; we shall have, by putting successively $\varepsilon = -\tfrac{1}{2}\varepsilon'$, $\varepsilon = \tfrac{1}{2}\varepsilon'$, and representing the corresponding values of s, by $s_{,}$, s',

[1965n′] $$s_{,} = A_0 - A_1 \,.\, (\tfrac{1}{2}\varepsilon') + A_2 \,.\, (\tfrac{1}{2}\varepsilon')^2 - A_3 \,.\, (\tfrac{1}{2}\varepsilon')^3 + \&c.;$$
$$s' = A_0 + A_1 \,.\, (\tfrac{1}{2}\varepsilon') + A_2 \,.\, (\tfrac{1}{2}\varepsilon')^2 + A_3 \,.\, (\tfrac{1}{2}\varepsilon')^3 + \&c.$$

[1965o] Hence $s' - s_{,} = 2 \,.\, \{A_1 \,.\, (\tfrac{1}{2}\varepsilon') + A_3 \,.\, (\tfrac{1}{2}\varepsilon')^3 + \&c.\} = A_1 \,.\, \varepsilon' + \tfrac{1}{4} A_3 \,.\, \varepsilon'^3 + \&c.;$ in which no even powers of ε' occur; and this represents the value of s included between the limits $-\tfrac{1}{2}\varepsilon'$ and $+\tfrac{1}{2}\varepsilon'$. Hence it is evident, that if we use the values of ψ, φ,
[1965p] corresponding to the middle latitude of the arc s, we may omit the term depending on ε^2, and [1965k] will become $s = \varepsilon - \tfrac{1}{2}\alpha\varepsilon \,.\, q \,.\, \{1 + 3 \,.\, \cos.\, 2\psi\} + \alpha\varepsilon \,.\, h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)$.

We may observe, that the last term, $\alpha\varepsilon \,.\, h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)$ [1966], is not in the
[1965q] original work. This term is produced by the term $\alpha h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)$, which I have introduced in [1965]. For the part of u' [1965c], depending on this term, is $h.\cos.2.(\varphi+\beta)$; which produces nothing in the terms [1965e], which are substituted in [1876], but affects the term $\alpha\varepsilon \,.\, u'_{,}$, by the quantity $\alpha\varepsilon \,.\, h \,.\, \cos.\, 2 \,.\, (\varphi + \beta)$.

[1966a] * (1421) Substituting $q_{,} = 1 + h \,.\, \cos.2 \,.\, (\varphi_{,} + \beta)$, or $h \,.\, \cos.2 \,.\, (\varphi_{,} + \beta) = q_{,} - 1$, [1965b, i], in [1965k], it becomes successively, by neglecting terms of the order α^2, and
[1966b] putting for brevity $-1 + 3 \,.\, \cos.\, 2\psi_{,} - 3\varepsilon \,.\, \sin.\, 2\psi_{,} = 2\lambda$,

$$s = \varepsilon - \tfrac{1}{2}\alpha\varepsilon \,.\, q_{,} \,.\, \{1 + 3 \,.\, \cos.\, 2\psi_{,} - 3\varepsilon \,.\, \sin.\, 2\psi_{,}\} + \alpha\varepsilon \,.\, (q_{,} - 1)$$
$$= \varepsilon - \tfrac{1}{2}\alpha\varepsilon \,.\, q_{,} \,.\, \{-1 + 3\cos.2\psi_{,} - 3\varepsilon \,.\, \sin.\, 2\psi_{,}\} - \alpha\varepsilon = \varepsilon - \alpha\varepsilon - \alpha\varepsilon \,.\, q_{,} \,.\, \lambda$$
[1966c] $$= \varepsilon \,.\, \{1 - \alpha - \alpha q_{,} \,.\, \lambda\} = (1 - \alpha) \,.\, \varepsilon \,.\, \{1 - \alpha q_{,} \,.\, \lambda\}.$$

In this equation, s is supposed to be expressed in parts of the radius, taken as unity; if this
[1966d] radius be equal to a toises, and s be also given in toises, we must change s into $\frac{s}{a}$, or multiply the preceding value of s by a, and we shall get,

[1966e] $$s = a \,.\, (1 - \alpha) \,.\, \varepsilon \,.\, \{1 - \alpha q_{,} \,.\, \lambda\}.$$

Corrected azimuth.

$$\varpi = [-2\alpha\varepsilon \,.\, h \,.\, \sin.\psi \,.\, \mathrm{tang.}\psi \,.\, \sin.2.(\varphi+\beta)].^*$$ [1967]

The observations of the azimuth angles, at the two extremities of the arc,

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For any intermediate arc s', we may change s, ε, λ, into s', ε', λ', respectively, and we have

$$s' = a.(1-\alpha).\varepsilon'.\{1-\alpha q_{,}.\lambda'\}.$$ [1966f]

Dividing this value of s', by that of s [1966e], neglecting α^2, we get,

$$\frac{s'\varepsilon}{s\varepsilon'} = 1+\alpha q_{,}.(\lambda-\lambda').$$ [1966g]

Hence $\alpha q_{,} = \dfrac{s'\varepsilon - s\varepsilon'}{s\varepsilon'.(\lambda-\lambda')} = \alpha.\{1+h.\cos.2.(\varphi_{,}+\beta)\}$ [1966a] may be computed; [1966h]
and then $a.(1-\alpha)$, from either of the equations [1966e, f]. In finding λ, λ', [1966b], we may, as in [1965m], use the values of ψ, φ, &c., corresponding to the middle of the arcs s, s', respectively, and neglect the term depending on ε.

* (1422) This equation, like [1965, 1966], is given differently by the author, the original being

$$\varpi = -2\alpha h.\varepsilon.\frac{\mathrm{tang.}^2\psi.(1+\cos.^2\psi)}{\cos.\psi}.\sin.2.(\varphi+\beta).$$ [1967a]

This difference arises from the term $\alpha h.\cos.2.(\varphi+\beta)$, introduced in [1965], as we shall soon see. If we substitute the values [1965g, f] in [1891], we get,

$$\varpi = -\frac{\alpha\varepsilon.\mathrm{tang.}\psi}{\cos.\psi}.\{-2h.\sin.2.(\varphi+\beta).\cos.^2\psi.\mathrm{tang.}\psi + 2h.\sin.2\psi.\sin.2.(\varphi+\beta)\}$$

$$= -\frac{\alpha\varepsilon.\mathrm{tang.}\psi}{\cos.\psi}.\{-2h.\sin.2.(\varphi+\beta).\cos.\psi.\sin.\psi + 4h.\sin.\psi.\cos.\psi.\sin.2.(\varphi+\beta)\}$$

$$= -\frac{\alpha\varepsilon.\mathrm{tang.}\psi}{\cos.\psi}.2h.\sin.2.(\varphi+\beta).\cos.\psi.\sin.\psi$$

$$= -2\alpha\varepsilon.h.\sin.\psi.\mathrm{tang.}\psi.\sin.2.(\varphi+\beta);$$ [1967b]

as in [1967]. The additional term $h.\cos.2.(\varphi+\beta)$, which I have introduced in [1965c], produces nothing in the expression $\left(\dfrac{ddu'}{d\varphi\, d\psi}\right)$ [1891]; but in the term $\left(\dfrac{du'}{d\varphi}\right)$, it produces $-2h.\sin.2.(\varphi+\beta)$. Substituting this in [1891], we get this additional term in ϖ, namely, $\dfrac{\alpha\varepsilon.\mathrm{tang.}^2\psi}{\cos.\psi}.2h.\sin.2.(\varphi+\beta)$; and by subtracting it from the terms computed in [1967b], we get the value of ϖ, corresponding to the expression of the author,
$-2\alpha h.\varepsilon.\dfrac{\mathrm{tang.}^2\psi}{\cos.\psi}.\sin.2.(\varphi+\beta).\left\{1+\dfrac{\sin.\psi.\cos.\psi}{\mathrm{tang.}\psi}\right\}$; the quantity between the [1967c]
braces being easily reduced to the form $1+\cos.^2\psi$, by putting $\mathrm{tang.}\psi = \dfrac{\sin.\psi}{\cos.\psi}$.

[1967′] will thus give the value of $\alpha h \,.\, \sin.\, 2 \,.\, (\varphi+\beta)$. Lastly, the degree, measured in a direction perpendicular to the meridian, is equal to*

Corrected value of the degree perpendicular to the meridian.

[1968] $$1^{\circ} + 1^{\circ} \,.\, \alpha \,.\, \{1 + h \,.\, \cos.\, 2 \,.\, (\varphi+\beta)\} \,.\, \sin.^2 \psi - [3^{\circ} \,.\, \alpha h \,.\, \cos.\, 2 \,.\, (\varphi+\beta)].$$

[1968′] Therefore the measure of this degree will give the value of $\alpha h . \cos. 2 . (\varphi+\beta)$.† Thus the osculatory ellipsoid will be determined by these different measures. It will be necessary, for an arc of considerable length, to notice the square

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* (1423) The length D of a degree, in a direction perpendicular to the meridian, is found nearly, by multiplying the radius R [1951] by 1°. Hence, by neglecting the marks below $\varphi_{,}$, $\psi_{,}$, $q_{,}$, as in [1965m], we have,

[1968a] $$D = 1^{\circ} \,.\, \left\{ 1 + \alpha u'_{,} - \alpha \,.\, \left(\frac{d u'_{,}}{d \psi}\right) \,.\, \text{tang.}\, \psi + \frac{\alpha \,.\, \left(\frac{d d u'_{,}}{d \varphi^2}\right)}{\cos.^2 \psi} \right\};$$

and by substituting the values deduced from [1965d, e, h], it becomes, by observing that $\sin.\, \psi \,.\, \cos.\, \psi \,.\, \text{tang.}\, \psi = \sin.^2 \psi$,

$$D = 1^{\circ} . \{1 - \alpha q . \sin.^2 \psi + \alpha h . \cos. 2 . (\varphi+\beta) + 2 \alpha q . \sin. \psi . \cos. \psi . \text{tang.} \psi - 4 \alpha h . \cos. 2 . (\varphi+\beta)\}$$

[1968b] $$= 1^{\circ} . \{1 + \alpha q . \sin.^2 \psi - 3 \alpha h \,.\, \cos.\, 2 \,.\, (\varphi+\beta)\};$$

which, by means of [1965b], becomes as in [1968]. The value given by the author, in the original work, is

[1968c] $$D = 1^{\circ} + 1^{\circ} \,.\, \alpha \,.\, \{1 + h \,.\, \cos.\, 2 \,.\, (\varphi+\beta)\} \,.\, \sin.^2 \psi + 4^{\circ} \,.\, \alpha h \,.\, \text{tang.}^2 \psi \,.\, \cos.\, 2 \,.\, (\varphi+\beta).$$

The difference of these two expressions arises from the term of u' [1965d] represented by $h \,.\, \cos.\, 2 \,.\, (\varphi+\beta)$, mentioned in the last note. This produces, in $\left(\frac{d d u'}{d \varphi^2}\right)$, the term $-4 h \,.\, \cos.\, 2 \,.\, (\varphi+\beta)$; and in [1968$a$], the terms,

$$1^{\circ} . \alpha h . \cos. 2 . (\varphi+\beta) . \left\{ 1 - \frac{4}{\cos.^2 \psi} \right\} = 1^{\circ} \,.\, \alpha h \,.\, \cos.\, 2 \,.\, (\varphi+\beta) \,.\, \{1 - 4 \,.\, (1 + \text{tang.}^2 \psi)\}$$

[1968d] $$= 1^{\circ} \,.\, \alpha h \,.\, \{-3 - 4 \,.\, \text{tang.}^2 \psi\}.$$

Adding these to [1968c], we get [1968].

† (1424) Putting $\alpha h \,.\, \cos.\, 2 \,.\, (\varphi+\beta) = x$, [1968] will be of the form $A + B x$;

[1968e] A, B, being functions of the latitude, &c., from which, and $\alpha q_{,}$ [1966h], x may be determined, when A, B, and the length of the degree are known; and we may combine a number of such expressions together, for greater accuracy, as in the next section of this work.

[1968f] From the results, obtained in [1966h, 1967′, 1968′], we may get the values of a, α, h, provided the observations be sufficiently accurate and diversified; hence the form of the osculatory ellipsoid will be known.

of ε,* in the value of ϖ; particularly if the azimuth angle does not vary in [1968″]
proportion to the measured arc, which is the case with the observations in
France; we must then add, to the preceding expression of the radius of the [1968‴]
ellipsoid, a term of the form $\alpha k \,.\, \sin.\psi \,.\, \cos.\psi \,.\, \sin.(\varphi + \beta')$, to obtain the
most general expression of that radius.†

39. Next to the sphere, the ellipsoidal figure is the most simple; and we
have already shown [1731′], that the earth and planets must have that form,
if we suppose, at their origin, they were in a fluid state, and that they have [1968⁗]
retained the same figure as they became solid. It is therefore natural to
compare the measured degrees of the meridian with this figure; but the
comparison of several observations has given, for the figure of the meridians,
very different ellipses, which vary so much from the observations, that they [1968v]
cannot be adopted. However, before we give up wholly the elliptical figure,
*we ought to determine that form of the ellipsis, in which the greatest error of
the measured degrees, is less than in any other figure of this kind.* We
must then examine whether this error is within the limits of the errors of
the observations. This may be done in the following manner.

Method of finding the ellipticity which will make the greatest error of a measured degree a minimum.

We shall put $a^{(1)}$, $a^{(2)}$, $a^{(3)}$, &c., for the measured degrees of the meridians;
$p^{(1)}$, $p^{(2)}$, $p^{(3)}$, &c., for the squares of the sines of the corresponding latitudes;‡ [1968vi]

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* (1424a) Terms of this order are included by using the values of ψ, φ, corresponding to the middle of the arc, as in [1965m, &c.].

† (1425) If we restrict ourselves to an osculatory *ellipsoid*, it will not be necessary to add the term here mentioned, because the expression [1965] contains all the terms as far as the order α, inclusively.

‡ (1426) Putting in [1795d], $c - \frac{2}{3}\alpha h.c = z$, $3\alpha h.c = y$, $p = \mu^2$, we shall have, [1969a]
by neglecting α^2, $z + py$ for the length of a degree; $\mu = \cos.\theta$ being nearly equal to the sine of the latitude [1795‴]. This agrees with [1969]; therefore the length a of a degree may be put under the form,

z is the equatorial length, y the increment in proceeding to the pole, p the square of the sine of the latitude.

$$a = z + py = z + y \,.\, \cos.^2\theta = z + y \,.\, \mu^2 = z + y \,.\, \sin.^2 \text{lat.} \qquad [1969b]$$

The length of a pendulum, in any latitude, is also expressed by the formula [1969b], supposing z to be the length of the pendulum at the equator, and $z + p$ its length at the pole, as is evident from [1804, &c.]. [1969b′]

Degree of the meridian, or length of a pendulum.

and we shall suppose, that in the required ellipsis, the degree of the meridian is expressed by the following formula,

[1969] $$z + p\,y.$$

In this expression, terms of the order α^2 are neglected; and as it is an object of curiosity
[1969c] to ascertain their effect, in the computation of a large arc of an elliptical meridian, we shall give the following investigation.

The radius of curvature of an arc of the meridian, in the latitude ψ, being represented by
[1969d] $R_{,}$, the polar semi-axis by k, the equatorial radius by $k' = k\,.\sqrt{(1+\lambda^2)}$, we shall have
[1969e] the value of $R_{,}$, as in [1585]. If we develop this formula, according to the powers of λ^2, neglecting terms of the order λ^6, and reducing by means of [6, 8] Int., we shall get,

[1969f] $$R_{,} = k\,.\,(1+\lambda^2)\,.\,\{1 - \tfrac{3}{2}\lambda^2\,.\cos.^2\psi + \tfrac{15}{8}\lambda^4\,.\cos.^4\psi\}$$

Radius of curvature. First form.

$$= k\,.\,(1+\lambda^2)\,.\,\{1 - \tfrac{3}{4}\lambda^2\,.\,(1+\cos.2\psi) + \tfrac{15}{64}\lambda^4\,.\,(3 + 4\,.\cos.2\psi + \cos.4\psi)\}$$

$$= k\,.\,(1+\lambda^2)\,.\,\{(1 - \tfrac{3}{4}\lambda^2 + \tfrac{45}{64}\lambda^4) - (\tfrac{3}{4}\lambda^2 - \tfrac{15}{16}\lambda^4)\,.\cos.2\psi + \tfrac{15}{64}\cos.4\psi\}$$

[1969g] $$= k\,.\,\{(1 + \tfrac{1}{4}\lambda^2 - \tfrac{3}{64}\lambda^4) - (\tfrac{3}{4}\lambda^2 - \tfrac{3}{16}\lambda^4)\,.\cos.2\psi + \tfrac{15}{64}\lambda^4\,.\cos.4\psi\}.$$

Multiplying this last expression by $d\psi$, we get ds, or the differential of the arc of the meridian s, corresponding to the increment of latitude $d\psi$. The integral of this, supposing it to commence at the equator, is

[1969h] $$s = k\,.\,\{(1 + \tfrac{1}{4}\lambda^2 - \tfrac{3}{64}\lambda^4)\,.\,\psi - (\tfrac{3}{8}\lambda^2 - \tfrac{3}{32}\lambda^4)\,.\sin.2\psi + \tfrac{15}{256}\lambda^4\,.\sin.4\psi\}.$$

Arc of the meridian. First form.

From this we obtain the following expression of the arc s' of the meridian, comprised between the latitudes ψ', ψ.

[1969i] $$s' = k.\{(1+\tfrac{1}{4}\lambda^2-\tfrac{3}{64}\lambda^4).(\psi'-\psi)-(\tfrac{3}{8}\lambda^2-\tfrac{3}{32}\lambda^4).(\sin.2\psi'-\sin.2\psi)+\tfrac{15}{256}\lambda^4.(\sin.4\psi'-\sin.4\psi)\}.$$

We may put this under another form, by substituting $k = k'.(1+\lambda^2)^{-\frac{1}{2}} = k'.(1-\tfrac{1}{2}\lambda^2+\tfrac{3}{8}\lambda^4)$, and reducing, which gives,

[1969k] $$s' = k'.\{(1-\tfrac{1}{4}\lambda^2+\tfrac{13}{64}\lambda^4).(\psi'-\psi)-(\tfrac{3}{8}\lambda^2-\tfrac{9}{32}\lambda^4).(\sin.2\psi'-\sin.2\psi)+\tfrac{15}{256}\lambda^4.(\sin.4\psi'-\sin.4\psi)\}.$$

[1969l] If we put $k' = k\,.\sqrt{(1+\lambda^2)} = k\,.\,(1+\varepsilon)$, ε will express the ellipticity in parts of the polar radius k, taken as unity, and we shall have $\lambda^2 = 2\,\varepsilon + \varepsilon^2$. Substituting this in [1969g, h, i], we get, by reduction, neglecting ε^3,

Second forms of the radius and arc.

[1969m] $$R_{,} = k\,.\,\{(1 + \tfrac{1}{2}\varepsilon + \tfrac{1}{16}\varepsilon^2) - \tfrac{3}{2}\varepsilon\,.\cos.2\psi + \tfrac{15}{16}\varepsilon^2\,.\cos.4\psi\};$$

[1969n] $$s = k\,.\,\{(1 + \tfrac{1}{2}\varepsilon + \tfrac{1}{16}\varepsilon^2)\,.\,\psi - \tfrac{3}{4}\varepsilon\,.\sin.2\psi + \tfrac{15}{64}\varepsilon^2\,.\sin.4\psi\};$$

[1969o] $$s' = k\,.\,\{(1 + \tfrac{1}{2}\varepsilon + \tfrac{1}{16}\varepsilon^2)\,.\,(\psi'-\psi) - \tfrac{3}{4}\varepsilon\,.\,(\sin.2\psi' - \sin.2\psi) + \tfrac{15}{64}\varepsilon^2.(\sin.4\psi' - \sin.4\psi)\}.$$

These will be varied a little, if we express the oblateness ε' in parts of the equatorial radius,

Putting $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, &c., for the errors of the observations, we shall have [1969']

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making $\frac{k}{k'} = 1 - \varepsilon' = \frac{1}{1+\varepsilon}$; or $\varepsilon' = 1 - \frac{1}{1+\varepsilon} = \frac{\varepsilon}{1+\varepsilon} = \frac{1}{\frac{1}{\varepsilon}+1}$; whence [1969p]

$\frac{1}{\varepsilon'} = \frac{1}{\varepsilon} + 1$. If we neglect ε^3, we shall have, $\varepsilon = \frac{1}{1-\varepsilon'} - 1 = \varepsilon' + \varepsilon'^2$, $k = k'.(1-\varepsilon')$, [1969p']

Third forms of the radius and arc.

$k\,\varepsilon = k'\,\varepsilon'$, $k\,\varepsilon^2 = k'\,\varepsilon'^2$. Substituting these in [1969m, n, o], we get,

$$R_{,} = k'.\{(1 - \tfrac{1}{2}\varepsilon' + \tfrac{1}{16}\varepsilon'^2) - \tfrac{3}{2}\varepsilon'.\cos.2\psi + \tfrac{15}{16}\varepsilon'^2.\cos.4\psi\}; \qquad [1969q]$$

$$s = k'.\{(1 - \tfrac{1}{2}\varepsilon' + \tfrac{1}{16}\varepsilon'^2).\psi - \tfrac{3}{4}\varepsilon'.\sin.2\psi + \tfrac{15}{64}\varepsilon'^2.\sin.4\psi\}; \qquad [1969r]$$

$$s' = k'.\{(1 - \tfrac{1}{2}\varepsilon' + \tfrac{1}{16}\varepsilon'^2).(\psi' - \psi) - \tfrac{3}{4}\varepsilon'.(\sin.2\psi' - \sin.2\psi) + \tfrac{15}{64}\varepsilon'^2.(\sin.4\psi' - \sin.4\psi)\}. \qquad [1969s]$$

If we put $\psi = 0$, $\psi' = \frac{1}{2}\pi$, in s' [1969o, s], we shall obtain the quadrantal arc of the elliptical meridian S, namely, [1969t]

Quadrantal arc of the meridian.

$$S = k.\tfrac{1}{2}\pi.\{1 + \tfrac{1}{2}\varepsilon + \tfrac{1}{16}\varepsilon^2\} \qquad [1969u]$$

$$= k'.\tfrac{1}{2}\pi.\{1 - \tfrac{1}{2}\varepsilon' + \tfrac{1}{16}\varepsilon'^2\}. \qquad [1969v]$$

If we put L equal to the latitude of the middle point of the measured arc, l equal to the difference of the latitudes of the extreme points of the arc, we shall have $\psi' = L + \frac{1}{2}l$, [1970a] $\psi = L - \frac{1}{2}l$; hence,

$$\sin.2\psi' = \sin.(2L + l) = \sin.2L.\cos.l + \cos.2L.\sin.l;$$

$$\sin.2\psi = \sin.(2L - l) = \sin.2L.\cos.l - \cos.2L.\sin.l; \qquad [1970b]$$

$$\sin.2\psi' - \sin.2\psi = 2.\cos.2L.\sin.l;$$

and in like manner,

$$\sin.4\psi' - \sin.4\psi = 2.\cos.4L.\sin.2l = 4.\cos.4L.\sin.l.\cos.l.$$

Substituting these in [1969i, k, o, s], we get the four following expressions of the arc of the meridian s', whose middle latitude is L, and difference of latitudes l. [1970c]

Arc of the meridian.

$$s' = k.\{(1 + \tfrac{1}{4}\lambda^2 - \tfrac{3}{64}\lambda^4).l - (\tfrac{3}{4}\lambda^2 - \tfrac{3}{16}\lambda^4).\sin.l.\cos.2L + \tfrac{15}{64}\lambda^4.\sin.l.\cos.l.\cos.4L\} \qquad [1970d]$$

$$= k'.\{(1 - \tfrac{1}{4}\lambda^2 + \tfrac{13}{64}\lambda^4).l - (\tfrac{3}{4}\lambda^2 - \tfrac{9}{16}\lambda^4).\sin.l.\cos.2L + \tfrac{15}{64}\lambda^4.\sin.l.\cos.l.\cos.4L\} \qquad [1970e]$$

$$= k.\{(1 + \tfrac{1}{2}\varepsilon + \tfrac{1}{16}\varepsilon^2).l - \tfrac{3}{2}\varepsilon.\sin.l.\cos.2L + \tfrac{15}{16}\varepsilon^2.\sin.l.\cos.l.\cos.4L\} \qquad [1970f]$$

$$= k'.\{(1 - \tfrac{1}{2}\varepsilon' + \tfrac{1}{16}\varepsilon'^2).l - \tfrac{3}{2}\varepsilon'.\sin.l.\cos.2L + \tfrac{15}{16}\varepsilon'^2.\sin.l.\cos.l.\cos.4L\}. \qquad [1970g]$$

Fourth form.

In the first term of each of these formulas, l is expressed in parts of the radius, taken as unity; but as it is usually given in degrees, we must change l into $\frac{l}{\rho}$, putting $\rho = 57^d,2957795$, or $\rho = 63°,66198$, for the arc equal to the radius, according as we [1970h] shall use the sexagesimal or centesimal division. If l be expressed in seconds, the corresponding values of ρ are $\rho = 206265^s$, or $\rho = 636619'',8$. Moreover, if we [1970h']

$p^{(1)}$, $p^{(2)}$, &c., [1969''] form an increasing progression.

the following equations, in which we shall suppose that $p^{(1)}$, $p^{(2)}$, $p^{(3)}$, &c., *form an increasing progression,*

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put e for the length of a degree, measured on the circumference of the equator, expressed in fathoms, metres, or any other linear measure, taken as unity, we shall have, by using the value of ρ corresponding to a degree, $k'=\rho.e$. Hence the formulas [1970e, g] will become,

[1970i]

Arc of the meridian. Fifth form.

[1970k] $$s'=e.\{(1-\tfrac{1}{4}\lambda^2+\tfrac{13}{64}\lambda^4).l-(\tfrac{3}{4}\lambda^2-\tfrac{9}{16}\lambda^4).\rho.\sin.l.\cos.2L+\tfrac{15}{64}\rho.\lambda^4.\sin.l.\cos.l.\cos.4L\}$$

[1970l] $$=e.\{(1-\tfrac{1}{2}\varepsilon'+\tfrac{1}{16}\varepsilon'^2).l.-\tfrac{3}{2}\varepsilon'.\rho.\sin.l.\cos.2L+\tfrac{15}{16}\varepsilon'^2.\rho.\sin.l.\cos.l.\cos.4L\}.$$

[1970m] Using ρ as above, and putting for brevity M, $N.\rho^{-1}$, $P.\rho^{-1}$, for the coefficients of $\psi'-\psi$, $\sin.2\psi'-\sin.2\psi$, $\cos.4\psi'-\cos.4\psi$, in either of the formulas [1969i, k, o, s];

[1970n] or for the coefficients of l, $\cos.2L$, $\cos.4L$, in either of the formulas [1970k, l], respectively, we shall get this general expression of s',

Arc of the meridian. Sixth form.

[1970o] $$s'=M.(\psi'-\psi)+N.(\sin.2\psi'-\sin.2\psi)+P.(\sin.4\psi'-\sin.4\psi)$$

[1970o'] $$=M.(\psi'-\psi)+N.\cos.2L+P.\cos.4L.$$

M, N, P, are expressed in terms of the semi-axis of the earth and its ellipticity, and we

[1970p] easily find by inspection that $P=\frac{5}{12}.\frac{N^2}{M}$, neglecting terms of the orders λ^6, ε^3, and $l^3\varepsilon^2$. The first of these forms is similar to that used by Mr. Airy, in the Philosophical Transactions for 1826, page 570; although he does not restrict himself to the consideration of an elliptical meridian, but supposes it to vary from that form, in terms of the second order of

[1970q] the ellipticity; and by this means P became independent of M, N, and his system of equations contains three unknown quantities M, N, P, to be determined by using the principle of the least squares. The quantity P being very small, we may, in [1970o, o'], compute its value, by using any ellipticity which does not differ much from that resulting from the same observations; and then connecting this term with the quantity s', making

[1970r] $s_{,}=s'-P.(\sin.4\psi'-\sin.4\psi)$, we shall get $s_{,}=M.(\psi'-\psi)+N.(\sin.2\psi'-\sin.2\psi)$, depending on two unknown quantities M, N, as in the method used by La Place [1970]. Instead of this method, we may put, in [1970k], $\lambda^2=\frac{1}{150}+\lambda'$, or in [1970l],

[1970s] $\varepsilon'=\frac{1}{300}+\varepsilon''$, and neglect the square and higher powers of λ', or ε''; by this means the system of equations [1970k, l] would depend on the two unknown quantities e, λ', or e, ε', only. This method is given by Mr. Ivory, in the Philosophical Magazine, London, 1828, page 345, using the formulas [1969s, 1970l].

After we have obtained the values of M, N, we may easily deduce the ellipticity. For if we compare [1969s, 1970o], using the value of ρ [1970h, h'], we shall get,

Ellipticity.

[1970t] $$M=k'.(1-\tfrac{1}{2}\varepsilon'+\tfrac{1}{16}\varepsilon'^2);\qquad N=-\tfrac{3}{4}\rho k'.\varepsilon';\qquad P=\tfrac{15}{64}\rho k'.\varepsilon'^2;$$

[1970t'] hence $-\frac{3}{4}\rho.\frac{M}{N}=\frac{1-\frac{1}{2}\varepsilon'+\frac{1}{16}\varepsilon'^2}{\varepsilon'}=\frac{1}{\varepsilon'}-\frac{1}{2}$, nearly, or $\frac{1}{\varepsilon'}=-\frac{3}{4}\rho.\frac{M}{N}+\frac{1}{2}$. If we

System of equations for the determination of the ellipticity.

$$a^{(1)} - z - p^{(1)} . y = x^{(1)},$$
$$a^{(2)} - z - p^{(2)} . y = x^{(2)},$$
$$a^{(3)} - z - p^{(3)} . y = x^{(3)}, \qquad (A)$$ [1970]
$$. \quad . \quad . \quad . \quad . \quad . \quad .$$
$$a^{(n)} - z - p^{(n)} . y = x^{(n)}, *$$

n being the number of the measured degrees.

We must eliminate from these equations, the two unknown quantities z and y, and we shall have $n-2$ equations of condition, between the n errors $x^{(1)}, x^{(2)}, \ldots . . x^{(n)}$. We must then determine the system of [1970′] equations, in which the greatest error, neglecting its sign, is less than in any other system.

First method of finding when the greatest error is a minimum.

We shall suppose, in the first place, that there is but one equation of condition between these errors, which we may represent in the following manner,

$$a = m\,x^{(1)} + n\,x^{(2)} + p\,x^{(3)} + \&c. ;$$ [1971]

a being positive. We shall have the system of values of $x^{(1)}$, $x^{(2)}$, &c., which gives the least value to the greatest error; by supposing them, [1971′] independent of their signs, to be equal to each other, and to the quotient of a divided by the sum of the coefficients m, n, p, &c., taken positively ;† [1971″]

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use the value of ρ [1970h'], corresponding to sexagesimal seconds, this will become,

Ellipticity.

$$\frac{1}{\varepsilon'} = -154699 . \frac{M}{N} + \tfrac{1}{2}, \qquad \text{whence} \qquad \frac{1}{\varepsilon} = -154699 . \frac{M}{N} - \tfrac{1}{2}.$$ [1970u]

If we compare the values of N, P, [1970t], we shall have, for the case of an elliptical meridian, $P = -\frac{5}{16}\varepsilon' . N$, or $P = -\frac{5}{16}\varepsilon . N$, nearly. [1970$v$]

* (1427) If a represent the length of a degree, and there be no error in the elliptical hypothesis, or in the observations, we shall have $a = z + py$ [1969], or $a - z - py = 0$. [1970w] If there be an error, producing, in the terms of the first member of this last equation, the quantity x, we shall have $a - z - py = x$, which is of the same form as the equations [1970].

† (1428) If all the quantities m, n, p, &c., [1971], were positive, as well as a, it is evident that the least possible values of the errors $x^{(1)}, x^{(2)} \ldots . . x^{(i)}$, would be found, by dividing a, by the sum of the coefficients $m + n + p + \&c.$, so that

[1971'''] each quantity having the same sign as the coefficient of that quantity in the proposed equation.

If we have two equations of condition between these errors, the system which
[1971''''] *will give the least possible value to the greatest of these errors, will be such, that by neglecting the signs, all these errors will be equal to each other, excepting one*, which will be less than the others, or at least will not exceed them. Supposing therefore that this error is $x^{(1)}$, we may determine it, in a function
[1971^{v}] of $x^{(2)}$, $x^{(3)}$, &c., by means of one of the proposed equations of condition; then by substituting this value of $x^{(1)}$, in the other equation of condition, we shall obtain another equation between $x^{(2)}$, $x^{(3)}$, &c.; which will be of the form,

[1972] $$a = m\,x^{(2)} + n\,x^{(3)} + \&c.$$

a being positive,* we shall have as above [1971''], the values of $x^{(2)}$, $x^{(3)}$,
[1972'] &c., by dividing a, by the sum of the coefficients m, n, &c., taken positively,

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[1971a] $$x^{(1)} = x^{(2)} = x^{(i)} = \frac{a}{m+n+p+\&c.}.$$

This value of the least error, is less when all the quantities m, n, p, &c., have the positive sign, than it is when any one, or more of them, are negative; because the denominator of this expression is decreased by using these negative values. We may however, in all cases, consider the quantities m, n, p, &c., as positive, by changing the sign of the error of the
[1971b] corresponding term. Thus if the second term of [1971] were of the form $-n\,x^{(2)}$, we might change the sign of $x^{(2)}$, putting instead of it $-x'^{(2)}$, and then $-n\,x^{(2)} = n\,x'^{(2)}$, might be substituted in [1971, &c.], and n considered positive, as in [1971''].

* (1429) To explain this by a simple case, we shall suppose the two equations of condition to be,

[1972a] $$a' = m'x^{(1)} + n'x^{(2)} + p'x^{(3)} + q'x^{(4)}; \qquad a'' = m''x^{(1)} + n''x^{(2)} + p''x^{(3)} + q''x^{(4)}.$$

Multiplying the first by m'', and the second by $-m'$, and adding the products, the sum
[1972b] will be free from $x^{(1)}$, as in [1972], $a = m\,x^{(2)} + n\,x^{(3)} + p\,x^{(4)}$; and the least value of $x^{(2)}$, $x^{(3)}$, $x^{(4)}$, independent of the signs, will be found, as in [1971''], by dividing a, by the
[1972c] sum $M = m + n + p$, taken positively. If these values, when substituted in [1972a], give for $x^{(1)}$, a quantity which is less than this computed value of $x^{(2)}$, $x^{(3)}$, $x^{(4)}$, neglecting the signs, the system of values will be those required. For if it be not, suppose that there is another system, in which the greatest value of every one of the terms $x^{(1)}$, $x^{(2)}$, &c., is less
[1972d] than $\frac{a}{M}$, so that $x^{(1)} < \frac{a}{M}$, $x^{(2)} < \frac{a}{M}$, $x^{(3)} < \frac{a}{M}$, $x^{(4)} < \frac{a}{M}$. Multiplying

and giving successively to the quotients the signs of m, n, &c. These values, being substituted in the expression of $x^{(1)}$, in terms of $x^{(2)}$, $x^{(3)}$, &c., will give the value of $x^{(1)}$; and if this value, neglecting its sign, do not [1972″] exceed that of $x^{(2)}$, this system of values will be that which we must adopt; but if it be greater, then the supposition that $x^{(1)}$ is the least error is not correct, and we must make successively the same supposition for $x^{(2)}$, $x^{(3)}$, [1972‴] &c., until we shall obtain a quantity which satisfies the conditions.

If we have three equations of condition, between these errors, the system which will give the least possible value to the greatest error, will be such, [1972⁗] that if we neglect the signs, all the errors will be equal excepting two, which will be less than the others. Therefore, by supposing $x^{(1)}$, $x^{(2)}$, to be these two errors, we may eliminate them from the third of the equations of [1972ᵛ] condition, by means of the other two, and we shall have an equation of condition between the errors $x^{(3)}$, $x^{(4)}$, &c.; which we shall represent in the following manner,

$$a = m\,x^{(3)} + n\,x^{(4)} + \&c., \qquad [1973]$$

a being positive. We shall obtain the values of $x^{(3)}$, $x^{(4)}$, &c., by dividing a by the sum of the coefficients m, n, &c., taken positively, and giving [1973′] successively to the quotients the signs of m, n, &c. These values, being substituted in the expressions of $x^{(1)}$ and $x^{(2)}$, in terms of $x^{(3)}$, $x^{(4)}$, &c., will give the values of $x^{(1)}$, $x^{(2)}$; and if these last values, neglecting their signs, do not exceed $x^{(3)}$, we shall have the system of errors which must be [1973″] adopted. But if one of these values exceed $x^{(3)}$, the supposition that $x^{(1)}$ and $x^{(2)}$ are the least errors, is not correct, and we must make the same supposition, upon another combination of the errors, $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, &c., taken two by two; until we shall obtain a combination, in which this supposition is correct. It is easy to extend this method to the case, where we shall [1973‴] have four, or a greater number of equations of condition, between the errors

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the three last expressions by m, n, p, respectively, and adding the products, we get,

$$m\,x^{(2)} + n\,x^{(3)} + p\,x^{(4)} < \frac{a}{M}\cdot(m+n+p). \qquad [1972e]$$

Substituting [1972b, c], we get, $a < \frac{a}{M}\cdot M$, or $a < a$, which is absurd; therefore we cannot put $x^{(2)} < \frac{a}{M}$, &c., [1972d].

$x^{(1)}$, $x^{(2)}$, $x^{(3)}$, &c. These errors being thus known, it will be easy to deduce from them the values of z and y.*

The method, which we have explained, applies to all questions of the same kind. Thus by having n observations of a comet, we may, by this
[1973''''] means, determine the parabolic orbit, in which the greatest error, neglecting its sign, is less than in any other orbit of that kind; and we may thereby ascertain, whether the parabolic hypothesis can satisfy these observations.
[1973v] When the number of observations is considerable, this method leads to long and tedious calculations, and we may, in the problem now under consideration, easily obtain the required system of errors, by the following method.

Second method of finding when the greatest error is a minimum.

Neglecting the signs, we shall suppose that $x^{(i)}$ is the greatest of the errors $x^{(1)}$, $x^{(2)}$, &c. We shall observe, in the first place, that there must
[1973vi] be another equal error, $x^{(i')}$, of a contrary sign to $x^{(i)}$; otherwise, by an appropriate change in the equation [1970],

[1974] $$a^{(i)} - z - p^{(i)} \cdot y = x^{(i)},$$

[1974'] we could diminish the error $x^{(i)}$, and still retain the property of its being the extreme error;† which is contrary to the hypothesis. We shall then

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[1973a] * (1430) Substituting the values of the errors, $x^{(1)}$, $x^{(2)}$, &c., in [1970], we shall have several linear equations in z, y, any two of which will give the values of z, y, by the usual rules.

[1974a] † (1431) If we vary z, y, by very small quantities, so that they may become $z + \delta z$, $y + \delta y$, and put $x_{,}^{(i)}$ for the corresponding value of $x^{(i)}$, we shall get, by successive reductions, from [1974],

[1974b] $$x_{,}^{(i)} = a^{(i)} - z - p^{(i)} \cdot y - \delta z - p^{(i)} \cdot \delta y, \qquad \text{or} \qquad x_{,}^{(i)} = x^{(i)} - \delta z - p^{(i)} \cdot \delta y,$$

and if $-\delta z$, $-\delta y$, be infinitely small, and of a *different* sign from $x^{(i)}$, the value $x_{,}^{(i)}$ will evidently be less than $x^{(i)}$, neglecting their signs. In like manner, any one of the errors $x^{(1)}$, $x^{(2)}$, &c., which has the same sign as $x^{(i)}$, will be decreased by means of the quantities δz, δy. But if any one, as $x^{(i')}$, has a different sign from $x^{(i)}$, the quantities $-\delta z$, $-p^{(i)}.\delta y$, must by hypothesis be of the same sign as $x^{(i')}$, and the quantity

[1974c] $$x_{,}^{(i')} = x^{(i')} - \delta z - p^{(i')} \cdot \delta y,$$

will be *greater* than $x^{(i')}$, neglecting the signs. Therefore if the values z, y, be so assumed as to make the *extreme* errors $x^{(i)}$, $x^{(i')}$, *equal to each other*, and of different signs, it will be useless to attempt to decrease this *maximum* error, by varying z, y; for the variations

observe that $x^{(i)}$ and $x^{(i')}$ being the two extreme errors, the one positive, the [1974″]
other negative, and of equal values, as we have just seen; there must also be a third error $x^{(i'')}$, which by neglecting its sign is equal to $x^{(i)}$. For if we subtract the equation corresponding to $x^{(i)}$ [1974], from the equation [1974‴]
corresponding to $x^{(i')}$ [1970], we shall have,

$$a^{(i')} - a^{(i)} - \{p^{(i')} - p^{(i)}\} \cdot y = x^{(i')} - x^{(i)}. \qquad [1975]$$

The second member of this equation, neglecting its sign, is the sum of the extreme errors; and it is evident, that by varying y, we may diminish that [1975′]
sum, and still retain the property of its being the greatest of the sums that can be obtained by the addition, or subtraction, of the errors $x^{(1)}$, $x^{(2)}$, &c., taken two by two; unless there be a third error $x^{(i'')}$, equal to $x^{(i)}$, neglecting [1975″]
the signs.* Now the sum of the extreme errors being diminished, and these

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δz, δy, which *decrease* $x^{(i)}$, will *increase* $x^{(i')}$. Hence we may conclude, that when we have obtained the values of z, y, which make the greatest term $x^{(i)}$ of the quantities $x^{(1)}$, $x^{(2)}$, &c., independent of its sign, a minimum, there will also be another of these quantities $x^{(i')}$, equal to $x^{(i)}$, but of a different sign; so that we shall have $x^{(i)} + x^{(i')} = 0$. [1974d]

* (1432) z and y being supposed, as in [1974a], to change into $z + \delta z$, $y + \delta y$, the values of $x^{(i)}$, $x^{(i')}$, $x^{(i'')}$, [1974b, c, &c.], will become respectively,

$$x_{,}^{(i)} = x^{(i)} - \delta z - p^{(i)} . \delta y; \quad x_{,}^{(i')} = x^{(i')} - \delta z - p^{(i')} . \delta y; \quad x_{,}^{(i'')} = x^{(i'')} - \delta z - p^{(i'')} . \delta y. \qquad [1975a]$$

These values of $x_{,}^{(i)}$, $x_{,}^{(i')}$, being substituted for $x^{(i)}$, $x^{(i')}$, in the second member of [1975], it becomes $x^{(i')} - x^{(i)} + (p^{(i)} - p^{(i')}) . \delta y$; so that, by giving an appropriate sign to δy, we may *decrease* the *numerical* value of the second member of [1975]. Now this second [1975b]
member, independent of its sign, is *the sum of the two maxima errors* $x^{(i)}$, $x^{(i')}$, since they have different signs [1974″]. Moreover, we may use the quantity δz, for the purpose of making $x_{,}^{(i)}$ equal to $x_{,}^{(i')}$, independent of its sign, or $x_{,}^{(i)} + x_{,}^{(i')} = 0$. For by adding the two [1975c]
first of the equations [1975a], and substituting the equations [1974d, 1975c], upon which the equality of these errors depends, we get $0 = -2\,\delta z - (p^{(i)} + p^{(i')}) . \delta y$, or $\delta z = -\frac{1}{2} . (p^{(i)} + p^{(i')}) . \delta y$. Hence δy being assumed as in [1975b], we get the corresponding value of δz, which will satisfy the equation [1975c], making $x_{,}^{(i)}$ equal to $x_{,}^{(i')}$, but of a different sign; the *numerical value being less than* that of $x^{(i)}$, $x^{(i')}$; and it is [1975d]
evident that $x_{,}^{(i)}$, $x_{,}^{(i')}$, will be the extreme values of the errors $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, &c., unless another of the errors, as $x^{(i'')}$, independent of the signs, be equal to $x^{(i)}$ or $x^{(i')}$. We may continue this process of reducing the numerical values of $x_{,}^{(i)}$, $x_{,}^{(i')}$, until one of the other errors, as $x_{,}^{(i'')}$, becomes *equal* to these last found values $x_{,}^{(i)}$, $x_{,}^{(i')}$, neglecting the signs. If we increase the numerical values of δz, δy, beyond this, the numerical value of $x_{,}^{(i'')}$ may

errors made equal, by means of the value of z, each of these errors will be
[1975‴] diminished, which is contrary to the hypothesis. Therefore there are three errors, $x^{(i)}$, $x^{(i')}$, $x^{(i'')}$, which, by neglecting the signs, are equal to each
[1975⁗] other; and of these, there is one which has a different sign from the other two.

Supposing this last to be $x^{(i')}$, then the number i' will fall between the two numbers i and i''. To prove this, let us suppose it not to be so, and
[1975ᵛ] that i' is either less or greater than i and i''. Subtracting the equation corresponding to i', successively from the two equations corresponding to i and i'' [1970], we shall have,

[1976]
$$a^{(i)}-a^{(i')}-\{p^{(i)}-p^{(i')}\}\,.\,y=x^{(i)}-x^{(i')}\,;$$
$$a^{(i'')}-a^{(i')}-\{p^{(i'')}-p^{(i')}\}\,.\,y=x^{(i'')}-x^{(i')}.$$

The second members of these equations are equal, and of the same sign; and by neglecting the signs, they are also the sums of the extreme errors.
[1976′] Now it is evident, that by varying the value of y, we may diminish each of these sums, since the coefficient of y has the same sign in the two first members;* and we may also, by varying z, retain the same value for $x^{(i')}$.
[1976″] Then $x^{(i)}$, $x^{(i'')}$, neglecting their signs, will become less than $x^{(i')}$, which will become the greatest of the errors, without having any one equal to it;† and in this case, as we have just seen, we may diminish the extreme error,‡

[1975e] exceed that of $x_{,}^{(i)}$, $x_{,}^{(i'')}$, and these last values will cease to be the maxima, which is contrary to the hypothesis. Hence there are three errors, $x^{(i)}$, $x^{(i')}$, $x^{(i'')}$, which are numerically equal, but one has a different sign from the other two.

* (1433) It is supposed, in [1975ᵛ], that i' is either greater than i, i'', or less than i, i'';
[1976a] and by [1969″], the quantities $p^{(1)}$, $p^{(2)}$, &c., are arranged according to the values of i; therefore $p^{(i')}$ must be either greater than $p^{(i)}$, $p^{(i'')}$, or less than both these quantities; and in both cases, $p^{(i)}-p^{(i')}$, $p^{(i'')}-p^{(i')}$, must have the same sign.

† (1434) This will be evident, by proceeding as in [1975c—e]; whence we shall find
[1976b] that the same values of z, y, which decrease $x^{(i)}$, $x^{(i'')}$, will also generally decrease the terms of the series $x^{(1)}$, $x^{(2)}$, &c., that are less than $x^{(i)}$, $x^{(i'')}$, but have the same sign.

‡ (1435) If we have three extreme errors, $x^{(i)}$, $x^{(i')}$, $x^{(i'')}$, of which $x^{(i')}$, independent
[1976c] of its sign, exceeds the other two, and has a different sign from it; we may, by appropriate changes in z and y, decrease the numerical value of this quantity, as in [1974′, &c.].

which is contrary to the hypothesis. Therefore the number i' must fall between the numbers i and i''. [1976‴]

We shall now determine which of the errors $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, &c., are the extreme errors. For this purpose, we must subtract the first of the equations [1970], successively from the others, and we shall obtain this system of equations, [1976⁗]

$$\begin{aligned} a^{(2)} - a^{(1)} - (p^{(2)} - p^{(1)}) \,.\, y &= x^{(2)} - x^{(1)}\,; \\ a^{(3)} - a^{(1)} - (p^{(3)} - p^{(1)}) \,.\, y &= x^{(3)} - x^{(1)}\,; \qquad (B) \\ a^{(4)} - a^{(1)} - (p^{(4)} - p^{(1)}) \,.\, y &= x^{(4)} - x^{(1)}\,; \\ \&\text{c.} \end{aligned} \qquad [1977]$$

Supposing y to be infinite, the first members of these equations will be negative ;* hence the value of $x^{(1)}$ will be greater than $x^{(2)}$, $x^{(3)}$, $x^{(4)}$, [1977′] &c. ; and by continually decreasing y, we shall finally obtain a value, that will render one of the first members positive ; which will become nothing, before it is positive. To find which of these members first becomes nothing, we shall compute the quantities,

$$\frac{a^{(2)} - a^{(1)}}{p^{(2)} - p^{(1)}}\,; \qquad \frac{a^{(3)} - a^{(1)}}{p^{(3)} - p^{(1)}}\,; \qquad \frac{a^{(4)} - a^{(1)}}{p^{(4)} - p^{(1)}}\,; \qquad \&\text{c.} \qquad [1978]$$

We shall put $\beta^{(1)}$ for the *greatest of these quantities*, and shall suppose that it is,

$$\beta^{(1)} = \frac{a^{(r)} - a^{(1)}}{p^{(r)} - p^{(1)}}\,. \qquad [1979]$$

If there be several values equal to $\beta^{(1)}$, we shall consider that corresponding

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* (1436) $p^{(1)}$, $p^{(2)}$, $p^{(3)}$, &c., are the squares of the sines of the latitudes [1968$^{\text{vi}}$], arranged according to the *magnitudes* of those quantities [1969″], so that $p^{(1)}$, $p^{(2)}$, &c., form an *increasing* progression, therefore the quantities $p^{(2)} - p^{(1)}$, $p^{(3)} - p^{(1)}$, &c., [1977a] must be *positive ;* and by putting y positive and equal to infinity, the first members of [1977] will become negative ; consequently the second members of those equations must also become negative ; and then $x^{(1)}$, *noticing the signs*, must be greater than $x^{(2)}$, $x^{(3)}$, &c. ; observing that in the remaining part of this article, as far as [1995], *the signs are taken into consideration, in estimating the relative values of quantities*, so that if $x^{(1)} = 2$, and $x^{(2)} = -4$, we should say that $x^{(1)}$ is greater than $x^{(2)}$, because $2 > -4$. [1977b] [1977c]

[1979′] to the highest value of r as the greatest.* Substituting $\beta^{(1)}$ for y, in the $r-1$ equation [1977], $x^{(r)}$ will become equal to $x^{(1)}$, and by *decreasing* y, $x^{(r)}$ will exceed $x^{(1)}$, and the first member of this equation will then become positive. If we decrease y, this member will increase more rapidly
[1979″] than the first members of the equations which precede it;† and since it became nothing when the preceding equations were negative, it is evident that in the successive diminutions of y, it will always exceed them; which
[1979‴] proves that $x^{(r)}$ will always exceed $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, ... $x^{(r-1)}$, when y is less than $\beta^{(1)}$.

The first members of the equations [1977], which follow that which is numbered $r-1$, will at first be negative; and as long as that takes place, $x^{(r+1)}$, $x^{(r+2)}$, &c., will be less than $x^{(1)}$, and therefore less than
[1979⁗] $x^{(r)}$, which becomes the greatest of all the errors $x^{(1)}$, $x^{(2)}$, ... $x^{(n)}$, when y begins to be less than $\beta^{(1)}$; but by continually decreasing y, we may obtain such a value of this variable quantity, that some one of the errors $x^{(r+1)}$, $x^{(r+2)}$, &c., will begin to exceed $x^{(r)}$.

To determine this value of y, we shall subtract the equation, numbered r in [1970], from the equations numbered $r+1$, $r+2$, &c., and we shall obtain,

[1980]
$$a^{(r+1)}-a^{(r)}-\{p^{(r+1)}-p^{(r)}\}\cdot y=x^{(r+1)}-x^{(r)};$$
$$a^{(r+2)}-a^{(r)}-\{p^{(r+2)}-p^{(r)}\}\cdot y=x^{(r+2)}-x^{(r)};$$
&c.

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* (1437) The reason for assuming the greatest value of r, is that $x^{(r)}$ will really become
[1977d] the greatest, if we continue to *decrease* y, putting for example $\beta^{(1)}-\delta\beta$ for y, $\delta\beta$ being positive [1979′]. Thus if we suppose the two quantities $\frac{a^{(r-t)}-a^{(1)}}{p^{(r-t)}-p^{(1)}}$, $\frac{a^{(r)}-a^{(1)}}{p^{(r)}-a^{(1)}}$, to
[1977e] be equal to $\beta^{(1)}$, we must assume the last of these expressions, which has the greatest exponent r, as the value of $\beta^{(1)}$. If we now substitute $y=\beta^{(1)}$, in the two equations [1977], which contain $a^{(r-t)}$, $a^{(r)}$, the first members will become nothing. Therefore if we put $y=\beta^{(1)}-\delta\beta$, $\delta\beta$ being positive, these two equations will become respectively, $(p^{(r-t)}-p^{(1)})\cdot\delta\beta=x^{(r-t)}-x^{(1)}$, $(p^{(r)}-p^{(1)})\cdot\delta\beta=x^{(r)}-x^{(1)}$. The difference of
[1977f] these two equations is $(p^{(r)}-p^{(r-t)})\cdot\delta\beta=x^{(r)}-x^{(r-t)}$, in which $p^{(r)}-p^{(r-t)}$ [1969″], and $\delta\beta$, are both positive, therefore the first member of this equation is positive, consequently $x^{(r)}>x^{(r-t)}$.

† (1438) This is evident, because the coefficient $p^{(r)}-p^{(1)}$, by which y is multiplied,
[1980a] in that equation, exceeds $p^{(2)}-p^{(1)}$, $p^{(3)}-p^{(1)}$, $p^{(4)}-p^{(1)}$ ..... $p^{(r-1)}-p^{(1)}$, [1977a, &c.], which are the coefficients of y in the equations that precede it.

We shall then compute the quantities,

$$\frac{a^{(r+1)}-a^{(r)}}{p^{(r+1)}-p^{(r)}}; \qquad \frac{a^{(r+2)}-a^{(r)}}{p^{(r+2)}-p^{(r)}}; \qquad \&c.$$ [1981]

We shall put $\beta^{(2)}$ *for the greatest of these quantities*, which we shall suppose to be $\frac{a^{(r')}-a^{(r)}}{p^{(r')}-p^{(r)}}$. If several of these quantities be equal to $\beta^{(2)}$, we shall [1982] take r' for the greatest of the numbers to which they correspond.* This being supposed, $x^{(r)}$ will be the greatest of the errors $x^{(1)}, x^{(2)}, \ldots\ldots x^{(n)}$, as [1982'] long as y is included between $\beta^{(1)}$ and $\beta^{(2)}$; but when we have decreased y, till it becomes equal to $\beta^{(2)}$, $x^{(r')}$ will begin to exceed $x^{(r)}$, and will become the greatest of the errors.

To determine within what limits this takes place, we shall compute the quantities,

$$\frac{a^{(r'+1)}-a^{(r')}}{p^{(r'+1)}-p^{(r')}}; \qquad \frac{a^{(r'+2)}-a^{(r')}}{p^{(r'+2)}-p^{(r')}}; \qquad \&c.$$ [1983]

Let the greatest of these quantities be $\beta^{(3)}$, which we shall suppose to be $\frac{a^{(r'')}-a^{(r')}}{p^{(r'')}-p^{(r')}}$. If several of these quantities be equal to $\beta^{(3)}$, we shall suppose [1984] r'' to be the greatest of the numbers to which they correspond; $x^{(r')}$ will be the greatest of the errors, from $y=\beta^{(2)}$ to $y=\beta^{(3)}$. When $y=\beta^{(3)}$, [1984'] then $x^{(r'')}$ begins to be the greatest error. Proceeding in this way, we shall form the two series,

Series of maxima errors, in La Place's first method.

$$x^{(1)}, \quad x^{(r)}, \quad x^{(r')}, \quad x^{(r'')}, \ldots\ldots x^{(n)}; \qquad (C)$$

$$\infty, \quad \beta^{(1)}, \quad \beta^{(2)}, \quad \beta^{(3)}, \ldots\ldots \beta^{(q)}, \quad -\infty.$$ [1985]

The first indicates the errors $x^{(1)}$, $x^{(r)}$, $x^{(r')}$, $x^{(r'')}$, &c., which become successively the greatest; the second series, composed of decreasing [1985'] quantities, indicates the limits of y, between which these errors are the greatest; thus $x^{(1)}$ *is the greatest error, from* $y=\infty$ *to* $y=\beta^{(1)}$; $x^{(r)}$ *is the greatest error, from* $y=\beta^{(1)}$ *to* $y=\beta^{(2)}$; $x^{(r')}$ *is the greatest error,* [1985''] *from* $y=\beta^{(2)}$ *to* $y=\beta^{(3)}$; *and so on.*

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* (1439) This is done upon the same principles as were used, relative to r and [1981a] $\beta^{(1)}$, [1977d—f].

We shall now resume the equations [1977], and shall suppose y to be
[1985‴] negative and infinite. The first members of these equations will be positive;* $x^{(1)}$ will then be the least of the errors $x^{(1)}$, $x^{(2)}$, &c.; and by
[1985⁗] continually *increasing* y, some of these members will become negative, and then $x^{(1)}$ will cease to be the least of the errors. If we apply here the same method which was used in the case of the greatest errors, putting $\lambda^{(1)}$ *for the*
[1985ᵛ] *least of the quantities*,

[1986] $$\frac{a^{(2)}-a^{(1)}}{p^{(2)}-p^{(1)}}; \qquad \frac{a^{(3)}-a^{(1)}}{p^{(3)}-p^{(1)}}; \qquad \frac{a^{(4)}-a^{(1)}}{p^{(4)}-p^{(1)}}; \qquad \&c.;$$

[1986′] supposing that quantity to be $\frac{a^{(s)}-a^{(1)}}{p^{(s)}-p^{(1)}}$, s being the greatest of the numbers which correspond to $\lambda^{(1)}$, when there are several quantities equal to $\lambda^{(1)}$; we shall find that $x^{(1)}$ will be the least of the errors, from $y=-\infty$
[1986″] to $y=\lambda^{(1)}$.† In like manner we shall put $\lambda^{(2)}$ *for the least of the quantities*,

[1987] $$\frac{a^{(s+1)}-a^{(s)}}{p^{(s+1)}-p^{(s)}}; \qquad \frac{a^{(s+2)}-a^{(s)}}{p^{(s+2)}-p^{(s)}}; \qquad \&c.;$$

[1987′] supposing that quantity to be $\frac{a^{(s')}-a^{(s)}}{p^{(s')}-p^{(s)}}$, s' being the greatest of the numbers which correspond to $\lambda^{(2)}$, when there are several quantities equal
[1987″] to $\lambda^{(2)}$; then $x^{(s)}$ will be the least of the errors, from $y=\lambda^{(1)}$ to $y=\lambda^{(2)}$; and in like manner for others. We shall, in this way, form the two series,

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* (1440) This is proved as in [1977a—c], the quantities $p^{(2)}-p^{(1)}$, $p^{(3)}-p^{(1)}$,
[1985a] $p^{(4)}-p^{(1)}$, &c., being positive, [1977b], and $y=-\infty$, the first members of [1977] must be positive and infinite.

† (1441) The calculations in [1985‴—1989], are exactly similar to those made in
[1985b] [1977′—1985], changing r into s, β into λ, and $-\delta\beta$ into $\delta\lambda$, and retaining the same accents on the letters; different signs being given to these two last expressions, because y decreases in [1979′], and increases in [1985⁗], making $\delta\lambda$ positive. By these changes in [1977f], we get $-(p^{(s)}-p^{(s-t)})\,.\,\delta\lambda=x^{(s)}-x^{(s-t)}$. Now $p^{(s)}-p^{(s-t)}$ and $\delta\lambda$ are both positive; therefore the first member of the preceding
[1985c] equation is negative, and $x^{(s)}<x^{(s-t)}$; whence we perceive, that if there be several equal values of $\lambda^{(1)}$, the least error must be considered as that which corresponds to the greatest exponent $x^{(s)}$, in order to satisfy the conditions [1986″].

Series of minima errors, in La Place's first method.

$$x^{(1)}, \quad x^{(s)}, \quad x^{(s')}, \quad x^{(s'')}, \ldots\ldots x^{(n)}; \qquad (D)$$

$$-\infty, \quad \lambda^{(1)}, \quad \lambda^{(2)}, \quad \lambda^{(3)}, \ldots\ldots \lambda^{(q')}, \qquad \infty. \qquad [1988]$$

The first series contains the errors $x^{(1)}$, $x^{(s)}$, $x^{(s')}$, &c., which become successively the least, as y is increased; the second series, forming an increasing progression, denotes the values of y, between which each of these errors is the least; thus $x^{(1)}$ *is the least of the errors, from* $y=-\infty$ *to* [1988′]
$y=\lambda^{(1)}$; $x^{(s)}$ *is the least of the errors, from* $y=\lambda^{(1)}$ *to* $y=\lambda^{(2)}$; $x^{(s')}$ *the least of the errors, from* $y=\lambda^{(2)}$ *to* $y=\lambda^{(3)}$; *and so on for others.* This [1989]
being premised,

The value of y *which appertains to the required ellipsis, will be one of the quantities* $\beta^{(1)}$, $\beta^{(2)}$, $\beta^{(3)}$, &c.; $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$, &c. *It will be in the first* [1989′]
series, if the two extreme errors of the same sign be positive. For these two errors being then the greatest, they will be in the series $x^{(1)}$, $x^{(r)}$, $x^{(r')}$, &c.; and since the same value of y renders them equal, they ought to be consecutive,* and the corresponding value of y must be one of the quantities [1989″]
$\beta^{(1)}$, $\beta^{(2)}$, &c.; because two of these errors cannot at the same time become maxima, and be equal to each other, except by using one of these quantities for y. We shall now investigate the method of ascertaining which of the quantities $\beta^{(1)}$, $\beta^{(2)}$, &c., must be taken for y.

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* (1142) The series $x^{(1)}$, $x^{(r)}$, $x^{(r')}$, &c., [1985], contains the *maxima* errors; and the series $x^{(1)}$, $x^{(s)}$, &c., [1988], the *minima* errors. If we put y equal to any one of [1989a]
the quantities $\beta^{(1)}$, $\beta^{(2)}$, &c., it will render two of the former errors equal to each other. Thus, if we put $y=\beta^{(2)}$, we shall get, from [1982],

$$y=\frac{a^{(r')}-a^{(r)}}{p^{(r')}-p^{(r)}}, \qquad \text{or} \qquad a^{(r')}-a^{(r)}-(p^{(r')}-p^{(r)}).y=0. \qquad [1989a']$$

Substituting this in the equation $a^{(r')}-a^{(r)}-\{p^{(r')}-p^{(r)}\}.y=x^{(r')}-x^{(r)}$, [1980], we obtain $0=x^{(r')}-x^{(r)}$, or $x^{(r')}=x^{(r)}$; therefore the value $y=\beta^{(2)}$ renders the two [1989b]
consecutive maxima errors $x^{(r')}$, $x^{(r)}$, equal to each other. In like manner, $y=\beta^{(3)}$ renders $x^{(r'')}$ equal to $x^{(r')}$, and so on for others. It is also evident, from the method by which $\beta^{(1)}$, $\beta^{(2)}$, $\beta^{(3)}$, &c., are computed, that no quantities except these can make the errors equal, and retain the property of maxima. Moreover, it follows from [1975″″], that there are two maxima and equal errors, having the *same* sign; and if they be *positive*, they [1989c]
must evidently be in the *positive* series $x^{(1)}$, $x^{(r)}$, &c., [1985]; but if they be both *negative*, they must be in the negative series $x^{(1)}$, $x^{(s)}$, &c., [1988], as in [1989′, &c.].

Supposing, for example, that this value is $\beta^{(3)}$; by what has been said, it appears that there must be, between $x^{(r')}$ and $x^{(r'')}$, an error, which will
[1989‴] be the minimum of all the errors,* since $x^{(r')}$ and $x^{(r'')}$ will be the maxima of these errors; therefore in the series $x^{(1)}$, $x^{(s)}$, $x^{(s')}$, &c., some one of the numbers s, s', &c., will be included between r' and r''. We
[1990] shall suppose this to be s. In order that $x^{(s)}$ may be the least of the errors, the value of y must be included between $\lambda^{(1)}$ and $\lambda^{(2)}$ [1989]; therefore if $\beta^{(3)}$ be comprised within these limits, it will be the required value of y, and it will be unnecessary to seek for any other. For if we subtract the equations [1970] corresponding to $x^{(s)}$, from those corresponding to $x^{(r')}$, $x^{(r'')}$, we shall have,

[1991]
$$a^{(r')} - a^{(s)} - \{p^{(r')} - p^{(s)}\} \cdot y = x^{(r')} - x^{(s)};$$
$$a^{(r'')} - a^{(s)} - \{p^{(r'')} - p^{(s)}\} \cdot y = x^{(r'')} - x^{(s)}.$$

If we suppose $y = \beta^{(3)}$, all the members of these equations will be
[1991′] positive; hence it is evident, that by increasing y, the quantity $x^{(r')} - x^{(s)}$ will increase;† and the sum of the extreme errors, taken positively, will then be augmented. If we decrease y, the quantity $x^{(r'')} - x^{(s)}$, will be
[1991″] increased, consequently also the sum of the extreme errors; therefore $\beta^{(3)}$ is the value of y, which gives the least of these sums. Hence it

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* (1443) If we put $y = \beta^{(3)}$, we may prove, by the method used in [1989*b*], that $x^{(r'')} = x^{(r')}$, both these quantities being maxima, and having the positive sign. Then it
[1989*d*] follows, from [1976‴], that there must be an equal error, as $x^{(s)}$, which is of a different sign, and is a minimum, s falling between r', r''.

† (1444) Since $s > r'$ [1990], we shall have $p^{(s)} > p^{(r')}$ [1969″]; consequently
[1991*a*] $-(p^{(r')} - p^{(s)})$, or $p^{(s)} - p^{(r')}$, is positive. Hence by *increasing* y, in the first equation [1991], it will increase the first member of that equation; therefore also the second member, or $x^{(r')} - x^{(s)}$, will be increased; that is, the sum of the two extreme errors, $x^{(r')}$, $x^{(s)}$, taken positively, will by this means be increased. Again, since $s < r''$ by [1990], it will follow, from [1969″], that $p^{(s)} < p^{(r'')}$; therefore $-(p^{(r'')} - p^{(s)})$, or $p^{(s)} - p^{(r'')}$, is *negative;* and by *decreasing* y, in the second of the equations [1991], the first member of that equation will be *increased;* hence the second member, or $x^{(r'')} - x^{(s)}$, which
[1991*b*] expresses the sum of the maxima errors $x^{(r'')}$, $x^{(s)}$, taken positively, will also be increased. Thus we see, that by *increasing* or *decreasing* y, the sum of two of the extreme maxima errors, neglecting the signs, will be *increased;* hence $\beta^{(3)}$ is the value of y which makes that sum, neglecting its sign, a minimum.

follows, that this value of y is the only one which satisfies the conditions of the problem.

We must examine, in this manner, the values of $\beta^{(1)}$, $\beta^{(2)}$, $\beta^{(3)}$, &c.; *which may be easily done by inspection; and if we find a value which satisfies the preceding conditions, we shall be sure of having obtained the required value of* y. [1991‴]

Determination of y, by La Place's first method.

If neither of these values of β *satisfies the conditions, then this value of* y *will be one of the terms of the series* $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$, &c. Supposing, for example, that it is $\lambda^{(2)}$; the two extreme errors $x^{(s)}$ and $x^{(s')}$ will then be negative,* and there will be, by what precedes, an intermediate error, which will be a maximum, and will therefore fall in the series $x^{(1)}$, $x^{(r)}$, $x^{(r')}$, &c. If we suppose it to be $x^{(r)}$, r will then necessarily be included between s and s'; $\lambda^{(2)}$ ought therefore to be included between $\beta^{(1)}$ and $\beta^{(2)}$.† If this be the case, it will prove that $\lambda^{(2)}$ is the required value of y. *In this way, we may try all the terms of the series* $\lambda^{(2)}$, $\lambda^{(3)}$, $\lambda^{(4)}$, &c., *until we obtain a term which satisfies the preceding conditions.* [1991⁗] [1991ᵛ] [1992]

When we have determined in this manner the value of y, we may easily obtain that of z. For that purpose, we shall suppose $\beta^{(2)}$ to be the value of y, and that the three extreme errors are $x^{(r)}$, $x^{(r')}$, $x^{(s)}$; then we shall have $x^{(s)} = -x^{(r)}$,‡ consequently, [1992′]

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* (1445) Multiplying the value of $\lambda^{(2)}$ [1987′] by $p^{(s')} - p^{(s)}$, we get,

$$a^{(s')} - a^{(s)} - (p^{(s')} - p^{(s)}) \cdot \lambda^{(2)} = 0. \qquad [1991c]$$

The difference of the two equations [1970], depending on the exponents s, s', is

$$a^{(s')} - a^{(s)} - (p^{(s')} - p^{(s)}) \cdot y = x^{(s')} - x^{(s)};$$

and by putting $y = \lambda^{(2)}$, the first member becomes equal to the first member of the preceding equation; therefore its second member, $x^{(s')} - x^{(s)}$, will be nothing, or $x^{(s')} = x^{(s)}$. These values $x^{(s')}$, $x^{(s)}$, being the extreme errors of the negative series [1988], with the *same* sign, there must be a third error [1975⁗], as $x^{(r)}$, having a different sign, and such that r must fall between s and s' [1976‴].

† (1446) $x^{(r)}$ is the greatest error, from $y = \beta^{(1)}$ to $y = \beta^{(2)}$ [1985″]; and we have proved, in the last note, that $y = \lambda^{(2)}$ gives the two maxima errors $x^{(s')} = x^{(s)}$, and r is supposed to fall between s and s'; hence $\lambda^{(2)}$ falls between $\beta^{(1)}$, $\beta^{(2)}$. [1991d]

‡ (1447) We have seen, in [1989a, b], that $y = \beta^{(2)}$ gives the maxima errors $x^{(r)}$, $x^{(r')}$, equal to each other, and of the same sign; and by [1975⁗] there must be another,

[1993]
$$a^{(r)} - z - p^{(r)} \cdot y = x^{(r)};$$
$$a^{(s)} - z - p^{(s)} \cdot y = -x^{(r)}.$$

Hence we deduce,

[1994]
$$z = \frac{a^{(r)} + a^{(s)}}{2} - \frac{\{p^{(r)} + p^{(s)}\}}{2} \cdot y;$$

Determination of z, by La Place's first method.

and we shall then have the greatest error $x^{(r)}$, by means of the equation,

[1995]
$$x^{(r)} = \frac{a^{(r)} - a^{(s)}}{2} + \frac{\{p^{(s)} - p^{(r)}\}}{2} \cdot y.$$

Method proposed by Boscovich, for the combination of a number of equations of condition, to obtain the most probable result for the ellipticity.

40. The ellipsis, determined in the preceding article, serves to ascertain whether the elliptical figure is within the limits of the errors of observation;
[1995'] but it does not determine, from the measured degrees, the figure which seems the most probable. It appears to me, that this last ellipsis ought to satisfy the following conditions.* *First*, that *the sum of the errors committed*

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as $x^{(s)}$, which is equal and of a different sign; hence $x^{(s)} = -x^{(r)}$. This being substituted
[1992a] in the equations [1970] containing $x^{(s)}$, $x^{(r)}$, produces [1993]. Half the sum of these equations gives z [1994], and their half difference is $x^{(r)}$ [1995].

* (1448) This method, proposed by Boscovich, and peculiarly well adapted to the present problem, is not now so much used as it ought to be; instead of it, the principle of
[1995a] making the sum of the squares of the errors a minimum, is generally adopted. This method of the least squares has already been explained in [815e—l, 849d—r], and is extremely well adapted to a set of observations, in which all the measured arcs are of nearly the same lengths, and subject to the same degree of uncertainty, from the imperfections of the methods
[1995b] of observation. But if the measure of one of these degrees should differ very much from the rest, the method of the least squares, applied in the usual manner, would give by far too
[1995c] great an influence to this defective observation, in the determination of the figure of the earth. This may be made to appear, from the consideration of the very simple case, of the earth being supposed to be perfectly spherical, consequently every degree z of an equal length, and $y = 0$ [1969]. If we suppose, upon this surface, an arc of *one* degree to be
[1995d] measured in Lapland, and represented by $a^{(1)}$; also an arc of *ten* degrees in India, and represented by $10 \cdot a^{(2)}$, we shall have $i^{(1)} = 1$, $i^{(2)} = 10$, [1996]. The system of equations [1970] will become $a^{(1)} - z = x^{(1)}$, $a^{(2)} - z = x^{(2)}$. Multiplying these, as in
[1995e] [1996], by the quantities $i^{(1)} = 1$, $i^{(2)} = 10$, we get the expression of the errors of the whole arcs, namely,

Imperfection of the usual method of the least squares, applied to geodetical observations.

[1995f]
$$a^{(1)} - z = x^{(1)}; \qquad 10a^{(2)} - 10z = 10 \cdot x^{(2)}.$$

in the measures of the whole arcs, ought to be nothing. *Second*, that *the sum of all these errors, taken positively, ought to be a minimum* *By considering,* [1995″]

To render the sum of the squares of these errors a minimum, we must, as in [815*f*, 849*k*], multiply these equations respectively by the coefficients of z, and put the sum of the products equal to nothing; hence we get $-a^{(1)} - 100 \,.\, a^{(2)} + 101 \,.\, z = 0$, or [1995*f*′] [1995*g*]

$$z = \tfrac{1}{101} \,.\, \{a^{(1)} + 100 \,.\, a^{(2)}\}. \qquad [1995h]$$

This differs very much from the rule usually adopted by astronomers, in taking the mean of any number of similar observations, depending on one unknown quantity, which gives,

$$z = \tfrac{1}{11} \,.\, \{a^{(1)} + 10 \,.\, a^{(2)}\}; \qquad [1995i]$$

making *each one* of the degrees in India have the same influence as *one* degree in Lapland; whereas in the expression [1995*h*], deduced from the principle of the least squares, each of the India degrees has *ten* times the influence of a degree in Lapland. This is unreasonable; for an arc of ten degrees, measured by one person, with the same instruments, and by the same method, is liable to the imperfection of the peculiar manner of observation of that single observer, and to the errors of *one* set of instruments; and it cannot be doubted that *ten* consecutive degrees, measured by *ten* different persons, of equal skill and carefulness in observing, each being furnished with instruments of the same completeness and accuracy, would be at least as satisfactory as ten consecutive degrees, measured by only one of these observers, with the same single set of instruments, notwithstanding the advantage, in this last measure, of requiring only two observations of the latitude throughout the whole arc. We shall therefore assume, as a principle, that *in the application of the method of the least squares to geodetical measures, we must suppose any arc of the length of* i *degrees, to have the same weight as* i *single degrees, measured separately, which would produce* i *equations in the system* [1970]. [1995*k*] [1995*l*] [1995*m*] Proposed correction of this method of the least squares. [1995*n*]

If we apply this principle to the system [1995*f*], in which there is one degree of the length $a^{(1)}$, and ten degrees each of the length $a^{(2)}$, the errors of which are $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, &c.; the whole will form a system of eleven equations, corresponding to [1970]; namely, [1995*o*]

$$a^{(1)} - z = x^{(1)}, \quad a^{(2)} - z = x^{(2)}, \quad a^{(2)} - z = x^{(3)}, \ldots\ldots a^{(2)} - z = x^{(11)}. \qquad [1995p]$$

Multiplying these, according to the principle of the least squares [815*f*], by the coefficients of z, and making the sum of the first members of these products equal to nothing, we shall get $-a^{(1)} - 10 \,.\, a^{(2)} + 11 \,.\, z = 0$. This gives $z = \tfrac{1}{11} \,.\, (a^{(1)} + 10 \,.\, a^{(2)})$, which is the same as the mean value found in [1995*i*], and agrees with the usual methods of astronomers. The same method may evidently be applied to a greater number of arcs; also to cases where the square, or higher powers, of the ellipticity of the meridian, are taken into consideration; and it is evident, that by this modification of the method of the least [1995*q*] [1995*r*]

[1995′′′] *in this manner, the whole arcs, instead of the degrees which have been deduced from them, we shall give to each of these degrees so much more influence, in the*

[1995s] squares, the larger arcs are prevented from having an undue influence, in the computation of the figure of the earth.

We shall now apply this method to the system of equations [1970], supposing, as in
[1995t] [1996], that the number of degrees of the arcs $a^{(1)}$, $a^{(2)}$, &c., are respectively represented
Symbols. by $i^{(1)}$, $i^{(2)}$, &c.; and using for brevity the following abridged symbols,

[1995u] $$i^{(1)}.a^{(1)}=a_1,\quad i^{(2)}.a^{(2)}=a_2,\quad \&c.;\qquad i^{(1)}.p^{(1)}=p_1,\quad i^{(2)}.p^{(2)}=p_2,\quad \&c.;$$

[1995v] $$I=i^{(1)}+i^{(2)}\ldots\ldots+i^{(n)};\qquad I.A=i^{(1)}.a^{(1)}+i^{(2)}.a^{(2)}\ldots\ldots+i^{(n)}.a^{(n)};$$

[1995v′] $$I.P=i^{(1)}.p^{(1)}+i^{(2)}.p^{(2)}\ldots\ldots+i^{(n)}.p^{(n)};$$

[1995w] $$IP.A'=i^{(1)}.a^{(1)}.p^{(1)}+i^{(2)}.a^{(2)}.p^{(2)}\ldots\ldots+i^{(n)}.a^{(n)}.p^{(n)};$$

[1995w′] $$IP.P'=i^{(1)}.p^{(1)}.p^{(1)}+i^{(2)}.p^{(2)}.p^{(2)}\ldots\ldots+i^{(n)}.p^{(n)}.p^{(n)}.$$

Now if we multiply the first of the equations [1970] by the coefficient of z, as in [1995f′], the product will be $-a^{(1)}+z+p^{(1)}.y=-x^{(1)}$; and i degrees of this measure will produce, in the resulting equation, the expression,

[1995x] $$-i^{(1)}.a^{(1)}+i^{(1)}.z+i^{(1)}.p^{(1)}.y=-i^{(1)}.x^{(1)}.$$

Changing the exponents of i, a, p, from 1 into 2, 3, 4, &c., we get the similar equations depending on the second, third, fourth, &c., of the equations [1970]; the sum of all these products, being put equal to nothing, gives the first of the equations, depending on the principle of the least squares. This sum, using the above symbols, is represented by

[1995y] $$-I.A+I.z+IP.y=0.$$

First fundamental equation. Dividing by $-I$, we finally get,

[1995z] $$A-z-P.y=0;$$

which is the same as the first equation [1997], used in the method of Boscovich, for obtaining the most probable result. In like manner, if we multiply the first of the equations [1970] by the coefficient of y, the resulting expression will be,

[1995α] $$-a^{(1)}.p^{(1)}+p^{(1)}.z+p^{(1)}.p^{(1)}.y=-p^{(1)}.x^{(1)}.$$

and i degrees of this measure will produce, in the resulting equation, the following expression,

[1995β] $$-i^{(1)}.a^{(1)}.p^{(1)}+i^{(1)}.p^{(1)}.z+i^{(1)}.p^{(1)}.p^{(1)}.y=-i^{(1)}.p^{(1)}.x^{(1)}.$$

Changing, as above, the exponent 1 into 2, 3, 4, &c.; adding together the whole of the
[1995β′] resulting equations, and putting the sum equal to nothing, we get the other equation

computation of the ellipticity of the earth, as the corresponding arc is of greater extent, which ought to be the case. The following is a very simple method of determining the ellipsis which satisfies these two conditions. [1995'''']

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depending on the principle of the least squares. If we use the above symbols, the equation will become $-IP.A'+IP.z+IP.P'.y=0$. Dividing this by $-IP$, it becomes of the following form, [1995γ]

Second fundamental equation.

$$A'-z-P'.y=0.$$ [1995δ]

Subtracting from this the equation [1995z], we get an equation, from which we obtain the following value of y; and then, from [1995z], we get z; namely,

Values of z, y.

$$z=\frac{A.P'-A'.P}{P'-P}; \qquad y=\frac{A'-A}{P'-P}.$$ [1995ε]

Instead of finding the equation [1995δ], we may proceed in the manner of the author, in [1998], by subtracting the equation [1995z] from each one of the equations [1970], and we shall get the following system, corresponding to [1998],

System like that in the method of Boscovich.

$$\begin{aligned}
\{a^{(1)}-A\}-\{p^{(1)}-P\}.y&=x^{(1)},\\
\{a^{(2)}-A\}-\{p^{(2)}-P\}.y&=x^{(2)},\\
\vdots\qquad\qquad&\quad\vdots\\
\{a^{(n)}-A\}-\{p^{(n)}-P\}.y&=x^{(n)}.
\end{aligned}$$ [1995ζ]

Multiplying each of these equations by the corresponding lengths of the arcs, in degrees, $i^{(1)}$, $i^{(2)}$, &c., we get a set of equations of the form,

$$i^{(1)}.\{a^{(1)}-A\}-i^{(1)}.\{p^{(1)}-P\}.y=i^{(1)}.x^{(1)}, \qquad \&c.;$$ [1995η]

multiplying each of these, by the corresponding coefficients of y, in the system [1970], namely $-p^{(1)}$, $-p^{(2)}$, &c., we get the following system of equations,

$$\begin{aligned}
-i^{(1)}.p^{(1)}.\{a^{(1)}-A\}+i^{(1)}.p^{(1)}.\{p^{(1)}-P\}.y&=-i^{(1)}.p^{(1)}.x^{(1)},\\
-i^{(2)}.p^{(2)}.\{a^{(2)}-A\}+i^{(2)}.p^{(2)}.\{p^{(2)}-P\}.y&=-i^{(2)}.p^{(2)}.x^{(2)},\\
\vdots\qquad\qquad\qquad&\quad\vdots\\
-i^{(n)}.p^{(n)}.\{a^{(n)}-A\}+i^{(n)}.p^{(n)}.\{p^{(n)}-P\}.y&=-i^{(n)}.p^{(n)}.x^{(n)}.
\end{aligned}$$ [1995θ]

The second members of this system are the same as in [1995β, &c.]; hence we find, as in [1995β'], that the sum of the first members of these expressions, being put equal to nothing, will give the second required equation, of the form $-C+D.y=0$. Substituting the value of y, deduced from this, in [1995z], we finally get, for the values of z, y, depending on the principle of the least squares, [1995ι]

Values of z, y, determined by the least squares, in the improved form.

$$z=A-P.\frac{C}{D}, \qquad y=\frac{C}{D},$$ [1995κ]

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We shall resume the equations [1970], multiplying them by the quantities
[1996] $i^{(1)}$, $i^{(2)}$, $i^{(3)}$, &c., which express the number of degrees contained in the

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We may take into consideration the square of the ellipticity, by means of any one of the values of s' [1969*i*—1970*o*], as for example that in [1970*o*]; and to conform to the present notation, we shall change successively s' into a_1, a_2, &c.; $\psi' - \psi$ into $i^{(1)}$, $i^{(2)}$, &c.;
[1995λ] $\sin. 2\psi' - \sin. 2\psi$ into $i^{(1)}.p^{(1)}$, $i^{(2)}.p^{(2)}$, &c.; $\sin. 4\psi' - \sin. 4\psi$ into $i^{(1)}.q^{(1)}$, $i^{(2)}.q^{(2)}$, &c.; by which means we shall get the following system of equations, to be used instead of [1995*x*, &c.],

Equations noticing the square of the ellipticity.

[1995μ]

$$
\begin{aligned}
-a_1 + Mi^{(1)} + Ni^{(1)}.p^{(1)} + Pi^{(1)}.q^{(1)} &= -i^{(1)}.x^{(1)},\\
-a_2 + Mi^{(2)} + Ni^{(2)}.p^{(2)} + Pi^{(2)}.q^{(2)} &= -i^{(2)}.x^{(2)},\\
\vdots \qquad\qquad & \qquad \vdots\\
-a_n + Mi^{(n)} + Ni^{(n)}.p^{(n)} + Pi^{(n)}.q^{(n)} &= -i^{(n)}.x^{(n)};
\end{aligned}
$$

in which $i^{(1)}.x^{(1)}$, $i^{(2)}.x^{(2)}$, &c., represent the errors of the whole measured arcs.

If M, N, P, be wholly independent of each other, as in [1970*q*], the principle of the least squares [1995*n*], would furnish three equations by which these quantities may be
[1995ν] determined. *First*, by putting the sum of the equations [1995μ] equal to nothing. *Second*, by multiplying each of the equations [1995μ] by the corresponding factor $p^{(1)}$, $p^{(2)}$, &c.,
[1995ξ] and putting the sum of these products equal to nothing. *Third*, by multiplying each of the equations [1995μ] by the corresponding factor $q^{(1)}$, $q^{(2)}$, &c., and putting the sum of these products equal to nothing.

[1995π] If we use the formula [1970*l*], putting as in [1970*s*], $\varepsilon' = \frac{1}{300} + \varepsilon''$, neglecting ε''^2; the system of equations [1995μ] may be put under the following form,

Another form of the same equation.

[1995ρ]

$$
\begin{aligned}
-a_1 + Mi^{(1)}.(1 + p^{(1)}.\varepsilon'') &= -i^{(1)}.x^{(1)},\\
-a_2 + Mi^{(2)}.(1 + p^{(2)}.\varepsilon'') &= -i^{(1)}.x^{(2)},\\
\vdots \qquad\qquad & \qquad \vdots\\
-a_n + Mi^{(n)}.(1 + p^{(n)}.\varepsilon'') &= -i^{(1)}.x^{(n)}.
\end{aligned}
$$

The sum of these equations being made equal to nothing gives, as in [1995ν], the first of the
[1995σ] equations depending on the principle of the least squares; and corresponds with [1997].
[1995τ] The second equation is found by multiplying each of the equations [1995ρ] by $p^{(1)}$, $p^{(2)}$, &c., and putting the sum of these products equal to nothing.

We shall hereafter find, in several instances, that the method of the least squares, when applied to a system of observations, in which one of the extreme errors is very great, does not
[1995υ] generally give so correct a result as the method proposed by Boscovich [1995″]. The reason is, that in the former method, this extreme error affects the result in proportion to the *second* power of the error; but in the other method, it is as the *first* power, and must therefore be less.

measured arcs respectively; and shall put A for the sum of the quantities $i^{(1)}.a^{(1)}$, $i^{(2)}.a^{(2)}$, &c., divided by the sum of the numbers $i^{(1)}$, $i^{(2)}$, [1996']
$i^{(3)}$, &c., [1995v]; also P for the sum of the quantities $i^{(1)}.p^{(1)}$, $i^{(2)}.p^{(2)}$, &c., divided by the sum of the numbers $i^{(1)}$, $i^{(2)}$, $i^{(3)}$, &c., [1995v']. [1996'']
Then the condition [1995''], that the sum of the errors $i^{(1)}.x^{(1)}$, $i^{(2)}.x^{(2)}$, &c., is nothing, gives,*

First fundamental equation of Boscovich.

$$0 = A - z - P.y.$$ [1997]

If we subtract this equation, from each of the equations [1970], we shall obtain a system of equations, of the following form,

System of equations of condition in the method of Boscovich.

$$\left.\begin{array}{l} b^{(1)} - q^{(1)}.y = x^{(1)} \\ b^{(2)} - q^{(2)}.y = x^{(2)} \\ b^{(3)} - q^{(3)}.y = x^{(3)} \\ \quad\&c. \end{array}\right\}; \qquad (O)$$ [1998]

We must compute the series of quotients $\frac{b^{(1)}}{q^{(1)}}$, $\frac{b^{(2)}}{q^{(2)}}$, $\frac{b^{(3)}}{q^{(3)}}$, &c., and [1998']
arrange them according to their magnitudes [$y^{(1)}$, $y^{(2)}$, $y^{(3)}$, &c.], beginning with the greatest;† then multiply the equations [1998] to which [1998'']
they correspond, by their respective numbers $i^{(1)}$, $i^{(2)}$, &c., and *place the equations, thus multiplied, in the same order as these quotients.* The first [1998''']
members of these equations, arranged in this manner, will be composed of a series of terms of this form,

System for the determination of y.

$$h^{(1)}.y - c^{(1)}, \quad h^{(2)}.y - c^{(2)}, \quad h^{(3)}.y - c^{(3)}, \quad \&c.; \qquad (P)$$ [1999]

in which we shall suppose $h^{(1)}$, $h^{(2)}$, &c., to be positive, which can be done by changing the signs of the terms, where y has a negative coefficient.

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* (1448a) Multiplying the equations [1970] by $i^{(1)}$, $i^{(2)}$, $i^{(3)}$, &c., and adding the products, the second member of the sum becomes nothing, by the hypothesis [1995'']. [1997a]
Dividing the first member by the sum of $i^{(1)}$, $i^{(2)}$, &c., using the values A, P, [1996'], it becomes as in [1997].

† (1449) *The signs in this case are to be noticed*, as in [1977c]. For convenience of [1998a]
reference, I have named these quantities $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, &c., which was not done in the original work. They represent the values of y, found by putting successively the errors $x^{(1)}$, $x^{(2)}$, &c., equal to nothing, and then arranging them according to their magnitudes. [1998b]

[1999'] These terms are the errors of the measured arcs, taken positively or negatively. This being premised,

It is evident, that by making y infinite, each term of this series becomes infinite; and by decreasing y, they decrease, and finally become negative; in the first place, the first term, then the second term, and so on for the others. As y decreases, the terms, which have once become negative,
[1999''] remain negative, and continually decrease. To obtain the value of y, which renders the sum of these terms, taken positively, a minimum, we must add
[1999'''] the quantities $h^{(1)}$, $h^{(2)}$, &c., until the sum begins to exceed the half sum of all these quantities; then, by putting this sum equal to F, we must
[2000] determine r, so that we may have,

Formula for finding y, in the method of Boscovich.

[2001]

$$h^{(1)}+h^{(2)}+h^{(3)}\,.\,.\,.\,.\,.+h^{(r)}>\tfrac{1}{2}F;$$

$$h^{(1)}+h^{(2)}+h^{(3)}\,.\,.\,.\,.\,.+h^{(r-1)}<\tfrac{1}{2}F.$$

[2001'] We must then put $y=\frac{c^{(r)}}{h^{(r)}}$, so that the error will be nothing for the degree which corresponds to that one of the equations [1998], whose first member, put equal to nothing, gives this value of y.*

To prove this, we shall suppose y to be increased by the quantity δy, so
[2001''] that $\frac{c^{(r)}}{h^{(r)}}+\delta y$ may be comprised between $\frac{c^{(r-1)}}{h^{(r-1)}}$ and $\frac{c^{(r)}}{h^{(r)}}$. The $r-1$ first terms of the series [1999] will be negative, as in the case of $y=\frac{c^{(r)}}{h^{(r)}}$; but by taking them with the sign $+$, their sum will decrease by the quantity,

[2002]
$$(h^{(1)}+h^{(2)}+h^{(3)}\,.\,.\,.\,.\,.+h^{(r-1)})\,.\,\delta y.$$

[2002'] The term r of this series, which is nothing when $y=\frac{c^{(r)}}{h^{(r)}}$, will become positive, and equal to $h^{(r)}\,.\,\delta y$; the sum of this term, and of the following ones, which are all positive, will increase by the quantity,

[2003]
$$(h^{(r)}+h^{(r+1)}+\&c.)\,.\,\delta y;$$

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* (1450) The above assumed value of $y=\frac{c^{(r)}}{h^{(r)}}$, being multiplied by $h^{(r)}$, gives by
[1999a] transposition $h^{(r)}\,.\,y-c^{(r)}=0$, which is the same as to put the first member of the equation [1998], depending on $h^{(r)}$, equal to nothing, as is evident from [1998', 1999].

but by supposition we have,*

$$h^{(1)}+h^{(2)}+h^{(3)}\ldots\ldots+h^{(r-1)}<h^{(r)}+h^{(r+1)}+\&c.\,; \qquad [2004]$$

therefore the whole sum of the terms of the series [1999], taken positively, will be increased, and as it is equal to the sum of the errors of the measured arcs, $i^{(1)}.x^{(1)}$, $i^{(2)}.x^{(2)}$, &c., taken all with the same sign $+$; this last [2004′] sum will be increased by the supposition of $y=\frac{c^{(r)}}{h^{(r)}}+\delta y$. It is easy to [2004″] prove, in the same manner, that by increasing y, so that it may be included between $\frac{c^{(r-1)}}{h^{(r-1)}}$ and $\frac{c^{(r-2)}}{h^{(r-2)}}$, or between $\frac{c^{(r-2)}}{h^{(r-2)}}$ and $\frac{c^{(r-3)}}{h^{(r-3)}}$, &c.; [2004‴] the sum of the errors, taken with the sign $+$, will be greater than when $y=\frac{c^{(r)}}{h^{(r)}}$.†

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* (1451) In [2000] we have $h^{(1)}+h^{(2)}\ldots\ldots+h^{(n)}=F$; and by the second [2004a] expression [2001], $h^{(1)}+h^{(2)}\ldots\ldots+h^{(r-1)}<\tfrac{1}{2}F$. Subtracting this from the preceding, we find $h^{(r)}+h^{(r+1)}+\&c.>\tfrac{1}{2}F$; and by comparing these two last expressions together, we evidently obtain [2004].

† (1452) It is demonstrated in [2004″], that the sum of the errors, taken positively, is greater when $y=\frac{c^{(r)}}{h^{(r)}}+\delta y$, than when $y=\frac{c^{(r)}}{h^{(r)}}$; and if we put [2004b]

$$\delta y=\frac{c^{(r-1)}}{h^{(r-1)}}-\frac{c^{(r)}}{h^{(r)}},$$

the first value, or $y=\frac{c^{(r)}}{h^{(r)}}+\delta y$, becomes $y=\frac{c^{(r-1)}}{h^{(r-1)}}$; hence $y=\frac{c^{(r-1)}}{h^{(r-1)}}$ makes the sum of the errors, taken positively, greater than when $y=\frac{c^{(r)}}{h^{(r)}}$. If we now put $y=\frac{c^{(r-1)}}{h^{(r-1)}}+\delta y$, we may prove, by the method [2002—2004], that this value of y makes the preceding sum greater than when $y=\frac{c^{(r-1)}}{h^{(r-1)}}$; and if we put $\delta y=\frac{c^{(r-2)}}{h^{(r-2)}}-\frac{c^{(r-1)}}{h^{(r-1)}}$, the value $y=\frac{c^{(r-1)}}{h^{(r-1)}}+\delta y$, will become $y=\frac{c^{(r-2)}}{h^{(r-2)}}$; therefore this value makes [2004c] the sum of the errors greater than when $y=\frac{c^{(r-1)}}{h^{(r-1)}}$; and so on for other values of y, as above.

[2004''''] We shall now decrease y by the quantity δy, so that $\frac{c^{(r)}}{h^{(r)}} - \delta y$ may be included between $\frac{c^{(r)}}{h^{(r)}}$ and $\frac{c^{(r+1)}}{h^{(r+1)}}$; the sum of the negative terms of the series [1999] will increase, by changing their sign, by the quantity,

[2005] $$(h^{(1)} + h^{(2)} \ldots\ldots + h^{(r)}) \cdot \delta y.$$

The sum of the positive terms of the same series, will decrease, by the quantity,

[2006] $$\{h^{(r+1)} + h^{(r+2)} + \&c.\} \cdot \delta y;$$

and since we have*

[2007] $$h^{(1)} + h^{(2)} + h^{(3)} \ldots\ldots + h^{(r)} > h^{(r+1)} + h^{(r+2)} + \&c.,$$

it is evident that the whole sum of the errors, taken with the sign $+$, will be increased. We shall find, in the same manner, that by decreasing y, so that it may be between $\frac{c^{(r+1)}}{h^{(r+1)}}$ and $\frac{c^{(r+2)}}{h^{(r+2)}}$; or between $\frac{c^{(r+2)}}{h^{(r+2)}}$ and $\frac{c^{(r+3)}}{h^{(r+3)}}$, &c.; the sum of the errors, taken with the sign $+$, is greater than when $y = \frac{c^{(r)}}{h^{(r)}}$;† therefore this value of y is that which renders

[2008] this sum a minimum.

Value of z, by Boscovich.

The value of y gives that of z, by means of the equation [1997],

[2009] $$z = A - P \cdot y.$$

The preceding analysis is founded upon the variation of the degrees of latitude from the equator to the poles, being proportional to the square of

[2009'] the sine of the latitude; and as this law of variation exists also for the force

[2009''] of gravity, it is evident that the same analysis may be applied to the observations of the lengths of a pendulum,‡ vibrating in a second of time.

---

* (1453) Subtracting the first of the expressions [2001] from $h^{(1)} + h^{(2)} \ldots + h^{(n)} = F$,

[2007a] [2004a], we obtain $h^{(r+1)} + h^{(r+2)} + \&c. < \frac{1}{2} F$. Comparing this with the same expression [2001], we get [2007].

† (1454) This demonstration is to be made as in [2004b—c].

‡ (1455) By [1804] the length of a pendulum varies as μ^2, or $\cos.^2 \theta$, in the same

[2008a] manner as the length of a degree [1969b].

41. We shall apply this method to the degrees of the terrestrial meridian which have already been measured. Among these degrees, we shall notice the seven following, which are given in parts of the standard rod or measure, used in the determination of the great arc from Dunkirk to [2009‴] Barcelona, for the purpose of ascertaining the fundamental unity of weights and measures. *We shall designate this rod, which is double the length of the toise used by Bouguer in Peru, by R;* and shall hereafter show how to find [2009⁗] the ratio of this measure to the metre.

Measured arcs of the meridian.

The *first* of these degrees is that of Peru, in the latitude $0° \, 0'$. Its length, according to Bouguer, is $25538^R, 85$;* the whole measured arc [2009v] from which this degree has been determined, is $3°, 4633$.

The *second* is that of the Cape of Good Hope, measured by La Caille, the middle of which corresponds to the latitude of $37°, 0093$. Its length [2009vi] is $25666^R, 65$.† The whole arc from which this degree has been determined, is $1°, 3572$.

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* (1456) The length of this degree of the ancient division of the quadrant, according to La Lande, is 56753 toises. Multiplying this by $\frac{90}{100}$, we get the length of a degree of the [2009a] centesimal division 51077,70 toises, or 25538,85 double toises, as above. The other lengths were deduced from the toises, in the same manner.

† (1457) Since the publication of this volume of the original work, several of these measures have been re-examined, and found to be erroneous, or liable to such mistakes as to [2009b] render them unfit for the purpose of determining the figure of the earth; as will evidently appear from the following remarks.

The degree measured at the Cape of Good Hope, by La Caille, from the account given of it by Captain G. Everest, in the first volume of the Memoirs of the Astronomical Society of London, seems liable to the objection of having been taken without using those precautions [2009c] which are now found to be absolutely necessary to obtain an accurate result. Moreover, the northern and southern extremities of the arc were bounded by high lands and mountains, which could not fail to have a sensible effect on the plumb line, used in the astronomical observations, for the determination of the latitudes of those extreme points. He estimates that a deviation of 4^s or 5^s, in each of these observations, would not be too great, when [2009d] the localities are duly taken into consideration; observing withal, that a greater attraction than this has actually been observed in some mountains. For instance, Bouguer found by observation, that the attraction of the mountain Chimboraço produced a deviation of $7^s, 5$; [2009e] and Dr. Maskelyne observed the attraction of the mountain Schehallien to be $5^s, 8$. The effect of this correction would be to increase the whole length of the arc measured by

Defects of the measures given by the author.

The *third* is that of Pennsylvania, measured by Mason and Dixon. Its
[2009vii] middle corresponds to the latitude of $43°,5556$; its length is $25599^R,60$; the whole measured arc is $1°,6435$.

The *fourth* is that of Italy, measured by Boscovich and Le Maire. Its
[2009viii] middle corresponds to the latitude $47°,7963$; its length is $25640^R,55$; the whole measured arc is $2°,4034$.

The *fifth* degree is that of France, lately measured by Delambre and
[2009ix] Mechain. Its middle corresponds to the latitude of $51°,3327$; its length is $25658^R,28$; the whole measured arc is $10°,7487$.

The *sixth* is that of Austria, measured by Liesganig. Its middle
[2009x] corresponds to the latitude of $53°,0926$; its length is $25683^R,30$; the whole measured arc is $3°,2734$.

The *seventh* is that of Lapland, measured by Clairaut, Maupertuis,
[2009xi] Le Monnier, &c. Its middle corresponds to $73°,7037$; by taking a mean between the several series of triangles, and noticing the refraction in the valuation of the celestial arc, I have fixed its length at $25832^R,25$; the whole measured arc is $1°,0644$.

La Caille nearly 9^s, to reduce the number of toises in a sexagesimal degree from 57037 to
[2009f] 56918, and the length of a centesimal degree from $25666^R,65$ to 25613^R nearly; which would make it agree with the most accurate measurements in the northern hemisphere.

From the remarks of the Baron Von Zach, in the *Mémoires de l'Académie des Sciences de Turin*, 1811—1812, and in the *Monatliche Correspondenz*, Vol. XXIII, page 153, it
[2009g] appears, that the measure in Austria, by Liesganig, ought to be rejected entirely, since no reliance whatever can be placed on its accuracy.

The degree measured in Lapland, by Maupertuis and his associates, has since been
[2009h] re-measured by Svanberg and others, and found to be less than the former estimate by about 200 toises, as will be seen in the table [2017l].

It is probable that the degrees in Pennsylvania and Italy, were not measured with all the
[2009i] precautions required by modern observers; and if these be rejected, there will remain only the degrees of Peru, France, and the corrected Lapland measure. To these we might add other modern observations, particularly that in England, and the very large arc measured in
[2009k] India, by Colonel Lambton. We shall hereafter give the result of this combination of degrees, after having gone through and explained the methods of the author, as they are applied to the measures he has collected in the table [2009v, &c.].

In the following table, these degrees are arranged according to the order of the latitudes.

Latitudes.		Lengths of the degrees.	
0°,0000	- - - - - - - -	$25538^R,85$	Measured arcs.
37 ,0093	- - - - - - - -	25666 ,65	
43 ,5556	- - - - - - - -	25599 ,60	
47 ,7963	- - - - - - - -	25640 ,55	
51 ,3327	- - - - - - - -	25658 ,28	[2010]
53 ,0926	- - - - - - - -	25683 ,30	
73 ,7037	- - - - - - - -	25832 ,25	

Hence the equations [1970] become,*

Resulting equations.

$$\begin{aligned}
25538^R,85 - z - y \,.\, 0,00000 &= x^{(1)}, \\
25666\ ,65 - z - y \,.\, 0,30156 &= x^{(2)}, \\
25599\ ,60 - z - y \,.\, 0,39946 &= x^{(3)}, \\
25640\ ,55 - z - y \,.\, 0,46541 &= x^{(4)}, \qquad (A') \\
25658\ ,28 - z - y \,.\, 0,52093 &= x^{(5)}, \\
25683\ ,30 - z - y \,.\, 0,54850 &= x^{(6)}, \\
25832\ ,25 - z - y \,.\, 0,83887 &= x^{(7)}.
\end{aligned}$$ [2011]

The two series [1985] become,†

$$\begin{array}{cccc} x^{(1)}, & x^{(2)}, & x^{(7)}, & \\ \infty, & 423,796, & 308,202, & -\infty; \end{array}$$ [2012]

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* (1458) The lengths of the degrees [2010] are to be inserted in [1970], instead of $a^{(1)}$, $a^{(2)}, \ldots\ldots a^{(7)}$. The values $p^{(1)}$, $p^{(2)} \ldots\ldots p^{(7)}$ represent the squares of the sines of the latitudes. [2010a]

† (1459) Subtracting the first of the equations [2011] from the others, we obtain the following system of equations (A), corresponding to [1977]. Dividing the constant terms of these equations, by the coefficients of $-y$, respectively, we get the column of numbers (B), representing the series [1978]; the greatest of these quantities, which is called $\beta^{(1)}$ [1979], is $\beta^{(1)} = \frac{a^{(2)} - a^{(1)}}{p^{(2)} - p^{(1)}} = 423,796$, making $r=2$, [1979]. The second equation [2011], corresponding to this value of r, being subtracted from the third, fourth, &c., equations of that system, gives the system (C), being the same as [1980]. Dividing the constant terms of these equations, by the coefficients of $-y$, we get the column of numbers marked (D); the greatest of these quantities, noticing the signs, as in [1977c], is [2011a] [2011b] [2011c]

and the two series [1988] become,*

[2013]	$x^{(1)}$,	$x^{(3)}$,	$x^{(5)}$,	$x^{(7)}$,	
	$-\infty$,	152,080,	483,087,	547,176,	∞.

[2013′] Hence it is evident, from § 39, that $y = 308^R,202$,† which gives $\frac{1}{277}$ for

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[2011d] 308,202; deduced from the last of these equations, which must therefore be the quantity marked $\beta^{(2)}$ [1982]. The exponents of x in this equation are 7 and 2, making $r' = 7$,
[2011e] $r = 2$; and as this is the last of these equations, the series will terminate; consequently the series [1985] will become as in [2012].

[2011f]

(A)	(B)	(C)	(D)
$127{,}80 - y\,.\,0{,}30156 = x^{(2)} - x^{(1)}$	423,796		
$60{,}75 - y\,.\,0{,}39946 = x^{(3)} - x^{(1)}$	152,080	$-67{,}05 - y\,.\,0{,}09790 = x^{(3)} - x^{(2)}$	-685
$101{,}70 - y\,.\,0{,}46541 = x^{(4)} - x^{(1)}$	218,	$-26{,}10 - y\,.\,0{,}16385 = x^{(4)} - x^{(2)}$	-159
$119{,}43 - y\,.\,0{,}52093 = x^{(5)} - x^{(1)}$	229,	$-\ 8{,}37 - y\,.\,0{,}21937 = x^{(5)} - x^{(2)}$	$-\ 38$
$144{,}45 - y\,.\,0{,}54850 = x^{(6)} - x^{(1)}$	263,	$16{,}65 - y\,.\,0{,}24694 = x^{(6)} - x^{(2)}$	67
$293{,}40 - y\,.\,0{,}83887 = x^{(7)} - x^{(1)}$	349,	$165{,}60 - y\,.\,0{,}53731 = x^{(7)} - x^{(2)}$	308,202

* (1459a) The least of the numbers, in the preceding column (B) is represented by
[2012a] $\lambda^{(1)}$ [1985v], hence $\lambda^{(1)} = 152,080$. The greatest exponent of x, in the corresponding
[2012b] equation, is 3, which is to be taken for s [1986′]. Hence $x^{(s)} = x^{(3)}$, $a^{(s)} = 25599^R,60$, $p^{(s)} = 0,39946$. Subtracting this value of $a^{(s)}$ from $a^{(4)}$, $a^{(5)}$, &c., [2011], we get the
[2012c] following system (E). In like manner, by subtracting $p^{(s)}$ from $p^{(4)}$, $p^{(5)}$, &c., [2011], we get the system (F). Dividing the first of these numbers, by the corresponding ones of the second, we obtain the system (G), representing the series of numbers [1987], the least
[2012c′] of which is to be taken for $\lambda^{(2)}$ [1986″]; hence $\lambda^{(2)} = 483,08$. The greatest exponent of a or p, in the corresponding equations (F) or (G), is 5, making $s' = 5$ [1987′]. Hence $x^{(s')} = x^{(5)}$, $a^{(s')} = a^{(5)} = 25658^R,28$, $p^{(s')} = p^{(5)} = 0,52093$. Subtracting
[2012d] these from $a^{(6)}$, $a^{(7)}$, $p^{(6)}$, $p^{(7)}$, respectively, we get the systems (H), (I). Dividing the first by the second, as above, we obtain the system (K), the last term of which, being
[2012e] the least of the series, gives $\lambda^{(3)} = 547,17$, and the greatest exponent of a or p, in the corresponding equation (H), gives $s'' = 7$, and the resulting error $x^{(7)}$. The series then terminates, and the system [1988] becomes as in [2013].

[2012f]

(E)	(F)	(G)	(H)	(I)	(K)
$a^{(4)} - a^{(3)} = 40^R,95$	$p^{(4)} - p^{(3)} = 0{,}06595$	620,			
$a^{(5)} - a^{(3)} = 58\ ,68$	$p^{(5)} - p^{(3)} = 0{,}12147$	483,08			
$a^{(6)} - a^{(3)} = 83\ ,70$	$p^{(6)} - p^{(3)} = 0{,}14904$	561,	$a^{(6)} - a^{(5)} = 25^R,02$	$p^{(6)} - p^{(5)} = 0{,}02757$	907,
$a^{(7)} - a^{(3)} = 232\ ,65$	$p^{(7)} - p^{(3)} = 0{,}43941$	529,	$a^{(7)} - a^{(5)} = 173\ ,97$	$p^{(7)} - p^{(5)} = 0{,}31794$	547,17

† (1460) If we take the first finite number of the series [2012] for y, and put

$$y = \beta^{(1)} = 423^R,796,$$

[2013a] it will make $x^{(1)} = x^{(r)}$, or $x^{(1)} = x^{(2)}$ [1979′]; but between these numbers

the ellipticity of the earth.* We then have, Least possible error of these measures.

$$x^{(2)} = x^{(7)} = -x^{(3)} = 48^R, 60.$$ [2014]

Hence it appears, that with any combination of the seven preceding degrees, and with any ratio of the polar and equatorial diameters of the earth we may select, it will be impossible to avoid an error of $48^R, 60$, in the measures of [2014′]

$x^{(1)}$, $x^{(2)}$, there is no number $x^{(s)}$, in the series [2013], in which s falls between 1 and 2, the exponents of the quantities $x^{(1)}$, $x^{(2)}$; therefore this value of y must be rejected. If we take the next term of the series [2012], and put $y = \beta^{(2)} = 308^R, 202$, we shall get $x^{(r')} = x^{(r)}$ [1989*b*], or $x^{(7)} = x^{(2)}$. Now in the series [2013] are the terms $x^{(3)}$, $x^{(5)}$, whose exponents 3, 5, fall between the exponents 2, 7, we have just found; so that [2013*b*] the condition [1975^{v}] will be satisfied, by taking for the third error, either of the quantities $x^{(3)}$, $x^{(5)}$, corresponding to $x^{(s)}$, $x^{(s')}$, [1988, 2013]. We cannot however take the last of these quantities $x^{(s')} = x^{(5)}$, because, by [1989], it is required, in order that $x^{(s')}$ should be the least of the errors, that the value of y should *fall between* $\lambda^{(2)}$ and $\lambda^{(3)}$ [1989], corresponding to 483,087 and 547,176 [2013]; and the assumed [2013*c*] value $y = 308^R, 202$ does not satisfy this condition. We must therefore take the other quantity $x^{(3)}$, which requires, by [1989], that the assumed value of y should fall between $\lambda^{(1)}$ and $\lambda^{(2)}$ [1988′], corresponding to 152,080 and 483,087 [2013]; and as this condition is satisfied, we may put $x^{(3)}$ for the least error, which is equal to the errors $x^{(2)}$, $x^{(7)}$, but of a different sign, [1975′′′′], so that we shall have $x^{(2)} = x^{(7)} = -x^{(3)}$, and [2013*d*] $y = 308^R, 202$.

* (1161) Having found, in the last notes, $x^{(r)} = x^{(2)}$, $x^{(s)} = x^{(3)}$, which correspond [2013*e*] to the second and third equations [2011]; we shall have, by comparing them with the same equations of the series [1970], $a^{(r)} = a^{(2)} = 25666^R, 65$; $a^{(s)} = a^{(3)} = 25599^R, 60$; $p^{(r)} = p^{(2)} = 0,30156$; $p^{(s)} = p^{(3)} = 0,39946$. Substituting these in [1994, 1995], [2013*f*] and also the value of y [2013*d*], we obtain,

$$z = 25633^R, 12 - y \,.\, 0,35051 = 25633^R, 12 - 108^R, 02 = 25525^R, 10\,;$$
$$x^{(r)} = \quad 33^R, 52 + y \,.\, 0,04895 = \quad 33^R, 52 + \; 15^R, 08 = \quad 48^R, 60.$$ [2013*g*]

This value of $x^{(r)}$, being substituted in [2013*e*], gives [2014]. Having obtained y, z, [2013*d*, *g*], the length a of a degree of the meridian [1969*b*] becomes,

$$a = 25525^R, 10 + 308^R, 202 \,.\, \mu^2.$$ [2013*h*]

From these values of z, y, we get $3z + \frac{5}{3}y = 77088^R, 97$, and the ellipticity [1795*f*],

$$\alpha h = \frac{y}{3z + \frac{5}{3}y} = \frac{308,202}{77088,97} = \tfrac{1}{250}\,;$$ [2013*i*]

which differs from $\frac{1}{277}$ given by the author, [2013′].

some of these degrees, in the elliptical hypothesis. Now this error is exactly on the least limit of those which might be considered as possible; and for that very reason, it is highly improbable that this particular error exists. We must therefore admit, in the elliptical hypothesis, much greater
[2014″] errors than $48^R, 60$. By attentively examining the measures of these degrees, it appears difficult to suppose, that in each of the degrees of Pennsylvania, the Cape of Good Hope, and Lapland, in which the greatest errors fall, there can be an error of $48^R, 60$; therefore it seems to follow, from the preceding measures, that the variations of the degrees of the
[2014‴] terrestrial meridian differ sensibly from the law of the square of the sine of the latitude, which is given by the hypothesis of the elliptical meridians.*

We shall now determine the most probable ellipsis, which results from these measures. Multiplying the equations [2011], respectively, by the quantities
[2014⁗] $i^{(1)}$, $i^{(2)}$, $i^{(3)}$, &c., which represent the number of degrees contained in the corresponding arcs, and dividing the sum by $i^{(1)}+i^{(2)}+i^{(3)}+\&c.$, we shall get, from the principle that the sum of the errors $i^{(1)}.x^{(1)}+i^{(2)}.x^{(2)}+\&c.$, is nothing, [1995″],†

[2015] $$0 = 25646^R, 80 - z - y \,.\, 0,43717.$$

[2014a] * (1462) The difficulty of reconciling the measures of these degrees of the meridian with the elliptical hypothesis, is somewhat diminished by using the corrected degree of Lapland [2009h]; by which means the error $48^R, 60$ would be decreased a few metres. If we reject the degrees of the Cape of Good Hope and Austria, on account of their incorrectness [2009f, g]; and combine, in the preceding manner, the remaining five degrees;
[2014b] the greatest error will be reduced to about one third of its value [2014]. If we confine ourselves to five of the most extensive and accurate measures, namely, those of Peru, India,
[2014c] France, England, and Lapland, the greatest error will be much more reduced, as will be seen hereafter, [2017r, &c.].

† (1463) The quantities $a^{(1)}$, $a^{(2)}$, &c., are contained in the first column of the system of equations [2011]; and the quantities $p^{(1)}$, $p^{(2)}$, &c., are the coefficients of $-y$, in the same system. The values $i^{(1)}$, $i^{(2)}$, &c., [2009^v, &c.], are in the first column of the annexed table; the second column contains the products $i^{(1)}.a^{(1)}$, $i^{(2)}.a^{(2)}$, &c.; and the third column the products $i^{(1)}.p^{(1)}$, &c. Dividing the sum of the
[2015a] numbers in the second column, $614341,62$, by that in the first, we get the term A of [1997], equal to $25646^R, 80$. Dividing, in like manner, the

Col. 1. $i^{(1)}$, &c.	Col. 2. $i^{(1)}.a^{(1)}$, &c.	Col. 3. $i^{(1)}.p^{(1)}$, &c.
3,4633	88448,69	0.00000
1,3572	34834,78	0,40928
1,6435	42072,94	0,65651
2,4034	61624,50	1,11856
10,7487	275793,15	5,59932
3,2734	84071,71	1,79546
1,0644	27495,85	0,89289
23,9539	614341,62	10.47202

Hence the equations [1998] of the preceding article become,

Computation by the method of Boscovich.

$$\begin{aligned}
-107^R,95 + y\,.\,0,43717 &= x^{(1)},\\
19\;,85 + y\,.\,0,13561 &= x^{(2)},\\
-\;47\;,20 + y\,.\,0,03771 &= x^{(3)},\\
-\;6\;,25 - y\,.\,0,02824 &= x^{(4)}, \qquad (O')\\
11\;,48 - y\,.\,0,08376 &= x^{(5)},\\
36\;,50 - y\,.\,0,11133 &= x^{(6)},\\
185\;,45 - y\,.\,0,40170 &= x^{(7)}.
\end{aligned}$$ [2016]

Hence we easily find, that the series of quantities $h^{(1)}$, $h^{(2)}$, $h^{(3)}$, &c., arranged according to the magnitude of the quotients, $\frac{b^{(1)}}{q^{(1)}}$, $\frac{b^{(2)}}{q^{(2)}}$, &c.,* is

0,06198, 0,42757, 0,36443, 1,51405, 0,90031, 0,18405, 0,06787. [2017]

The equations [2016], taken in the order 3, 7, 6, 1, 5, 2, 4, correspond to the quantities [2017]. The sum of the three first of these quantities, is less than the half sum of all of them; and the sum of the four first, exceeds this half sum; therefore we have $x^{(1)}=0$; which gives $y = 246^R,93$; [2017′] consequently $z = 25538^R,85$; so that the expression of the degree of the meridian is,

Degree of the meridian.

$$25538^R,85 + 246^R,93\,.\,\sin.^2\theta\,;$$ [2018]

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sum of the numbers in the third column, 10,47202, we obtain the term P of [1997], equal to 0,43717. Hence the equation [1997] becomes nearly as in [2015]; subtracting this from each of the equations [2011], we get the system of equations [2016], which [2015b] corresponds to [1998].

* (1464) Comparing the systems [1998] and [2016], we find $b^{(1)} = -107^R,95$, $b^{(2)} = 19^R,85$, $b^{(3)} = -47^R,20$, $b^{(4)} = -6^R,25$, $b^{(5)} = 11^R,48$, $b^{(6)} = 36^R,50$, $b^{(7)} = 185^R,45$; $q^{(1)} = -0,43717$, $q^{(2)} = -0,13561$, $q^{(3)} = -0,03771$, $q^{(4)} = 0,02824$, $q^{(5)} = 0,08376$, $q^{(6)} = 0,11133$, $q^{(7)} = 0,40170$. Hence we deduce,

$$\frac{b^{(1)}}{q^{(1)}} = 247, \quad \frac{b^{(2)}}{q^{(2)}} = -146, \quad \frac{b^{(3)}}{q^{(3)}} = 1252, \quad \frac{b^{(4)}}{q^{(4)}} = -221,$$
$$\frac{b^{(5)}}{q^{(5)}} = 137, \quad \frac{b^{(6)}}{q^{(6)}} = 328, \quad \frac{b^{(7)}}{q^{(7)}} = 461.$$ [2016a]

These terms, arranged according to their magnitudes, are 1252, 461, 328, 247, 137, —146, [2016b] —221; and they correspond respectively to the equations [2016] numbered 3, 7, 6, 1, [2016c]

[2018′] θ being the latitude. *Hence we deduce $\frac{1}{312}$ for the oblateness of the earth*,* *and then the error of the degree of Lapland becomes* $86^R,26$.† This error is
[2018″] much too great to be admitted ;‡ which confirms what we have said [2014‴], that the earth varies sensibly from an elliptical figure.§

5, 2, 4. Now if we arrange the equations [2016] in the order of these numbers, changing the signs when the coefficient of y is negative, we shall have the following system of equations,

[2016d]

$$\begin{aligned}
0,03771\,.\,y - 47,20 &= x^{(3)},\\
0,40170\,.\,y - 185,45 &= -x^{(7)},\\
0,11133\,.\,y - 36,50 &= -x^{(6)},\\
0,43717\,.\,y - 107,95 &= x^{(1)},\\
0,08376\,.\,y - 11,48 &= -x^{(5)},\\
0,13561\,.\,y + 19,85 &= x^{(2)},\\
0,02824\,.\,y + 6,25 &= -x^{(4)}.
\end{aligned}$$

$$\begin{array}{|r|}
\hline
h^{(1)}=0,03771\times1,6435=0,06198\\
h^{(2)}=0,40170\times1,0644=0,42757\\
h^{(3)}=0,11133\times3,2734=0,36443\\
h^{(1)}+h^{(2)}+h^{(3)}=0,85398\\
h^{(4)}=0,43717\times3,4633=1,51405\\
h^{(1)}+h^{(2)}+h^{(3)}+h^{(4)}=2,36803\\
h^{(5)}=0,08376\times10,7487=0,90031\\
h^{(6)}=0,13561\times\ 1,3572=0,18405\\
h^{(7)}=0,02824\times\ 2,4034=0,06787\\
h^{(1)}+h^{(2)}\ldots\ldots+h^{(7)}=F=3,52026\\
\hline
\end{array}$$

[2016e] Multiplying the coefficients of y, in this system, by the quantities $i^{(3)}$, $i^{(7)}$, $i^{(6)}$, $i^{(1)}$, $i^{(5)}$, $i^{(2)}$, $i^{(4)}$, respectively, we shall obtain the products $h^{(1)}$, $h^{(2)}$, &c., [1999], as in the adjoined table, which agrees with [2017]. The sum of these products is $F=3,52026$ [2004a], making $\frac{1}{2}F=1,76013$. Now by the same table we have

[2016f] $$h^{(1)}+h^{(2)}+h^{(3)}=0,85398<\tfrac{1}{2}F, \quad\text{and}\quad h^{(1)}+h^{(2)}+h^{(3)}+h^{(4)}=2,36803>\tfrac{1}{2}F;$$

hence we get $h^{(r)}=h^{(4)}$, $r=4$, [2001]; and from [2001′], we find that the value of y will be obtained, by putting the fourth of the equations [2016d] equal to nothing; which
[2016g] gives $0,43717\,.\,y-107^R,95=x^{(1)}=0$, or $y=246^R,93$. Substituting $x^{(1)}=0$, in the first equation [2011], we get $z=25538^R,85$; hence the degree of the meridian [1969b] becomes as in [2018].

* (1465) Substituting the values of z, y, [2016g], in the formula of the ellipticity [2013i],
[2016h] we get $\alpha h=\dfrac{246,93}{77028,10}=\frac{1}{312}$, as above.

[2016i] † (1466) This is easily deduced from the last of the equations [2016], substituting y [2016g], from which we get $x^{(7)}=185,45-99,19=86,26$.

[2016k] ‡ (1466a) An error of this magnitude did however actually exist, as has been stated in [2009h].

§ (1467) The results of later and more accurate observations do not vary so much from
[2017a] the elliptical hypothesis, as the author has here supposed. This will appear from the following calculations, made with the measures of the arcs of the meridian of Peru, India, France, England, and Lapland. The measures of these arcs are given, in column 6 of the

The operations made by Delambre and Mechain, to obtain the measure of the terrestrial meridian, included between the parallels of latitude of [2018‴] Dunkirk and Barcelona, leave no doubt on this point, when we take into

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Earth does not vary so much from an elliptical figure as is supposed by the author.

following table A, in English fathoms, at the standard temperature of 62° Fahrenheit; and the latitudes of the extreme limits of the arcs are expressed in columns 2, 3, according to the sexagesimal notation; being the same as those in Mr. Airy's table, page 570 of the [2017b] Transactions of the Royal Society of London for 1826. The latitude of the middle point of each measured arc, is in column 4. The lengths of the arcs, in sexagesimal degrees, are in column 5. The mean length of each of the degrees, in English fathoms, is in column 7.

We shall, in the first place, use the mean length of each of the measured degrees, and the mean of the latitudes of the extreme points of the arc; neglecting, as the author has usually done, the terms depending on the square of the ellipticity. With these measures, using the formula [1969b], $a = z + y \cdot \sin.^2 \text{lat.}$, we obtain the system of equations in the annexed [2017c] table B, similar to [1970]. We shall now combine these equations by the method of the least squares, improved in the manner pointed out in [1995n, &c.].

Multiplying the equations in table B, by the lengths of the arcs $i^{(1)}$, $i^{(2)}$, &c., table A, column 5, we get the system of equations in table C, corresponding to the formula [1995x], changing the signs of the terms. The sum of these, put equal to nothing, and divided by the coefficient of $-z$, gives,

$$60655^f,8 - z - y \cdot 0,32803 = 0, \qquad [2017d]$$

corresponding to [1995z]. Subtracting this from each of the equations in table B, we get the system of equations in table D, similar to [1995ζ]; multiplying these by the corresponding lengths of the arcs in degrees $i^{(1)}$, $i^{(2)}$, &c., we obtain the system of products in table E, [2017e] similar to [1995η]. Lastly, multiplying these products in table E, by the corresponding coefficients of $-y$ in table B, representing $p^{(1)}$, $p^{(2)}$, &c., we obtain the system in table F, corresponding to [1995θ], whose sum, put equal to nothing, gives the equation [1995ι],

$$-1235^f,3 + y \cdot 2,11408 = 0, \qquad \text{whence} \qquad y = 584^f,3. \qquad [2017f]$$

Substituting this value of y in [2017d], we get $z = 60655^f,8 - 191^f,7 = 60464^f,1$; [2017g] hence the general expression of the length of a degree [2017c] becomes,

$$60464^f,1 + 584^f,3 \cdot \sin.^2 \text{lat.}; \qquad [2017h]$$

and the ellipticity $\varepsilon = \frac{y}{3z + \frac{5}{3}y} = \frac{1}{312}$ [1795f]. We shall find hereafter [2017y], [2017i] that this result would not be essentially varied, by taking into consideration the whole length of the measured arcs, and noticing the terms of the order of the square of the ellipticity. The preceding value of y, being substituted in the table D, gives the errors of the measured [2017k] degrees $x^{(1)}$, $x^{(2)}$, &c.

[2018''''] consideration the great accuracy of this measure. The following are the principal results of these operations,

Measured degrees of the meridian.

[2017l]

TABLE. A.

Col. 1. Place.	Col. 2. Extreme Latitudes. ψ	Col. 3. Extreme Latitudes. ψ'	Col. 4. Middle Latitude. L	Col. 5. Whole measured arc, in degrees, $i^{(1)}$, $i^{(2)}$, &c.	Col. 6. Whole measured arc, in fathoms, a_1, a_2, &c.	Col. 7. Mean deg. in fathoms, $a^{(1)}$, &c.
	d m s	d m s	d m s	d		
Peru,	— 0 02 31,22	3 04 31,9	1 31 00,34	3,117533	188510	60467,7
India,	8 09 38,39	18 03 23,6	13 06 31,00	9,895892	598630	60492,8
France,	38 39 56,11	51 02 09,2	44 51 02,65	12,370303	751567	60755,7
England,	50 37 05,27	53 27 29,89	52 02 17,58	2,840172	172751	60824,1
Sweden,	65 31 30,27	67 08 49,55	66 20 09,91	1,622022	98870	60954,8

[2017m]

TABLE B. [1970]

$$\begin{array}{l}
f \\
60467,7 - z - y\,.\,0,00070 = x^{(1)} \\
60492,8 - z - y\,.\,0,05144 = x^{(2)} \\
60755,7 - z - y\,.\,0,49740 = x^{(3)} \\
60824,1 - z - y\,.\,0,62161 = x^{(4)} \\
60954,8 - z - y\,.\,0,83890 = x^{(5)} \\
\hline
303495,1 - 5z - y\,.\,2,01005 = 0
\end{array}$$

TABLE C. [1995η]

$$\begin{array}{l}
f \qquad\qquad d \\
188510 - z\,.\,3,117533 - y\,.\,0,002184 = i^{(1)}\,.\,x^{(1)} \\
598630 - z\,.\,9,895892 - y\,.\,0,509016 = i^{(2)}\,.\,x^{(2)} \\
751567 - z\,.\,12,370303 - y\,.\,6,152927 = i^{(3)}\,.\,x^{(3)} \\
172751 - z\,.\,2,840172 - y\,.\,1,765474 = i^{(4)}\,.\,x^{(4)} \\
98870 - z\,.\,1,622022 - y\,.\,1,360716 = i^{(5)}\,.\,x^{(5)} \\
\hline
1810328 - z\,.\,29,845922 - y\,.\,9,790317 = 0 \\
\hline
[1995z] \quad 60655,8 - z - y\,.\,0,32803 = 0
\end{array}$$

[2017n]

TABLE D. [1995ζ]

$$\begin{array}{l}
f \\
-188,1 + y\,.\,0,32733 = x^{(1)} \\
-163,0 + y\,.\,0,27659 = x^{(2)} \\
99,9 - y\,.\,0,16937 = x^{(3)} \\
168,3 - y\,.\,0,29358 = x^{(4)} \\
299,0 - y\,.\,0,51087 = x^{(5)}
\end{array}$$

TABLE E.

$$\begin{array}{l}
f \\
-586,4 + y\,.\,1,02046 \\
-1613,0 + y\,.\,2,73710 \\
1235,8 - y\,.\,2,09516 \\
478,0 - y\,.\,0,83383 \\
485,0 - y\,.\,0,82863
\end{array}$$

TABLE F. [1995θ]

$$\begin{array}{l}
0,4 - y\,.\,0,00071 \\
83,0 - y\,.\,0,14080 \\
-614,7 + y\,.\,1,04213 \\
-297,1 + y\,.\,0,51832 \\
-406,9 + y\,.\,0,69514 \\
\hline
0 = -1235,3 + y\,.\,2,11408 \quad [1995\iota]
\end{array}$$

The same observations were computed by Mr. Airy, so as to include terms of the order of the square of the ellipticity, by means of the formula [1970o], supposing M, N, P, to be [2017n'] independent of each other. The following system [2017o], similar to that in table C, is given by him, changing however M into $\frac{M'}{100000}$, in order to avoid large numbers; he having expressed the coefficients of M, or the whole length of the measured arcs, in sexagesimal seconds; consequently the coefficients of M' represent $\frac{1}{100000}$ of the measured arc, in sexagesimal seconds. I have used the same formula as Mr. Airy [1970o], in order that the difference of the results of his calculation, by the usual method [815c—l], and by that which I have proposed in [1995λ], may be more easily perceived.

Latitudes.		Distances from the parallel of latitude of Montjoui to the parallels of the following places,		
Montjoui - - - -	$45^{\circ},958281 = \theta^{(1)}$,			
Carcassonne - -	$48\ ,016790 = \theta^{(2)}$,	Carcassonne -	$52749^{R},48 = a^{(2)}$,	Arcs of the meridian, measured in France.
Evaux - - - -	$51\ ,309414 = \theta^{(3)}$,	Evaux - -	$137174\ ,03 = a^{(3)}$,	
Pantheon at Paris -	$54\ ,274614 = \theta^{(4)}$,	Pantheon - -	$213319\ ,77 = a^{(4)}$,	[2019]
Dunkirk - - - -	$56\ ,706944 = \theta^{(5)}$.	Dunkirk - -	$275792\ ,36 = a^{(5)}$.	

$$-188510^{f} + M'.\,0,112231 + N.\,0,1086 + P.\,0,216 = -i^{(1)}.\,x^{(1)};$$
$$-598630 + M'.\,0,356252 + N.\,0,3084 + P.\,0,412 = -i^{(2)}.\,x^{(2)};$$
$$-751567 + M'.\,0,445331 + N.\,0,0023 - P.\,0,837 = -i^{(3)}.\,x^{(3)}; \qquad [2017o]$$
$$-172751 + M'.\,0,102246 - N.\,0,0241 - P.\,0,175 = -i^{(4)}.\,x^{(4)};$$
$$-\ 98870 + M'.\,0,058393 - N.\,0,0383 - P.\,0,009 = -i^{(5)}.\,x^{(5)}.$$

From these Mr. Airy obtained, by the method explained in [849k], the values,

$$M = 16,88164, \qquad N = -9358, \qquad P = 267\ ; \qquad [2017p]$$

hence the length of an arc of the meridian, in fathoms, is expressed by,

$$16,88164\,.\,\text{seconds in}\,(\psi' - \psi) - 9358\,.\,(\sin.2\psi' - \sin.2\psi) + 267\,.\,(\sin.4\psi' - \sin.4\psi). \qquad [2017q]$$

Mr. Airy found the lengths of the several arcs, from this formula, to differ from those given in the above table, by -4, $+4$, -19, $+35$, $+63$ fathoms. He also [2017r] computed the ellipticity to be $\frac{1}{278,6}$ [1970u], by putting $M=16,88164$, $N=-9358$, [2017r′] [1969o, 1970o, 2017q]. We shall soon see that this ellipticity is probably too great, by nearly one tenth part, and that the result of the method I have proposed [1995λ—τ] will [2017s] agree much better with these observations. For if we find the three fundamental equations, by the method explained in [1995ν], we get,

$$-1810328 + M'.\,1,074453 + N.\,0,35690 - P.\,0,39300 = 0,$$
$$-\ 598948 + M'.\,0,356900 + N.\,0,40287 + P.\,0,60850 = 0, \qquad [2017t]$$
$$668370 - M.\,0,393000 + N.\,0,60850 + P.\,2,76623 = 0.$$

Hence $M' = 1687659,4$, $M = 16,876594$, $N = -8368,2$, $P = -10,0$; and [2017u] the length of an arc of the meridian is expressed by,

$$16^{f},876594\,.\,\text{seconds in}\,(\psi' - \psi) - 8368^{f},2\,.\,(\sin.2\psi' - \sin.2\psi)$$
$$-10^{f},0\,.\,(\sin.4\psi' - \sin.4\psi). \qquad [2017v]$$

These numbers differ considerably from those of Mr. Airy [2017p, q], particularly in the value of P, which he makes above twenty-six times as great as by my method; and it is in [2017w]

These distances being represented by $a^{(2)}$, $a^{(3)}$, $a^{(4)}$, $a^{(5)}$; the latitudes
[2019'] by $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$, $\theta^{(4)}$, $\theta^{(5)}$; and we shall put $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $x^{(4)}$, $x^{(5)}$,

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consequence of his using this excessively great value of P, that he supposes the curve of the meridian to differ essentially from an elliptical form. The value of P, corresponding to an
[2017x] elliptical meridian [1970v], is $P = -\frac{5}{16}\varepsilon \,.\, N = \frac{5}{16} \cdot \frac{1}{312} \cdot 8368{,}2 = 8^f{,}4$ nearly.

[2017y] Substituting M, N, [2017u], in [1970u], we get $\frac{1}{\varepsilon} = 311{,}5$, $\frac{1}{\varepsilon'} = \frac{1}{312{,}5}$; hence the ellipticity of the earth, expressed in parts of the polar radius, is $\frac{1}{311{,}5}$, or $\frac{1}{312{,}5}$ part
[2017z] of the equatorial radius. The corresponding values of Mr. Airy are $\frac{1}{278{,}6}$, or $\frac{1}{279{,}6}$, respectively. If we compute the whole lengths of these five measured arcs, by means of the formula [2017v], we shall find the errors of these measures, expressed in fathoms, to be
[2017α] $+13$, -17, $+11$, -9, $+2$, respectively. The sum of the positive errors is equal to the sum of the negative ones, as must necessarily be the case, in this method of calculation; because the first of the equations [2017t] is found, by putting the sum of the errors of all the measured arcs equal to nothing [1995v]. These errors are much less than
[2017β] those, found by using Mr. Airy's formula [2017r]; the extreme errors of the two formulas being in the ratio of 17 to 63, neglecting the signs.

If we substitute the elliptical value of $P = 8^f{,}4$ [2017x], in the equations [2017o], the
[2017γ] terms depending on this quantity, will be $1^f{,}8$, $3^f{,}4$, $-7^f{,}0$, $-1^f{,}5$, $-0^f{,}1$; which are far within the limits of the errors of the observations [2017α], and may therefore be neglected; or we may connect these terms with the measured arcs, by the method [1970r]; observing that they would vary only a fraction of a fathom, by taking for the ellipticity any
[2017δ] quantity between $\frac{1}{300}$ and $\frac{1}{330}$, instead of using $\frac{1}{312}$. In this way the system [2017t] will be reduced to the two first equations, in which we must put $P = 8^f{,}4$, and they will become nearly,

[2017ε]
$$-1810331 + M' \,.\, 1{,}074453 + N \,.\, 0{,}35690 = 0\,;$$
$$-598943 + M' \,.\, 0{,}356900 + N \,.\, 0{,}40287 = 0.$$

[2017ζ] Hence we get $M' = 1687681$, $M = 16{,}87681$, $N = -8415{,}2$; and ε is obtained from [1970u] $\frac{1}{\varepsilon} = -154699 \,.\, \frac{M}{N} - \frac{1}{2} = 309{,}8$; therefore the ellipticity, expressed in
Oblateness.
[2017η] parts of the polar radius, is $\frac{1}{309{,}8}$, and in parts of the equatorial radius, is $\frac{1}{310{,}8}$. Then the formula [1970v], gives $P = -\frac{5}{16}\varepsilon \,.\, N = 8^f{,}4$ nearly, as above; and the length of an arc of the meridian is expressed by,

[2017θ]
$$16^f{,}87681 \,.\, \text{sexagesimal seconds in } (\psi' - \psi) - 8415^f{,}2 \,.\, (\sin.2\psi' - \sin.2\psi)$$
$$+ 8^f{,}4 \,.\, (\sin.4\psi' - \sin.4\psi).$$

for the errors to which these latitudes are liable. These errors may be attributed, either to the observed latitudes of the places, or to the geodetical measures, the defects of which have an influence on the latitudes [2019″] of the parallels, whose distances, from that of Montjoui, are supposed to be

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If we compute the lengths of the whole arcs of the preceding measures, by the formula [2017δ], we shall find that the errors are -12, $+18$, -17, $+9$, $+1$; which agree nearly as well with the observations, as those computed in [2017a], where the meridian is not restricted to an elliptical form. *Hence it appears that this strictly elliptical form of the meridian is more conformable to these observations than the irregular figure obtained by Mr. Airy's calculation.* [2017ι]

These observations agree so well with the hypothesis of an elliptical meridian, that it makes but very little difference what method is used in combining them. For the results of the calculations in [2017i, y, ξ] are nearly the same; and we shall find upon examination, that if we use the common method of applying the principle of the least squares, putting P equal to its elliptical value $8^s,4$, the results will be nearly the same as those just mentioned. This may be easily proved, by putting $P=8^s,4$ in the two first fundamental equations given by Mr. Airy, in page 570 of his paper abovementioned; by which means the ellipticity will be changed from $\frac{1}{278,6}$ to $\frac{1}{307}$, and the results will nearly agree with [2017ξ]. The same also appears from the calculation of Mr. Ivory, in the Philosophical Magazine for 1828, page 346; where, by means of the formulas [1970l, s], he obtains the ellipticity $\frac{1}{309}$, using the common method of combining the equations. Finally, if we apply the method of Boscovich [1996, &c.] to the system of equations in table B [2017m], we shall find, that the value of y is to be obtained by putting $x^{(2)}=0$, in the second equation of table D [2017n], which will give $y=\frac{163}{0,27659}=589,3$; and then, from [2017$d$], we shall get, [2017$\kappa$] [2017$\lambda$] [2017$\mu$]

$$z=60655,8-y\,.\,0,32803=60462,5\,; \qquad [2017\nu]$$

consequently the length of a degree of the meridian will be expressed by

$$60462^f,5+589^f,3\,.\,\sin.^2\,\text{lat.}$$

This differs but little from [2017h], and the ellipticity, resulting from these numbers, computed as in [2017i], is $\frac{1}{310}$ nearly. [2017ξ]

If we put $\psi=0$, $\psi'=90^d=32400^s$, in [2017δ], we shall obtain the length of a quadrant of the meridian 5468087, in English standard fathoms, at the temperature of 62° Fahrenheit. We may reduce any of the measures given in this article in standard fathoms, to the standard metre, by using the ratio of these measures given in the Philosophical Transactions for 1818, 1826, pages 109, 569; from which it appears, that the standard metre of 0° centigrade, was found by Captain Kater to be equal to $39^{\text{inches}},37079$ of the English standard, at the temperature of 62° Fahrenheit. [2017π] [2017ρ]

$a^{(2)}$, $a^{(3)}$, &c. The terrestrial arc, comprised between the equator and the parallel of latitude of Montjoui, is, by what has been said, nearly equal to,*

[2020] $$s\,.\,\{\theta^{(1)}+x^{(1)}-\tfrac{3}{4}\rho\,.\,\sin.\,2\,\theta^{(1)}\}\,;$$

[2020′] s being the mean length of a degree, and ρ the oblateness of the earth, reduced to degrees. The arc comprised between the equator and the parallel of Carcassonne, is

[2021] $$s\,.\,\{\theta^{(2)}+x^{(2)}-\tfrac{3}{4}\rho\,.\,\sin.\,2\,\theta^{(2)}\}\,;$$

therefore the arc included between the two parallels of Carcassonne and Montjoui, is

[2022] $$s\,.\,\{\theta^{(2)}-\theta^{(1)}+x^{(2)}-x^{(1)}-\tfrac{3}{4}\rho\,.\,(\sin.\,2\,\theta^{(2)}-\sin.\,2\,\theta^{(1)})\}.$$

Putting this equal to $a^{(2)}$, we shall obtain,†

[2023] $$\theta^{(2)}-\theta^{(1)}+x^{(2)}-x^{(1)}-\tfrac{3}{2}\rho\,.\,\sin.\,(\theta^{(2)}-\theta^{(1)})\,.\,\cos.\,(\theta^{(2)}+\theta^{(1)})=\frac{a^{(2)}}{s}.$$

The parallels of the other places, compared with that of Montjoui, furnish
[2023′] three similar expressions; and by substituting the corresponding numerical values, we shall obtain the four following equations :‡

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* (1468) If we neglect terms of the order of the square of the ellipticity ε, the expression of the length of an arc of the meridian, included between the equator and the latitude θ, will
[2020a] be $k.\{(1+\frac{1}{2}\varepsilon).\theta-\frac{3}{4}\varepsilon.\sin.2\theta\}$ [1969n], or $k.(1+\frac{1}{2}\varepsilon).\{\theta-\frac{3}{4}\varepsilon.\sin.2\theta\}$. Putting $\theta=90^d$, we get the length of the quadrantal arc, equal to $k.(1+\frac{1}{2}\varepsilon).90^d$; hence the mean length of one degree is $k.(1+\frac{1}{2}\varepsilon)$. If we represent this by s, as in [2020′],
[2020b] and change the ellipticity ε into ρ, to conform to the present notation, we shall get the length of the arc [2020a], $s.\{\theta-\frac{3}{4}\rho.\sin.2\theta\}$. From this formula we easily obtain the expressions [2020, 2021], supposing the errors of the measured arcs to be $s.x^{(1)}$, $s.x^{(2)}$, respectively. The difference of these two arcs is the value of $a^{(2)}$ [2022].

† (1469) Putting [2022] equal to $a^{(2)}$, and dividing by s, we get,

[2023a] $$\theta^{(2)}-\theta^{(1)}+x^{(2)}-x^{(1)}-\tfrac{3}{4}\rho\,.\,\{\sin.\,2\,\theta^{(2)}-\sin.\,2\,\theta^{(1)}\}=\frac{a^{(2)}}{s}.$$

Substituting $\sin.\,2\,\theta^{(2)}-\sin.\,2\,\theta^{(1)}=2\sin.\,(\theta^{(2)}-\theta^{(1)})\,.\,\cos.\,(\theta^{(2)}+\theta^{(1)})$, [26] Int., it becomes as in [2023].

‡ (1470) These equations are deduced from [2023]; but if we have a table of natural sines, it will be easier to deduce them from [2023a], using the values [2019], and changing successively $a^{(2)}$ into $a^{(3)}$, $a^{(4)}$, &c., we shall get very nearly the numbers in [2024].

$$\left.\begin{array}{l} 2^{\circ},058509 + x^{(2)} - x^{(1)} - \rho \,.\, 0,0045829 = \dfrac{52749^{R},48}{s} \\ 5^{\circ},351133 + x^{(3)} - x^{(1)} - \rho \,.\, 0,0054036 = \dfrac{137174^{R},03}{s} \\ 8^{\circ},316333 + x^{(4)} - x^{(1)} + \rho \,.\, 0,0007152 = \dfrac{213319^{R},77}{s} \\ 10^{\circ},748663 + x^{(5)} - x^{(1)} + \rho \,.\, 0,0105491 = \dfrac{275792^{R},36}{s} \end{array}\right\} ; \qquad (B)$$ [2024]

If we apply to these equations, the first method, given at the beginning of § 39,* we find, that in the elliptical hypothesis, which gives a *minimum* for [2024′]

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* (1471) This method will lead to a formula, similar to [1972]; observing that ρ, s, [2024], take the place of z, y, [1970], and must be exterminated, as is observed in [1970′]. In the first place we shall multiply the equations [2024] respectively by the quantities $\frac{275792^{R},36}{52749^{R},48}$, $\frac{275792^{R},36}{137174^{R},03}$, $\frac{275792^{R},36}{213319^{R},77}$, 1, and we shall get,

$$\begin{array}{l} 10^{\circ},762590 + 5,228342 \,.\, \{x^{(2)} - x^{(1)}\} - \rho \,.\, 0,0239610 = 275792^{R},36 \,.\, s^{-1}\,; \\ 10^{\circ},758606 + 2,010528 \,.\, \{x^{(3)} - x^{(1)}\} - \rho \,.\, 0,0108641 = 275792^{R},36 \,.\, s^{-1}\,; \\ 10^{\circ},751840 + 1,292859 \,.\, \{x^{(4)} - x^{(1)}\} + \rho \,.\, 0,0009246 = 275792^{R},36 \,.\, s^{-1}\,; \\ 10^{\circ},748663 + 1,000000 \,.\, \{x^{(5)} - x^{(1)}\} + \rho \,.\, 0,0105491 = 275792^{R},36 \,.\, s^{-1}. \end{array}$$ [2024a]

Subtracting each of these equations from that immediately preceding it, we shall eliminate s, and shall get,

$$\begin{array}{l} 0^{\circ},003984 - 3,217814 \,.\, x^{(1)} + 5,228342 \,.\, x^{(2)} - 2,010528 \,.\, x^{(3)} - \rho \,.\, 0,0130969 = 0\,; \\ 0^{\circ},006766 - 0,717669 \,.\, x^{(1)} + 2,010528 \,.\, x^{(3)} - 1,292859 \,.\, x^{(4)} - \rho \,.\, 0,0117887 = 0\,; \\ 0^{\circ},003177 - 0,292859 \,.\, x^{(1)} + 1,292859 \,.\, x^{(4)} - x^{(5)} \qquad - \rho \,.\, 0,0096245 = 0. \end{array}$$ [2024b]

Multiplying these equations respectively by 1, $\frac{130969}{117887}$, $\frac{130969}{96245}$, we obtain,

$$\begin{array}{l} 0^{\circ},0039840 - 3,217814 \,.\, x^{(1)} + 5,228342 \,.\, x^{(2)} - 2,010528 \,.\, x^{(3)} - \rho \,.\, 0,0130969 = 0\,; \\ 0^{\circ},0075168 - 0,797309 \,.\, x^{(1)} - 1,436328 \,.\, x^{(4)} + 2,233638 \,.\, x^{(3)} - \rho \,.\, 0,0130969 = 0\,; \\ 0^{\circ},0043232 - 0,398519 \,.\, x^{(1)} + 1,759306 \,.\, x^{(4)} - 1,360787 \,.\, x^{(5)} - \rho \,.\, 0,0130969 = 0. \end{array}$$ [2024c]

Subtracting the first from the second, and the third from the second, we obtain the following,

$$\begin{array}{l} 0^{\circ},0035328 = -2,420505 \,.\, x^{(1)} + 5,228342 \,.\, x^{(2)} + 1,436328 \,.\, x^{(4)} - 4,244166 \,.\, x^{(3)}\,; \\ 0^{\circ},0031936 = \quad 0,398790 \,.\, x^{(1)} + 3,195634 \,.\, x^{(4)} - 2,233638 \,.\, x^{(3)} - 1,360787 \,.\, x^{(5)}. \end{array}$$ [2024d]

In order to compute the minimum error, we must make different suppositions, putting successively either one of the quantities $x^{(1)}$, $x^{(2)}$, &c., for the least error, and making

[2025] the greatest error, $x^{(1)} = x^{(4)} = -x^{(3)} = -x^{(5)} = 4'',43$; $x^{(2)} = 3'',99$;

[2025′] the oblateness $\rho = \frac{1}{150,6}$;* and the degree corresponding to the mean parallel of latitude, equal to $25649^R,8$. These observations have been made with so much care and accuracy, that it is not possible that these errors, small as they are, can really exist; it appears therefore that we ought to attribute them, in part at least, to causes which make the figure of the

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[2024e] the other four equal to each other, according to the directions in [1971″—1972‴]. In the first place, we shall suppose that $x^{(1)}$, independent of its sign, is the least of these errors; then the first of the equations [2024d] gives,

[2024f]
$$x^{(1)} = -0^\circ,0014595 + 2,1600 \,.\, x^{(2)} - 1,7534 \,.\, x^{(3)} + 0,5934 \,.\, x^{(4)}.$$

Substituting this in the second of the equations [2024d], we get the equation corresponding to [1972],

[2024g]
$$0^\circ,0037756 = 0,8614 \,.\, x^{(2)} - 2,9328 \,.\, x^{(3)} + 3,4323 \,.\, x^{(4)} - 1,3608 \,.\, x^{(5)}.$$

Dividing, as in [1972′], this first member, $0^\circ,0037756$, by the sum of the coefficients of $x^{(2)}$, $-x^{(3)}$, $x^{(4)}$, $-x^{(5)}$, which is $8,5873$, we get,

$$x^{(2)} = -x^{(3)} = x^{(4)} = -x^{(5)} = 0^\circ,00044.$$

Substituting these in [2024f], we get $x^{(1)} = 0^\circ,0052$; and as this exceeds the preceding value of $x^{(2)} = 0^\circ,00044$, the supposition made in [2024e], that the error $x^{(1)}$ is the
[2024h] least, is not correct; and we must, as in [1972‴], take the next in order, $x^{(2)}$, and suppose it to be the least of the errors, independent of its sign. In this case, the second of the equations [2024d] will give, as in [1972′], by dividing the constant quantity of the first member, 0,0031936, by the sum of the coefficients of $x^{(1)}$, $x^{(4)}$, $-x^{(3)}$, $-x^{(5)}$,
[2024i] in the second member, which is 7,188849, $x^{(1)} = -x^{(3)} = x^{(4)} = -x^{(5)} = 0^\circ,000444$. Substituting this in the first of the equations [2024d], we obtain $x^{(2)} = 0^\circ,000399$. This, being less than the preceding value of $x^{(1)}$, agrees with the hypothesis [2024h]; and it will therefore be unnecessary to examine the other cases, in which $x^{(3)}$, $x^{(4)}$, $x^{(5)}$, are supposed to be the least of the errors [1972‴].

* (1472) Substituting the values of $x^{(1)}$, $x^{(4)}$, $x^{(5)}$, [2024i] in the last of the equations
[2025a] [2024b], we get $0^\circ,003177 + 0,000888 - \rho \,.\, 0,0096245 = 0$, whence $\rho = 0^\circ,42236$. To reduce this from degrees [2020′] to parts of the radius, we must divide it by $63^\circ,66198$ [1970h], and it will become $\rho = \frac{0,42236}{63,66198} = \frac{1}{150,7}$, being nearly as in [2025′]. Substituting $x^{(4)} - x^{(1)} = 0$ [2024i], and $\rho = 0,42236$ [2025a] in the
[2025b] third of the equations [2024], we get $8^\circ,316333 + 0^\circ,000302 = 213319^R,77 \,.\, s^{-1}$; hence $s = 25649^R,8$, as in [2025′].

earth differ from that of an ellipsoid. But what proves the fact beyond doubt is, that the oblateness $\frac{1}{150,6}$, which this combination of errors [2025″] gives to the earth, does not agree, either with the phenomena of gravity, or with the observations of the precession and nutation; which do not allow [2025‴] us to suppose that the earth has a greater oblateness than in the case of homogeneity, namely $\frac{1}{230}$.*

If we put in [2024] $\rho = \frac{1}{230}$, or in degrees† $\rho = 0^\circ,276791$; [2025⁗] supposing also,

$$s = \frac{10000}{1^\circ . y}, \qquad [2026]$$

we shall get the following system of equations,‡

$$\begin{aligned}
0^\circ,000000 - z - y \,.\ 0^\circ,000000 &= -x^{(1)};\\
2^\circ,057240 - z - y \,.\ 5^\circ,274948 &= -x^{(2)};\\
5^\circ,349637 - z - y \,.\, 13^\circ,717403 &= -x^{(3)};\\
8^\circ,316531 - z - y \,.\, 21^\circ,331977 &= -x^{(4)};\\
10^\circ,751583 - z - y \,.\, 27^\circ,579236 &= -x^{(5)}.
\end{aligned} \qquad [2027]$$

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* (1473) In the case of homogeneity, the ellipticity of the earth is $\frac{1}{231,7}$ of the equatorial radius, or $\frac{1}{230,7}$ of the polar radius, [1592a]. This quantity is nearly equal to the term $\frac{5}{4}\alpha\varphi$, in the expression of the radius [1648]; and by [1800′], the oblateness of the earth, in the most probable hypothesis of the density, is less than this. The above [2026a] remarks, relative to the limit of the ellipticity of the earth, derived from the observations of the precession and nutation, are conformable to the results obtained in [3418].

† (1474) Multiplying $\rho = \frac{1}{230}$, expressed in parts of the radius, by the value of the radius in degrees, $63^\circ,66198$ [1970h], it becomes $\rho = 0^\circ,276791$. In the original, it [2026b] was printed $0^\circ,27691$, by a typographical mistake.

‡ (1475) If we suppose $z = x^{(1)}$, it may be put for the sake of symmetry, under the [2027a] form of the first of the equations [2027]. Substituting this value of $x^{(1)}$ in the equations [2024], also $\rho = 0,276791$ [2025⁗], and for $\frac{1}{s}$ its value $\frac{1^\circ . y}{10000}$ [2026], we [2027b] shall get the four last of the equations [2027]. It may be observed, that in the value of s, [2026], the numerator was erroneously printed 100000 in the original work, instead of 10000.

These equations are similar to the equations [1970], the only difference
[2027′] being in the signs of the errors $x^{(1)}$, $x^{(2)}$, &c. Making use of the second method explained in § 39, [1974, &c.], the two series [1985] become,*

[2028]
$$\begin{array}{ccccc} -x^{(1)}, & -x^{(2)}, & -x^{(3)}, & -x^{(5)}, & \\ \infty, & 0^{\circ},390002, & 0^{\circ},389981, & 0^{\circ},389699, & -\infty; \end{array}$$

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* (1476) Comparing the equations [2027] with [1970], we obtain,

[2028a]
$$\begin{array}{lll} a^{(1)}=0^{\circ},000000, & a^{(2)}=2^{\circ},057240, & a^{(3)}=5^{\circ},349637, \\ & a^{(4)}=8^{\circ},316531, & a^{(5)}=10^{\circ},751583. \\ p^{(1)}=0^{\circ},000000, & p^{(2)}=\ 5^{\circ},274948, & p^{(3)}=13^{\circ},717403, \\ & p^{(4)}=21^{\circ},331977, & p^{(5)}=27^{\circ},579236. \end{array}$$

Hence the series of numbers [1978] becomes,

[2028b] $$0,390002, \qquad 0,389989, \qquad 0,389862, \qquad 0,389843.$$

[2028c] The greatest of these quantities is $0,390002=\dfrac{a^{(2)}-a^{(1)}}{p^{(2)}-p^{(1)}}=\beta^{(1)}$ [1979]; the greatest exponent of a being $r=2$. The least of these quantities is $\lambda^{(1)}=0,389843=\dfrac{a^{(5)}-a^{(1)}}{p^{(5)}-p^{(1)}}$
[2028c′] [1986′]; the greatest exponent of a or p being $s=5$. Having found

[2028d] $$a^{(r)}=a^{(2)}=2^{\circ},057240, \qquad p^{(r)}=p^{(2)}=5^{\circ},274948,$$

we must subtract these from $a^{(3)}$, $a^{(4)}$, $a^{(5)}$, and $p^{(3)}$, $p^{(4)}$, $p^{(5)}$, respectively, to obtain the numerators and denominators of the series [1981]; which series is represented by 0,389981, 0,3898, 0,3898. The greatest of these numbers is,

[2028e] $$\beta^{(2)}=\frac{a^{(r')}-a^{(r)}}{p^{(r')}-p^{(r)}}=\frac{a^{(3)}-a^{(2)}}{p^{(3)}-p^{(2)}}=0,389981 \qquad [1982];$$

making $r'=3$. Hence $a^{(r')}=5^{\circ},349637$, $p^{(r')}=13^{\circ},717403$. Subtracting these from $a^{(4)}$, $a^{(5)}$, $p^{(4)}$, $p^{(5)}$, respectively, we obtain the numerators and denominators of the series [1983]. This series is represented by 0,389603, 0,389699. The greatest of
[2028f] these is, as in [1984], $\beta^{(3)}=\dfrac{a^{(r'')}-a^{(r')}}{p^{(r'')}-p^{(r')}}=\dfrac{a^{(5)}-a^{(3)}}{p^{(5)}-p^{(3)}}=0,389699$; whence $r''=5$.
This is the last term of the series; and by substituting the values of $x^{(r)}$, $x^{(r')}$, $x^{(r'')}$, $\beta^{(1)}$, $\beta^{(2)}$, $\beta^{(3)}$, in [1985], we obtain the system [2028], observing as in [2027′], that
[2028g] the signs of $x^{(1)}$, $x^{(2)}$, &c., are different in [2027] from what they are in [1970]. Lastly, as $\lambda^{(1)}$ [2028c′] corresponds to $s=5$, which is the greatest exponent of a or p, there will be no other value $\lambda^{(2)}$, $\lambda^{(3)}$, &c., of the series [1988]; and by substituting this value of $\lambda^{(1)}$ in the series [1988], it becomes as in [2029].

and the two series [1988] become,

$$\begin{array}{ccc} -x^{(1)}, & -x^{(5)}, & \\ -\infty, & 0^{\circ},389843, & \infty. \end{array}$$ [2029]

Hence we deduce,*

$$-x^{(1)} = -x^{(5)} = x^{(3)} = 9'',98\,;$$ [2030]

$$y = 0,389843\,;$$ [2030']

and the degree, on the parallel of 50°, is $25651^{R},33$.† [2031]

An error of $9'',98$ is much greater than can be supposed to exist; therefore these measures will not allow us to suppose the oblateness to be $\frac{1}{230}$, and still less will they agree with a smaller oblateness. It is therefore proved satisfactorily, that the earth varies very sensibly from an elliptical figure. But it is very remarkable, that the measures lately made in France, and in England, with great precision, in the direction of the meridian, and in the perpendicular to the meridian, both indicate an osculatory ellipsoid, in which the ellipticity is $\frac{1}{150}$, and the mean length of a degree [2025b], $25649^{R},8$. [2031'] [2031''] [2031'''] [2032]

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* (1477) As there is only one term $\lambda^{(1)}$ in the lower series [2029], we must put $y=\lambda^{(1)}=0,389843$, corresponding to $s=5$; and by proceeding as in [1991c, d], we shall find that $y=\lambda^{(1)}$ gives, in the series [2029], $-x^{(1)}=-x^{(5)}$. The exponents of these two quantities are 1, 5, and between them there are, in the series [2028], two quantities $-x^{(2)}$, $-x^{(3)}$; one of which must be equal to $x^{(1)}$ or $x^{(5)}$, as in [1975'''', 1976''']. To ascertain which of these two quantities is to be selected, we shall observe, that $x^{(r')}=x^{(3)}$ [1985''] is the greatest error from $y=\beta^{(2)}=0,389931$ to $y=\beta^{(3)}=0,389699$; and as $y=\lambda^{(1)}=0,389843$ [2029a] falls between these values, it will make $x^{(3)}$ a maximum; hence, as in [1975''''], we shall have the equation $-x^{(1)}=-x^{(5)}=x^{(3)}$ [2030].. Substituting $-x^{(3)}=x^{(1)}$, $z=x^{(1)}$, [2027a], in the third of the equations [2027], and dividing by 2, we get, [2029a] [2029b]

$$x^{(1)}=z=2^{\circ},674818-y\,.\,6^{\circ},858701=0^{\circ},001002=10'',02\,;$$ [2029c]

which agrees nearly with [2030].

† (1478) If we put, in [2022], $x^{(1)}=x^{(2)}=0$, $\theta^{(1)}=49^{\circ},5$, $\theta^{(2)}=50^{\circ},5$, we shall get s for the length of a degree of the meridian, included between the latitudes $49^{\circ},5$, $50^{\circ},5$; hence, by [2026, 2030'], it will become,

$$s=\frac{10000}{1^{\circ}.\,y}=\frac{10000}{0,389843}=25651^{R},3, \quad [2031].$$ [2030a]

To represent, with these elements [2032], the measures of the degrees between Dunkirk and the Pantheon, the Pantheon and Evaux, Evaux and Carcassonne, Carcassonne and Montjoui; it is only necessary to alter the
[2032′] observed latitudes about $4'',4$.* The degree perpendicular to the meridian,
[2032″] in the latitude of $56^{\circ},3144$, becomes $25837^{R},6$;† and by some very
[2032‴] exact operations made in England, it has been found to be $25833^{R},4$. From this near agreement it appears, that the great oblateness of the osculatory ellipsoid, in France, does not depend on the attractions of the Pyrenees, and the other mountains in the south of France; but that it
[2032⁗] arises from much more extensive attractions, the effect of which is sensible in the north of France, and in England, as well as in Austria and in Italy. For all the degrees, measured in this part of the surface of the earth, are

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* (1479) This agrees with [2025], where $x^{(1)}=4'',43$, $x^{(2)}=3'',99$, $x^{(3)}=-4'',43$,
[2031a] $x^{(4)}=4'',43$, $x^{(5)}=-4'',43$. If these values, and those of $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$, $\theta^{(4)}$, $\theta^{(5)}$, [2019], be substituted in $\theta^{(1)}+x^{(1)}$, $\theta^{(2)}+x^{(2)}$, $\theta^{(3)}+x^{(3)}$, $\theta^{(4)}+x^{(4)}$, $\theta^{(5)}+x^{(5)}$, they will give the corrected latitudes of Montjoui, &c., corresponding to the values $\rho=\frac{1}{150,6}$ and $s=25649^{R},8$, as is evident from [2025, &c.].

† (1480) If we suppose this part of the surface of the earth to be represented by an osculatory ellipsoid of revolution, and fix the origin of the radius at the centre of the generating ellipsis, the radius of the ellipsoid will evidently be independent of ϖ [1965];
[2032a] consequently we may put $h=0$ in [1966], which expresses the length of an arc of the meridian, corresponding to the difference of latitude ε; and if we neglect terms of the order
[2032b] ε^2, it will become $\varepsilon-\frac{1}{2}\alpha\varepsilon-\frac{3}{2}\alpha\varepsilon.\cos.2\psi$, or $\varepsilon.(1-\frac{1}{2}\alpha).\{1-\frac{3}{2}\alpha.\cos.2\psi\}$; whence the mean length of a degree $\varepsilon.(1-\frac{1}{2}\alpha)$; and if we use the value of this
[2032c] degree [2025b], we get $1^{\circ}.(1-\frac{1}{2}\alpha)=25649^{R},8$. Substituting this, and $\varepsilon=1^{\circ}$,
[2032d] in [2032b], we have $25649^{R},8.\{1-\frac{3}{2}\alpha.\cos.2\psi\}$, for the length of a degree of the meridian, in the latitude ψ. Substituting the same values of h, $1^{\circ}.(1-\frac{1}{2}\alpha)$, [2032a, c], in [1968], we obtain, by successive reductions, the length of a degree of the perpendicular to the meridian, in the latitude ψ, neglecting terms of the order α^2,

$$1^{\circ}.\{1+\alpha.\sin.^2\psi\}=1^{\circ}.(1-\tfrac{1}{2}\alpha).\{1+\tfrac{1}{2}\alpha+\alpha.\sin.^2\psi\}$$

[2032e]
$$=25649^{R},8.\{1+\tfrac{1}{2}\alpha+\alpha.\sin.^2\psi\}.$$

Putting $h=0$, in the expression of the radius [1965], it becomes $1-\alpha.\sin.^2\psi$; which at the equator is 1, and at the poles is $1-\alpha$; hence the ellipticity is α. Supposing, as in
[2032f] [2031‴], $\alpha=\frac{1}{150}$, this expression of the degree of the perpendicular to the meridian [2032e] becomes $25837^{R},6$, as in [2032″].

represented, within 8^R or 9^R, by the osculatory ellipsoid, we have just [2032v]
mentioned [2032].* [2033]

It appears also, by the azimuths observed, upon the arc of the meridian from Dunkirk to Montjoui, that the osculatory ellipsoid is not accurately a solid of revolution. If we apply the formulas of § 38, and the preceding [2033′] methods, to these observations, we may determine the osculatory ellipsoid, which satisfies the observations of the azimuths, as well as those of the latitudes. We shall not however discuss this point, but shall merely remark, that the measure of a perpendicular to the meridian of the observatory, made in the widest part of France, using the same methods as in the measure of [2033″] the meridian, observing at several places the azimuths and the latitudes, would furnish important data for the determination of the excentricity of this ellipsoid, in the directions of the parallels of latitude; and it is therefore to be desired, that this new measure should be added to the preceding. The observations of the azimuth, already made, prove that the meridians are [2033‴] not similar; and if we compare the degree of the Cape of Good Hope† with the degrees measured in the northern hemisphere of the earth, we shall find that there is reason to suppose the northern and southern hemispheres differ [2033⁗] from each other. The figure of the earth is therefore very complex, as is natural to suppose it would be, when we take into consideration the great inequalities of its surface, the different density of the parts which cover it, [2033v] and the irregularities in the shores and in the bottom of the ocean.

To determine the length of a quadrant of the terrestrial meridian, from the arc comprised between Dunkirk and Montjoui, we must adopt some hypothesis relative to the figure of the earth; and notwithstanding all the irregularities [2033vi]

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* (1481) Substituting $\rho = \frac{1}{150}$ [2031‴] in [2032d], we get the corresponding length of a degree of the meridian, in the latitude ψ, equal to $25649^R,8 - 256^R,5 \,.\, \cos. 2\psi$; and by putting successively $\psi = 47^\circ,7963$, $\psi = 53^\circ,0926$ [2009viii, 2009x], we get $25632^R,1$, $25674^R,7$, for the degrees of Italy and Austria, instead of the observed values [2033a] $25640^R,55$, $25683^R,30$ [2010]; the differences being $8^R,45$, $8^R,60$, as in [2032v].

† (1481a) The imperfections of the measure at the Cape of Good Hope [2009d, &c.], render it wholly unfit for the purpose for which it is here used; and we have not, in the [2033b] southern hemisphere, any observations made with sufficient accuracy to justify the assertion, that there is any essential difference in the figure of the meridians, in the northern and southern hemispheres.

of its surface, the most simple and natural supposition is that of an ellipsoid of revolution. Making use of this hypothesis, a quadrant of the meridian
[2033vii] will be nearly equal to one hundred times the arc included between Dunkirk and Montjoui, divided by the number of its degrees, if the middle of the arc correspond to 50° of latitude:* but it is a little to the north of it; therefore
[2033viii] we must apply a small correction, depending upon the oblateness of the earth. We have selected the ellipticity which results from the comparison of the arc measured in France, with that at the equator. This last measure is preferred to any other, on account of its position, its distance, and extent; as well as for the care which several excellent observers have taken in
[2034] measuring it. The ellipticity derived from this comparison is $\frac{1}{334}$.† Hence the quadrant of the meridian, deduced from the arc measured between

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* (1482) This evidently appears, by supposing the errors $x^{(2)}$, $x^{(1)}$, to be nothing, and $\theta^{(1)}$, $\theta^{(2)}$, to be the latitudes of Montjoui and Dunkirk. For the arc of the meridian, contained between the parallels of latitude of those places [2023], becomes

[2033c] $$s\,.\,\{\theta^{(2)}-\theta^{(1)}-\tfrac{3}{2}\rho\,.\,\sin.\,(\theta^{(2)}-\theta^{(1)})\,.\,\cos.\,(\theta^{(2)}+\theta^{(1)})\}.$$

Now if the latitude $\theta^{(2)}$ be as much above 50°, as $\theta^{(1)}$ is below it, we shall have $\cos.\,(\theta^{(2)}+\theta^{(1)})=\cos.\,100^{\circ}=0$; and the preceding expression will be reduced to $s\,.\,\{\theta^{(2)}-\theta^{(1)}\}$. Multiplying this by 100°, and dividing the product by the difference of
[2033d] the two latitudes $\theta^{(2)}-\theta^{(1)}$, according to the directions [2033vii], it becomes $100^{\circ}.s$; and as s is by hypothesis [2020′] equal to the mean length of a degree, $100^{\circ}.s$ must be the length of the whole quadrant, which is therefore found correctly in the above hypothesis. This value of s corresponds with the assumed value of c [1590a].

† (1483) If in the expression of the length of a degree of the meridian,

$$1^{\circ}.\,(1-\tfrac{1}{2}\alpha)\,.\,(1-\tfrac{3}{2}\alpha\,.\,\cos.\,2\,\psi), \qquad [2032b],$$

[2033e] we put successively, $\psi=0$, $\psi=51^{\circ},3327$, we shall get the lengths of a degree at the equator and in France. Substituting the values [2009v, 2009ix], we have,

[2033f]
$$25538^{R},85=1^{\circ}.\,(1-\tfrac{1}{2}\alpha)\,.\,(1-\tfrac{3}{2}\alpha);$$
$$25658^{R},28=1^{\circ}.\,(1-\tfrac{1}{2}\alpha)\,.\,(1-\tfrac{3}{2}\alpha.\cos.\,102^{\circ},6654)=1^{\circ}.\,(1-\tfrac{1}{2}\alpha)\,.\,(1+\alpha\,.\,0{,}0628).$$

Dividing the second by the first, neglecting α^2, we obtain,

[2033g]
$$\frac{25658{,}28}{25538{,}85}=\frac{1+\alpha\,.\,0{,}0628}{1-\frac{3}{2}\alpha}=1+\alpha\,.\,1{,}5628;$$

hence $\alpha=\frac{1}{334}$.

Dunkirk and Montjoui, is equal to* 2565370^R. *The metre, being the ten millionth part of this length, is therefore equal to* [2034′] Metre.

$$0^R,256537 = 0^{\text{toise}},513074 = 1^{\text{metre}} \qquad [2009''''].$$ [2035]

The toise is that used in the measure of Peru, referred to the temperature of sixteen and a quarter degrees of a mercurial thermometer, divided into a hundred degrees, from the temperature of freezing water to that of boiling water, under a pressure of a column of mercury of the height of seventy-six centimetres. By means of this value, it will be easy to reduce all the preceding measures into metres, and also those which are expressed in toises. Toise. [2035′] [2035″]

Whatever be the figure of the earth, we find by observation, that the degrees decrease, from the poles to the equator, in both hemispheres; which requires a corresponding increase in the radii of the earth; consequently a [2035‴]

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* (1484) Multiplying $\rho = \frac{1}{334}$ [2034], by the radius in degrees $63^\circ,66198$ [1970*h*], we get $\rho = 0^\circ,190605$; substituting this in the last of the equations [2024], neglecting $x^{(5)}$, $x^{(1)}$, we get $10^\circ,748663 + 0^\circ,002010 = 275792^R,36 \,.\, s^{-1}$; whence $s = 25653^R,49$, representing the mean length of a degree [2020′]; therefore the quadrantal arc of the meridian is $100 \,.\, s = 2565349^R$; which differs a little from the estimate in [2034′]. This difference arises from the neglect of the terms of the order ρ^2, as will be evident by repeating the calculation, using the formula [1969*o*], in which those terms are retained. For if we neglect terms of the order ε^3, we may easily reduce [1969*o*] to the following form, [2034*a*] [2034*b*] [2034*c*]

$$k \,.\, (1 + \tfrac{1}{2}\varepsilon + \tfrac{1}{16}\varepsilon^2) \,.\, \{\psi' - \psi - (\tfrac{3}{4}\varepsilon - \tfrac{3}{8}\varepsilon^2) \,.\, (\sin. 2\psi' - \sin. 2\psi) + \tfrac{15}{64}\varepsilon^2 \,.\, (\sin. 4\psi' - \sin. 4\psi)\}.$$ [2034*d*]

Putting, in this expression, $\psi = \theta^{(1)}$, $\psi' = \theta^{(5)}$ [2019], $\varepsilon = \rho$, $k \,.\, (1 + \frac{1}{2}\varepsilon + \frac{1}{16}\varepsilon^2) = s$, and for brevity $A = \sin. 2\,\theta^{(5)} - \sin. 2\,\theta^{(1)} + \frac{5}{8}\sin. 4\,\theta^{(5)} - \frac{5}{8}\sin. 4\,\theta^{(1)}$; then connecting together the terms of the order ρ^2, it will become, [2034*e*]

$$s \,.\, \{\theta^{(5)} - \theta^{(1)} - \tfrac{3}{4}\rho \,.\, (\sin. 2\,\theta^{(5)} - \sin. 2\,\theta^{(1)}) + \tfrac{3}{8} A \,.\, \rho^2\}.$$ [2034*f*]

The term $\frac{3}{8} A \,.\, \rho^2 = -0^\circ,000094$, is easily found, from the values of $\theta^{(1)}$, $\theta^{(5)}$ [2019], and $\rho = \frac{1}{334}$ [2034], observing that it is multiplied by the expression of the radius $63^\circ,66198$ [1970*h*], to reduce it to degrees; connecting this with the two preceding terms, computed in [2034*b*], we get,

$$10^\circ,748663 + 0^\circ,002010 - 0^\circ,000094 = 275792^R,36 \,.\, s^{-1};$$ [2034*g*]

hence $s \,.\, 1^\circ = 25653^R,7$. Multiplying this by 100, we have the whole quadrant of the meridian, $100 \,.\, s$ [2020′], equal to 2565370^R nearly [2034′]. Comparing this with [1969*u*], we get $S = k \,.\, \frac{1}{2}\pi \,.\, (1 + \frac{1}{2}\rho + \frac{1}{16}\rho^2) = 2565370^R$. [2034*h*]

flattening in the direction of the poles. To prove this, we shall suppose, for greater simplicity, that the earth is a spheroid of revolution. In this case, the radius of curvature of the meridian, at the pole, is in the direction
[2035''''] of the axis of revolution; and it continually decreases, till it becomes perpendicular to this axis, when it will be in the plane of the equator. These radii form, by their mutual intersections, the evolute of the terrestrial meridian; of which the first extreme tangent is in the polar axis, and the
[2035$^{\text{v}}$] second is in the axis of the equator. We shall put a and a' for these two tangents, measured from the point of intersection of the polar axis, with the diameter of the equator. Then taking this point, for the centre of the earth; also R, R', for the radii of curvature of the meridian, at the north
[2035$^{\text{vi}}$] pole, and at the equator; r, r', for the radii, drawn from the centre of the earth, to these two points, we shall evidently have,*

[2035a] (1485) Dividing [2034h] by the coefficient of k, namely $1{,}570796\,.\,(1+\frac{1}{2}\rho+\frac{1}{16}\rho^2)$, neglecting ρ^3, and substituting $\rho=\frac{1}{334}$ [2034], we get successively,

$$k=1633165^R\,.\,(1-\tfrac{1}{2}\rho+\tfrac{3}{16}\rho^2)=1633165^R-2444^R+2^R$$

[2035a']
$$=1630723^R=3261446^{\text{toises}}.$$

[2035b] Dividing this by $0{,}513074$, we get in metres $k=6356677^{\text{met.}}$. Hence the oblateness $\frac{1}{334}k=19032^{\text{met.}}$; adding this to k, we get the equatorial radius $k'=6375709^{\text{met.}}$. Some of these quantities differ a little from the values used in [389b], owing to the neglect of small fractional parts in the numerical calculation.

* (1486) In the annexed figure, $PD'B'A'E$ is an arc of the meridian; CP the polar semi-axis; CE the equatorial radius; $GABDH$ the evolute of the meridian; HP the radius of curvature at P; GE the radius of curvature at E; DD', BB', AA', the radii of curvature
[2037a] corresponding to the infinitely near points D', B', A'. The lines Bb, Dd, are drawn perpendicular to PH; and Bb', Dd', perpendicular to CE. About C as a centre, with the radius CD', is described the circular arc $D'B''$, cutting BB' in B''. Then by the above notation [2035$^{\text{vi}}$], we shall have $HP=R$, $EG=R'$, $CP=r$, $CE=r'$,
[2037b] $HC=a$, $CG=a'$. Now it is evident, that

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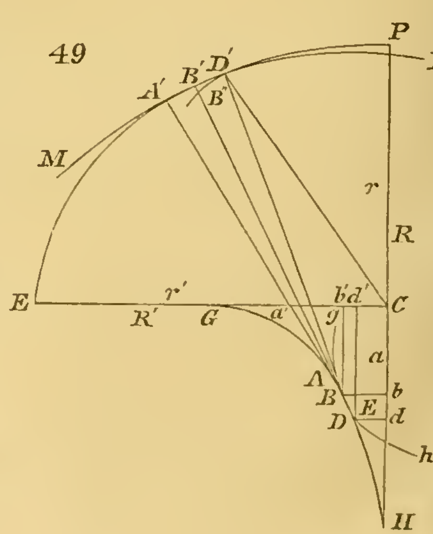

$$r = R - a\,; \qquad r' = R' + a'. \qquad [2036]$$

Hence we deduce,

$$r' - r = a + a' - (R - R'). \qquad [2037]$$

The evolute is convex* towards the polar axis, because the radii of curvature and the degrees of the meridian decrease from the pole to the equator; moreover, the whole arc of the evolute is less than the sum $a + a'$ of the two extreme tangents.† Now $R - R'$ is equal to this arc; therefore [2037′]
$r' - r$ is a positive quantity. If we put r'' for the radius drawn from the [2037″]
centre of the earth to the south pole, we shall find, in like manner, that
$r' - r''$ is positive;‡ therefore $2r'$ is greater than $r + r''$, or the diameter [2037‴]
of the equator exceeds the polar axis; in other words, the earth is flattened at the poles.

If we consider an infinitely small arc of the meridian as an arc of a circle; and suppose the circumference, of which this arc makes a part, to

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$CP = HP - HC$, $CE = EG + CG$; and if we substitute the preceding values of these quantities, we shall get the equations [2036]. Subtracting the first of these from the second, we obtain [2037].

* (1487) It is easy to prove that the evolute HBG is *convex* towards HC, if the [2037c]
radii of curvature HP, DD', GE, decrease from the pole to the equator. For if we suppose the involute $PB'E$ to be described by the winding of a thread about the evolute HDG, from H towards D; and that, at the moment of arriving at D, the evolute becomes *concave* of the form hDg; the radius DD', instead of *decreasing*, by winding about the arc DB, will *increase*, by unwinding from the arc Dh. Hence the [2037d]
radius of curvature will *increase*, in proceeding towards the equator; and as this is contrary to observation, the supposition of a *concave* evolute is inadmissible.

† (1488) In the rectangular triangle BED, we have the hypotenuse BD less than the sum of the two sides BE, DE, or $b'd'$, bd; that is, $BD < bd + b'd'$. Taking the integrals of these quantities, from the point H to G of this convex evolute [2037c], we shall get $\int BD < \int bd + \int b'd'$; but $\int BD = \text{arc}\, HG$, $\int bd = HC = a$,
$\int b'd' = CG = a'$; hence arc $HG < a + a'$, as in [2037′]. But by the nature of the [2037e]
evolute, we have arc $HG = HP - EG = R - R'$; hence $R - R' < a + a'$, or
$a + a' - (R - R') > 0$. Substituting this in [2037], we get $r' - r > 0$ [2037″]. [2037f]

‡ (1489) In [2037f] we have $r' - r > 0$; and in like manner we get $r' - r'' > 0$.
The sum of these two expressions gives $2r' - r - r'' > 0$, or $2r' > r + r''$. Hence [2037g]
the equatorial diameter $2r'$ exceeds the polar axis $r + r''$.

be described; the extremity of this infinitely small arc, nearest to the pole,
[2037''''] will be nearer than the other extremity to that point of the circumference of the circle which is nearest the centre of the earth.* Hence it is easy to perceive, that the radius of the earth, drawn from its centre to the first
[2037v] extremity, is less than the radius drawn to the second extremity; or in other words, the terrestrial radii increase from the poles to the equator.

[2037vi] Since $a+a'$ is less than $2.(R-R')$,† we shall have $r'-r$ less than $R-R'$; therefore the difference between the equatorial and the polar radius of the earth is less than the difference of the corresponding radii of curvature; consequently the degrees of the meridian increase, from
[2037vii] the equator to the poles, in a greater ratio than that in which the terrestrial radii decrease. It is easy to extend the same reasoning to the case where the earth is not a solid of revolution.

42. *We shall now consider the observed lengths of pendulums vibrating*
[2037viii] *in a second of time.* Among them we shall select the fifteen following.‡

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* (1490) We shall suppose $MA'B'D'N$ to be the circumference of a circle, whose centre B is the same as the centre of curvature of the arc $A'B'D'$, and N the point of this circle which is nearest to the centre of the earth C, so that the lines drawn from C to
[2037h] different points of this circle, increase with the distance from the point N. Then it is evident that the extremity D' of the arc $A'D'$, nearest to the pole P, is nearer to the point N than the other extremity A' of the same arc; hence it follows, that CD' is less than CB', as in [2037''''].

† (1491) Having $bd<BD$, and $b'd'<BD$, their sum gives $bd+b'd'<2BD$; and their integrals, taken from the point H to G, give $\int bd+\int b'd'<2.\int BD$; hence
[2037i] $a+a'<2.\text{arc}\,HG<2.(R-R')$ [2037e, f]. Substituting this expression of $a+a'$ in [2037], we get $r'-r<R-R'$ [2037vi].

‡ (1491a) Since the first publication of this work, many improvements have been made in the methods of taking these observations, and in reducing them, on account of the differences of temperature, density, &c., by Captain Kater, Mr. Bessel, and others. We shall in the first place go through the calculations of the author, with the observations he has selected, and shall correct his results for some errors of calculation; we shall then give a
[2038a] collection of the most important modern observations, and shall combine them together, upon the principles explained in the preceding articles, so as to obtain the ellipticity of the earth, which agrees best with the whole of them, taken collectively.

The *two first* have been determined by Bouguer, the one at the equator in Peru, the other at Porto-Bello; the *third* at Pondicherry, by Le Gentil; the *fourth* has been deduced from that of London, by comparing the number of oscillations of an invariable pendulum, transported by Campbell from London to Jamaica; the *fifth* has been determined by Bouguer, at Petit-Goave; the *sixth* by La Caille, at the Cape of Good Hope; the *seventh* by Darquier, at Toulouse; the *eighth* by Liesganig, at Vienna, in Austria; the *ninth* at Paris, by Bouguer; the *tenth* at Gotha, by Zach; the *eleventh* has been deduced from that of Paris, by the difference of oscillations of an invariable pendulum, transported from London to Paris; the *twelfth* and *fourteenth* have been deduced, in the same manner, from that of Paris, by the observations of Mallet, at Petersburgh and Ponoi; the *thirteenth* has been obtained, in a similar way, from that of Paris, by Grischow, at Arensgberg; lastly, the *fifteenth* has been determined, in like manner, by the French Academicians, [Maupertuis, Clairaut, &c.], who measured the degree of the meridian in Lapland.

Observations of the pendulum. [2037ix] [2037x]

The nine absolute measures have the advantage of being determined by the same method; which consists in observing the oscillations of a weight, suspended at the lower extremity of a very fine thread, or wire, of about a metre in length, and supported by a clasp at the other end.* All these measures may be considered as having been observed at the level of the sea. [2037xi]

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* (1491*b*) In the Philosophical Transactions of the Royal Society of London, for 1818, Captain Kater has given an improved method of suspension, depending on the well known principle, that if the point of suspension be changed to its centre of oscillation, its former point of suspension will become the new centre of oscillation. [2038*b*]

Remarkable property of an inverted pendulum.

This property of a pendulum is easily deduced from the expression of the distance l, of the centre of oscillation from the point of suspension, given in [293′], $l=\frac{C}{mh}$; in which m is the mass of the body; h the distance of its centre of gravity from the point of suspension; $C=\int dm.(z''^2+y''^2)$ its momentum of inertia, about the axis of x'' [288′, &c.]; z'', y'', the rectangular co-ordinates of the particle dm. The axis of z'' being drawn from the origin, or point of suspension, through the centre of gravity. When the centre of oscillation is taken for the point of suspension, h changes into $l-h$; z'' into $l-z''$; y'' remaining unchanged; and if we suppose l, C, to become l', C', for this new point of suspension, we shall have, from [2038*c*], $l'=\frac{C'}{m.(l-h)}$. The value of C' is easily deduced from C [2038*d*], by substituting $l-z''$ for z'', and observing, that by the nature [2038*c*] [2038*d*] [2038*e*] [2038*f*]

I have reduced them to a vacuum, and to the same temperature; so that even upon the supposition that there is some uncertainty in the absolute
[2037xii] length of the pendulum, vibrating in a second, yet the uniformity of the method would give with precision the law of the variations of this length, which is one of the principal objects of inquiry. The other eight measures have been deduced, by comparing an invariable pendulum, observed at Paris, and then transported into the places corresponding to these measures.

I have referred these measures to the length of the pendulum observed at Paris, by Bouguer, taking it for unity. These ratios express also the ratio
[2037xiii] of the weight of a body, transported successively to those several places, to its weight at Paris, taken for the unity of weight.

Observations of a pendulum vibrating in one second.

[2038]

Places.	Latitudes.	Lengths of a second pendulum.
Peru, - - - - - -	0°,00 - - - - - -	0,99669,
Porto Bello, - - -	10 ,61 - - - - - -	0,99689,
Pondicherry, - -	13 ,25 - - - - - -	0,99710,
Jamaica, - - - -	20 ,00 - - - - - -	0,99745,
Petit-Goave, - - -	20 ,50 - - - - - -	0,99728,
Cape of Good Hope,	37 ,69 - - - - - -	0,99877,
Toulouse, - - -	48 ,44 - - - - - -	0,99950,
Vienna, - - - -	53 ,57 - - - - - -	0,99987,
Paris, - - - - -	54 ,26 - - - - - -	1,00000,
Gotha, - - - -	56 ,63 - - - - - -	1,00006,
London, - - - -	57 ,22 - - - - - -	1,00018,
Petersburgh, - - -	64 ,72 - - - - - -	1,00074,
Arensgberg, - - -	66 ,60 - - - - - -	1,00101,
Ponoi, - - - - -	74 ,22 - - - - - -	1,00137,
Lapland, - - - -	74 ,53 - - - - - -	1,00148.

The equations [1970] corresponding to these observations, are

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of the centre of gravity, we have $\int z''.dm = h.m$, and $C = lh.m$, [2038c]; whence we obtain,

$$C' = \int dm.(l^2 - 2lz'' + z''^2 + y''^2) = l^2.\int dm - 2l.\int z''.dm + \int dm.(z''^2 + y''^2)$$

[2038g] $$= l^2.m - 2lh.m + C = l^2.m - 2lh.m + lh.m = lm.(l-h).$$

[2038h] Substituting this in l' [2038f], we get $l' = \frac{lm.(l-h)}{m.(l-h)} = l$; which was to be proved.

Resulting equations.

$$\begin{aligned}
x^{(1)} &= 0{,}99669 - z - y\,.\,0{,}00000,\\
x^{(2)} &= 0{,}99689 - z - y\,.\,0{,}02752,\\
x^{(3)} &= 0{,}99710 - z - y\,.\,0{,}04270,\\
x^{(4)} &= 0{,}99745 - z - y\,.\,0{,}09549,\\
x^{(5)} &= 0{,}99728 - z - y\,.\,0{,}10016,\\
x^{(6)} &= 0{,}99877 - z - y\,.\,0{,}31142,\\
x^{(7)} &= 0{,}99950 - z - y\,.\,0{,}47551,\\
x^{(8)} &= 0{,}99987 - z - y\,.\,0{,}55596, \qquad (A'')\\
x^{(9)} &= 1{,}00000 - z - y\,.\,0{,}56672,\\
x^{(10)} &= 1{,}00006 - z - y\,.\,0{,}57624,^{*}\\
x^{(11)} &= 1{,}00018 - z - y\,.\,0{,}61244,\\
x^{(12)} &= 1{,}00074 - z - y\,.\,0{,}72307,\\
x^{(13)} &= 1{,}00101 - z - y\,.\,0{,}74909,\\
x^{(14)} &= 1{,}00137 - z - y\,.\,0{,}84478,\\
x^{(15)} &= 1{,}00148 - z - y\,.\,0{,}84829.
\end{aligned}$$ [2039]

The two series [1985] become,†

$x^{(1)}$,	$x^{(3)}$,	$x^{(4)}$,	$x^{(6)}$,	$x^{(13)}$,	$x^{(15)}$,	
∞,	$+0{,}0096019$,	$+0{,}0066300$,	$+0{,}0061131$,	$+0{,}0051180$,	$+0{,}0047379$,	.....$-\infty$; [2040]

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* (1492) There is a mistake in this coefficient of y; it ought to be 0,60339, instead of 0,57624. Probably this arose from taking out the log. sine of the latitude 56° 63′, equal to 9,8802999, instead of 9,8902999; a mistake which is very easily made in using Callet's tables of log. sines of the centesimal division; because the three first figures 9,89 are marked at the top of the page, and 9,88 at the bottom. We shall, in the first place, go through the calculation, using the same numbers as the author; and shall afterwards, in notes 1501—1505, point out the corrections to be made for this mistake. All the other coefficients of y [2039] agree with the squares of the sines of the corresponding latitudes [2038]. The numbers in the first column of the system [2039], are the lengths of the pendulum [2038], the whole being the same as in [1970, 2009′]. [2039a] [2039b]

† (1493) Subtracting the first of the equations [2039] from the other equations of that system, we shall obtain the values of $x^{(2)} - x^{(1)}$, $x^{(3)} - x^{(1)}$, &c., as in the column C of the following table, corresponding to [1977]. Dividing the constant terms of this system, by the coefficients of y, we obtain the series [1978] $\frac{20}{2752}$, $\frac{41}{4270}$, $\frac{76}{9549}$, &c.; of which the second term is the greatest, and therefore must be put equal to $\beta^{(1)}$ [1979], or [2040a]

$$\beta^{(1)} = \frac{a^{(r)} - a^{(1)}}{p^{(r)} - p^{(1)}} = \frac{41}{4270} = 0{,}0096019, \qquad [2040b]$$

corresponding to $r = 3$, or $x^{(3)}$. Hence the value of $x^{(3)}$ must be subtracted from those of $x^{(4)}$, $x^{(5)}$, &c., [2039], to obtain $x^{(4)} - x^{(3)}$, $x^{(5)} - x^{(3)}$, &c., [1980],

and the two series [1988] become,*

[2041]

$$x^{(1)}, \qquad x^{(14)}, \qquad x^{(15)},$$

$$-\infty, \qquad +0{,}0055399, \qquad +0{,}0313390, \ldots\ldots +\infty.$$

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as in the column D. Dividing the constant terms of these equations by the corresponding coefficients of y, we obtain the series [1981], represented by $\frac{35}{5279}$, $\frac{18}{5746}$, $\frac{167}{26872}$, &c.; of which the first is the greatest, and must be put equal to $\beta^{(2)}$ [1982]. Hence

[2040c]
$$\beta^{(2)} = \frac{a^{(r')} - a^{(3)}}{p^{(r')} - p^{(3)}} = \tfrac{35}{5279} = 0{,}0066300,$$

corresponding to $r' = 4$, and $x^{(4)}$. Therefore we must subtract the fourth equation from the fifth, sixth, &c., [2039], to obtain $x^{(5)} - x^{(4)}$, $x^{(6)} - x^{(4)}$, &c., as in the column E. Dividing the constant terms by the coefficients of y, we obtain the series $-\frac{17}{467}$, $\frac{132}{21593}$, &c.; of which the second is the greatest, and must be put equal to $\beta^{(3)}$,

[2040d] as in [1984]. Hence $\beta^{(3)} = \dfrac{a^{(r'')} - a^{(4)}}{p^{(r'')} - p^{(4)}} = \tfrac{132}{21593} = 0{,}0061131$, making $r'' = 5$,

and corresponding to $x^{(5)}$. Then subtracting the fifth equation [2039] from the following ones, we obtain $x^{(6)} - x^{(5)}$, $x^{(7)} - x^{(5)}$, &c., as in the column F. Dividing by the

[2040e] coefficients, as before, we have $\frac{73}{16409}$, $\frac{110}{24454}$, &c.; the greatest of which is the seventh, making $\beta^{(4)} = \frac{224}{43767} = 0{,}0051180$, corresponding to $r'''' = 13$, and $x^{(13)}$. Lastly, subtracting the thirteenth equation [2039] from the following, we obtain the series G; and dividing by the coefficients, we have $\frac{36}{9569}$, $\frac{47}{9920}$; the last of which, being the greatest. is put equal to $\beta^{(5)} = \frac{47}{9920} = 0{,}0047379$, corresponding to $x^{(r)} = 15$ and $x^{(15)}$,

[2040f] Substituting these values of $\beta^{(1)}$, $\beta^{(2)}$, &c.; $x^{(r)}$, $x^{(r')}$, &c., in [1985], it becomes as in [2040].

[2040g]

	C.	D.	E.	F.	G.
$x^{(n)}$	$x^{(n)} - x^{(1)} =$	$x^{(n)} - x^{(3)} =$	$x^{(n)} - x^{(4)} =$	$x^{(n)} - x^{(6)} =$	$x^{(n)} - x^{(13)} =$
$x^{(2)}$	$0{,}00020 - y.0{,}02752$				
$x^{(3)}$	$0{,}00041 - y.0{,}04270$				
$x^{(4)}$	$0{,}00076 - y.0{,}09549$	$0{,}00035 - y.0{,}05279$			
$x^{(5)}$	$0{,}00059 - y.0{,}10016$	$0{,}00018 - y.0{,}05746$	$-0{,}00017 - y.0{,}00467$		
$x^{(6)}$	$0{,}00208 - y.0{,}31142$	$0{,}00167 - y.0{,}26872$	$+0{,}00132 - y.0{,}21593$		
$x^{(7)}$	$0{,}00281 - y.0{,}47551$	$0{,}00240 - y.0{,}43281$	$0{,}00205 - y.0{,}38002$	$0{,}00073 - y.0{,}16409$	
$x^{(8)}$	$0{,}00318 - y.0{,}55596$	$0{,}00277 - y.0{,}51326$	$0{,}00242 - y.0{,}46047$	$0{,}00110 - y.0{,}24454$	
$x^{(9)}$	$0{,}00331 - y.0{,}56672$	$0{,}00290 - y.0{,}52402$	$0{,}00255 - y.0{,}47123$	$0{,}00123 - y.0{,}25530$	
$x^{(10)}$	$0{,}00337 - y.0{,}57624$	$0{,}00296 - y.0{,}53354$	$0{,}00261 - y.0{,}48075$	$0{,}00129 - y.0{,}26482$	
$x^{(11)}$	$0{,}00349 - y.0{,}61244$	$0{,}00308 - y.0{,}56974$	$0{,}00273 - y.0{,}51695$	$0{,}00141 - y.0{,}30102$	
$x^{(12)}$	$0{,}00405 - y.0{,}72307$	$0{,}00364 - y.0{,}68037$	$0{,}00329 - y.0{,}62758$	$0{,}00197 - y.0{,}41165$	
$x^{(13)}$	$0{,}00432 - y.0{,}74909$	$0{,}00391 - y.0{,}70639$	$0{,}00356 - y.0{,}65360$	$0{,}00224 - y.0{,}43767$	
$x^{(14)}$	$0{,}00468 - y.0{,}84478$	$0{,}00427 - y.0{,}80208$	$0{,}00392 - y.0{,}74929$	$0{,}00260 - y.0{,}53336$	$0{,}00036 - y.0{,}09569$
$x^{(15)}$	$0{,}00479 - y.0{,}84829$	$0{,}00438 - y.0{,}80559$	$0{,}00403 - y.0{,}75280$	$0{,}00271 - y.0{,}53687$	$0{,}00047 - y.0{,}09920$

* (1494) The series [2040a] corresponds to [1986], and the least term is put equal to

[2041a] $\lambda^{(1)}$ [1986′]; hence $\lambda^{(1)} = \dfrac{a^{(s)} - a^{(1)}}{p^{(s)} - p^{(1)}} = \tfrac{468}{84478} = 0{,}0055399$, corresponding to the

We easily find, from § 39, that* $y = 0{,}0055399$. We then get, [2042]

$$x^{(6)} = -x^{(1)} = -x^{(14)} = 0{,}00018\,; \quad [2043]$$

$$z = 0{,}99687. \quad [2043']$$

Therefore, in whatever manner we combine the fifteen preceding measures, we cannot avoid an error of at least $0{,}00018$, in the hypothesis that the variations of gravity increase from the equator to the poles, in proportion to [2043''] the square of the sine of the latitude. This quantity is within the limits of the errors to which these observations are liable, and we see that it is much less than the corresponding error of the measures of the degrees of the meridian. This confirms what the theory has indicated; namely, that the [2043'''] terms of the expression of the terrestrial radius, which cause the earth to vary from an elliptical figure, are much less sensible in the lengths of a pendulum, than in the lengths of the degrees of the meridian [1777''].†

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fourteenth term, or $s = 14$, and $x^{(14)}$. Subtracting the fourteenth equation [2039] from the fifteenth, we get $0{,}00011 - 0{,}00351\,.\,y = x^{(15)} - x^{(14)}$; and as this is the only remaining equation, we shall obtain $\lambda^{(2)}$ [1987'], in dividing by the coefficient of y, as above. Hence $\lambda^{(2)} = \frac{11}{351} = 0{,}0313390$, corresponding to $s' = 15$, and $x^{(15)}$. Substituting [2041b] this in [1988], we get the series [2041].

* (1495) Proceeding according to the method explained in [1991''''], we shall suppose that $y = \lambda^{(1)} = 0{,}0055399$ [2041a]. Substituting this in the thirteenth equation of the [2042a] preceding column C [2040g], $x^{(14)} - x^{(1)} = 0{,}00468 - y\,.\,0{,}84478$, its second member vanishes, and we get $x^{(1)} = x^{(14)}$ for the two extreme *negative* errors, corresponding to this [2042b] value of y. Comparing these with $x^{(i)}$, $x^{(i'')}$, [1976''], we get $i = 1$, $i'' = 14$; and if this value of y be rightly assumed, the other equal positive error $x^{(i')}$, must have for its exponent i' [1976'''], one of the intermediate terms of the series of exponents [2040, 1985], $r = 3$, $r' = 4$, $r'' = 6$, $r''' = 13$. Now $x^{(r'')} = x^{(6)}$ [1985''] is the greatest positive [2042c] error, while y falls between $\beta^{(3)} = 0{,}0061131$ and $\beta^{(4)} = 0{,}0051180$ [2040d, e]; and as this is the case with the value assumed in [2042a], we shall have $x^{(6)} = -x^{(1)} = -x^{(14)}$, [2042d] as in [2043]. Substituting this value of $x^{(1)}$, and y [2042a], in the fifth equation C [2040g], we get successively,

$$x^{(6)} - x^{(1)} = 2\,x^{(6)} = 0{,}00208 - y\,.\,0{,}31142 = 0{,}00208 - 0{,}00172 = 0{,}00036\,; \quad [2042e]$$

hence we obtain $x^{(6)}$, $x^{(1)}$, as in [2043]. Lastly, the first of the equations [2039] gives $z = 0{,}99669 - x^{(1)} = 0{,}99669 + 0{,}00018 = 0{,}99687$, as in [2043'].

† (1496) The minimum value of the greatest error of the measured degrees is $48^{R},60$ [2014], and the mean length of a degree $25653^{R},70$ [2034g]; hence this error, [2043a]

We have seen in § 34, that in the elliptical hypothesis, the oblateness of
[2043''''] the earth is found by subtracting y, from the product of $\frac{5}{2}$ by the ratio of the centrifugal force to gravity ;* this ratio is $\frac{1}{289}$ [1594a] ; therefore,

Formula for the oblateness.

[2043ᵛ] $$\text{The oblateness} = 0{,}00865 - y.$$

[2044] Substituting the preceding value of y, we get $\frac{1}{321{,}48}$ for the ellipticity of the earth, which makes the greatest error of the preceding measures a minimum.

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expressed in parts of a degree, is $\frac{4860}{2565370} = 0{,}00189$, which is ten times as great as the maximum error of the pendulum [2043]. This result is not however confirmed by later observations ; for we shall see in [2055z, &c.], that the discrepancies among the observations
[2043b] of the length of the pendulum, are greater than in those of the best observations of the measured arcs of the meridian.

[2044a] * (1497) The general expression of the length of a pendulum [1969b] is $z + y.\sin.^2\text{lat.}$ At the equator this becomes z, and at the pole it is $z + y$; their difference y is equal to $\alpha\varepsilon$ [1804''] ; the length of the pendulum at the equator being unity ; but $\alpha\varepsilon$, or y, being
[2044b] of the order α, we may, by neglecting terms of the order α^2, suppose the length of the pendulum at Paris to be equal to unity, as in [2037ˣⁱⁱⁱ] ; and the ellipticity αh, deduced
[2044c] from [1806], is $\alpha h = \frac{5}{2}\alpha\varphi - \alpha\varepsilon = \frac{5}{2}\alpha\varphi - y$. Now $\alpha\varphi$ [1771'] represents the ratio of the centrifugal force to the gravity at the equator, and it is computed in [1787c] to be equal to $0{,}0034675 = \frac{1}{288{,}4}$. If we had compared the centrifugal force with the gravity
[2044d] at the pole, it would have decreased the value of $\alpha\varphi$, in the ratio of the polar to the equatorial radius, or in the ratio $\frac{230}{231}$, nearly [1592''] ; by which means it would become
[2044e] $0{,}0034525 = \frac{1}{289{,}6}$. Now as terms of the order α^2 are neglected in [2044c], we shall come within the limits of the accuracy of this formula, by using either of these values of $\alpha\varphi$; and it is generally best in such cases to use the mean value, as we have seen in many
[2044f] instances, [1875h, 1891', 1965m] ; therefore we may put $\alpha\varphi = \frac{1}{289}$, or $\frac{5}{2}\alpha\varphi = 0{,}00865$, and then the oblateness [2044c] becomes as in [2043ᵛ] ; or as it may be written,

Formula to compute the oblateness by observations of a pendulum.

[2044g] $$\text{The oblateness} = 0{,}00865 - \frac{\text{Excess of the length of the pendulum at the pole above that at the equator}}{\text{Length of the pendulum}}.$$

The length of the pendulum [1804''], to be used in this formula, is that at the equator ; but as quantities of the order α^2 are neglected, it will be within the limits of the errors of the formula to use this length for any latitude whatever, from the equator to the pole. If in this we substitute the value of $y = 0{,}00554$, nearly [2042], we shall get,

$$\text{The oblateness} = 0{,}00865 - 0{,}00554 = 0{,}00311 = \tfrac{1}{321}, \qquad \text{nearly as above.}$$

We shall now compute, by the method of § 40, the most probable ellipsis which results from these measures. If we add together the equations [2039], and divide their sum by 15, we shall obtain,*

$$0 = 0{,}99923 - z - y \,.\, 0{,}43529 \;; \qquad [2045]$$

which is the equation of condition necessary to make the sum of the errors equal to nothing. The equations [1998] will then become,

$$\begin{aligned}
-0{,}00254 + y \,.\, 0{,}43529 &= x^{(1)}, \\
-0{,}00234 + y \,.\, 0{,}40777 &= x^{(2)}, \\
-0{,}00213 + y \,.\, 0{,}39259 &= x^{(3)}, \\
-0{,}00178 + y \,.\, 0{,}33980 &= x^{(4)}, \\
-0{,}00195 + y \,.\, 0{,}33513 &= x^{(5)}, \\
-0{,}00046 + y \,.\, 0{,}12387 &= x^{(6)}, \\
0{,}00027 - y \,.\, 0{,}04022 &= x^{(7)}, \\
0{,}00064 - y \,.\, 0{,}12067 &= x^{(8)}, \qquad (O'') \qquad [2046] \\
0{,}00077 - y \,.\, 0{,}13143 &= x^{(9)}, \\
0{,}00083 - y \,.\, 0{,}14095 &= x^{(10)}, \\
0{,}00095 - y \,.\, 0{,}17715 &= x^{(11)}, \\
0{,}00151 - y \,.\, 0{,}28778 &= x^{(12)}, \\
0{,}00178 - y \,.\, 0{,}31380 &= x^{(13)}, \\
0{,}00214 - y \,.\, 0{,}40949 &= x^{(14)}, \\
0{,}00225 - y \,.\, 0{,}41300 &= x^{(15)}.
\end{aligned}$$

(1498) The mistake mentioned in [2039a], does not affect the results [2040—2044]. For the coefficients of y, in the equations containing $x^{(10)}$ in columns C, D, E, F, [2040g], become $-y \,.\, 0{,}60339$, $-y \,.\, 0{,}56069$, $-y \,.\, 0{,}50790$, $-y \,.\, 0{,}29197$; and as these numerical values are increased, the corresponding terms of the expressions [2044h]
[1978, 1981, 1983, &c.], will be *decreased*. Now as the *largest* numbers of these series are selected for $\beta^{(1)}$, $\beta^{(2)}$, $\beta^{(3)}$, $\beta^{(4)}$, $\beta^{(5)}$, in the series [2040], they cannot be affected by this error, and all these terms will remain unaltered. In like manner the series [2041], containing the values of $\lambda^{(1)}$, $\lambda^{(2)}$, [2041a, b], is not to be altered. For the terms of the series [2040a], depending on this equation will be changed from $\frac{337}{57624}$ to $\frac{337}{60339} = 0{,}00558$, [2044$i$]
which is greater than the assumed value of $\lambda^{(1)}$ [2041a]; and as $\lambda^{(1)}$ must be the least of these quantities [1985^{v}], the series [2041] will not be affected by this error; consequently the expressions [2042, 2043, 2044], will remain unchanged.

* (1499) The sum of the equations [2039] put equal to nothing, as in [1997], gives $14{,}98839 - 15\,z - y \,.\, 6{,}52939 = 0$. Dividing this by 15, we get [2045]. Subtracting [2045a]

[2046′] Hence we may easily find, that the series of quantities $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, &c., of § 40, is*

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this successively from each of the equations [2039], we get the system [2046], corresponding to [1998].

* (1500) Comparing the systems of equations [1998, 2046], we get the values of $b^{(1)}$, $b^{(2)}$, &c., $q^{(1)}$, $q^{(2)}$, &c., in columns I, K, of the annexed table. The column L

[2046a] represents the values of $\frac{b^{(1)}}{q^{(1)}}$, $\frac{b^{(2)}}{q^{(2)}}$, &c. These quotients are named $y^{(1)}$, $y^{(2)}$, &c., according to their magnitudes, as in [1998a]. Thus the greatest term of column L, being the quotient of $b^{(7)}$ divided by $q^{(7)}$, is put equal to $y^{(1)}=0{,}0067131$; the next is $y^{(2)}=0{,}0058886$,

[2046b] being the quotient of $b^{(10)}$ by $q^{(10)}$, &c. The exponents of b or q, in this last series, are given in column M. The equations [2046] are arranged in column N, according to the

[2046c] order of the quantities $y^{(1)}$, $y^{(2)}$, &c., as in [1998a, 1999], the signs being changed, when necessary to make the coefficients of y positive [1999, &c.]; observing that $i^{(1)}$, $i^{(2)}$, &c., are all equal to unity [1998″]. The coefficients of y, in column N, are $h^{(1)}$, $h^{(2)}, \ldots\ldots h^{(15)}$. Their sum is equal to F [2004a]; hence $F=4{,}06894$, and

[2046d] $\frac{1}{2}F=2{,}03447$. Now the sum of the first seven coefficients of y, in column N, is 1,80459,

[2046e] representing the sum of $h^{(1)}$, $h^{(2)}, \ldots\ldots h^{(7)}$; which is less than $\frac{1}{2}F$. To this add $h^{(8)}=0{,}41300$, and the sum becomes $2{,}21759>\frac{1}{2}F$; therefore we have, as in [2001], $h^{(r)}=h^{(8)}$, $r=8$; and then, as in [2001′],

[2046f]
$$y=\frac{c^{(8)}}{h^{(8)}}=\frac{b^{(15)}}{q^{(15)}}=0{,}0054479.$$

Substituting this value of y in [2045], we get,

$$z=0{,}99923-y\,.\,0{,}43529=0{,}99923-0{,}00237=0{,}99686.$$

These quantities differ from those given by the author in [2048]; the origin of this difference will be pointed out in the following note.

[2046g]

I.	K.	L.	M.	N [1999].
$b^{(1)}=-0{,}00254$	$q^{(1)}=-0{,}43529$	$y^{(4)}=0{,}0058352$	7	$y\,.\,0{,}04022-0{,}00027=-x^{(7)}$
$b^{(2)}=-0{,}00234$	$q^{(2)}=-0{,}40777$	$y^{(6)}=0{,}0057385$	10	$y\,.\,0{,}14095-0{,}00083=-x^{(10)}$
$b^{(3)}=-0{,}00213$	$q^{(3)}=-0{,}39259$	$y^{(9)}=0{,}0054255$	9	$y\,.\,0{,}13143-0{,}00077=-x^{(9)}$
$b^{(4)}=-0{,}00178$	$q^{(4)}=-0{,}33980$	$y^{(13)}=0{,}0052384$	1	$y\,.\,0{,}43529-0{,}00254=x^{(1)}$
$b^{(5)}=-0{,}00195$	$q^{(5)}=-0{,}33513$	$y^{(5)}=0{,}0058186$	5	$y\,.\,0{,}33513-0{,}00195=x^{(5)}$
$b^{(6)}=-0{,}00046$	$q^{(6)}=-0{,}12387$	$y^{(15)}=0{,}0037136$	2	$y\,.\,0{,}40777-0{,}00234=x^{(2)}$
$b^{(7)}=0{,}00027$	$q^{(7)}=0{,}04022$	$y^{(1)}=0{,}0067131$	13	$y\,.\,0{,}31380-0{,}00178=-x^{(13)}$
$b^{(8)}=0{,}00064$	$q^{(8)}=0{,}12067$	$y^{(11)}=0{,}0053037$	15	$y\,.\,0{,}41300-0{,}00225=-x^{(15)}$
$b^{(9)}=0{,}00077$	$q^{(9)}=0{,}13143$	$y^{(3)}=0{,}0058586$	3	$y\,.\,0{,}39259-0{,}00213=x^{(3)}$
$b^{(10)}=0{,}00083$	$q^{(10)}=0{,}14095$	$y^{(2)}=0{,}0058886$	11	$y\,.\,0{,}17715-0{,}00095=-x^{(11)}$
$b^{(11)}=0{,}00095$	$q^{(11)}=0{,}17715$	$y^{(10)}=0{,}0053627$	8	$y\,.\,0{,}12067-0{,}00064=-x^{(8)}$
$b^{(12)}=0{,}00151$	$q^{(12)}=0{,}28778$	$y^{(12)}=0{,}0052471$	12	$y\,.\,0{,}28778-0{,}00151=-x^{(12)}$
$b^{(13)}=0{,}00178$	$q^{(13)}=0{,}31380$	$y^{(7)}=0{,}0056724$	4	$y\,.\,0{,}33980-0{,}00178=x^{(4)}$
$b^{(14)}=0{,}00214$	$q^{(14)}=0{,}40949$	$y^{(14)}=0{,}0052260$	14	$y\,.\,0{,}40949-0{,}00214=-x^{(14)}$
$b^{(15)}=0{,}00225$	$q^{(15)}=0{,}41300$	$y^{(8)}=0{,}0054479$	6	$y\,.\,0{,}12387-0{,}00046=x^{(6)}$
				$4{,}06894=F.$

0,0067131,	0,0058886,	0,0058586,	0,0058352,	0,0058186,
0,0057385,	0,0056724,	0,0054479,	0,0054255,	0,0053627,
0,0053037,	0,0052471,	0,0052384,	0,0052260,	0,0037136.

[2047]

The equations [2046] correspond to them in the order 7, 10, 9, 1, 5, 2, 13, 15, 3, 11, 8, 12, 4, 14, 6; the sum of the six first is less than the half sum of all these quantities, and the sum of the seven first exceeds that half sum; the seventh quantity corresponds to the thirteenth of the equations [2039]; therefore we have, by § 40, $x^{(13)}=0$, consequently,* [2047′] [2047″]

$$y = 0,0056724\,; \qquad z = 0,99676\,; \qquad [2048]$$

which give $\frac{1}{335,78}$ for the ellipticity of the earth. This agrees, in a remarkable manner, with the ellipticity deduced from the measures in France and at the equator. It appears therefore, by observations of the pendulum, that the earth is much less flattened than in the case of homogeneity; and that the ratio of the axes cannot be supposed greater than that of 320 to 321,† which gives the least errors in the preceding lengths of the pendulum [2044]. The most probable ellipsis, deduced from these [2048′] [2048″]

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* (1501) In addition to the mistake mentioned in [2039a], the author has, in this part of the calculation, made another of considerable importance, by using the numbers $y^{(1)}$, $y^{(2)}$, &c., [2047], in forming the expressions [2001]; instead of $h^{(1)}$, $h^{(2)}$, &c., corresponding to the coefficients of y, in the column N [2046g]. For the whole sum of the quantities [2047] is 0,0824899, its half 0,0412449; the sum of the six first terms is 0,0358526, which is less than the preceding half sum; the sum of the first *seven* terms is 0,0415250, which exceeds this half sum, as is observed in [2047″]. In consequence of this, the author supposes, as in [2001′], that the error of the *seventh* equation of the column N [2046g] is equal to nothing, or $x^{(13)}=0$, making $y=y^{(7)}=0,0056724$; and then, from [2045], $z=0,99676$, as in [2048]. Both these values are erroneous, and the error affects the formulas [2049, 2052, 2053, 2054], and also the ellipticity of the earth, computed in [2048′], by means of the formula [2043$^{\text{v}}$], using y [2048]; from which he gets, [2048a] [2048b] [2048c]

$$0,00865 - y = 0,0029776 = \tfrac{1}{335}. \qquad [2048d]$$

This ellipticity becomes $\frac{1}{315}$, when the two preceding errors are corrected, as will be seen in [2054l]. [2048e]

† (1502) We shall find, in [2055q, w, 2056i, o], that the later and more numerous observations of the pendulum make the oblateness greater than is here stated. [2048f]

[2048‴] observations, is that in which the axes are in the ratio of 335 to 336; the expression of the length of the pendulum is then, by what precedes,*

[2049] $$0,99676 + 0,0056724 \,.\, \sin.^2 \psi\,; \qquad (e)$$

ψ being the latitude.

[2050] We must multiply this expression by the actual length of a pendulum at the equator, to obtain its length in any place whose latitude is ψ. Bouguer has found this length at the equator, equal to $0^{\text{met.}},739615$, but there is reason to believe that this method makes it too great, because on account of the thickness of the wire, and the small resistance it opposes to the flexion, the centre of the oscillations must be a little below the point of suspension. Borda, who has determined the length of a pendulum, vibrating in a second, at the observatory of Paris, by a very accurate method, has

[2051] found it equal to $0^{\text{met.}},741887$. Dividing this by

[2052] $$0,99676 + 0,0056724 \,.\, \sin.^2 \psi,$$

[2053] ψ being here the latitude of the observatory, we get $0^{\text{met.}},741905$.† This is the factor by which we must multiply the formula [2049], to obtain the actual length of a pendulum at any place; hence this length is expressed by‡

Length of a pendulum.

[2054] $$0^{\text{met.}},739502 + 0^{\text{met.}},004208 \,.\, \sin.^2 \psi.$$

[2049a] * (1503) The ellipticity is here assumed as in [2048′], and the values of z, y, as in [2048]; these being substituted in the expression of the length of a pendulum $z + y \,.\, \sin.^2 \theta$, or $z + y \,.\, \sin.^2 \psi$ [1969b], it becomes as in [2049]. The corrected value of this expression will be given in [2054m].

† (1504) The latitude of Paris being $\psi = 54^{\circ},26$ [2038], we have,

$$0,0056724 \,.\, \sin.^2 \psi = 0,00321.$$

[2052a] Substituting this in [2049], we get $0,99676 + 0,00321 = 0,99997$ for the length of the pendulum at Paris; and as this length, in metres, was found by observation to be $0^{\text{met.}},741887$, we shall obtain the expression of the formula [2049] in metres, by multiplying it by $\frac{741887}{999970} = 0,741909$; hence it becomes as in [2054] nearly.

[2053a] ‡ (1505) Substituting, in the tenth equation [2039], 0,60339 for 0,57624, we shall correct for the mistake mentioned in [2039a]. By this means the coefficient of $-y$, in the equation [2045a], will be increased, by the difference of these numbers 0,02715, and the coefficient of $-y$, in [2045], will be increased by $\frac{1}{15} \,.\, 0,02715 = 0,00181$; hence that equation will become,

[2054a] $$0 = 0,99923 - z - y \,.\, 0,43710.$$

We shall here remark, that the same anomalies which have been observed in the measures of different degrees of the arc of the meridian included between Dunkirk and Barcelona, arising without doubt from the irregularity of the parts of the earth, are also perceived in the observed lengths of pendulums. For [2054']

Subtracting this successively from the equations [2039], we get the corrected values of [2046], from which we deduce the corrected values of $q^{(1)}$, $q^{(2)}$, &c., inserted in column K′ of the following table; the values of $b^{(1)}$, $b^{(2)}$, &c., in column I′, being the same as in column I of the similar table [2046g]. Hence we find $\frac{b^{(1)}}{q^{(1)}}$, $\frac{b^{(2)}}{q^{(2)}}$, &c., in column L′. These are called according to their magnitudes $y^{(1)}$, $y^{(2)}$, &c., corresponding to the seventh, ninth, first, &c., of the equations [2046], and these equations are arranged, in column N′, according to the magnitudes of $y^{(1)}$, $y^{(2)}$, &c., the coefficients of y being positive. Half the sum of these coefficients is equal to $\frac{1}{2}F = 2{,}04533$. The sum of the first six of these coefficients is $1{,}66364 < \frac{1}{2}F$; the sum of the first seven is equal to $2{,}07483 > \frac{1}{2}F$. Hence, by [2001], we must put the error of the seventh equation of column N′ equal to nothing, or $x^{(15)} = 0$; which gives $y \,.\, 0{,}41119 - 0{,}00225 = 0$, consequently $y = 0{,}005472$. Substituting this in [2054a], we get, [2054b] [2054c] [2054d] [2054e]

$$z = 0{,}99923 - y \,.\, 0{,}43710 = 0{,}99684\,; \qquad [2054f]$$

and the ellipticity [2043^{v}] $0{,}00865 - y = 0{,}003178 = \frac{1}{315}$; so that the ratio of the axes of the earth is nearly as 314 to 315, instead of 335 to 336, found by La Place [2048′]. Now substituting these values of z, y, in the length of a pendulum $z + y \,.\, \sin.^2\psi$ [2049a], it becomes $0{,}99684 + 0{,}005472 \,.\, \sin.^2\psi$, which is the corrected value of [2049]. In the latitude of Paris, where $\psi = 54^{\circ},26$ [2038], this becomes, [2054g] [2054h]

$$0{,}99684 + 0{,}00310 = 0{,}99994\,;$$

and as this, expressed in metres, is $0^{\text{met.}},741887$ [2051], it is evident that the formula [2054h] may be reduced to metres, by multiplying it by $\frac{741887}{999940} = 0^{\text{met.}},741931$, by which means it will become $0^{\text{met.}},739586 + 0^{\text{met.}},004060 \,.\, \sin.^2\psi$, which is to be used instead of [2054]. We shall now, for the convenience of reference, collect together these corrected values of the formulas [2048, 2048‴, 2049, 2054]. [2054i]

Values corrected for the mistakes of the author in his computation.

$$y = 0{,}005472\,; \qquad z = 0{,}99684\,; \qquad [2054k]$$

$$\text{Ellipticity } \tfrac{1}{315}\,; \qquad \text{Ratio of the axes of the earth } \tfrac{314}{315}\,; \qquad [2054l]$$

$$\text{Length of the pendulum } 0{,}99684 + 0{,}005472 \,.\, \sin.^2\psi\,; \qquad [2054m]$$

$$\text{Length of the pendulum in metres } 0^{\text{met.}},739586 + 0^{\text{met.}},004060 \,.\, \sin.^2\psi\,; \qquad [2054n]$$

which are to be used as the corrected results of the calculations of the author [2045—2054]; with the same system of observations [2038].

[2054″] Grischow observed at Petersburg and at Arensgberg, under latitudes differing but very little from each other, and found variations in these lengths, sensibly

[2054o]

P′.	K′.	L′.	M′.	N′.
$b^{(1)} = -0,00254$	$q^{(1)} = -0,43710$	$y^{(3)} = 0,00581$	7	$y \,.\, 0,03841 - 0,00027 = -x^{(7)}$
$b^{(2)} = -0,00234$	$q^{(2)} = -0,40958$	$y^{(5)} = 0,00571$	9	$y \,.\, 0,12962 - 0,00077 = -x^{(9)}$
$b^{(3)} = -0,00213$	$q^{(3)} = -0,39440$	$y^{(9)} = 0,00540$	1	$y \,.\, 0,43710 - 0,00254 = x^{(1)}$
$b^{(4)} = -0,00178$	$q^{(4)} = -0,34161$	$y^{(13)} = 0,00521$	5	$y \,.\, 0,33694 - 0,00195 = x^{(5)}$
$b^{(5)} = -0,00195$	$q^{(5)} = -0,33694$	$y^{(1)} = 0,00579$	2	$y \,.\, 0,40958 - 0,00234 = x^{(2)}$
$b^{(6)} = -0,00046$	$q^{(6)} = -0,12568$	$y^{(15)} = 0,00366$	13	$y \,.\, 0,31199 - 0,00178 = -x^{(13)}$
$b^{(7)} = 0,00027$	$q^{(7)} = 0,03841$	$y^{(4)} = 0,00703$	15	$y \,.\, 0,41119 - 0,00225 = -x^{(15)}$
$b^{(8)} = 0,00064$	$q^{(8)} = 0,11886$	$y^{(10)} = 0,00539$	11	$y \,.\, 0,17534 - 0,00095 = -x^{(11)}$
$b^{(9)} = 0,00077$	$q^{(9)} = 0,12962$	$y^{(2)} = 0,00594$	3	$y \,.\, 0,39140 - 0,00213 = x^{(3)}$
$b^{(10)} = 0,00083$	$q^{(10)} = 0,16629$	$y^{(14)} = 0,00499$	8	$y \,.\, 0,11886 - 0,00064 = -x^{(8)}$
$b^{(11)} = 0,00095$	$q^{(11)} = 0,17534$	$y^{(8)} = 0,00542$	12	$y \,.\, 0,28597 - 0,00151 = -x^{(12)}$
$b^{(12)} = 0,00151$	$q^{(12)} = 0,28597$	$y^{(11)} = 0,00528$	14	$y \,.\, 0,40768 - 0,00214 = -x^{(14)}$
$b^{(13)} = 0,00178$	$q^{(13)} = 0,31199$	$y^{(6)} = 0,00570$	4	$y \,.\, 0,34161 - 0,00178 = x^{(4)}$
$b^{(14)} = 0,00214$	$q^{(14)} = 0,40768$	$y^{(12)} = 0,00525$	10	$y \,.\, 0,16629 - 0,00083 = -x^{(10)}$
$b^{(15)} = 0,00225$	$q^{(15)} = 0,41119$	$y^{(7)} = 0,00547$	6	$y \,.\, 0,12568 - 0,00046 = x^{(6)}$
				$4,09066 = F.$

(1506) Having corrected the calculations of the oblateness of the earth, by the author, using the ancient observations of the pendulum, contained in the table [2038]; we shall now proceed to a more complete investigation of the subject, with the latest and best observations

[2055a] of the lengths of a pendulum, vibrating in one second in different parts of the world. These observations have been made by Biot, Kater, Sabine, Duperrey, Svanberg, Freycinet, Brisbane, Hall, Foster, Goldingham, &c. The details of the observations are scattered in many works; particularly in the Transactions of the Royal Societies of London and Paris, Base du Système Metrique, Sabine's Work on the Figure of the Earth, the London Philosophical

Observations of the pendulum, by Biot, Kater, Sabine, &c.

[2055b] Magazine, the Connoisance des Tems, &c. In the Philosophical Magazine, are several papers on the subject, by Mr. Galbraith, and Mr. Ivory; particularly in the third volume, for the year 1828, where Mr. Ivory has collected the observations most deserving of notice, and

[2055c] has pointed out their discrepancies. We have, in this note, made a free use of this collection of Mr. Ivory, referring also in most cases to the original authorities; observing that there will sometimes be found a slight difference in the results of an observation, when reduced to the same standard and temperature by methods of equal authority, but differing a little from each other. The observations of Captain Sabine, published in his work abovementioned, were afterwards corrected by him in the Philosophical Transactions.

From these various sources, we have collected 52 of the latest and best of the observations made in the places mentioned in the column B of the following table. The latitude ψ,

[2055d] corresponding to any place, is given in column C; the length l of a pendulum vibrating in a second, expressed in English inches, at the standard temperature of 62° Fahrenheit, is in

[2055e] column D; the square of the sine of the latitude is in column E; the fourth power of that sine is in column F; the product of the square of the sine of the latitude, by the quantity

[2055f] $l' = l - 39^{\text{inches}}$, is in column G.

greater than what ought to follow from the preceding law of the variation of the pendulum, from the equator to the poles. [2054‴]

[2055f']

A.	B.	C.	D.	E.	F.	G.	H.	I.	K.	L.		M.		N.	
No.	Stations.	Latitude ψ.	Observed lengths of the pendulum in inches, l.	$\sin.^2\psi$.	$\sin.^4\psi$.	l' is equal to l minus 39 inches. $l'.\sin.^2\psi$.	System of equations [1998] for all the 52 observations.	Values of y, or $y^{(1)}$, $y^{(2)}$, &c. 52 Obs.	Values of $y^{(1)}$, $y^{(2)}$, &c. 44 Obs.	Excess of the calculated lengths, in hundred thousandth parts of an inch. 52 Observ's. Least squ. $z=39{,}01612$. $y=0{,}20208$. $\varepsilon=\frac{1}{288}$.		44 Observ's. Least squ. $z=39{,}01307$. $y=0{,}20644$. $\varepsilon=\frac{1}{297}$.		44 Observ's. Boscovich. $z=39{,}01228$. $y=0{,}20805$. $\varepsilon=\frac{1}{301}$.	
		d m s								—	+	—	+	—	+
1	Rawak,	0 1 34 S.	39,01479	,00000	,00000	,00000	— 0,08655 + y.0,42171	0,2052	0,2029		133	172		251	
2	St. Thomas,	0 24 41 N.	39,02074	,00005	,00000	,00000	— 0,08060 + y.0,42166	0,1911		461					
3	Galapagos,	0 32 19 N.	39,01717	,00009	,00000	,00000	— 0,08417 + y.0,42162	0,1996		103					
4	Maranham,	2 31 43 S.	39,01213	,00195	,00000	,00002	— 0,08921 + y.0,41976	0,2125	0,2092		438		134		56
5	Ascension, (D.)	7 55 48 S.	39,02360	,01903	,00036	,00045	— 0,07774 + y.0,40268	0,1931		363					
6	" (S.)	7 55 12 S.	39,02410	,01899	,00036	,00046	— 0,07724 + y.0,40272	0,1918		414					
7	Sierra Leone,	8 29 28 N.	39,01997	,02180	,00048	,00044	— 0,08137 + y.0,39991	0,2035	0,2013		56	240		315	
8	Trinidad,	10 38 56 N.	39,01888	,03415	,00117	,00064	— 0,08246 + y.0,38756	0,2128	0,2092		414		124		51
9	Bahia,	12 59 1 S.	39,02433	,05052	,00255	,00123	— 0,07701 + y.0,37119	0,2075	0,2046		200	83		154	
10	Madras,	13 4 9 N.	39,02338	,05113	,00261	,00120	— 0,07796 + y.0,37058	0,2104	0,2070		307		25	46	
11	Guam,	13 27 51 N.	39,03423	,05421	,00294	,00185	— 0,06711 + y.0,36750	0,1826		715					
12	Jamaica,	17 56 7 N.	39,03503	,09483	,00899	,00332	— 0,06631 + y.0,32688	0,2029	0,2004		25	238		302	
13	I. of France,	20 9 56 S.	39,04788	,11884	,01412	,00569	— 0,05346 + y.0,30287	0,1765		774					
14	" (D.)	20 9 23 S.	39,04674	,11874	,01410	,00555	— 0,05460 + y.0,30297	0,1802		662					
15	I. of Mowi,	20 52 7 N.	39,04737	,12690	,01610	,00601	— 0,05397 + y.0,29481	0,1831		561					
16	San Blas, (H.)	21 32 24 N.	39,03776	,13480	,01817	,00509	— 0,06358 + y.0,28691	0,2216	0,2153		559		314		257
17	" (F.)	21 32 24 N.	39,03881	,13480	,01817	,00523	— 0,06253 + y.0,28691	0,2179	0,2123		455		209		152
18	Rio Janeiro, (H.)	22 55 22 S.	39,04374	,15170	,02301	,00664	— 0,05760 + y.0,27001	0,2133	0,2084		304		65		7
19	" (F.)	22 55 13 S.	39,04370	,15167	,02300	,00663	— 0,05764 + y.0,27004	0,2134	0,2085		307		68		10
20	Paramatta, (D.)	33 48 43 S.	39,07751	,30966	,09589	,02400	— 0,02383 + y.0,11205	0,2127	0,2035		119	51		80	
21	" (B.)	33 48 43 S.	39,07696	,30966	,09589	,02383	— 0,02438 + y.0,11205	0,2176	0,2066		174		4	25	
22	Port Jackson, (F.)	33 51 34 S.	39,08044	,31042	,09636	,02497	— 0,02090 + y.0,11129	0,1878	0,1879	159		329		357	
23	" (D.)	33 51 40 S.	39,07899	,31045	,09638	,02452	— 0,02235 + y.0,11126	0,2009	0,1961	13		183		212	
24	Cape of G. Hope,	33 55 15 S.	39,07815	,31142	,09698	,02434	— 0,02319 + y.0,11029	0,2103	0,2019		90	79		108	
25	Formentara,	38 39 56 N.	39,09119	,39034	,15237	,03677	— 0,00715 + y.0,03137	0,2279	0,2009		81	54		70	
26	New York,	40 42 43	39,10120	,42544	,18100	,04305	— 0,00014 + y.0,00373		0,2015		89	30		40	
27	Toulon,	43 7 20	39,10996	,46725	,21832	,05138	0,00862 — y 0,04554	0,1893	0,1855		58	43		47	
28	Figeac,	44 36 45	39,11319	,49324	,24329	,05583	0,01185 — y.0,07153	0,1657	0,1183		261		170		171
29	Bordeaux,	44 50 26	39,11301	,49722	,24723	,05619	0,01167 — y.0,07551	0,1545	0,0868		359		271		272
30	Clermont,	45 46 48	39,11809	,51361	,26380	,06065	0,01675 — y.0,09190	0,1823	0,1671		182		101		105
31	Paris,	8 50 14	39,12929	,56677	,32123	,07328	0,02795 — y.0,14506	0,1927	0,1965		136		79		91
32	Shanklin,	50 37 24	39,13606	,59752	,35703	,08130	0,03472 — y.0,17581	0,1974	0,2032		81		36		54
33	Dunkirk,	51 2 10	39,13769	,60457	,36550	,08324	0,03635 — y.0,18286	0,1988	0,2019		60		19		37
34	London,	51 31 8	39,13929	,61280	,37552	,08536	0,03795 — y.0,19109	0,1986	0,2042		67		29		48
35	Falkland I. (D.)	51 31 44 S.	39,13945	,61297	,37573	,08548	0,03811 — y.0,19126	0,1993	0,2052		54		16		36
36	" (F.)	51 35 18 S.	39,13720	,61398	,37697	,08424	0,03586 — y.0,19227	0,1865	0,1857		299		262		282
37	Arbury Hill,	52 12 55 N.	39,14223	,62460	,39013	,08884	0,04089 — y.0,20289	0,2015	0,2080		11	22			0
38	Clifton,	53 27 43	39,14593	,64555	,41673	,09421	0,04459 — y.0,22384	0,1992	0,2039		61		41		66
39	Leith, (B.)	55 58 37	39,15538	,68693	,47187	,10674	0,05404 — y.0,26522	0,2038	0,2090	45		50		18	
40	" (K.)	55 58 41	39,15556	,68695	,47190	,10686	0,05422 — y.0,26524	0,2041	0,2098	62		67		36	
41	Portsoy,	57 40 59	39,16161	,71420	,51008	,11542	0,06027 — y.0,29249	0,2061	0,2113	116		110		74	
42	Stockholm,	59 20 34	39,16541	,74000	,54760	,12240	0,06407 — y.0,31829	0,2013	0,2048		25		37		83
43	I. of Brassa,	60 9 42	39,16929	,75244	,56616	,12738	0,06795 — y.0,33073	0,2055	0,2098	112		89		46	
44	Unst, (B.)	60 45 25	39,17141	,76135	,57965	,13050	0,07007 — y.0,33964	0,2063	0,2107	144		117		73	
45	" (K.)	60 45 28	39,17151	,76137	,57968	,130 8	0,07017 — y.0,33966	0,2066	0,2111	153		126		83	
46	Drontheim,	63 25 54	39,17456	,79995	,63992	,13964	0,07322 — y.0,37824	0,1936	0,1947		321		365		415
47	Hare Isle,	70 26 17	39,19840	,88789	,78835	,17616	0,09706 — y.0,46618	0,2082	0,2115	285		204		139	
48	Hammerfort,	70 40 5	39,19475	,89041	,79283	,17341	0,09341 — y.0,46870	0,1993	0,2011		130		213		278
49	Port Bowen,	73 13 39	39,20347	,91673	,84039	,18653	0,10213 — y.0,49502	0,2063	0,2091	210		115		46	
50	Greenland,	74 32 19	39,20335	,92893	86291	,18890	0,10201 — y.0,50722	0,2011	0,2031		49		149		220
51	Melville Island,	74 47 12	39,20700	,93114	,86702	,19275	0,10566 — y.0,50943	0,2074	0,2129	271		170		99	
52	Spitzbergen,	79 49 58	39,21469	,96884	,93865	,20800	0,11335 — y.0,54713	0,2072	0,2098	279		161		84	

[2055′′′′] These anomalies in the variation of gravity disappear almost entirely, at

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These observations furnish a system of equations, similar to that in [1970], or in [2039], of the following forms:

[2055g]
$$\begin{aligned}
x^{(1)} &= 39^{\text{inches}},01479 - z - 0,00000\,.\,y,\\
x^{(2)} &= 39\quad ,02074 - z - 0,00005\,.\,y,\\
x^{(3)} &= 39\quad ,01717 - z - 0,00009\,.\,y,\\
&\vdots\\
x^{(52)} &= 39\quad ,21469 - z - 0,96884\,.\,y\,;
\end{aligned}$$

$x^{(1)}$, $x^{(2)}$, ... $x^{(52)}$, being the errors of the observations, or rather the differences between [2055h] the calculated and observed lengths of the pendulums. If we compute the values of z, y, upon the principle that the sum of the squares of these errors is a minimum; we must, in the [2055i] first place, put the sum of these equations equal to nothing, as in [815f]; then dividing this sum by 52, the coefficient of z, we shall get the equation marked [2055l]. To obtain a second equation, we must multiply each of the equations [2055g] by the corresponding coefficient of y, and take the sum of these products; then dividing by 21,92880, the [2055k] coefficient of z, we get the equation [2055m]. These products are given in columns D, E, F, G, of the foregoing table.

[2055l] $$0 = 39^{\text{inches}},10134 - z - 0,42171\,.\,y,$$

[2055m] $$0 = 39\quad ,14854 - z - 0,65528\,.\,y.$$

The difference of these two equations is $0 = 0,04720 - 0,23357\,.\,y$; whence we find [2055n] $y = 0,20208$. Substituting this in either of the equations [2055l, m], we shall get [2055o] $z = 39^{\text{inches}},01612$; therefore we have, for the length of a pendulum in any latitude, the [2055p] following expression, $39^{\text{inches}},01612 + 0^{\text{inches}},20208\,.\,\sin.^2\text{lat.}$ [1969b]. The oblateness, deduced from the formula [2044g], is

[2055q] $$0,00865 - \frac{y}{z} = 0,00865 - 0,00518 = 0,00347 = \tfrac{1}{288}.$$

This result agrees nearly with that obtained by Captain Sabine from his observations; but [2055r] differs very much from the value $\frac{1}{310}$, or $\frac{1}{312}$, deduced from the measures of the degrees of the meridian [2017i, y, ʒ, ξ], and from the two lunar equations $\frac{1}{304}$, [5593, 5602].

Computed oblateness by the combination of all the late observations.

The method of Boscovich [1995″, &c.], applied to the same 52 observations, gives a result, which agrees a little better with those obtained from the measured degrees, and from the [2055s] lunar equations. For the equation [1997], in the method of Boscovich, is the same as [2055l]; and by subtracting it successively from each of the equations [2055g], we obtain the system corresponding to [1998], as in column H of the table. Putting successively each [2055t] of these errors $x^{(1)}$, $x^{(2)}$, &c., equal to nothing, we get the corresponding values of $y^{(1)}$, $y^{(2)}$, &c., as in column I of the same table. The equations in column H, are to be arranged

great distances; so that we only perceive the law of variation proportional to

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according to the magnitudes of $y^{(1)}$, $y^{(2)}$, &c., as in [1998″], making the coefficients of y positive [1999′]. Half the sum of all these coefficients is 7,31654, and we have $h^{(1)}+h^{(2)}\ldots\ldots+h^{(23)}=7{,}30117<\tfrac{1}{2}F$, and $h^{(1)}+h^{(2)}\ldots\ldots+h^{(24)}=7{,}70108>\tfrac{1}{2}F$; [2055u]
hence $r=24$ [2061]. The equation corresponding to this value of r, is

$$0=0{,}39991\,.\,y-0{,}08137, \qquad \text{whence} \qquad y=0{,}20347. \qquad [2055v]$$

Substituting this in [2055l], we get $z=39{,}01554$, $\frac{y}{z}=0{,}00521$; consequently the [2055w]
oblateness $0{,}00865-0{,}00521=0{,}00344=\frac{1}{291}$, and the length of the pendulum in any latitude becomes $39^{\text{inches}}{,}01554+0^{\text{inches}}{,}20347\,.\,\sin.^2\,\text{lat.}$ [2055x]

The least value of y, in column I of the table, is $y=0^{\text{inches}}{,}1545$, the greatest is $y=0^{\text{inches}}{,}2279$. If we suppose $z=39^{\text{inches}}{,}016$, which is nearly the value found above, [2055y]
the oblateness $0{,}00865-\frac{y}{z}$ corresponding to these two values of y, will be $\frac{1}{213}$ and $\frac{1}{356}$. [2055z]
These represent the extreme values of the oblateness, corresponding to the whole of these 52 observations, and to this value of z; neglecting the observation made at New York, because the constant term of this equation becomes almost insensible, and falls within the limits of the errors of the observations. With a different value of z, these extreme values [2056a]
of the oblateness would be different, but there would still be found the same great discrepancy between them. Now when observations differ so much from each other, we cannot place great confidence in the accuracy of the result of any combination of them, unless the number of observations be very great; and it is therefore desirable to obtain many more observations, particularly near the equator, where the most remarkable variations have been [2056b]
found. The exact length of the equatorial pendulum is of the greatest importance in this investigation; for it is by comparing it with the observations made near the pole, that we obtain the oblateness. This is evident by the inspection of the coefficients of y, in the equations in column H of the table; which are great near the equator, and in the polar circles; but are small, and almost insensible, in the middle latitudes; so that the observations [2056c]
in the middle latitudes have comparatively but little influence in determining the oblateness.

The excess of the length of the pendulum, computed by the formula [2055p], above the observed length, is given, for each observation, in column L of the table. The discrepancies of these results are very great, particularly in the observations near the equator; and several [2056d]
of them are neglected by Mr. Ivory, in his computation of the oblateness of the earth; because they differ very much from all the other observations; namely, those numbered 2, 3, 5, 6, 11, 13, 14 and 15. If we neglect the same observations, and combine [2056e]
together the remaining 44 equations of the system [2055g], by the principle of the least squares, we shall get, instead of [2055l, m], the two following equations,

Oblateness computed from a selection of the best observations.

$$0=39^{\text{inches}}{,}11381-z-0{,}48800\,.\,y, \qquad [2056f]$$
$$0=39\phantom{^{\text{inches}}}{,}15076-z-0{,}66699\,.\,y. \qquad [2056g]$$

the square of the sine of the latitude. We have seen, in § 33, that if the expression of the radius of the earth be [1775],

[2055] $$1+\alpha.\{Y^{(2)}+Y^{(3)}+Y^{(4)}+\&c.\},$$

[2056h] Hence we get $z=39^{\text{inches}},01307$, $y=0^{\text{inches}},20644$, $\frac{y}{z}=0,00529$; and the
[2056i] corresponding oblateness $0,00865-0,00529=0,00336=\frac{1}{297}$; the length of the
[2056k] pendulum being represented by the formula $39^{\text{inches}},01307+0^{\text{inches}},20644\,.\,\sin.^2\text{lat.}$

If we now combine the same 44 observations by the method of Boscovich, as in [2055s—x], we shall get the values of $y^{(1)}$, $y^{(2)}$, &c., in column K. The values of $h^{(1)}$, $h^{(2)}$, &c., deduced from a system of equations similar to that in column H, and arranged according to the magnitudes of $y^{(1)}$, $y^{(2)}$, &c., give

$$h^{(1)}+h^{(2)}\,.\,.\,.\,.\,.+h^{(44)}=11,21997=F;$$

[2056l] $$h^{(1)}+h^{(2)}\,.\,.\,.\,.\,.+h^{(16)}=5,50677<\tfrac{1}{2}F;$$

$$h^{(1)}+h^{(2)}\,.\,.\,.\,.\,.+h^{(17)}=5,64337>\tfrac{1}{2}F;$$

[2056m] hence $r=17$. The equation corresponding to this value of r, is $0=0,13660.y-0,02842$,
[2056n] whence $y=0^{\text{inches}},20805$; substituting this in [2056f], we get $z=39^{\text{inches}},01228$,
[2056o] $\frac{y}{z}=0,00533$, and the oblateness [2044g] $0,00865-0,00533=0,00332=\frac{1}{301}$. The
[2056p] corresponding length of the pendulum is $39^{\text{inches}},01228+0^{\text{inches}},20805\,.\,\sin.^2\text{lat.}$

The excess of the computed length of the pendulum, above its observed length, is given
[2056q] in column M of the table, for the formula [2056k]; and in column N, for the formula [2056p]. These errors are generally much less than those in column L, where all the observations are included. The mean error of a single observation, in column L, neglecting
[2056r] the signs, is 227, or $0^{\text{inches}},00227$; in column M, $0^{\text{inches}},00124$; and in column N, $0^{\text{inches}},00123$. Hence it appears, that the average error of a single observation, using the oblateness $\frac{1}{288}$, and the length of the pendulum [2055p], deduced from the whole
[2056s] 52 observations, is nearly double of that, where only 44 of the best observations are combined by the method of Boscovich, with the oblateness $\frac{1}{301}$. The mean of the errors given in these columns, is less than the limit of the error to which these observations are liable, when made in the same place, by different observers, and with different instruments. This difference has been found sometimes to be more than $0^{\text{inches}},002$; and
[2056t] we have, at present, no better means of diminishing the effect of such errors, in the computation of the oblateness of the earth, than by increasing greatly the number of observations. In making these calculations, it is best not to restrict ourselves to the observations of one man, or to one set of instruments, in which there may be a constant
[2056u] source of error; but to combine all the best observations together; and if a few should differ very much from the rest, they ought to be wholly rejected. We may remark, that the

the expression of the length of a pendulum, vibrating in a second, will be [1772],

$$L + \alpha L . \{ Y^{(2)} + 2 Y^{(3)} + 3 Y^{(4)} + \&c. \} + \tfrac{5}{2} \alpha \varphi . L . (\mu^2 - \tfrac{1}{3}). \qquad [2056]$$

Now as observations on the pendulum make this length nearly proportional

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errors of the eight anomalous observations which we have rejected are increased about 0,00280, by using the same data as in column M; and they are still further increased about 0,00070, in column N.

From what has been said, we may conclude, that with the observations of the pendulum we now possess, no very great degree of accuracy can be obtained in the determination of the oblateness of the earth; but instead of being dissatisfied with this result, we ought to feel [2056v] some degree of surprise, that by means of the very small excess of the polar over the equatorial pendulum, *which may be considered as a base line, of less than a quarter of an inch in length, we can determine, within a fraction of a mile, the difference between the polar and the equatorial radius of the earth.* Various causes have been assigned, for these differences in the observations of the pendulum; as the local attractions of the neighboring bodies; the [2056w] peculiar action of the substance, composing the stratum of the earth, over which the pendulum is placed; and the magnetic action of the earth upon the pendulum. It has been proposed to ascertain the magnetic action, by making the pendulum vibrate successively in the plane of the magnetic meridian, and in a plane perpendicular to it; in order to ascertain whether there [2056x] is any difference in the times of vibration, in these two situations. Biot suggested, that some part of these differences might possibly be owing to a gradual decrease of the value of y, from the pole to the equator; but it has been shown, by Mr. Ivory, that this apparent [2056y] decrease may be avoided, by changing a little the value of z.

From the preceding observations of the pendulum, it appears, that the oblateness does not differ much from $\frac{1}{300}$, and may possibly be a little more; on the other hand, the measured [2056z] degrees of the meridian, and the two lunar equations, make it less, [2055r]. We may therefore adopt, as very near to the true value, the ratio $\frac{1}{300}$, proposed by La Lande, in his Astronomy, about forty years since; and which I have always used in calculating the moon's parallax in occultations and eclipses. With this ellipticity, using the first of the equations [2017t], we find the polar radius to be 3950 miles, and the equatorial radius 3963 [2056α] miles, nearly.

Mayer's method of combining many observations.

Mayer used another method of combining many equations like those in the system H. It consists in changing the signs of all the equations in which the coefficient of y is negative, then adding the whole of them together, and putting the sum equal to nothing. This process [2056β] is very simple, and requires no explanation; when applied to the system H, it produces nearly the same result as in [2055n], &c.

[2056′] to μ^2, $Y^{(2)}$ must be nearly equal to $-h\,.\,(\mu^2-\frac{1}{3})$;* and as the earth revolves about one of its principal axes, $Y^{(2)}$ must be of the form [1763], $-h\,.(\mu^2-\frac{1}{3})+h''''.\,(1-\mu^2)\,.\,\cos.2\varpi$. Hence the observations upon the pendulum prove, that h'''' is very small in comparison with h; also that
[2056″] $Y^{(3)}$, $Y^{(4)}$, &c., are very small in comparison with $Y^{(2)}$; so that they may be neglected, in the expression of the radius of the earth, and also in those of the gravity and the parallax ;† but at the same time, the different measures
[2056‴] of the degrees of the meridian, indicate that these terms become sensible in the expression of these degrees, on account of the magnitude of the coefficients, by which they are multiplied.

[2056⁗] 43. *We shall now consider Jupiter, whose very perceptible oblateness has been determined with great accuracy.* If we, in the first place, suppose the planet to be homogeneous, we may determine its ellipticity by the equation [1574],

On the oblateness of Jupiter.

[2057]
$$0=\frac{9\lambda+2q\,.\,\lambda^3}{9+3\lambda^2}-\text{arc. tang.}\,\lambda,$$

$\sqrt{1+\lambda^2}$ being the ratio of the axis of the equator to that of the pole [1592′]. To deduce λ from this equation, we must determine q. Now if
[2057′] we put D for the distance of the fourth satellite from the centre of Jupiter, and T for the time of its revolution about the planet, expressed in parts of
[2057″] a day; the centrifugal force‡ of this satellite will be equal to the mass of

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[2057a] * (1507) This is the form under which the terms depending on μ^2 appear, in the general value of $Y^{(i)}$ [1761].

† (1508) The parallax is proportional to the radius [1775]; and if we compare the terms $Y^{(i)}$ of this expression with that of gravity [1769], or with that of the length of a
[2057b] degree [1777], we shall find that they have the factors 1, $i-1$, i^2+i-1, respectively; and when i is 3, or 4, this last term becomes considerably larger than the two others; therefore the effect of this term must be most easily perceived in the measures of the degrees.

‡ (1509) This *centrifugal* force is equal and opposite to the *centripetal* force of the satellite towards Jupiter; and this last force, by the common theory of gravity, is equal to
[2058a] $\frac{M}{D^2}$. Now the general expression of the centrifugal force φ, corresponding to the radius ρ, the velocity v, and time of revolution τ, is $\varphi\propto\frac{v^2}{\rho}$ [54′]. But it is evident, that v is

Jupiter M, divided by D^2. But this centrifugal force, is to the centrifugal force, arising from the rotatory motion of Jupiter, at the distance 1 from the axis of rotation, as $\frac{D}{T^2}$ is to $\frac{1}{t^2}$, t being the time of rotation of Jupiter, expressed in fractions of a day; hence we obtain, [2057‴] [2058]

$$g=\frac{M.T^2}{t^2.D^3}. \qquad [2059]$$

Now we have $M=\frac{4}{3}\pi.k^3.(1+\lambda^2)$ [1567], therefore,* [2060]

$$\frac{g}{\frac{4}{3}\pi}=\frac{k^3.(1+\lambda^2).T^2}{t^2.D^3}=q. \qquad [2061]$$

We shall suppose with Newton, in conformity with the measures of Pound, that the distance of the fourth satellite is equal to 26,63 semi-diameters of the equator of Jupiter; which gives† $\frac{k.\sqrt{1+\lambda^2}}{D}=\frac{1}{26{,}63}$; then we have, [2062]

$$t=0^{\text{day}},41377; \qquad T=16^{\text{days}},68902; \qquad [2063]$$

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directly proportional to the circumference of the described circle, or to its radius, and inversely proportional to the time of revolution; hence $v\propto\frac{\rho}{\tau}$, consequently $\varphi\propto\frac{\rho}{\tau^2}$, or, as it may be expressed, $\varphi=m.\frac{\rho}{\tau^2}$; in which the quantity m may be determined, by observing, that when $\rho=1$, and $\tau=t$, we shall have, as in [1569′], $\varphi=g$; whence $g=m.\frac{1}{t^2}$, or $m=gt^2$. Substituting this in φ, we get $\varphi=gt^2.\frac{\rho}{\tau^2}$. Now putting $\rho=D$, and $\tau=T$, we obtain the centrifugal force of the satellite, $\varphi=gt^2.\frac{D}{T^2}$. Putting this equal to its value $\frac{M}{D^2}$ [2058a], we get $\frac{M}{D^2}=gt^2.\frac{D}{T^2}$; whence we easily obtain g [2059]. [2058b] [2058c]

* (1510) Substituting M [1567] in [2059], putting the density $\rho=1$, dividing by $\frac{4}{3}\pi$, and using the value of q [1573′], we get [2061]. [2061a]

† (1511) The equatorial radius of Jupiter is $k.\sqrt{(1+\lambda^2)}$ [1574a]; and the distance of the fourth satellite D [2057′], expressed in terms of this radius, is $\frac{D}{k.\sqrt{(1+\lambda^2)}}=26{,}63$, as in [2062]. The time t [2063] corresponds to $9^h\ 55^m\ 50^s$. [2062a]

therefore we find,*

[2064] $$q = 0{,}0861450 \,.\, (1+\lambda^2)^{-\frac{1}{2}},$$

Oblateness of Jupiter, supposing it to be homogeneous.

and the equation in λ [2057] becomes,†

[2065] $$0 = 9\lambda + \frac{0{,}172290 \,.\, \lambda^3}{\sqrt{1+\lambda^2}} - (9+3\lambda^2) \,.\, \text{ang. tang.}\, \lambda\,;$$

hence we deduce $\lambda = 0{,}481$, *consequently the axis of the pole being taken*
[2066] *for unity, the axis of the equator will be* 1,10967.

[2066′] According to the observations of Pound, as quoted by Newton, Jupiter's equatorial axis is 1,0771. Short, by observation, made this axis equal to 1,0769. Lastly, from the theory of the motions of the nodes and perijoves of the satellites of Jupiter, it appears that this axis is equal to 1,0747 ;‡ and we shall hereafter see, in the development of this theory, that it is

Measures of Jupiter by Pound and Short.

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* (1512) The third power of [2062], divided by $\sqrt{(1+\lambda^2)}$, gives

[2063a] $$\frac{k^3 \,.\, (1+\lambda^2)}{D^3} = \frac{1}{(26{,}63)^3 \,.\, (1+\lambda^2)^{\frac{1}{2}}}\,;$$

hence [2061] becomes $q = \frac{T^2}{t^2} \,.\, \frac{1}{(26{,}63)^3 \,.\, (1+\lambda^2)^{\frac{1}{2}}}$. Substituting t, T, [2063], we obtain [2064].

† (1513) Multiplying [2057] by $9+3\lambda^2$, and substituting q [2064], we get [2065]. Putting this into numbers, we must express ang. tang. λ in parts of the radius; and if it be given in degrees of the centesimal division, we must divide it by $63^{\circ},66198$ [1970h'], and then it becomes,

[2065a] $$0 = 9\lambda + \frac{0{,}172290 \,.\, \lambda^3}{(1+\lambda^2)^{\frac{1}{2}}} - (9+3\lambda^2) \,.\, 0{,}01570796 \,.\, \text{ang. tang.}\, \lambda.$$

This may be solved by approximation; supposing successively $\lambda = 0{,}482$, $\lambda = 0{,}483$, and using the arithmetical process of double position, we finally get $\lambda = 0{,}4826$, which differs a little from that given by the author [2066]. This value of λ makes the equatorial
[2065b] semi-axis $\sqrt{(1+\lambda^2)} = 1{,}1104$, instead of 1,10967, given by the author [2066], the polar semi-axis being unity.

‡ (1514) This subject is treated of in Book viii, § 27 [7159′], and the ratio of the axes
[2066a] is found to be 0,9286992 to 1, differing a little from that mentioned above. According to the late measures of Struve, the apparent diameters, at the mean distance from the earth, are $35^s{,}538$ and $38^s{,}327$, which are as 0,9272 to 1.

Struve's measures of Jupiter.

determined with much greater accuracy by this method, than by direct admeasurement. *All these results concur in proving that Jupiter is less oblate than in the case of homogeneity; therefore its density, like that of the earth, increases from the surface to the centre.* [2067]

Jupiter must be denser near its centre than at its surface.

We have seen, in § 30, that if the planets, in their primitive state, had been fluid, as is natural to suppose, the limits of their ellipticity would be $\frac{5}{4}\alpha\varphi$ and $\frac{1}{2}\alpha\varphi$ [1732‴]; so that the polar axis being 1, the axis of the equator would be comprised between $1+\frac{5}{4}\alpha\varphi$ and $1+\frac{1}{2}\alpha\varphi$. [2067′] The first of these limits corresponds to the case of homogeneity [1732‴]; and as this limit is, by what precedes, 1,10967 [2066], we have $\frac{5}{4}\alpha\varphi=0{,}10967$, which gives 1,10967 and 1,04387,* for the two [2068] limits between which the axis of the equator must be included. Now the preceding values, which are given both by direct admeasurement, and by the motion of the nodes of the orbits of the satellites of Jupiter, are comprised within these limits; therefore the theory of gravity is, in this respect, [2068′] perfectly accordant with observations.

It follows also from § 30, that if Jupiter and the earth were supposed fluid, [2068″] *and their respective densities, at distances from their centres proportional to their diameters, were in a constant ratio, the law of their ellipticities would be the same;*† and the ellipticity being the excess of the axis of the equator above that of the pole, taken for unity, the ratio of the ellipticity of Jupiter

Important example in which the ellipticities are equal.

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* (1515) If we put $\sqrt{(1+\lambda^2)}=1{,}1104$ [2065*b*] equal to $1+\frac{5}{4}\alpha\varphi$, we shall have $\frac{5}{4}\alpha\varphi=0{,}1104$, whence $1+\frac{1}{2}\alpha\varphi=1{,}04416$; so that the limits are 1,1104 [2066*b*] and 1,04416, which differ a little from [2068].

† (1516) If h, a, ρ, refer to the earth; h', a', ρ', to Jupiter; then the ellipticity αh of [2068*a*] the earth will depend on the equation [1732]; and the ellipticity $\alpha h'$ of Jupiter will depend an equation similar to [1732], in which the symbols a, h, ρ, are changed into a', h', ρ', respectively; by which means it will become,

$$\frac{ddh'}{da'^2}=\frac{6h'}{a'^2}\cdot\left(1-\frac{\rho'.a'^3}{3.\int\rho'.a'^2da'}\right)-\frac{2\rho'.a'^2}{\int\rho'.a'^2da'}\cdot\frac{dh'}{da'}. \qquad [2068b]$$

If we suppose the ratio of the greatest value of a' to the greatest value of a, to be represented by m, and then put generally $a'=ma$, also $\rho'=m'\rho$ [2068″], m, m', being [2068*c*] independent of a, a', we shall have $da'=m.da$, $\int\rho'.a'^2da'=m^3m'.\int\rho.a^2da$, &c. Substituting these in the equation [2068*b*], multiplying by m^2, and reducing, we get,

$$\frac{ddh'}{da^2}=\frac{6h'}{a^2}\cdot\left(1-\frac{\rho.a^3}{3.\int\rho.a^2da}\right)-\frac{2\rho.a^2}{\int\rho.a^2da}\cdot\frac{dh'}{da}. \qquad [2068d]$$

to that of the earth would the same, whatever be the law of the densities.
[2068‴] Now in the case of homogeneity, the ellipticities are, by what has been said, and by § 19, as 0,10967 is to 0,00433441 ;* therefore by supposing the ellipticity of Jupiter equal to 0,0747, as it appears to be, by the motion
[2068⁗] of the nodes of the satellites [2066′, &c.], we shall have $\frac{1}{338{,}72}$ for the ellipticity of the earth, corresponding to the same law of density. This
[2069] ellipticity will be $\frac{1}{323{,}17}$, if we adopt the ellipticity of Jupiter, resulting

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This is of the same form as [1732], and becomes identical by putting $h' = n\,h$, and then dividing by n; observing that this quantity n is supposed to be independent of a. At the
[2068e] surfaces of the bodies, h, h', become $h_{,}$, $h'_{,}$, respectively, [1721d], and $h'_{,} = n\,h_{,}$. Dividing
[2068f] the preceding value of h' by that of $h'_{,}$, we get $\frac{h}{h_{,}} = \frac{h'}{h'_{,}}$. Now accenting the symbols in [1732m‴, n], to obtain the values corresponding to Jupiter, we shall have,

[2068g]
$$H' = \frac{\int_0^1 \rho' . d . \left(a'^5 . \frac{h'}{h'_{,}}\right)}{a'^2 . \int_0^1 \rho' . d\,a'^3};$$

the factor a'^2 without the sign $\int$ in the denominator, being inserted to render it homogeneous in a', as it evidently ought to be, from [1732n]; this factor a'^2 being represented by 1^2, or
[2068h] unity, in the value of H [1732m‴]. Substituting the values of a', ρ', $\frac{h'}{h'_{,}}$, [2068e, f], in H' [2068g], and neglecting the factors m', m^5, which occur in the numerator and
[2068i] denominator, we get $H' = \frac{\int_0^1 \rho . d . \left(a^5 . \frac{h}{h_{,}}\right)}{a^2 . \int \rho . d . a^3}$, and by resubstituting the factor $a^2 = 1$ in the denominator, it becomes $H' = H$ [1732m‴]; and then the ellipticity of Jupiter [1732n] becomes $\varepsilon' = \frac{\alpha\varphi'}{2 - \frac{6}{5}H}$. Comparing this with [1732n], which corresponds to
[2068k] the earth, we get $\frac{\varepsilon'}{\varepsilon} = \frac{\alpha\varphi'}{\alpha\varphi}$, a constant quantity, depending on the ratio of $\alpha\varphi'$ to $\alpha\varphi$.

[2069a] * (1517) The ellipticity of Jupiter, supposing it to be homogeneous, is, according to the calculations of La Place, 0,10967 [2068]; that of the earth, in the same hypothesis, 0,00433441 [1592″]. Hence, from [2068k], we have generally, upon the hypothesis of
[2069b] the densities assumed in [2068″], $\frac{\varepsilon'}{\varepsilon} = \frac{0{,}10967}{0{,}00433441}$, or $\varepsilon = \varepsilon' . 0{,}03952$. Putting $\varepsilon' = 0{,}0747$ [2066′, &c.], it becomes $\varepsilon = 0{,}002952 = \frac{1}{338}$ [2068⁗]. If we use Pound's
[2069c] measure, $\varepsilon' = 0{,}0771$ [2066′], we get $\varepsilon = 0{,}0771 \times 0{,}03952 = 0{,}00305 = \frac{1}{328}$ [2069]. These numbers would vary a little, if we were to use the corrected value 0,1104 [2066b] instead of 0,10967.

from the measure of Pound. These results agree with those given by the observations of the pendulum ;* *thus the analogy of Jupiter with the earth,* [2069′] *concurs with these observations, in proving that the oblateness of the terrestrial spheroid is less than* $\frac{1}{230}$, *and even less than* $\frac{1}{300}$; in the fifth book, we shall see this result confirmed by the phenomena of the precession of the [2069″] equinoxes, and the nutation of the axis of the earth.

We shall treat of the figure of the moon in the fifth book ; taking into consideration the motions of the lunar spheroid about its centre of gravity, which is the only phenomenon that gives us any insight into this figure, [2069‴] since it differs too little from that of a sphere to be determined by direct observations.

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* (1518) The author here refers to the result obtained by him in [2044] ; but it is evident, [2069*d*] from [2056*i*, *o*], that the ellipticity is greater than this, but less than $\frac{1}{230}$, as in [2069′].

CHAPTER VI.

ON THE FIGURE OF THE RING OF SATURN.

44. The Ring of Saturn is a very thin circular crown; its centre is the
[2069'''] same as that of the planet, and its width appears to be about one third of the diameter of Saturn; the distance from the inner edge to the surface of the planet, is nearly equal to this width.* *The surface of the ring is divided*
[2069v] *into two nearly equal parts, by a faint concentrical band, which proves that the ring is formed of two concentrical rings; and perhaps of a greater number, if we confide in the observations of Short, who assures us that he saw, with a large telescope, the surface of the outer ring divided into concentrical bands.*

The ring is composed of several concentrical bands.

[2069vi] *We shall suppose, as in the preceding researches, that an infinitely thin stratum of fluid, spread upon the surface of these rings, would be in equilibrium, by means of the forces acting upon it.* For it is contrary to all probability, to suppose that these rings sustain themselves, about Saturn, merely by the
[2069vii] cohesion of their particles; because the parts nearest the planet, being incessantly acted upon by the force of gravity, would be detached from the rings by insensible degrees, so that they would finally be destroyed; as is
[2069viii] the case in all those works of nature, in which there is not sufficient power to resist the action of foreign causes. We shall determine the figure of these rings by the conditions of the equilibrium of this fluid.

We may conceive each ring to be produced, by the revolution of an oval
[2069ix] *figure, like the ellipsis, moving perpendicularly to its plane, about the centre*

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* (1519) According to the late measures of Struve, the diameters of the outer ring, at the mean distance from the earth, are $40^s,095$, $35^s,289$; those of the inner ring $34^s,475$,
[2070a] $26^s,668$; the equatorial diameter of the planet $17^s,991$; the width of the outer ring $2^s,403$, of the inner ring $3^s,903$; space between the rings $0^s,407$. The void spaces between the rings and the planet have sometimes been supposed to be of different magnitudes on opposite sides of the planet; but this is probably an optical illusion.

of Saturn, situated on the continuation of the axis of this figure. We shall suppose this axis to be very small in comparison with the distance of its centre from the centre of the planet. We have seen, in § 11 of the second book, [2069x] that x, y, z, being the three rectangular co-ordinates of a point attracted by a spheroid, and V being the sum of the particles of the spheroid, divided by their distances from this point, we shall [459],

$$0=\left(\frac{d\,d\,V}{d\,x^2}\right)+\left(\frac{d\,d\,V}{d\,y^2}\right)+\left(\frac{d\,d\,V}{d\,z^2}\right). \qquad [2070]$$

The spheroid being formed by the revolution of a curve about an axis, which we shall take for the axis of z; if we put* $r^2=x^2+y^2$, V will [2071] become a function of z and r, since this function must remain the same, when r and z are given; therefore we shall have,†

$$\left(\frac{d\,d\,V}{d\,x^2}\right)=\frac{y^2}{r^3}\cdot\left(\frac{d\,V}{d\,r}\right)+\frac{x^2}{r^2}\cdot\left(\frac{d\,d\,V}{d\,r^2}\right);$$

$$\left(\frac{d\,d\,V}{d\,y^2}\right)=\frac{x^2}{r^3}\cdot\left(\frac{d\,V}{d\,r}\right)+\frac{y^2}{r^2}\cdot\left(\frac{d\,d\,V}{d\,r^2}\right). \qquad [2072]$$

Hence the preceding equation [2070] becomes

$$0=\frac{1}{r}\cdot\left(\frac{d\,V}{d\,r}\right)+\left(\frac{d\,d\,V}{d\,r^2}\right)+\left(\frac{d\,d\,V}{d\,z^2}\right); \qquad [2073]$$

which is the equation corresponding to a spheroid of revolution.

If we put $r=a+u$, a being the distance from the centre of Saturn [2073'] to the centre of the generating figure of the ring, we shall have,‡

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* (1520) r represents the distance of the attracted point from the axis of z; and its projections, on the planes of $z\,x$, $z\,y$, are represented by x, y, respectively, making [2070b] $r^2=x^2+y^2$.

† (1521) In [1430], V is considered as a function of a, b, c; which, by putting [2072a] $r^2=b^2+c^2$ [1558σ—τ], is transformed into [1558ψ]. If we now change a, b, c, into z, x, y, respectively, the equation [1430] will change into [2070], r^2 [2072a] will become as in [2071], the expressions [1558φ, χ] will give [2072], and [1558ψ] will change into [2073], [2072b] representing the original equation [2070].

‡ (1522) Putting $a+u=r$, we get $\left(\frac{d\,u}{d\,r}\right)=1$; and if we suppose V to be a [2072c] function of u instead of r, we shall have

Fundamental equation for the determination of V.

[2074] $$0=\frac{1}{a+u}\cdot\left(\frac{dV}{du}\right)+\left(\frac{ddV}{du^2}\right)+\left(\frac{ddV}{dz^2}\right); \qquad (1)$$

and if we suppose the co-ordinates u, z, to be very small in comparison with the radius a, we shall have nearly,

Second form.

[2075] $$0=\left(\frac{ddV}{du^2}\right)+\left(\frac{ddV}{dz^2}\right);$$

[2075'] *which is the equation corresponding to a cylinder of an infinite length, on each side of the origin of u and z;* and we see that this case is nearly that of the ring, when the attracted point is near its surface.*

This equation gives, by integration,†

[2076] $$V=\varphi\,(u+z\,.\sqrt{-1})+\psi\,(u-z\,.\sqrt{-1});$$

[2074a] $$\left(\frac{dV}{dr}\right)=\left(\frac{dV}{du}\right)\cdot\left(\frac{du}{dr}\right), \quad \text{or} \quad \left(\frac{dV}{dr}\right)=\left(\frac{dV}{du}\right).$$

In like manner, from this last equation, we get, $\left(\frac{ddV}{dr^2}\right)=\left(\frac{ddV}{du^2}\right)\cdot\left(\frac{du}{dr}\right)=\left(\frac{ddV}{du^2}\right).$ Substituting these in [2073], we get [2074]; and if a be very large, we may neglect the term divided by $a+u$, and it becomes as in [2075].

* (1523) By increasing a, the curvature of the ring decreases, and its figure, for a short
[2075a] distance on each side of the attracted point, is nearly cylindrical. When a is infinite, the first term of [2074] vanishes, and the equation becomes as in [2075], which corresponds to the limiting cylindrical form of the ring.

† (1524) Putting $\varphi'(u)$ for the differential of $\varphi(u)$ divided by du, $\varphi''(u)$ for the
[2076a] differential of $\varphi'(u)$ divided by du, &c.; then supposing the general value of V, corresponding to [2075], to be a function of $u+nz$ represented by $V=\varphi\,(u+nz)$, n being a constant quantity; we shall have, by taking the partial differentials of V relative to u and z,

[2076b] $$\left(\frac{dV}{du}\right)=\varphi'(u+nz); \qquad \left(\frac{dV}{dz}\right)=n\,.\,\varphi'(u+nz);$$
$$\left(\frac{ddV}{du^2}\right)=\varphi''(u+nz); \qquad \left(\frac{ddV}{dz^2}\right)=n^2\,.\,\varphi''(u+nz).$$

Substituting these two last expressions in [2075], it becomes $(1+n^2)\,.\,\varphi''(u+nz)=0$, which is satisfied by putting $1+n^2=0$, or $n=\pm\sqrt{-1}$; and according as we use the upper or the lower sign, we shall have, for V, an arbitrary function of $u+z\,.\sqrt{-1}$,
[2076c] represented by $\varphi(u+z\,.\sqrt{-1})$; or an arbitrary function of $u-z\,.\sqrt{-1}$, represented

$\varphi(u)$ and $\psi(u)$ being arbitrary functions of u. We may put this expression of V under the following form,* [2076']

$$V=f(u+z.\sqrt{-1})+\sqrt{-1}.F(u+z.\sqrt{-1})$$
$$+f(u-z.\sqrt{-1})-\sqrt{-1}.F(u-z.\sqrt{-1});$$ [2077]

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by $\psi(u-z.\sqrt{-1})$. Now the equation [2075] being linear in V, it will also be satisfied by taking for V the sum of these two functions, as in [2076]; and as this is the requisite number of arbitrary functions for an equation of the second degree [1558s], it will be the complete integral. If in [2076] we change $\varphi(u+z.\sqrt{-1})$ into

$$f(u+z.\sqrt{-1})+\sqrt{-1}.F(u+z.\sqrt{-1}),$$

and $\psi(u-z.\sqrt{-1})$ into $f(u-z.\sqrt{-1})-\sqrt{-1}.F(u-z.\sqrt{-1})$, it will become as in [2077]; which we shall investigate in a different manner in the next note. [2076d]

* (1525) The equation [1430], neglecting wholly the terms depending on c, and then putting $a=z$, $b=u$, becomes as in [2075]. Making the same changes in the integral of this equation [1558κ], and connecting in one line the terms depending on $v^{(0)}$, and in another those depending on $v^{(1)}$, we get, [2077a]

$$V=v^{(0)}-\frac{z^2}{1.2}.\left(\frac{d\,d\,v^{(0)}}{d\,u^2}\right)+\frac{z^4}{1.2.3.4}.\left(\frac{d^4\,v^{(0)}}{d\,u^4}\right)-\&c. \qquad [2077b]$$

$$+z.v^{(1)}-\frac{z^3}{1.2.3}.\left(\frac{d\,d\,v^{(1)}}{d\,u^2}\right)+\frac{z^5}{1.2.3.4.5}.\left(\frac{d^4\,v^{(1)}}{d\,u^4}\right)-\&c.; \qquad [2077c]$$

$v^{(0)}$, $v^{(1)}$, being independent of z; or in other words, they are functions of u only; which we shall represent by $v^{(0)}=f(u)$, $v^{(1)}=\sqrt{-1}.F'(u)$. Now from Taylor's theorem [617] we have [2077d]

$$\tfrac{1}{2}.f(u+z.\sqrt{-1})=\tfrac{1}{2}.\left\{f(u)+z.\sqrt{-1}.f'(u)-\frac{1}{1.2}.z^2.f''(u)-\frac{1}{1.2.3}.z^3.\sqrt{-1}.f'''(u)+\&c.\right\}, \qquad [2077e]$$

$$\tfrac{1}{2}.f(u-z.\sqrt{-1})=\tfrac{1}{2}.\left\{f(u)-z.\sqrt{-1}.f'(u)-\frac{1}{1.2}.z^2.f''(u)+\frac{1}{1.2.3}.z^3.\sqrt{-1}.f'''(u)+\&c.\right\}. \qquad [2077f]$$

The sum of these two equations is given in [2077g]; and by substituting $v^{(0)}=f(u)$ [2077d], it becomes as in [2077h]. Now changing f into F in [2077e, f], and taking the difference of the resulting expressions, we get the equation [2077i], which, by substituting $v^{(1)}=\sqrt{-1}.F'(u)$ [2077d], becomes as in [2077k];

$$\tfrac{1}{2}.f(u+z.\sqrt{-1})+\tfrac{1}{2}.f(u-z.\sqrt{-1}) \qquad [2077f']$$

$$=f(u)-\frac{1}{1.2}.z^2.f''(u)+\frac{1}{1.2.3.4}.z^4.f''''(u)-\&c. \qquad [2077g]$$

$$=v^{(0)}-\frac{1}{1.2}.z^2.\left(\frac{d^2\,v^{(0)}}{d\,u^2}\right)+\frac{1}{1.2.3.4}.z^4.\left(\frac{d^4\,v^{(0)}}{d\,u^4}\right)-\&c.; \qquad [2077h]$$

[2077′] $f(u)$ and $F(u)$ being real functions of u. *If the generating figure of the cylinder be composed of two equal and similar parts, on each side of the*
[2077″] *axis of* u; *the expression of* V *will remain unchanged, when we change the sign of* z; *hence we have, in this case,**

[2078] $$V = f(u + z.\sqrt{-1}) + f(u - z.\sqrt{-1}).$$

To determine the function $f(u)$, *it will only be necessary to find the value*
[2078′] *of* V *when* $z=0$, *or when the attracted point is on the continuation of the axis of* u; and we shall soon see, that the determination of this function depends on the quadrature of curves.

The value of V, *corresponding to a cylinder, is to be considered only as an*
[2078″] *approximation, relative to a ring*; but by substituting it in the equation [2074], it is easy to deduce from it, by approximation, a more accurate value of V. If in this equation we put

[2079] $$u + z.\sqrt{-1} = s; \qquad u - z.\sqrt{-1} = s';$$

it will become,†

[2077h′] $$\tfrac{1}{2}.F(u+z.\sqrt{-1} - \tfrac{1}{2}.F(u-z.\sqrt{-1})$$

[2077i] $$= z.\sqrt{-1}.F'(u) - \frac{1}{1.2.3}.z^3.\sqrt{-1}.F'''(u) + \frac{1}{1.2.3.4.5}.z^5.\sqrt{-1}.F^{v}(u) - \&c.$$

[2077k] $$= z.v^{(1)} - \frac{1}{1.2.3}.z^3.\left(\frac{d^2 v^{(1)}}{du^2}\right) + \frac{1}{1.2.3.4.5}.z^5.\left(\frac{d^3 v^{(1)}}{du^4}\right) + \&c.$$

The sum of the expressions [2077h, k], is equal to the value of V [2077b, c]; therefore V will be represented, as in [2077], by the sum of the first members of the same equations [2077f′, h′].

* (1526) Since the change of the sign of z does not alter the value of V [2077″], we may write $-z$ for z in [2077], and subtract the resulting expression from [2077], the remainder will be equal to nothing; hence

[2078a] $$0 = 2.\sqrt{-1}.F(u+z.\sqrt{-1}) - 2.\sqrt{-1}.F(u-z.\sqrt{-1}).$$

Subtracting the half of this from [2077], we get [2078]. When $z=0$, the value [2078] becomes $V=2.f(u)$; whence the form of the function f may be obtained, by means of the value of V corresponding to $z=0$. Then changing u into $u \pm z.\sqrt{-1}$, we shall get the general values of the functions which occur in [2078].

† (1527) From the expressions of s, s', [2079], we get

[2079a] $$u = \tfrac{1}{2}.(s+s'), \qquad z = \tfrac{1}{2}.\sqrt{-1}.(s'-s);$$

$$0 = 2 \cdot \left(\frac{ddV}{ds\,ds'}\right) + \frac{1}{2a+s+s'} \cdot \left\{ \left(\frac{dV}{ds}\right) + \left(\frac{dV}{ds'}\right) \right\}. \qquad [2080]$$

If we put

Assumed form of V.

$$V = V' + \frac{1}{a} \cdot V'' + \frac{1}{a^2} \cdot V''' + \&c.\,; \qquad [2081]$$

we shall obtain, by comparing the similar powers of $\frac{1}{a}$, the following equations,*

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substituting these in V, which is considered as a function of u, z, [2074], it becomes a function of s, s'. Moreover the partial differentials of [2079] relative to u, z, give

$$\left(\frac{ds}{du}\right) = 1, \qquad \left(\frac{ds'}{du}\right) = 1, \qquad \left(\frac{ds}{dz}\right) = \sqrt{-1}, \qquad \left(\frac{ds'}{dz}\right) = -\sqrt{-1}. \qquad [2079b]$$

Now supposing V to contain u, z, only as they are found in s, s', we shall get the partial differentials, to be substituted in [2074], by using the values [2079b]; in the following manner. First we have

$$\left(\frac{dV}{du}\right) = \left(\frac{dV}{ds}\right) \cdot \left(\frac{ds}{du}\right) + \left(\frac{dV}{ds'}\right) \cdot \left(\frac{ds'}{du}\right) = \left(\frac{dV}{ds}\right) + \left(\frac{dV}{ds'}\right);$$

$$\left(\frac{dV}{dz}\right) = \left(\frac{dV}{ds}\right) \cdot \left(\frac{ds}{dz}\right) + \left(\frac{dV}{ds'}\right) \cdot \left(\frac{ds'}{dz}\right) = \sqrt{-1} \cdot \left\{ \left(\frac{dV}{ds}\right) - \left(\frac{dV}{ds'}\right) \right\}. \qquad [2079c]$$

Taking the differentials of these two expressions, and making the same substitutions, we get,

$$\left(\frac{ddV}{du^2}\right) = \left(\frac{ddV}{ds^2}\right) \cdot \left(\frac{ds}{du}\right) + \left(\frac{ddV}{ds\,ds'}\right) \cdot \left\{ \left(\frac{ds}{du}\right) + \left(\frac{ds'}{du}\right) \right\} + \left(\frac{ddV}{ds'^2}\right)$$

$$= \left(\frac{ddV}{ds^2}\right) + 2 \cdot \left(\frac{ddV}{ds\,ds'}\right) + \left(\frac{ddV}{ds'^2}\right); \qquad [2079d]$$

$$\left(\frac{ddV}{dz^2}\right) = \sqrt{-1} \cdot \left[\left(\frac{ddV}{ds^2}\right) \cdot \left(\frac{ds}{dz}\right) + \left(\frac{ddV}{ds\,ds'}\right) \cdot \left\{ \left(\frac{ds'}{dz}\right) - \left(\frac{ds}{dz}\right) \right\} - \left(\frac{ddV}{ds'^2}\right) \cdot \left(\frac{ds'}{dz}\right) \right]$$

$$= -\left(\frac{ddV}{ds^2}\right) + 2 \cdot \left(\frac{ddV}{ds\,ds'}\right) - \left(\frac{ddV}{ds'^2}\right). \qquad [2079e]$$

The sum of these two last equations is $\left(\frac{ddV}{du^2}\right) + \left(\frac{ddV}{dz^2}\right) = 4 \cdot \left(\frac{ddV}{ds\,ds'}\right)$; substituting [2079$f$] this and $\left(\frac{dV}{du}\right)$ [2079c] in [2074], then dividing by 2, we get [2080]; observing that the sum of the two equations [2079], added to $2a$, make $2a + 2u = 2a + s + s'$.

* (1528) If we use the characteristic Σ of finite integrals, the assumed value [2081] becomes $V = \Sigma \cdot a^{-i+1} \cdot V^{(i)}$. Substituting this in [2080], we get successively, by putting [2081a]

Equations to find V', V'', &c.

[2082]
$$0=2.\left(\frac{ddV'}{ds\,ds'}\right);$$
$$0=2.\left(\frac{ddV''}{ds\,ds'}\right)+\tfrac{1}{2}.\left\{\left(\frac{dV'}{ds}\right)+\left(\frac{dV'}{ds'}\right)\right\};$$
$$0=2.\left(\frac{ddV'''}{ds\,ds'}\right)+\tfrac{1}{2}.\left\{\left(\frac{dV''}{ds}\right)+\left(\frac{dV''}{ds'}\right)\right\}-\frac{(s+s')}{4}.\left\{\left(\frac{dV'}{ds}\right)+\left(\frac{dV'}{ds'}\right)\right\}.$$

These equations, being integrated, will give the values of V', V'', &c.* To
[2082′] determine the arbitrary functions, *we shall suppose, for greater simplicity, that the generating figure of the ring is equal and similar on each side of the axis of* u; *this reduces the arbitrary functions of each of the quantities*

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for brevity $s+s'=2\sigma$, and using $\frac{1}{2a+2\sigma}=\tfrac{1}{2}a^{-1}.(1-a^{-1}.\sigma+a^{-2}.\sigma^2-a^{-3}.\sigma^3+\&c.)$,

$$0=2\Sigma.a^{-i+1}.\left(\frac{ddV^{(i)}}{ds\,ds'}\right)+\Sigma.\frac{1}{2a+s+s'}.a^{-i+1}.\left\{\left(\frac{dV^{(i)}}{ds}\right)+\left(\frac{dV^{(i)}}{ds'}\right)\right\}$$

[2081b]
$$=2\Sigma.a^{-i+1}.\left(\frac{ddV^{(i)}}{ds\,ds'}\right)+\tfrac{1}{2}\Sigma.\left\{a^{-i}-a^{-i-1}.\sigma+a^{-i-2}.\sigma^2-\&c.\right\}.\left\{\left(\frac{dV^{(i)}}{ds}\right)+\left(\frac{dV^{(i)}}{ds'}\right)\right\}.$$

Connecting together the terms multiplied by a^{-i+1}, we may put the preceding expression under the form,

[2081c]
$$0=\Sigma.a^{-i+1}.\left\{\begin{array}{l}2.\left(\frac{ddV^{(i)}}{ds\,ds'}\right)+\tfrac{1}{2}.\left\{\left(\frac{dV^{(i-1)}}{ds}\right)+\left(\frac{dV^{(i-1)}}{ds'}\right)\right\}\\ \qquad -\tfrac{1}{2}\sigma.\left\{\left(\frac{dV^{(i-2)}}{ds}\right)+\left(\frac{dV^{(i-2)}}{ds'}\right)\right\}\\ \qquad +\tfrac{1}{2}\sigma^2.\left\{\left(\frac{dV^{(i-3)}}{ds}\right)+\left(\frac{dV^{(i-3)}}{ds'}\right)\right\}-\&c.\end{array}\right\};$$

in this all the values of V must be neglected in which the index is nothing or negative, as in [2081]. To satisfy this last equation, the coefficients of a^{-i+1} must be put equal to nothing. Now putting successively $i=1$, $i=2$, $i=3$, &c., we obtain the equations [2082].

* (1529) Multiplying the first of the equations [2082] by $\tfrac{1}{2}ds$, and integrating relatively to s, adding, to complete the integral, an arbitrary function of s', represented by $\psi'(s')$; because in this integration s' is considered as constant; we get $\left(\frac{dV'}{ds'}\right)=\psi'(s')$. In like manner, multiplying this by ds', and integrating, adding the arbitrary function $\varphi(s)$,
[2082a] to complete the integral, we obtain $V'=\varphi(s)+\psi(s')$. Substituting this in the second of the equations [2082], and integrating in a similar manner, we get V'' in functions of s, s'. In like manner we obtain V''', V'''', &c., from the other equations.

V', V'', &c., *to one.* To obtain these functions, it will only be necessary to find their values, when the attracted point is situated on the continuation of the axis of* u. We shall consider a circular arc, parallel to the plane passing through the axis of u, perpendicular to the generating figure, supposing the centre of this arc [2083b] to be in the right line passing through the centre of Saturn, perpendicular to this plane. We shall then put y for the height of this centre above the plane; $a+x$ for the radius of the circular arc; ϖ for the angle which this radius makes with the plane of the generating figure, that passes through the attracted point; and $a+u$ for the distance of this attracted point from the centre of Saturn. This being premised, we shall have, for the sum of the particles of the circular arc, divided by their respective distances from the attracted point,†

[2082′] [2082″] [2082‴]

$$\int \frac{(a+x)\,.\,d\varpi}{\sqrt{(a+u)^2-2\,.\,(a+u)\,.\,(a+x)\,.\,\cos.\varpi+(a+x)^2+y^2}}; \qquad [2083]$$

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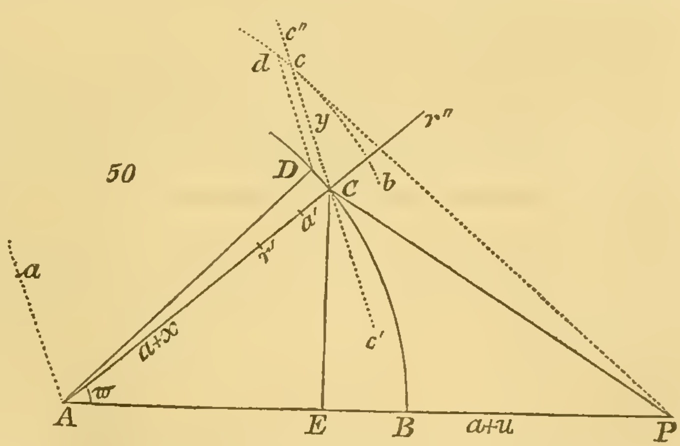
* (1530) When the generating figure is equal and similar on each side of the axis of u, it is evident that we may change z into $-z$, without altering the value of V; and by this alteration of the sign of z, s [2079] changes into s', and s' into s. By this means the value of V' [2082a] becomes $V'=\varphi(s')+\psi(s)$, s and s' being independent; this last value of V' is made to agree with [2082a], generally, by supposing $\varphi(s)=\psi(s)$, $\varphi(s')=\psi(s')$; so that the two functions φ, ψ, are reduced to one. Similar remarks may be made relative to the value of V'', &c. These functions φ, ψ, are generally most easily found, in the simple case of $z=0$, or $s=s'$ [2079]; that is, where the attracted point is situated on the continuation of the axis $a+u$. For after finding $\varphi(u)$ for that point, we have $\varphi(s)$, by changing u into s. [2082b] [2082c]

† (1531) In the figure annexed, A is the centre of Saturn, P the attracted point, situated on the axis AP; Aa perpendicular to the plane of the figure, is the axis about which the generating curve revolves, to form the ring. A circular arc of the ring, described about the centre a, is represented by bcd, which is parallel to the plane of [2083a]



the integral being taken from $\varpi=0$ to ϖ equal to the whole circumference 2π. We must multiply this integral by dy, and again integrate relatively
[2083′] to y, from* $y=-\varphi(x)$ to $y=\varphi(x)$; $y^2=\{\varphi(x)\}^2$ being the
[2083″] equation of the generating figure of the ring. Finally, to obtain the value of V, we must multiply this last integral by dx, and then integrate relatively
[2083‴] to x, from $x=-k$ to $x=k'$; $-k$ and k' being the limits of the values of x.† These various integrations cannot be made rigorously; but we can obtain their values in series, developed according to the powers of $\frac{1}{a}$, which is sufficiently accurate for the present investigation; but as
[2083⁗] V becomes infinite when a is supposed to be infinite,‡ we must, instead of V,

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[2083b] the figure, and is orthographically projected upon this plane, in the circular arc BCD; $cC=dD=aA$ are perpendicular to this plane, and we shall suppose Cc, Dd, to be infinitely near to each other; CE is perpendicular to the axis AP. Then we have
[2083b′] $AP=a+u$, $AB=AC=a+x$, $Cc=y$, angle $PAC=\varpi$, angle $CAD=d\varpi$; hence the arc $CD=cd=(a+x)\,.\,d\varpi$, which may be taken to represent the mass of
[2083c] the particle of the ring cd. This mass, divided by the distance Pc, represents the part of V [1428⁗] depending on this particle, or $dV=\frac{(a+x)\,.\,d\varpi}{Pc}$. Now

[2083c′] $$Pc^2=PC^2+Cc^2=PC^2+y^2;$$

and in the triangle PAC, we have, by [62] Int.,

[2083d] $$PC^2=AP^2-2AP.AC.\cos.PAC+AC^2=(a+u)^2-2.(a+u).(a+x).\cos.\varpi+(a+x)^2;$$

hence we obtain Pc, and by substituting it in the preceding value of dV, and integrating, we get the expression of V [2083]. Moreover, it is evident from the figure, that this
[2083e] integral must be taken from $\varpi=0$ to $\varpi=2\pi$, to obtain the part of V corresponding to the whole circumference of the circle.

* (1532) Supposing the co-ordinate Cc, fig. 50, to be continued on both sides of C,
[2083f] till it meet the surface of the ring in c' and c''; we shall have, from the equation of the surface
[2083g] [2083′], $Cc'=-\varphi(x)$, $Cc''=\varphi(x)$; these being the extreme values of y, between which the integration relatively to y is to be made.

† (1533) Supposing the line ACr'' to cut the surface of the ring in the points r', r'',
[2083h] and that $Aa'=a$, $Ar'=a-k$, $Ar''=a+k'$; the values of x [2082⁗], corresponding to those points, are $x=-k$, $x=k'$, respectively, which are evidently the limits of the integration, relative to x.

‡ (1534) Supposing a to be infinite, we may neglect x, u, y, in comparison with a, and the denominator of the integral expression [2083] becomes,

compute $\left(\frac{dV}{du}\right)$, which is never infinite. It is evident that the expression of $\left(\frac{dV}{du}\right)$ will give that of $\left(\frac{dV}{dz}\right)$,* consequently we shall have the [2083v] attractions of the ring, parallel to the axes of u and z.

The dimensions of the generating figure of the ring of Saturn, are so small, in comparison with the diameter of the ring, that we may neglect the [2083vi] terms divided by a. Now if we substitute, in the preceding integral, the value of $\cos.\varpi$ in a series, $1-\frac{1}{2}\varpi^2+\&c.$ [44] Int., and suppose $a\varpi=t$, [2083vii] it becomes, neglecting the terms divided by a,†

$$\int\frac{dt}{\sqrt{(u-x)^2+y^2+t^2}}. \qquad [2084]$$

$$\{a^2-2\,a^2.\cos.\varpi+a^2\}^{\frac{1}{2}}=a.\{2-2.\cos.\varpi\}^{\frac{1}{2}}; \qquad [2083i]$$

consequently an element of that integral becomes of the order

$$\frac{a.d\varpi}{a.\{2-2.\cos.\varpi\}^{\frac{1}{2}}}=\frac{d\varpi}{\{2-2.\cos.\varpi\}^{\frac{1}{2}}}; \qquad [2083k]$$

which increases infinitely when ϖ is infinitely small, because the denominator vanishes. Now this does not happen with the expression $\left(\frac{dV}{du}\right)$, deduced from [2083]. For the element of this integral is

$$\frac{(a+x).\{(a+u)-(a+x).\cos.\varpi\}.d\varpi}{\{(a+u)^2-2.(a+u).(a+x).\cos.\varpi+(a+x)^2+y^2\}^{\frac{3}{2}}}; \qquad [2083l]$$

which, by neglecting as above u, x, y, and then dividing the numerator and denominator by $a^2.(1-\cos.\varpi)$, becomes $\frac{d\varpi}{a.\{1-\cos.\varpi\}^{\frac{1}{2}}}$; which is infinitely smaller than the element of V [2083k], because it is divided by the quantity a. Hence we perceive the [2083m] justness of the remarks in [2083″], which may also be confirmed by integration.

* (1535) An example of this is given in [2089—2093]. For after having computed the value of $-\left(\frac{dV}{du}\right)$ [2089], we can obtain from it the expression of $-f'(u)$ [2091a], [2083n] and then $-f'(u\pm z.\sqrt{-1})$ [2092a]. Substituting these in [2091], we shall get $-\left(\frac{dV}{dz}\right)$ [2093].

† (1536) Substituting $\cos.\varpi=1-\frac{1}{2}\varpi^2+\&c.$ [44] Int., in the denominator of [2083], it becomes, by neglecting terms of the orders ϖ^4, $u\varpi^2$, $x\varpi^2$, and using t [2083vii], [2084a]

Its differential being taken, relatively to u, and divided by du, is

[2085] $$-\int \frac{(u-x).dt}{\{(u-x)^2+y^2+t^2\}^{\frac{3}{2}}}.$$

The integral relative to ϖ must be taken from $\varpi=0$ to $\varpi=2\pi$; π being the semi-circumference of a circle whose radius is unity. Now this integration
[2085′] is evidently the same, as if we were to take it from $\varpi=-\pi$ to $\varpi=\pi$;
[2085″] which, in the case of a being infinite, is the same as to integrate relatively
[2085‴] to t, from $t=-\infty$ to $t=\infty$;* and then the integral becomes,†

[2086] $$-\frac{2.(u-x)}{(u-x)^2+y^2};$$

consequently, when the attracted point is situated in the axis of u, we shall have,‡

[2087] $$-\left(\frac{dV}{du}\right)=2.\iint\frac{(u-x).dy.dx}{(u-x)^2+y^2}=4.\int dx.\text{arc. tang.}\left(\frac{\varphi(x)}{u-x}\right).$$

$$\{(a+u)^2-2.(a+u).(a+x)+(a+x)^2+y^2+(a+u).(a+x).\varpi^2\}^{\frac{1}{2}}$$
$$=\{[(a+u)-(a+x)]^2+y^2+a^2\varpi^2\}^{\frac{1}{2}}=\{(u-x)^2+y^2+a^2\varpi^2\}^{\frac{1}{2}}$$
[2084b] $$=\{(u-x)^2+y^2+t^2\}^{\frac{1}{2}}.$$

[2084c] Substituting this and $a.d\varpi=dt$ in [2083], it becomes as in [2084]. Now x, y, t, are independent of u; we may therefore take the partial differential of [2084] relatively to u, and it will become of the form [2085].

* (1537) This integral is to be taken through the whole circumference of the circle bcd,
[2085a] fig. 50, page 499, and it is immaterial at what point we begin; so that the limits may be from $\varpi=-\pi$ to $\varpi=\pi$; and as $t=a\varpi$ [2083^{vii}], the limits of t will be $-a\pi$
[2085b] and $a\pi$; which, when $a=\infty$ [2085″], become $-\infty$ and $+\infty$, as in [2085‴]

† (1538) This integral is easily deduced from the integral in Vol. I, page 303, note 356,
[2086a] $\int_{-\infty}^{\infty}\frac{Ar'.dz'}{(r'^2+z'^2)^{\frac{3}{2}}}=\frac{2A}{r'}$, by putting $z'=t$, $r'^2=(u-x)^2+y^2$, $A=-\frac{(u-x)}{r'}$; observing that in this integration, u, x, y, are supposed to be constant.

‡ (1539) Having obtained the integral relatively to t [2086], we must multiply it by $dy.dx$, and prefix the double sign of integration, relatively to dy, dx, and we shall get, by changing the sign, the first value of $-\left(\frac{dV}{du}\right)$ [2087]. The integration, relatively to y, is
[2087a] obtained by putting $y=(u-x).t'$, $dy=(u-x).dt'$, x being considered constant

If we suppose the generating figure of the ring to be an ellipsis, and represent [2087′]
its equation by the following expression,*

$$\lambda^2 y^2 = k^2 - x^2 ; \qquad [2088]$$

we shall find,†

$$-\left(\frac{dV}{du}\right) = \frac{4\pi.\lambda}{\lambda^2-1} \cdot \left\{ u - \sqrt{u^2 - k^2 . \frac{(\lambda^2-1)}{\lambda^2}} \right\}. \qquad [2089]$$

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in this integration; hence $-\left(\frac{dV}{du}\right) = 2 . \int dx . \int \frac{dt'}{1+t't'}$. But by [51] Int.,

$$\int \frac{dt'}{1+t't'} = \text{arc. tang. } t' = \text{arc. tang.} \left(\frac{y}{u-x}\right); \qquad [2087b]$$

to which we must add the quantity $\text{arc. tang.} \left(\frac{\varphi(x)}{u-x}\right)$, to make it vanish at the first limit [2083′] $y = -\varphi(x)$; then at the second limit $y = \varphi(x)$, it becomes

$$\int \frac{dt'}{1+t't'} = 2 . \text{arc. tang.} \left(\frac{\varphi(x)}{u-x}\right); \qquad [2087c]$$

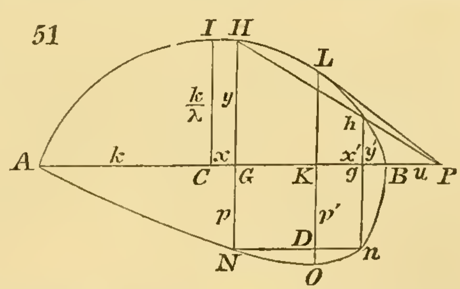
consequently, as in (2087],

$$-\left(\frac{dV}{du}\right) = 4 . \int dx . \text{arc. tang.} \left(\frac{\varphi(x)}{u-x}\right). \qquad [2087d]$$

* (1540) Putting, in the equation of the ellipsis [378n], $\frac{a^2}{b^2} \cdot y^2 = a^2 - x^2$, the greatest semi-axis $a = k$, and the least semi-axis $b = \frac{k}{\lambda}$, it becomes as in [2088]. [2088a]

† (1541) In finding the integral of the expression [2087], relatively to x, it will be convenient to use, for brevity, the symbol $t = \text{tang.} p = \frac{(k^2 - x^2)^{\frac{1}{2}}}{\lambda . (u-x)}$; the greatest values [2089a]
of t, p, being represented by the accented letters t', p', respectively. For the purpose of [2089b]
illustration, we shall refer to the annexed figure, in which AHB represents the generating semi-ellipsis, whose semi-axes are

51



$$CA = CB = k, \qquad CI = \frac{k}{\lambda}; \qquad [2089c]$$

absciss $CG = x$; ordinate $GH = y$, or $Cg = x'$, $gh = y'$; P the attracted point, through which is drawn the line PhH, cutting the ellipsis in h, H. Then we have $CP = u$, $PG = u - x$; hence by using [2088], we have $\text{tang.}\, GPH = \frac{GH}{PG} = \frac{y}{u-x} = \frac{(k^2 - x^2)^{\frac{1}{2}}}{\lambda.(u-x)} = \text{tang.} p$; so that the angle [2089d]

[2089′] The value of $-\left(\frac{dV}{du}\right)$, relative to any attracted point whatever, is, by what has been said,

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$GPH = p$. Now from [2083′, 2088], we have $\varphi(x) = y = \lambda^{-1}\cdot(k^2 - x^2)^{\frac{1}{2}}$; and substituting this in [2087], we get

[2089e] $$-\left(\frac{dV}{du}\right) = 4\,.\int dx\,.\,\text{arc. tang.}\,\frac{(k^2-x^2)^{\frac{1}{2}}}{\lambda.(u-x)} = 4\,.\int dx\,.\,p\,;$$

[2089f] the limits of this integral being the least and greatest values of x [2083‴], which are evidently $-k$ and k. The integral $\int dx\,.\,p$, may be represented by the area of the curve AOB; in which the ordinate $GN = p$ corresponds to the absciss $CG = x$; and there is also another equal and parallel ordinate gn, corresponding to the absciss $Cg = x'$. The greatest ordinate of this curve is $OK = p'$; and if this be continued, it will meet the ellipsis in the point L of the tangent PL, drawn through the attracted point P; making the
[2089g] angle $CPL = p'$. The area of the curve AOB may also be found, by supposing $KD = p$ to be the absciss corresponding to the ordinate $Nn = x' - x$, and its integral
[2089g′] will be represented, in the usual manner, by $\int dp\,.\,(x'-x)$, the integral being taken from $p = 0$ to $p = p'$. Substituting this for $\int dx\,.\,p$, in [2089e], we get

[2089h] $$-\left(\frac{dV}{du}\right) = 4\,.\int dp\,.\,(x'-x)\,;$$

in which we must substitute the values of x', x, p, in terms of t. Now by squaring the expression of t [2089a], we get a quadratic equation in x, represented by

$$(u-x)^2\,.\,\lambda^2 t^2 = k^2 - x^2\,;$$

from which we obtain,

[2089i] $$x = (1+\lambda^2 t^2)^{-1}\,.\,\{u\,.\,\lambda^2 t^2 \pm [k^2 - (u^2-k^2)\,.\,\lambda^2 t^2]^{\frac{1}{2}}\}.$$

The upper sign gives the value of x', the lower that of x; their difference is

[2089k] $$x' - x = 2\,.\,(1+\lambda^2 t^2)^{-1}\,.\,\{k^2 - (u^2-k^2)\,.\,\lambda^2 t^2\}^{\frac{1}{2}}.$$

Substituting this, and $dp = \frac{dt}{1+tt}$ [2089a], [51] Int., in [2089h], we get

[2089l] $$-\left(\frac{dV}{du}\right) = 8\,.\int\frac{dt}{1+tt}\,.\,\frac{\{k^2-(u^2-k^2)\,.\,\lambda^2 t^2\}^{\frac{1}{2}}}{1+\lambda^2 t^2}.$$

[2089m] To avoid the radicals, we shall put $t = a\,.\,\sin.z$; and shall use $a^2 = \frac{k^2}{\lambda^2\,.\,(u^2-k^2)}$, for brevity; by which means we get

[2089n] $$\{k^2 - (u^2-k^2)\,.\,\lambda^2 t^2\}^{\frac{1}{2}} = \{k^2 - k^2\,.\,\sin.^2 z\}^{\frac{1}{2}} = k\,.\,\cos.z\,;$$

hence, by successive reductions, we obtain,

$$-\left(\frac{dV}{du}\right)=-f'(u+z.\sqrt{-1})-f'(u-z.\sqrt{-1})\,;^{*}$$ [2090]

$f'(u)$ being the differential of $f(u)$ divided by du. Putting these values

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$$-\left(\frac{dV}{du}\right)=8.\int\frac{a\,dz.\cos.z}{1+a^2.\sin.^2z}\cdot\frac{k.\cos.z}{1+\lambda^2a^2.\sin.^2z}=8\,a\,k.\int\frac{(1-\sin.^2z).dz}{(1+a^2.\sin.^2z).(1+\lambda^2a^2.\sin.^2z)}$$

$$=\frac{8k}{\lambda^2-1}\cdot\frac{(1+\lambda^2a^2)}{a}\cdot\int\frac{dz}{1+\lambda^2a^2.\sin.^2z}-\frac{8k}{\lambda^2-1}\cdot\frac{(1+a^2)}{a}\cdot\int\frac{dz}{1+a^2.\sin.^2z}\,;$$ [2089o]

the last expression being easily derived from the preceding, by reducing to a common denominator. At the first limit of these integrals, $p=0$, $t=0$, $z=0$, [2089g', a, m]; at the second limit, $p=p'$ [2089g'], corresponding to the point L, where we have $x'-x=0$; therefore the radical of [2089k], which is represented by $k.\cos.z$ [2089n], is equal to nothing, consequently $z=\frac{1}{2}\pi$; hence the limits of z are 0 and $\frac{1}{2}\pi$. Now we have found in note 934, pages 18, 19, $\int_0^{\pi}\frac{dq}{D^2-\cos.^2q}=\frac{\pi}{D.(D^2-1)^{\frac{1}{2}}}$; and as the elements of the integral are the same for $q=\frac{1}{2}\pi+q'$, as for $q=\frac{1}{2}\pi-q'$, we may take half this quantity for the integral between the limits 0, $\frac{1}{2}\pi$; hence [2089p]

$$\int_0^{\frac{1}{2}\pi}\frac{dq}{D^2-\cos.^2q}=\frac{\pi}{2D.(D^2-1)^{\frac{1}{2}}}.$$ [2089q]

Multiplying this by D^2-1, putting $\cos.^2q=1-\sin.^2q$, changing q into z, and putting $D^2-1=\frac{1}{a^2}$, we get, $\int_0^{\frac{1}{2}\pi}\frac{dz}{1+a^2.\sin.^2z}=\frac{\frac{1}{2}\pi}{(1+a^2)^{\frac{1}{2}}}$; and by changing a into λa, $\int_0^{\frac{1}{2}\pi}\frac{dz}{1+\lambda^2a^2.\sin.^2z}=\frac{\frac{1}{2}\pi}{(1+\lambda^2a^2)^{\frac{1}{2}}}$. Substituting these in [2089o], we find, by successive reductions, and resubstituting the value of a^2 [2089m], the same value as in [2089], [2089r]

$$-\left(\frac{dV}{du}\right)=\frac{8k}{\lambda^2-1}\cdot\frac{(1+\lambda^2a^2)}{a}\cdot\frac{\frac{1}{2}\pi}{(1+\lambda^2a^2)^{\frac{1}{2}}}-\frac{8k}{\lambda^2-1}\cdot\frac{(1+a^2)}{a}\cdot\frac{\frac{1}{2}\pi}{(1+a^2)^{\frac{1}{2}}}$$

$$=\frac{4\pi k}{\lambda^2-1}\cdot\left\{\left(\frac{1+\lambda^2a^2}{a^2}\right)^{\frac{1}{2}}-\left(\frac{1+a^2}{a^2}\right)^{\frac{1}{2}}\right\}=\frac{4\pi k}{\lambda^2-1}\cdot\left\{\frac{\lambda u}{k}-\frac{\lambda.\left\{u^2-k^2.\left(\frac{\lambda^2-1}{\lambda^2}\right)\right\}^{\frac{1}{2}}}{k}\right\}$$

$$=\frac{4\pi\lambda}{\lambda^2-1}\cdot\left\{u-\left(u^2-k^2.\frac{(\lambda^2-1)}{\lambda^2}\right)^{\frac{1}{2}}\right\}.$$ [2089s]

* (1542) The expressions [2090, 2091] are easily deduced from [2078]; and as in [1387a], they express the attractions, parallel to the axes of u, z, respectively. [2090a]

[2090′] of $-\left(\frac{dV}{du}\right)$ equal to each other, when $z=0$, we shall obtain the value of $f'(u)$.* We also have

[2091] $$-\left(\frac{dV}{dz}\right)=-\sqrt{-1}\,.\,f'(u+z\,.\sqrt{-1})+\sqrt{-1}\,.\,f'(u-z\,.\sqrt{-1})\,;$$

$-\left(\frac{dV}{du}\right)$, $-\left(\frac{dV}{dz}\right)$, express the attractions of the ring, parallel to the
[2091′] axes of u and z, and in the direction towards the centre of the generating figure. Hence it is easy to perceive, that in the case where this figure is an ellipsis, these attractions are,†

[2092] $$-\left(\frac{dV}{du}\right)=\frac{2\pi.\lambda}{\lambda^2-1}\,.\left\{\begin{array}{l} u+z.\sqrt{-1}-\sqrt{(u+z.\sqrt{-1})^2-k^2.\left(\frac{\lambda^2-1}{\lambda^2}\right)} \\ +u-z.\sqrt{-1}-\sqrt{(u-z.\sqrt{-1})^2-k^2.\left(\frac{\lambda^2-1}{\lambda^2}\right)} \end{array}\right\};$$

Attraction of the ring parallel to the axes of u, z.

and

[2093] $$-\left(\frac{dV}{dz}\right)=\frac{2\pi\lambda.\sqrt{-1}}{\lambda^2-1}\,.\left\{\begin{array}{l} u+z.\sqrt{-1}-\sqrt{(u+z.\sqrt{-1})^2-k^2.\left(\frac{\lambda^2-1}{\lambda^2}\right)} \\ -(u-z.\sqrt{-1})+\sqrt{(u-z.\sqrt{-1})^2-k^2.\left(\frac{\lambda^2-1}{\lambda^2}\right)} \end{array}\right\}.$$

Elliptical figure of the ring.

If the attracted point be upon the surface of the spheroid, where we have

[2094] $$u^2+\lambda^2 z^2=k^2, \qquad [2095a],$$

Attraction at the surface.

they will become,‡

[2095] $$\frac{4\pi.u}{\lambda+1}\,; \qquad \text{and} \qquad \frac{4\pi.\lambda z}{\lambda+1}.$$

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* (1543) Putting $z=0$ in [2090], it becomes $-\left(\frac{dV}{du}\right)=-2.f'(u)$; and by

[2091a] substituting this in [2089], we get $-f'u=\frac{2\pi\lambda}{\lambda^2-1}\,.\left\{u-\left(u^2-k^2.\frac{(\lambda^2-1)}{\lambda^2}\right)^{\frac{1}{2}}\right\}.$

† (1544) Changing successively u into $u+z.\sqrt{-1}$, $u-z.\sqrt{-1}$, in [2091a],

[2092a] we get $-f'(u+z.\sqrt{-1})$, $-f'(u-z.\sqrt{-1})$; substituting these in [2090, 2091], we get [2092, 2093], respectively.

‡ (1545) The co-ordinates of the attracted point are $a+u$, z; those of the elliptical surface $a+x$, y [2082′′′′, &c.]; therefore when the attracted point is situated

45. *We shall now suppose the ring to be a homogeneous fluid mass, whose generating figure is an ellipsis.* We shall put a for the distance from the centre of this ellipsis to the centre of Saturn, this distance being very great, in comparison with the dimensions of the ellipsis; and we shall suppose that the ring revolves, in its plane, about Saturn; g being the centrifugal force arising from the rotatory motion, at the distance 1 from the axis of rotation. This force, corresponding to the particle of the ring whose co-ordinates are u and z, will be $(a+u)\cdot g$;* and by multiplying it by the element of its direction, the product will be $(a+u)\cdot g\,du$. The attraction of Saturn upon the same particle is $\frac{S}{(a+u)^2+z^2}$, S being the mass of Saturn. Multiplying it by the element of its direction, which is equal to

Hypothesis of an elliptical figure of the ring.

[2095]

[2095′]

[2095″]

$$-d\cdot\sqrt{(a+u)^2+z^2},\qquad [2095'''']$$

we shall have, by neglecting the squares of u and z,

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upon the surface, we shall have $x=u$, $y=z$; and the equation [2088] will become $k^2=u^2+\lambda^2z^2$. Substituting this in the radical [2092, 2093], we get, by successive reductions, [2095a]

$$\left\{(u\pm z\cdot\sqrt{-1})^2-k^2\cdot\left(\frac{\lambda^2-1}{\lambda^2}\right)\right\}^{\frac{1}{2}}=\left\{(u\pm z\cdot\sqrt{-1})^2-(u^2+\lambda^2z^2)\cdot\left(\frac{\lambda^2-1}{\lambda^2}\right)\right\}^{\frac{1}{2}}$$

$$=\left\{\frac{u^2}{\lambda^2}\pm 2uz\cdot\sqrt{-1}-\lambda^2z^2\right\}^{\frac{1}{2}}=\frac{u}{\lambda}\pm\lambda z\cdot\sqrt{-1}.\qquad [2095b]$$

Hence [2092] becomes $\frac{2\pi\lambda}{\lambda^2-1}\cdot\left\{2u-\frac{2u}{\lambda}\right\}=\frac{4\pi\lambda.u}{\lambda^2-1}\cdot\frac{\lambda-1}{\lambda}=\frac{4\pi.u}{\lambda+1}$, as in [2095]; and [2093] changes into

$$\frac{2\pi\lambda\cdot\sqrt{-1}}{\lambda^2-1}\cdot\{2z\cdot\sqrt{-1}-2\lambda z\cdot\sqrt{-1}\}=\frac{4\pi\lambda.z}{\lambda^2-1}\cdot(\lambda-1)=\frac{4\pi\lambda.z}{\lambda+1},\quad [2095].\qquad [2095b']$$

If the ellipsis become a circle, whose radius is $k=r'$, and area $A=\pi.r'^2$, we shall have $\lambda=1$ [2088]; and by putting, in [2095], $z=0$, $u=r'$, we shall have the whole attraction in the direction of the radius r' equal to

$$\frac{4\pi.u}{\lambda+1}=2\pi.r'=\frac{2.A}{r'},\qquad [498].\qquad [2095c]$$

* (1546) The distance of the attracted point of the ring, whose co-ordinates are $a+u$, z, from the axis of rotation, is $a+u$; and this, being multiplied by g, gives the centrifugal force $(a+u)\cdot g$, tending to increase that distance, [1569b]. [2095d]

[2096] $$-\frac{S\,.\,du}{a^2}+\frac{2S\,.\,u\,du}{a^3}-\frac{S\,.\,z\,dz}{a^3}.*$$

The attractions which the same particle suffers from the ring, multiplied by the elements of the directions $-du$ and $-dz$, give the products †

[2097] $$-\frac{4\pi\,.\,u\,du}{\lambda+1}; \qquad \text{and} \qquad -\frac{4\pi\,.\,z\,dz\,.\,\lambda}{\lambda+1}.$$

Equation of equilibrium at the surface.

Now the general equation of equilibrium makes the sum of these products equal to nothing; therefore we have,‡

[2098] $$0=\left\{\frac{S}{a^2}-ag\right\}.du+\left\{\frac{4\pi}{\lambda+1}-\frac{2S}{a^3}-g\right\}.u\,du+\left\{\frac{4\pi\,.\,\lambda}{\lambda+1}+\frac{S}{a^3}\right\}.z\,dz\,;$$

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* (1547) The co-ordinates of the attracted point being $a+u$, z, its distance from the origin of the co-ordinates, or from the centre of Saturn, is $\{(a+u)^2+z^2\}^{\frac{1}{2}}$. Hence

[2096a] the whole attractive force of the planet is $\frac{S}{(a+u)^2+z^2}$; and as this force tends to decrease the distance, the element of its direction, or differential of the distance, by which it is to be multiplied, must be *negative* and equal to

$$-d\,.\,\{(a+u)^2+z^2\}^{\frac{1}{2}}=-\{a\,du+u\,du+z\,dz\}\,.\,\{(a+u)^2+z^2\}^{-\frac{1}{2}};$$

by which means the product becomes,

[2096b] $$\{-S\,.\,a\,du-S\,.\,u\,du-S\,.\,z\,dz\}\,.\,\{(a+u)^2+z^2\}^{-\frac{3}{2}}.$$

If we retain only terms of the second degree, in u, z, and their differentials, we may neglect u^2, z^2, in this last factor, and it will become,

$$\left\{(a+u)^2+z^2\right\}^{-\frac{3}{2}}=a^{-3}\,.\left\{1+\frac{2u}{a}\right\}^{-\frac{3}{2}}=a^{-3}\,.\left\{1-\frac{3u}{a}\right\};$$

substituting this in [2096b], it becomes as in [2096].

† (1548) The attractive forces [2095] tend to *decrease* the co-ordinates u, z; therefore

[2097a] they must be multiplied by $-du$, $-dz$, respectively, to obtain the corresponding elements [2097].

‡ (1549) We have seen, in [1615a, b, c, &c.], that the sum of several forces F, F', F'', &c.; multiplied by the elements of their directions df, df', df'', &c.; may be reduced to three forces P, Q, R; parallel to the axes x, y, z, respectively; making

[2097b] $$\Sigma\,.\,F\,.\,df=P\,.\,dx+Q\,.\,dy+R\,.\,dz\,;$$

so that the equation of equilibrium [1563π, ρ] is equivalent to $\Sigma\,.\,F\,.\,df=0$; which represents that the sum of each of the forces, multiplied by the element of its direction, is nothing. In the present example, these products are given in [2095''', 2096, 2097]; their sum, put equal to nothing, and changing the signs, is as in [2098].

which is the differential equation of the generating figure of the ring. But we have supposed that this figure is an ellipsis, whose equation is

$$u^2+\lambda^2 z^2=k^2\,; \qquad [2098']$$

consequently its differential is $0=u\,d\,u+\lambda^2.z\,d\,z$; by comparing this differential equation with the preceding, we shall obtain the two following equations,* [2098'']

$$g=\frac{S}{a^3} \qquad [2099]$$

Conditions necessary for an elliptical figure.

$$\frac{\dfrac{4\pi\lambda}{\lambda+1}+\dfrac{S}{a^3}}{\dfrac{4\pi}{\lambda+1}-\dfrac{3S}{a^3}}=\lambda^2. \qquad [2100]$$

The first of these equations determines the rotatory motion of the ring;† *the second gives the ellipticity of the generating figure.* If we put $e=\frac{S}{4\pi.a^3}$, [2101] the second of these equations gives,‡

$$e=\frac{\lambda.(\lambda-1)}{(\lambda+1).(3\lambda^2+1)}\,; \qquad [2102]$$

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* (1550) The differential of the assumed equation of the ring [2094] is

$$0=u\,d\,u+\lambda^2.z\,d\,z\,; \qquad [2098a]$$

which ought to agree with [2098]. The former contains only the two terms, multiplied by $u\,d\,u$, $z\,d\,z$; and to make it agree with [2098], the other term, depending on $d\,u$, ought to vanish from [2098]; hence $\frac{S}{a^2}-a\,g=0$, or $g=\frac{S}{a^3}$, as in formula [2099]. [2098b] Substituting this in [2098], and dividing by the coefficient of udu, we have

$$0=u\,d\,u+\frac{\dfrac{4\pi\lambda}{\lambda+1}+\dfrac{S}{a^3}}{\dfrac{4\pi}{\lambda+1}-\dfrac{3S}{a^3}}.z\,dz\,; \qquad [2098c]$$

and by putting the coefficient of zdz equal to λ^2, as in [2100], it becomes as in [2098a].

† (1551) This is shown in [2110]. [2100a]

‡ (1552) Substituting $\frac{S}{a^3}=4\pi.e$ [2101], in [2100]; rejecting the factor 4π from each term of the numerator and denominator; and then multiplying by the denominator, we

[2102′] e being positive, we see that λ must exceed unity. The axis of the ellipsis, directed towards Saturn, is equal to $2k$, and represents the width of the ring; the axis perpendicular to it, $\frac{2k}{\lambda}$ [2089c], is the thickness of the
[2103] ring; therefore the thickness is less than the width.

The ring must be oblate.

[2103′] *We also see that* e *is nothing when* $\lambda = 1$, *or when* $\lambda = \infty$;* *hence it follows, that for the same value of* e, *there are two values of* λ; *but we can make choice of the greatest, which gives the most flattened ring. The value of* e *is susceptible of a maximum, which corresponds nearly to* $\lambda = 2{,}594$.†
[2103″]

Two elliptical figures satisfy the equilibrium.

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get $\frac{\lambda}{\lambda+1} + e = \lambda^2 . \left\{ \frac{1}{\lambda+1} - 3e \right\}$. Multiplying by $\lambda+1$, we have
[2102a] $\lambda + (\lambda+1) . e = \lambda^2 . \{1 - 3 . (\lambda+1) . e\}$, or $(3\lambda^3 + 3\lambda^2 + \lambda + 1) . e = \lambda^2 - \lambda$; whence we obtain e [2102].

* (1553) If $\lambda = 1$, the expression [2102] becomes $e = 0$; and as it may be put
[2102b] under the form $e = \frac{1}{\lambda} . \left\{ \frac{1 - \frac{1}{\lambda}}{\left(1+\frac{1}{\lambda}\right) . \left(3 + \frac{1}{\lambda^2}\right)} \right\}$, it also evidently becomes $e = 0$, when $\lambda = \infty$; observing that the factor between the braces is then equal to $\frac{1}{3}$. Again, since S, a, are positive, e [2101] must be positive; which cannot be, unless the factor $\lambda - 1$ [2102] is positive, or $\lambda > 1$. Hence we see that when $\lambda = 1$, $e = 0$; when $\lambda > 1$, e is positive; and when $\lambda = \infty$, e again becomes nothing; so that between $\lambda = 1$ and
[2102c] $\lambda = \infty$, there must be a value λ', which will make e a maximum, represented by e'; and we shall see, in the next note, that there is only one of these maximum values of e. Hence it is evident, that for any positive value of $e < e'$, there will be two values of λ, the one
[2102d] greater, and the other less than λ'.

† (1554) The differential of [2102], put equal to nothing, gives the maximum of λ. The denominator of this expression is $(3\lambda^3 + 3\lambda^2 + \lambda + 1)^2$; and its numerator, divided
[2103a] by $d\lambda$, and put equal to nothing, is $-3\lambda^4 + 6\lambda^3 + 4\lambda^2 + 2\lambda - 1 = 0$. From a few trials, we find that λ is nearly equal to $2{,}594$, as in [2103″]. No other value of λ,
[2103b] greater than unity [2102′], will satisfy the preceding equation [2103a]. For if we divide it by the factor which we have just found, $-\lambda + 2{,}594 = 0$, we get

[2103c] $$3 . \lambda^3 + 1{,}78 . \lambda^2 + 0{,}62 . \lambda - 0{,}39 = 0,$$

nearly; and when $\lambda > 1$, the terms of this expression depending on λ, must exceed $3 + 1{,}78 + 0{,}62$, or $5{,}4$; which cannot be reduced to nothing by the constant term $-0{,}39$; therefore no other value of λ will satisfy the proposed conditions of the maximum of e. This value $\lambda = 2{,}594$, being substituted in [2102], gives e' as in [2104].

In this case $e = 0{,}0543026$; which is the greatest possible value of e. [2104]
Putting R for the radius of the body of Saturn, and ρ for its mean density,
that of the ring being taken for unity, we shall have [1430k],* [2105]

$$S = \tfrac{4}{3}\pi\rho \,.\, R^3 ; \qquad [2106]$$

consequently [2101, 2106],

$$e = \frac{\rho \,.\, R^3}{3\,a^3} ; \qquad [2107]$$

therefore the greatest value that ρ can possibly have, is

$$\rho = 0{,}1629078 \,.\, \frac{a^3}{R^3}. \qquad [2108]$$

The difficulty of obtaining the true ratio of a to R, on account of the smallness of these quantities, and the effect of irradiation, prevents us from [2108']
determining exactly the limit of ρ. If we suppose, for the inner ring, that [2108'']
$\frac{a}{R} = 2$, *which varies but little from the truth* [2070a], *we shall get* $\rho = \frac{13}{10}$ *nearly, for the greatest possible density.* [2109]

The irradiation must increase considerably the apparent widths of the rings, consequently their real widths must be much less than their apparent; [2109']
perhaps also it makes several distinct rings appear as one, in the same manner as a telescope of a small magnifying power makes the whole of the rings appear as one connected mass; therefore we cannot determine with accuracy the figures of the rings which surround this planet. [2109''] We may also observe, that the smallness of the width and thickness of any one of the rings, in comparison with its distance from the centre of Saturn, serves to increase the accuracy of the application of the preceding theory to the figure of the ring, and to render more probable the explanation we have given of [2109''']
the manner in which it can be sustained about the planet, by the laws of the equilibrium of fluids.

It is easy to determine the time of rotation of each ring, by means of its distance a from the centre of the generating curve to the centre of Saturn. For [2109'''']

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* (1555) From [2107], we have $\rho = 3\,e \,.\, \frac{a^3}{R^3}$; substituting the maximum value of [2108a]
e [2104], we get the maximum value of ρ [2108]; and if we put $a = 2\,R$ [2108''], it becomes $\rho = 1{,}3$, as in [2109].

The ring revolves in the same time as a satellite at the same distance. *the centrifugal force g, arising from its rotatory motion, being equal to*

[2109v] $\frac{S}{a^3}$ [2099], *it is evident that this motion is the same as that of a satellite, placed at the distance a from the centre of Saturn;* hence it follows, that the period*

[2110] *of this motion is* $0^{day},44$ *for the inner ring, which agrees with observation.*

46. *The preceding theory will also subsist, when the generating ellipsis*

[2110′] *varies in magnitude and position, throughout the whole extent of the generating circumference of the ring, which may be supposed of unequal widths in its*

Rings of irregular forms. *different parts; we may also suppose it to be a curve of double curvature, provided that all these variations of magnitude and position are sensible only at much greater distances from the given point of the surface under consideration,*

[2110″] *than the diameter of the generating curve, passing through this point.* These inequalities are indicated by the appearances and disappearances of the ring, in which the two anses appear differently. It may also be observed, that these inequalities are necessary to maintain the ring in its equilibrium about

[2110‴] Saturn. For if it were perfectly similar in all its parts, the equilibrium would be troubled by the smallest force; such as the attraction of a comet,

[2110⁗] or satellite; and the ring would finally be precipitated upon the body of Saturn. To prove this, we shall suppose the ring to be a circle whose radius is r, and centre at the distance z from the centre of Saturn, situated in the plane of the ring. It is evident that the result of the attraction of

[2110v] Saturn, upon this circle, will be in the direction of the line z, which connects

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* (1556) Supposing T to denote the time of revolution of a satellite, at the distance a from the centre of Saturn; and t to be the time of rotation of a ring of the same radius a; we shall get the centrifugal force g of the ring, by changing in [2059] M into S, and D

[2109a] into a; hence $g=\frac{S.T^2}{t^2.a^3}$. Putting this equal to the value [2099], and dividing by $\frac{S}{t^2.a^3}$, we get $T^2=t^2$; whence $T=t$, as in [2109v]. We shall see, in Book VIII, that the time of revolution of the outer satellite is $79^{days},3296$ [7669], and its distance

[2110a] from Saturn, expressed in semi-diameters of that planet, $59,154$ [7717]. The time of revolution of a satellite, at the distance of two of these semi-diameters, computed by Kepler's

[2110b] rule [387], will therefore be expressed by $79^{days},3296\times\left(\frac{2}{59,154}\right)^{\frac{3}{2}}=0^{day},49$, instead of $0^{day},44$, given by the author [2110].

Action of Saturn upon a circular ring.

the two centres.*. If we put ϖ for the angle which the radius r makes with the continuation of the line z, we shall have,†

$$-\frac{d}{dz}\cdot\int_0^{2\pi}\frac{S\,.\,r\,d\varpi}{\sqrt{r^2+2rz\,.\cos.\varpi+z^2}},$$ [2111]

for the attraction of Saturn upon the ring, resolved in a direction parallel to to z; the integral being taken from $\varpi=0$ to ϖ equal to the whole [2111′] circumference of the circle; and the differential being taken relatively to z. Putting A for this attraction; the centre of the ring will be moved as if all [2111″] its mass were collected at this centre, and it were then acted upon by the [2111‴] force A, directed towards the centre of Saturn.

Putting c for the number whose hyperbolic logarithm is unity, we have‡

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* (1557) In the annexed figure, C is the centre of the ring $EBAB'$, D the centre of Saturn. The diameter DC cuts the ring, in the points A, E; and we have $CE=CB=CB'=r$, $CD=z$, angle $BCA=\varpi$. [2110c] Then if we make $AB'=AB$, it is evident that the parts of the attraction of Saturn, upon two points, situated at B, B', resolved in the direction BB', mutually destroy each other; leaving only a force in the direction CD. The same takes place in all other parts of the ring similarly situated, above and below the line AE; so that the whole attraction of Saturn, on the ring, may be reduced to a force in the [2110d] direction CD.

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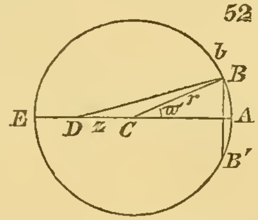

† (1558) We have $CB=r$, $CD=z$, $DCB=\pi-\varpi$; and by [62] Int., $DB^2=CB^2-2\,CB\,.\,CD\,.\cos.DCB+CD^2=r^2+2rz\,.\cos.\varpi+z^2$. Now the [2111a] mass of Saturn S, divided by DB, gives the part of V [1457a], depending on the action of Saturn, upon a point placed at B. Multiplying this by the differential of the arc $Bb=rd\varpi$, we get the part of V, resulting from the action of Saturn upon Bb, equal to $\frac{S.rd\varpi}{DB}$. Substituting DB [2111a], and integrating, through the whole circumference of the ring, we obtain V. Then taking the differential, relatively to z, and dividing by $-dz$, we get $-\left(\frac{dV}{dz}\right)$, or the attraction in the direction z [1387a], as in [2111]. [2111b] The factor r, in the numerator of this formula, was not in the original work; and the second [2111c] members of [2115, 2116] are also multiplied by r.

‡ (1559) Substituting $2\,.\cos.\varpi=c^{\varpi.\sqrt{-1}}+c^{-\varpi.\sqrt{-1}}$ [12] Int., in the denominator of the first member of [2112], it becomes,

[2112] $$\frac{1}{\{r^2+2rz\,.\cos.\varpi+z^2\}^{\frac{1}{2}}}=\frac{1}{r\,.\left\{1+\frac{z}{r}\,.\,c^{\varpi.\sqrt{-1}}\right\}^{\frac{1}{2}}.\left\{1+\frac{z}{r}\,.\,c^{-\varpi.\sqrt{-1}}\right\}^{\frac{1}{2}}}.$$

If we put

[2113] $$\frac{1}{\left\{1+\frac{z}{r}\,.\,c^{\varpi.\sqrt{-1}}\right\}^{\frac{1}{2}}}=1+\alpha\,.\frac{z}{r}\,.\,c^{\varpi.\sqrt{-1}}+\alpha'.\frac{z^2}{r^2}\,.\,c^{2\varpi.\sqrt{-1}}+\&c.,$$

we shall have,

[2114] $$\frac{1}{\left\{1+\frac{z}{r}\,.\,c^{-\varpi.\sqrt{-1}}\right\}^{\frac{1}{2}}}=1+\alpha\,.\frac{z}{r}\,.\,c^{-\varpi.\sqrt{-1}}+\alpha'.\frac{z^2}{r^2}\,.\,c^{-2\varpi.\sqrt{-1}}+\&c.$$

If we multiply these two series by each other, and then their product by $d\varpi$, taking the integral, from $\varpi=0$ to ϖ equal to the whole circumference, represented by 2π, we shall have,*

[2115] $$\int\frac{r\,d\varpi}{\sqrt{r^2+2rz\,.\cos.\varpi+z^2}}=2\pi\,.\left\{1+\alpha^2\,.\frac{z^2}{r^2}+\alpha'^2\,.\frac{z^4}{r^4}+\&c.\right\};$$

[2112a] $$\left\{r^2+rz\,.\,c^{\varpi.\sqrt{-1}}+rz\,.\,c^{-\varpi.\sqrt{-1}}+z^2\right\}^{\frac{1}{2}}=r\,.\left\{1+\frac{z}{r}\,.\,c^{\varpi.\sqrt{-1}}\right\}^{\frac{1}{2}}.\left\{1+\frac{z}{r}\,.\,c^{-\varpi.\sqrt{-1}}\right\}^{\frac{1}{2}},$$

as is easily proved by multiplication and reduction; hence we get the second member of [2112]. These factors, developed by the binomial theorem, become as in [2113, 2114]; the second being deduced from the first, by changing ϖ into $-\varpi$.

* (1560) In performing the multiplication of the two series [2113, 2114], we may neglect all the terms of the products which contain ϖ. For if it contain a term of the form $Bc^{n\varpi.\sqrt{-1}}$, it will also have $B\,c^{-n\varpi.\sqrt{-1}}$, and their sum will be

[2115a] $$B\,.\left\{c^{n\varpi.\sqrt{-1}}+c^{-n\varpi.\sqrt{-1}}\right\}=2\,B\,.\cos.\,n\,\varpi,\qquad [12]\ \text{Int.};$$

as is evident from the similarity of the functions [2113, 2114], and the equality of the corresponding coefficients. This produces, in [2111], an integral depending on

[2115a′] $$\int_0^{2\pi}d\varpi\,.\cos.n\varpi,$$

which, by [1483b], is equal to nothing; therefore we may neglect such quantities, and multiply the term depending on $c^{n\varpi.\sqrt{-1}}$ [2113], by that containing $c^{-n\varpi.\sqrt{-1}}$ [2114],

hence we deduce,

$$A = -\frac{4\pi . S . z}{r^2} . \left\{ \alpha^2 + 2\,\alpha'^2 . \frac{z^2}{r^2} + \&c. \right\} .$$ [2116]

This quantity is negative whatever be the value of z; therefore *the centre of Saturn repels the centre of this circular homogeneous ring; and whatever be* [2116'] *the relative motion of the second centre about the first, the curve it describes, by* Saturn repels the centre of a circular ring. *this motion, is convex towards Saturn; the centre of this circular ring must therefore recede more and more from the centre of the planet, until its* [2116''] *circumference shall finally come in contact with the surface of the planet.*

A ring, perfectly similar in all its parts, would be composed of an infinite number of circles similar to that we have mentioned; therefore the centre [2116'''] of this ring would be repelled by that of Saturn, however little those centres might be separated. In consequence of this action, the ring would finally be brought in contact with the surface of Saturn.

Hence it follows, that the separate rings which surround the body of Saturn, are irregular solids, of unequal widths in the different parts of their Irregular form of the rings necessary for their preservation. *circumferences; so that their centres of gravity do not coincide with their centres of figure.* These centres of gravity may be considered as so many satellites,*

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by which means the product of these functions becomes $1 + \alpha^2 . \frac{z^2}{r^2} + \alpha'^2 . \frac{z^4}{r^4} + \&c. = C,$ [2115b] for brevity. This is multiplied by $\frac{1}{r}$ in the second member of [2112], and by substitution in the first member of [2115], it becomes $\int_0^{2\pi} C . d\varpi = C . \int_0^{2\pi} d\varpi = 2\pi . C,$ as in [2115]; its differential relative to z, multiplied by $-\frac{S}{dz}$, gives A [2111, 2111'']. All the terms of the factor of A [2116], between the braces, being square positive numbers, [2115c] their sum must be positive, and the sign of A must therefore be negative; corresponding to a repulsive instead of an attractive force.

* (1561) For the purpose of illustration, we shall suppose the ring to be a solid circular line $XYX'D'$, composed of particles of variable densities, symmetrically placed on each side of the axis $X'CGX$, passing through the centre of gravity G of the ring, and the point C, which is the centre of the planet, and of the ring; the parts of the ring on the side X being denser than those on the opposite side towards X', so that the centre of gravity of the ring is distant from the [2116a]

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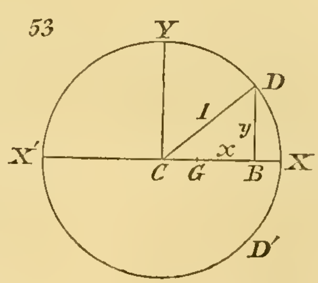

[2116''''] *which move about the centre of Saturn, at distances depending on the inequalities of the parts of each ring, and with velocities of rotation equal to those of their respective rings.*

In the investigation of the figures of the rings, we have neglected their [2116v] attractions on each other; supposing the intervals between them to be sufficiently great, to prevent this force from having a sensible influence on their figures. It will however be easy to notice this attraction;* and we

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centre of its figure, by the distance $CG=X$. Taking the line CX for the axis of x, [2116b] and the perpendicular line CY for the axis of y; we shall have, for the co-ordinates of a particle m of the ring, situated at D, $CB=x$, $DB=y$; the radius of the ring $CX=CD$ being taken for unity. The attraction of the mass M of the planet, upon a single particle of the ring, being represented by M, its action on the particle m, situated at the point D, will be $M.m$, in the direction DC; which may be resolved into the two forces $M.mx$, $M.my$, parallel to the axes x, y, respectively. The sums of all these forces, [2116c] throughout the whole circumference of the ring, are $M.\int mx$, $M.\int my$, respectively; and if we change, in [126, 127], Σ into $\int$, putting $\int m=m'$ for the whole mass of the ring, we shall have $\int mx=m'.X$, $\int my=m'.Y$; consequently the preceding sums [2116d] become $Mm'.X$, $Mm'.Y$. Now since the axis of x passes through the centre of gravity G, we shall have $Y=0$, $X=CG$; therefore the action of Saturn, upon the ring, is reduced to the force $M.m'.CG$, attracting the centre of gravity G towards C, instead of repelling it, as it does when the ring is perfectly homogeneous [2116']. The [2116e] attraction of the planet tends therefore to keep the centre of gravity of the ring near to that of the planet, and by this means the motions become stable.

* (1562) To give some idea of this attraction, we shall take the case of the ring AHB, [2116f] treated of in [2089a, &c.]; and shall suppose that it is also acted upon by a second ring, similarly placed; whose generating figure $A'H'B'$ is situated in the same plane as AHB; the line CC', connecting the centres of these generating figures, when continued beyond C', being supposed to pass through [2116g] the centre of Saturn. In this second ellipsis, we shall accent the symbols u, z, λ, used in [2092, &c.]; so that the action of this second ellipsis, upon a point without the surface, whose co-ordinates are u', z', will be represented by the following formulas [2116k], similar to [2092, 2093], in which, for brevity, we have put R, R', for the radicals,

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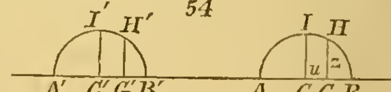

[2116h]
$$\left\{ (u' \pm z'.\sqrt{-1})^2 - k'^2.\frac{(\lambda'^2-1)}{\lambda'^2} \right\}^{\frac{1}{2}}$$

[2116i] the upper sign corresponding to R, the lower to R'; the origin of the co-ordinates u', z', being at the centre C' of this second ellipsis.

can, without much trouble, satisfy ourselves, that the generating figure of each ring will yet be elliptical, if the rings be very much flattened. But the stability of the equilibrium of the rings requires that their figures should be [2116vi]
irregular; moreover, as they have different rotatory motions, and incessantly change their relative positions, their reciprocal action must be extremely

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$$-\left(\frac{dV}{du'}\right)=\frac{2\pi.\lambda'}{\lambda'^2-1}.\{2u'-R-R'\};$$
$$-\left(\frac{dV}{dz'}\right)=\frac{2\pi\lambda'.\sqrt{-1}}{\lambda'^2-1}.\{2z'.\sqrt{-1}-R+R'\}.$$ [2116k]

If the attracted point be situated at any point H of the surface of the first ellipsis, corresponding to the co-ordinates $CG=u$, $GH=z$, we shall have, by putting

$$CC'=a, \qquad u'=a+u, \qquad z'=z, \qquad m^2=a^2-k'^2+k'^2.\lambda'^{-2},$$ [2116l]

the following value of R [2116h, &c.], developed according to the powers of u, z,

$$R=\{(a+u+z.\sqrt{-1})^2-k'^2+k'^2.\lambda'^{-2}\}^{\frac{1}{2}}=\{m^2+2au+u^2-z^2+2.(a+u).z.\sqrt{-1}\}^{\frac{1}{2}}$$
$$=m+\frac{a}{m}.u+\frac{a}{m}.z.\sqrt{-1}+\&c.$$ [2116m]

Changing $z.\sqrt{-1}$ into $-z.\sqrt{-1}$, we get $R'=m+\frac{a}{m}.u-\frac{a}{m}.z.\sqrt{-1}+\&c.$, [2116i]. Hence if we neglect terms of the second degree in u, z, we shall have,

$$-R-R'=-2m-2.\frac{a}{m}.u; \qquad -R+R'=-2.\frac{a}{m}.z.\sqrt{-1};$$ [2116n]

substituting these in [2116k], we get,

$$-\left(\frac{dV}{du'}\right)=\frac{2\pi.\lambda'}{\lambda'^2-1}.\left\{2.(a+u)-2m-2.\frac{a}{m}.u\right\};$$
$$-\left(\frac{dV}{dz'}\right)=\frac{2\pi.\lambda'}{\lambda'^2-1}.\left\{-2z+2.\frac{a}{m}.z\right\};$$ [2116o]

which are similar to the forces, produced by the first ring [2095]. They are to be multiplied respectively by $-du$, $-dz$, as in [2096'], and the products added to [2098]. Now the addition of these terms does not alter the form of the expression [2098]; since it will still depend on the three terms du, udu, zdz; we may make it satisfy the differential equation of the ellipsis [2098''], by putting, as in [2098b], the coefficient of [2116p]
du equal to nothing. The same reasoning will apply to any number of rings, provided the dimensions of the ring u, z, are so very small, in comparison with m, that we can neglect terms of the second order in u, z.

[2116vii] variable; so that it is not necessary to take it into consideration, in the investigation of their permanent figures.*

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* (1563) If the figures of the rings were irregular, their permanent figures would depend on their mean actions upon each other; and the differences between these mean
[2116q] values, and their actual attractions, might cause oscillations in the fluids upon their surfaces; in the same manner as the irregular attractions of the sun and moon, upon the earth, produce the tides.

CHAPTER VII.

ON THE FIGURE OF THE ATMOSPHERES OF THE HEAVENLY BODIES.

47. A RARE, *transparent, elastic and compressible fluid, surrounding a body, upon which it rests, is called its atmosphere.* We can conceive that there is an atmosphere of this kind about each of the heavenly bodies. The existence of such atmospheres, about all these bodies, is highly probable; and as it regards the sun, the earth, and several of the planets, it is indicated by observations.* In proportion as the atmospherical fluid is elevated above the surface of the body, it becomes rarer, by means of its elasticity, which dilates it, as it becomes less compressed. But if the parts of its surface were elastic, it would continue to expand, and would finally be dissipated in the region of space. It is therefore necessary that the elasticity of the atmospherical fluid should diminish, in a greater proportion than the weight which compresses it; and that there should exist a state of rarity, in which the fluid is without elasticity. It must be in this state at the surface of the atmosphere.

Atmosphere. [2116viii] [2116ix]

All the atmospherical strata must finally acquire the same rotatory motion as the body which they surround. For the friction of these strata against each other, and against the surface of the body, must accelerate the slowest

[2116x] All the strata have the same rotatory velocity.

* (1563a) Dr. Wollaston, in a paper published in the Philosophical Transactions of 1822, has fully confirmed the theory of the finite extent of the solar atmosphere, by observations of the planet Venus, made near the superior conjunction in 1821; having followed the planet till it came within $53^m\ 15^s$ of the sun's centre, without discovering any sensible irregularity in the motion, from the double horizontal refraction, which the ray of light would suffer, in passing through the solar atmosphere, if it extended to that distance from the sun's centre. [2116r]

motions, and retard the most rapid, until a perfect equality is established among them.

At the surface of the atmosphere, the fluid is retained only by its gravity ;
[2116xi] so that the figure of its surface is such, that the resultant of the centrifugal force, and the attractive force of the body, is perpendicular to it. For the smallness of the density of the atmosphere allows us to neglect the attraction of its particles. We shall now determine the figure of this surface ; and for
[2116xii] this purpose, we shall put V for the sum of the particles of the spheroid which the atmosphere surrounds, divided by their respective distances from
[2116xiii] any particle dM of that atmosphere ; r for the distance of this particle from the centre of gravity of the spheroid ; θ the angle which r makes with the axis of rotation of the spheroid ; and ϖ the angle which the plane drawn
[2116xiv] through this axis and the radius r, makes with a fixed meridian upon the surface of the spheroid. Then if n be the angular rotatory velocity of the spheroid, the centrifugal force of the particle dM will be $n^2 r\,.\,\sin.\,\theta$.*
[2116xv] The element of its direction will be $d\,.\,(r\,.\,\sin.\,\theta)$; therefore the integral
[2116xvi] of this force multiplied by the element of its direction, will be $\frac{1}{2}n^2r^2.\sin.^2\theta$; and by putting ρ for the density of the particle dM, and Π for the pressure it suffers ; we shall have, by § 22,†

[2117] $$\int\frac{d\,\Pi}{\rho} = \text{constant} + V + \tfrac{1}{2}\,n^2\,r^2\,.\,\sin.^2\,\theta, \qquad (1)$$

Π being a function of ρ.

If the spheroid differ but little from a sphere, the expression of V is of the form,

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[2117a] * (1564) This is the same as is found in [352a—b], changing R into r; by which means the distance of the particle from the axis of rotation becomes $r\,.\,\sin.\,\theta$; and the
[2117a′] centrifugal force, in the direction of this line $n^2r\,.\sin.\,\theta$. Multiplying this by the differential of the distance, we obtain $n^2\,.\,(r\,.\,\sin.\,\theta)\times d\,.\,(r\,.\,\sin.\,\theta)$; and then, by integration, we
[2117b] get the expression $\frac{1}{2}\,n^2\,.\,(r\,.\,\sin.\,\theta)^2$ [2116xvi].

[2117c] † (1565) The integral of [1615] is $\int\frac{d\Pi}{\rho} = \text{constant} + \int(F.df + F'.df' + \&c.)$. The part of the integral of the second member, depending on the attraction of the spheroid,
[2117d] is V [1616xii] ; and that on the centrifugal force, $\frac{1}{2}\,n^2r^2\,.\,\sin.^2\,\theta$; hence we get [2117], which is the same as [349], changing p into Π.

$$V=\frac{m}{r}+\frac{U^{(2)}}{r^3}+\frac{U^{(3)}}{r^4}+\&c.\,;*$$ [2118]

m being the mass of the spheroid, and $U^{(i)}$ an integral and rational function of μ, $\sqrt{1-\mu^2}.\sin.\varpi$, $\sqrt{1-\mu^2}.\cos.\varpi$, which satisfies the equation [2118′] of partial differentials [1460],

$$0=\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dU^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddU^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i.(i+1).U^{(i)}\,;$$ [2119]

μ being equal to $\cos.\theta$ [1434′]. If we substitute this value of V in the equation [2117], we shall obtain the equation of any stratum of the [2119′] atmosphere having the same density, [137‴, &c., or 1615′, &c.].

At the exterior surface, $\Pi=0$; if we neglect the excentricity of the [2119″] spheroid, and put α equal to the ratio of the centrifugal force to the gravity at the equator, upon the surface of the spheroid, whose radius we shall take [2119‴] for unity; the equation [2117] will become,†

$$c=\frac{2}{r}+\alpha r^2.\sin.^2\theta.$$ [2120]

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* (1566) The term $U^{(1)}$ vanishes from [1459], as is evident by substituting $Y^{(1)}=0$ [1745], in [1466]; hence we get $V=\frac{U^{(0)}}{r}+\frac{U^{(2)}}{r^3}+\frac{U^{(3)}}{r^4}+\&c.$ When r is infinite, [2118a] it is reduced to its first term $V=\frac{U^{(0)}}{r}$; but by [1392, 1390′], it is then equal to the mass M, divided by the distance of the attracted point from the centre of gravity of the spheroid; which, in the present notation, is represented by r; and M is changed into m; so that we shall have $V=\frac{m}{r}$. Comparing this with the preceding expression [2118b] $V=\frac{U^{(0)}}{r}$, we get $U^{(0)}=m$; and then, by substitution in the general value of V, [2118a], it becomes as in [2118].

† (1567) Neglecting the excentricity of the spheroid, we may put $y=0$ in [1464], and we shall have generally $Y^{(i)}=0$; whence $U^{(i)}=0$ [1466], and $V=\frac{m}{r}$ [2118]; consequently the equation [2117] will become $\int\frac{d\Pi}{\rho}=\text{constant}+\frac{m}{r}+\tfrac{1}{2}n^2r^2.\sin.^2\theta.$ [2120a] The integral in the first member must be a function of Π, as in [137‴, &c.]; and at the external surface, where $\Pi=0$, it becomes constant; hence if we divide

Putting R equal to the polar radius of the atmosphere, we shall have*

[2121] $c=\frac{2}{R}$, therefore

Equation of the surface of the atmosphere.

[2122]
$$\frac{2}{R}=\frac{2}{r}+\alpha r^2 . \sin.^2 \theta.$$

[2122′] To obtain the ratio of the two axes of the atmosphere, we shall put R' for the radius of its equator, and the preceding equation will give,†

[2123]
$$\alpha R'^3=2 . \left(\frac{R'-R}{R}\right).$$

The maximum ratio of the equatorial radius to the polar radius, is $\frac{3}{2}$.

The greatest possible value of R' is that which extends to the point where the centrifugal force becomes equal to gravity. Now in this case‡ $\frac{1}{R'^2}=\alpha R'$,

[2124] or $\alpha R'^3=1$; consequently,

[2125]
$$\frac{R'}{R}=\frac{3}{2}.$$

[2120b] the whole equation by $\frac{1}{2}m$, and put $\frac{n^2}{m}=\alpha$, it will become as in [2120], c being a constant quantity. At the equator, where $\sin.\theta=1$, and $r=1$ [2119‴], the centrifugal force $n^2 r . \sin.\theta$ [2116^{xv}] becomes n^2, and the gravity $\frac{m}{r^2}=m$; hence

[2120c] $\frac{n^2}{m}=\alpha$ expresses the ratio of the centrifugal force at the equator to the gravity, as in [2119″].

[2121a] * (1568) At the pole $\theta=0$, $r=R$, and [2120] becomes $c=\frac{2}{R}$; substituting this in [2120], it becomes as in [2122].

[2122a] † (1569) At the equator, $\theta=100°$, $r=R'$; hence [2122] gives $\frac{2}{R}-\frac{2}{R'}=\alpha R'^2$. Multiplying this by R', it becomes as in [2123].

‡ (1570) At the point where the centrifugal force begins to exceed the gravity, the fluid would begin to fly off; this point must therefore be the limit of the equatorial axis. Now at the distance R', upon the plane of the equator, where $\sin.\theta=1$, the centrifugal force [2117a′] is $n^2 R'$, and the gravity at that point $\frac{m}{R'^2}$. Putting these quantities equal to each

[2123a] other, then substituting $n^2=\alpha m$ [2120c], and dividing by $m R'^{-2}$, we get $\alpha R'^3=1$

This ratio of R' to R is the greatest possible; for by making $\alpha R'^3 = 1 - z$, [2125']
z being necessarily positive or zero, we shall have,* $\frac{R'}{R} = \frac{3-z}{2}$. [2126]

The greatest radius of the atmosphere is that of the equator, for the differential of the equation of the surface gives†

$$d r = \frac{\alpha r^4 . d \theta . \sin. \theta . \cos. \theta}{1 - \alpha r^3 . \sin.^2 \theta}. \qquad [2127]$$

The denominator of this fraction is always positive. For the centrifugal [2127']
force, resolved in the direction of the radius r, is $\alpha m r . \sin.^2 \theta$;‡ and this
must be less than gravity, which is equal to $\frac{m}{r^2}$; therefore r increases [2127'']
with θ from the pole to the equator.

[2124]; substituting this in [2123], we have $1 = \frac{2R' - 2R}{R}$, or $R' = \frac{3}{2} R$, as in [2125].

* (1570) The expressions of α, R', are both positive, and the maximum value of
$\alpha R'^3$ is 1 [2124]; hence in general we have $\alpha R'^3$ positive and less than unity; therefore [2125a]
z is positive and less than unity [2125']. Now substituting $\alpha R'^3 = 1 - z$ [2125'] in
the first member of [2123], we get $1 - z = 2 . \frac{R'}{R} - 2$, or $\frac{R'}{R} = \frac{3-z}{2}$, as in [2126]. [2125b]

† (1571) Taking the differential of [2122], r, θ, being variable, we have

$$0 = -\frac{2 d r}{r^2} + 2 \alpha . r d r . \sin.^2 \theta + 2 \alpha . r^2 d \theta . \sin. \theta . \cos. \theta. \qquad [2127a]$$

Dividing by the coefficient of $d r$, and reducing, we get $d r$ [2127].

‡ (1572) This expression is the same as in [352b], changing R into r [2117a], and
substituting $n^2 = \alpha m$ [2123a]. Now as the gravity $\frac{m}{r^2}$, in the direction of the radius, [2128a]
must exceed the centrifugal force in the same direction, we shall have

$$\frac{m}{r^2} - \alpha m . r . \sin.^2 \theta > 0.$$

Multiplying this by $\frac{r^2}{m}$, we get $1 - \alpha r^3 . \sin.^2 \theta > 0$, or positive, as in [2127']; and [2128b]
as the numerator of [2127] is positive, the expression of $d r$ must have the same sign as
$d\theta$; or in other words it must increase in proceeding from the pole to the equator.

We shall put the equation of the surface of the atmosphere [2122] under the following form,*

Surface of the atmosphere. Second form.

[2128] $$r^3 - \frac{2r}{\alpha R . \sin.^2 \theta} + \frac{2}{\alpha . \sin.^2 \theta} = 0.$$

The values of r, to be used in this problem, must always be positive, and
[2128'] such that $1 - \alpha r^3 . \sin.^2 \theta$ may exceed zero [2128b]. Now there can be but one root of this kind. For if we put p, p', p'', for the three values of r, corresponding to [2128], we shall have $p'' = -p - p'$, because the coefficient of r^2 in this equation is nothing; and if we suppose p, p', to be
[2128''] positive and less than† $\frac{1}{(\alpha . \sin.^2 \theta)^{\frac{1}{3}}}$, we shall have p'' negative and less than
[2128'''] $\frac{2}{(\alpha . \sin.^2 \theta)^{\frac{1}{3}}}$; consequently the product $-p . p' . p''$ will be less than $\frac{2}{\alpha . \sin.^2 \theta}$. But by the theory of equations, that product must be equal to this quantity; therefore the preceding supposition is impossible; and the

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[2128c] * (1573) Multiplying [2122] by $\frac{r}{\alpha . \sin.^2 \theta}$, and transposing the first term, we get the equation [2128].

† (1574) From [2128b] we have $\alpha r^3 . \sin.^2 \theta < 1$; dividing by $\alpha . \sin.^2 \theta$, and extracting the cube root, we get $r < (\alpha . \sin.^2 \theta)^{-\frac{1}{3}}$; and if there be two positive roots
[2128d] p, p', each of them must be less than $(\alpha . \sin.^2 \theta)^{-\frac{1}{3}}$, and their sum

$$p + p' < 2 . (\alpha . \sin.^2 \theta)^{-\frac{1}{3}}.$$

Now by the theory of equations, the coefficient of r^2, in [2128], is equal to the sum of the
[2128e] three roots of that equation, or $p + p' + p'' = 0$; and by the same theory, the product of the three roots is equal to the constant term of the same equation, changing its sign; hence
[2128f] $-p . p' . p'' = \frac{2}{\alpha . \sin.^2 \theta}$. From [2128e], we find that $p'' = -(p + p')$ would be
[2128g] negative, and, independent of its sign, less than $2 . (\alpha . \sin.^2 \theta)^{-\frac{1}{3}}$; hence the product of these three values of p, p', p'', [2128d, g], give $-p . p' . p'' < \frac{2}{\alpha . \sin.^2 \theta}$; and as this is
[2128h] inconsistent with the preceding value of $-p . p' . p''$ [2128f], it must follow, that there is but one positive value of r, which will satisfy the conditions of the problem.

equation in r can have but one positive root, which satisfies the problem; or [2128''''] in other words, the atmosphere has but one figure of equilibrium.

If we apply these results to the solar atmosphere, we shall find, First. *That this atmosphere can extend no farther than to the orbit of a planet, whose* [2128v] *periodical revolution is performed in the same time as the sun's rotatory motion* The solar atmosphere differs from the zodiacal light. *about its axis* [2124]; *or in twenty-five days and a half. Therefore it does not extend so far as the orbits of Mercury and Venus, and we know that the zodiacal light extends much beyond them.* Second. *The ratio of the polar to* [2128vi] *the equatorial diameter of the solar atmosphere cannot be less than* $\frac{2}{3}$ [2125], *and the zodiacal light appears under the form of a very flat lens, the apex of which is in the plane of the solar equator. Therefore the fluid which reflects to us the zodiacal light is not the atmosphere of the sun; and since it surrounds that planet, it must revolve about it, according to the same laws as the planets;* [2128vii] *perhaps this is the reason why its resistance to their motions is so insensible.*

FOURTH BOOK.

ON THE OSCILLATONS OF THE SEA AND ATMOSPHERE.

THE action of the sun and moon, upon the sea and atmosphere, produces oscillations in these fluid masses; and it is interesting to ascertain the law of these motions. The oscillations of the sea are known under the names
[2128viii] of the *ebb* and *flow*, and they are very sensible in our ports. Those of the atmosphere are small, and difficult to be observed, because they are combined with the irregular winds which continually agitate the atmosphere. We shall, in this book, consider these various motions.

CHAPTER I.

THEORY OF THE EBB AND FLOW OF THE SEA.

1. WE *shall resume the general equations of the motions of the sea*, given
Symbols. in the last chapter of the first book, using the following symbols,

[2128ix] $1 =$ The polar semi-axis of the earth;

[2128x] $\gamma =$ The depth of the sea, supposing it to be small in comparison with this semi-axis, and to be represented by a function of θ, ϖ, or μ, ϖ;

[2128xi] $\theta =$ The complement of the latitude of a particle dm, at the surface of the sea, when in a state of equilibrium, in which it would be placed, if the motions of the sun and moon were excluded;

[2128xii] $\mu = \cos.\theta =$ sin. latitude of the particle dm [2135′];

[2128xiii] $\varpi =$ The longitude of the same particle dm, when in a state of equilibrium: this longitude being estimated from a fixed meridian upon the surface of the earth;

$\alpha y =$ The elevation of the particle dm above the surface of equilibrium, [2128xiv]
when in a state of motion [326″]; in which case θ changes into
$\theta + \alpha u$, and ϖ into $\varpi + \alpha v$ [323v]; observing that y is negative when [2128xv]
the fluid is depressed below its situation in a state of equilibrium;

$n t =$ The mean rotatory motion of the earth; [2128xvi]

$g =$ The force of gravity. [2128xvii]

Then we have the two following equations,*

General equations for all parts of the fluid, in motion; whether upon or below the surface; θ, ϖ, being considered as the variable quantities.

$$y = -\left(\frac{d \,.\, \gamma u}{d\theta}\right) - \left(\frac{d \,.\, \gamma v}{d\varpi}\right) - \frac{\gamma u \,.\, \cos.\theta}{\sin.\theta}\,; \qquad (1) \quad [2129]$$

$$d\theta \,.\left\{\left(\frac{ddu}{dt^2}\right) - 2n \,.\left(\frac{dv}{dt}\right).\sin.\theta\,.\cos.\theta\right\}$$
$$+ d\varpi \,.\left\{\sin.^2\theta\,.\left(\frac{ddv}{dt^2}\right) + 2n\,.\left(\frac{du}{dt}\right).\sin.\theta\,.\cos.\theta\right\} = -g\,.\,\mathrm{d}y + \mathrm{d}V'\,; \qquad (2) \quad [2130]$$

the differentials $\mathrm{d}y$, $\mathrm{d}V'$, *refer only to the variable quantities* θ *and* ϖ. *The function* $\alpha\mathrm{d}V'$ *expresses, as we have seen in* § 35 *of the first book* [326iv], *the sum of the products of all the forces which disturb the state of equilibrium of the* [2130′]
particle dm, *by the elements of their directions, retaining only the differentials* $d\theta$ *and* $d\varpi$ [328′]. These forces are, *first*, the action of the sun and the moon; and we shall get the part of $\alpha\mathrm{d}V'$ corresponding to this action, by [2130″]
dividing the masses of the sun and moon, by their distances from the particle dm, respectively; then adding these quotients, and taking the differential [2130‴]

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* (1575) The equation [2129] is the same as [347]; and [338], divided by r^2, becomes,

$$\delta\theta\,.\left\{\left(\frac{ddu}{dt^2}\right) - 2n\,.\left(\frac{dv}{dt}\right).\sin.\theta\,.\cos.\theta\right\} + \delta\varpi\,.\left\{\sin.^2\theta\,.\left(\frac{ddv}{dt^2}\right) + 2n\,.\left(\frac{du}{dt}\right).\sin.\theta\,.\cos.\theta\right\}$$
$$= \frac{-g\,.\,\delta y + \delta V'}{r^2}\,. \qquad [2130a]$$

Now from [342v] it follows, that if we consider only the quantities θ, ϖ, as variable, this equation [2130a′]
will be the same as the equation [325], which corresponds to every part of the fluid, whether upon the surface, or below it. Therefore [2130a] will also correspond to all parts of the fluid, supposing δy, $\delta V'$, in the second member, to be affected only by the differentials of θ, ϖ. Moreover, as the radius r differs but little from unity, because the ellipticity of the [2130b]
earth is less than $\frac{1}{300}$, and the depth of the sea very small, we may put $r = 1$; and then [2130a] becomes as in [2130].

of the sum, relatively to the variable quantities θ and ϖ. Now if we put r for the distance of the body L from the centre of the earth, v for its
[2130''''] declination, and ψ its right ascension; we shall have its distance from the particle dm, by § 23 of the third book, nearly equal to*

[2131] $$\sqrt{r^2-2r.\{\cos.\theta\,.\,\sin.\mathrm{v}+\sin.\theta\,.\,\cos.\mathrm{v}\,.\,\cos.(nt+\varpi-\psi)\}+1}\ ;$$

[2131'] the angle $nt+\varpi$ being counted, like the angle ψ, from the vernal equinox; hence the part of $\alpha\,\mathrm{d}V'$ corresponding to the action of the body L, is obtained by taking the differential of the following expression, relatively to θ and ϖ,

[2132] $$\frac{L}{\sqrt{r^2-2r.\{\cos.\theta\,.\,\sin.\mathrm{v}+\sin.\theta\,.\,\cos.\mathrm{v}\,.\,\cos.(nt+\varpi-\psi)\}+1}}\,.$$

But as we have supposed the centre of gravity of the earth to be immoveable,
[2132'] we must transfer to the particle dm, in a contrary direction, the force which affects this centre, by the action of L;† and we have seen, in § 23

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* (1576) If we suppose, as in the figure page 247, C to be the centre of the spheroid,
[2130c] CAX the axis of revolution, M the place of the particle dm, S the place of the attracting body, $CXEB$ the plane passing through the vernal equinox, from which the angles ϖ, ψ,
[2130d] are counted; we shall have, by the present notation, $CS=r$, $CM=1$, nearly; the angles $XCM=\theta$, $MAB=nt+\varpi$, $SDE=\psi$, $SCX=100^\circ-\mathrm{v}$; which
[2130e] correspond respectively to s, r, θ, ϖ, ψ, v, in the notation used in [1620a, &c.]. If we make these changes in the value of $SM=f$ [1625c'], putting also

$$\cos.(\psi-\varpi)=\cos.(\varpi-\psi),$$

it will become as in [2131]. Dividing the mass L by this distance f, we get the part of
Angular distance from the meridian. $\alpha\mathrm{d}V'$ [2130''] depending on the body L, as in [2132], which is similar to [1626]. From this notation, it appears, that when the angles MAB, SDE are equal, or

[2131a] $$nt+\varpi-\psi=0,$$

the attracting body is on the meridian *above* the horizon; when

[2131b] $$MAB=200^\circ+SDE, \qquad \text{or} \qquad nt+\varpi-\psi=200^\circ,$$

[2131c] the body is on the meridian *below* the horizon; and generally $nt+\varpi-\psi$ represents the distance of the attracting body from the superior meridian.

† (1577) This is given in [1622], and by substituting [1624, 1631a], the force becomes
[2133a] $\frac{S}{f}-\frac{S}{s^2}.r\delta-\frac{S}{s}$. Now putting L for S, and altering the notation, as in [2130d, e], we get

of the third book, that this is equivalent to the subtraction of the following expression from that in [2132],

$$\frac{L}{r}+\frac{L}{r^2}\cdot\{\cos.\theta\,.\,\sin.\mathrm{v}+\sin.\theta\,.\,\cos.\mathrm{v}\,.\,\cos.(nt+\varpi-\psi)\};$$ [2133]

Therefore we shall obtain the value of $\alpha\mathrm{d}V'$ depending on the action of L, by subtracting the function [2133] from [2132], and taking the differential of the remainder, considering θ, ϖ, as the only variable quantities. Now [2133'] by § 23 of the third book, this difference may be developed in a series, descending relative to the powers of r; so that if we denote it by*

$$\frac{\alpha Z^{(0)}}{r}+\frac{\alpha Z^{(2)}}{r^3}+\frac{\alpha Z^{(3)}}{r^4}+\frac{\alpha Z^{(4)}}{r^5}+\&c.$$ [2134]

$Z^{(i)}$ will be a rational and integral function of μ, $\sqrt{1-\mu^2}\,.\,\sin.\varpi$, and $\sqrt{1-\mu^2}\,.\,\cos.\varpi$, of the degree i, subjected to the following equation of partial differentials, [2134']

$$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Z^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}+\frac{\left(\frac{d\,d\,Z^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,Z^{(i)};$$ [2135]

in which $\mu=\cos.\theta$ [1616xxi]. [2135']

$\alpha\,\mathrm{d}V'$ is composed also of the attraction exerted upon the particle dm, [2135''] by the aqueous stratum whose internal radius is unity, and external radius $1+\alpha y$. Now it is evident, that to determine it, we must divide each of the [2135'''] particles of the stratum, by its distance from the particle dm, and then

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$\frac{L}{f}-\frac{L}{r^2}\,.\,\delta-\frac{L}{r}$ for the whole force; being equal to the difference of the expressions [2133b] [2132, 2133]; observing that δ [1629], changing the notation as in [2130d, e], becomes

$$\delta=\cos.\theta\,.\,\sin.\mathrm{v}+\sin.\theta\,.\,\cos.\mathrm{v}\,.\,\cos.(n\,t+\varpi-\psi).$$ [2133c]

* (1578) Changing, as in [2130e], S, s, r, into L, r, 1, respectively, the expression of the force [1631, 1631', 1631a] will become

$$\frac{L}{f}-\frac{L}{r}-\frac{L}{r^2}\,.\,\delta=\frac{L}{r^2}\,.\left\{P^{(2)}+\frac{1}{r}\,.\,P^{(3)}+\frac{1}{r^2}\,.\,P^{(4)}+\&c.\right\};$$ [2134a]

and by putting $LP^{(i)}=\alpha Z^{(i)}$, in the second member, it becomes as in [2134]. Multiplying [1630] by $\frac{L}{\alpha}$, and substituting $\frac{LP^{(i)}}{\alpha}=Z^{(i)}$, we get [2135].

take the differential of the sum of these quotients, relatively to θ and ϖ.
[2135''''] We must also transfer to this particle, in a contrary direction, the action of the stratum upon the centre of gravity of the earth; but as this centre does
[2135v] not change its situation, by means of the attraction and pressure of the different parts of the earth [404'], it is evident that this action must be neglected.

Case in which the earth has no rotatory motion,

2. *We shall first consider the case in which the earth has no rotatory motion,*
[2135vi] *consequently we shall have* $n=0$. *We shall also suppose the earth to be*

and the depth of the sea is constant.

spherical, and the depth of the sea γ *to be equal to a constant quantity* l. We shall then determine the oscillations that would be produced by the action of
[2135vii] the sun and moon. The equation [2129] of the preceding article becomes, by
[2135viii] putting $\cos.\theta=\mu$, and $\gamma=l$,*

[2136]
$$\frac{y}{l}=\left(\frac{d.(u.\sqrt{1-\mu^2})}{d\mu}\right)-\left(\frac{dv}{d\varpi}\right);$$

and the equation [2130] becomes,†

[2137]
$$d\theta.\left(\frac{ddu}{dt^2}\right)+d\varpi.(1-\mu^2).\left(\frac{ddv}{dt^2}\right)=-g.\left(\frac{dy}{d\mu}\right).\left(\frac{d\mu}{d\theta}\right).d\theta-g.\left(\frac{dy}{d\varpi}\right).d\varpi$$
$$+\left(\frac{dV'}{d\mu}\right).\left(\frac{d\mu}{d\theta}\right).d\theta+\left(\frac{dV'}{d\varpi}\right).d\varpi.$$

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* (1579) The differential of $\cos.\theta=\mu$ [2135'], being divided by $\sin.\theta=\sqrt{(1-\mu^2)}$,
[2136a] gives $d\theta=-\frac{d\mu}{\sin.\theta}=-\frac{d\mu}{\sqrt{(1-\mu^2)}}$. Substituting this in $-\left(\frac{d.\gamma u}{d\theta}\right)-\frac{\gamma u.\cos.\theta}{\sin.\theta}$, it becomes $\left(\frac{d.\gamma u}{d\mu}\right).\sqrt{(1-\mu^2)}-\frac{\gamma u.\mu}{\sqrt{(1-\mu^2)}}$, which is equal to $\left(\frac{d.\{\gamma u.\sqrt{(1-\mu^2)}\}}{d\mu}\right)$; as is easily proved, by developing this last expression, after dividing the numerator into two factors, γu and $\sqrt{(1-\mu^2)}$. Substituting this in [2129], we get,

[2136b]
$$y=\left(\frac{d.\{\gamma u.\sqrt{(1-\mu^2)}\}}{d\mu}\right)-\left(\frac{d.\gamma v}{d\varpi}\right);$$

in which the constant depth $\gamma=l$ may be brought from under the signs of differentiation. Dividing this by l, we get [2136].

† (1580) The rotatory motion nt [2128xvi] being nothing, gives $n=0$. Then [2130]
[2137a] becomes $d\theta.\left(\frac{ddu}{dt^2}\right)+d\varpi.\sin.^2\theta.\left(\frac{ddv}{dt^2}\right)=-g.\mathrm{d}\,y+\mathrm{d}\,V'$; and as the sign d. in the second member affects only θ, ϖ, [2130'], we shall have,

[2137b]
$$\mathrm{d}\,y=\left(\frac{dy}{d\theta}\right).d\theta+\left(\frac{dy}{d\varpi}\right).d\varpi;\qquad \mathrm{d}\,V'=\left(\frac{dV'}{d\theta}\right).d\theta+\left(\frac{dV'}{d\varpi}\right).d\varpi;$$

now we have $\left(\frac{d\mu}{d\theta}\right) = -\sqrt{1-\mu^2}$ [2136a]; therefore, by comparing the [2137′]
coefficients of $d\theta$, $d\varpi$, we shall find,*

$$\left(\frac{ddu}{dt^2}\right) = g\cdot\left(\frac{dy}{d\mu}\right)\cdot\sqrt{1-\mu^2} - \left(\frac{dV'}{d\mu}\right)\cdot\sqrt{1-\mu^2}\,; \qquad [2138]$$

$$\left(\frac{ddv}{dt^2}\right) = -\frac{g\cdot\left(\frac{dy}{d\varpi}\right)}{1-\mu^2} + \frac{\left(\frac{dV'}{d\varpi}\right)}{1-\mu^2}\,; \qquad [2139]$$

therefore†

$$\left(\frac{d.\left\{\left(\frac{ddu}{dt^2}\right)\cdot\sqrt{1-\mu^2}\right\}}{d\mu}\right) = g\cdot\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dy}{d\mu}\right)\right\}}{d\mu}\right\} - \left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dV'}{d\mu}\right)\right\}}{d\mu}\right\}; \qquad [2140]$$

$$\left(\frac{d.\left(\frac{ddv}{dt^2}\right)}{d\varpi}\right) = -g\cdot\frac{\left(\frac{ddy}{d\varpi^2}\right)}{1-\mu^2} + \frac{\left(\frac{ddV'}{d\varpi^2}\right)}{1-\mu^2}. \qquad [2141]$$

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hence, by substitution in [2137a], we get,

$$d\theta\cdot\left(\frac{ddu}{dt^2}\right) + d\varpi\,.\sin.^2\theta\,.\left(\frac{ddv}{dt^2}\right)$$
$$= -g\cdot\left(\frac{dy}{d\theta}\right)\cdot d\theta - g\cdot\left(\frac{dy}{d\varpi}\right)\cdot d\varpi + \left(\frac{dV'}{d\theta}\right)\cdot d\theta + \left(\frac{dV'}{d\varpi}\right)\cdot d\varpi. \qquad [2137c]$$

But as y and V' are now supposed to be functions of μ, instead of θ, they will only contain θ, as it is found in μ; so that we shall have,

$$\left(\frac{dy}{d\theta}\right) = \left(\frac{dy}{d\mu}\right)\cdot\left(\frac{d\mu}{d\theta}\right), \qquad \left(\frac{dV'}{d\theta}\right) = \left(\frac{dV'}{d\mu}\right)\cdot\left(\frac{d\mu}{d\theta}\right), \qquad \sin.^2\theta = 1-\mu^2\,; \qquad [2137d]$$

hence, by substitution, we get [2137].

* (1581) The equation [2137] was successively deduced from [2130, 338, 325, 296]; the quantities $d\theta$, $d\varpi$, [2137], taking the place of the variations δx, δy, δz, [296], whose coefficients are to be put separately equal to nothing [296′]. Now if we substitute [2138a] [2137′] in [2137], we shall find that the coefficient of $d\theta$, put equal to nothing, will give [2138]. Also the coefficient of $d\varpi$, put equal to nothing, and divided by $(1-\mu^2)$, will give [2139].

† (1582) Multiplying [2138] by $\sqrt{(1-\mu^2)}$, and taking its first differential relative [2139a] to μ, we get [2140]. The differential of [2139], relative to ϖ, gives [2141].

The preceding expression of $\frac{y}{l}$ gives,*

[2142] $$\left(\frac{d\,d\,y}{d\,t^2}\right)=l\,.\left\{\frac{d\,.\left\{\left(\frac{d\,d\,u}{d\,t^2}\right).\sqrt{1-\mu^2}\right\}}{d\,\mu}\right\}-l\,.\left\{\frac{d\,.\left(\frac{d\,d\,v}{d\,t^2}\right)}{d\,\varpi}\right\};$$

therefore we shall have,

[2143] $$\left(\frac{d\,d\,y}{d\,t^2}\right)=l\,g\,.\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,y}{d\,\mu}\right)\right\}}{d\,\mu}\right\}+\frac{l\,g\,.\left(\frac{d\,d\,y}{d\,\varpi^2}\right)}{1-\mu^2}$$
$$-l\,.\left\{\frac{d\,.\left\{(1-\mu^2).\left(\frac{dV'}{d\,\mu}\right)\right\}}{d\,\mu}\right\}-\frac{l\,.\left(\frac{d\,d\,V'}{d\,\varpi^2}\right)}{1-\mu^2}.$$

To integrate this equation, we shall put [1464]

[2144] $$y=Y^{(0)}+Y^{(1)}+Y^{(2)}+Y^{(3)}+\&c.\,;$$

$Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}$, &c., being rational and integral functions of

[2144′] $$\mu,\qquad \sqrt{1-\mu^2}\,.\,\sin.\,\varpi,\qquad \sqrt{1-\mu^2}\,.\,\cos.\,\varpi\,;$$

and of such forms that we shall have generally [1465],

[2145] $$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)\right\}}{d\,\mu}\right\}+\frac{\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,Y^{(i)}.$$

[2145′] The part of V' relative to the spherical stratum of fluid, whose internal radius is unity, and external radius $1+\alpha y$, is, by § 14 of the third book, supposing the density of the sea equal to unity,†

[2146] $$4\pi\,.\left\{Y^{(0)}+\tfrac{1}{3}\,Y^{(1)}+\tfrac{1}{5}\,Y^{(2)}+\tfrac{1}{7}\,Y^{(3)}+\&c.\right\};$$

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* (1583) The second differential of [2136], taken relatively to d, and divided by $d\,t^2$,
[2143a] gives [2142]; observing that y, u, v, are the only quantities which vary with t, μ being the same for all values of t, since it represents the sine of the latitude of the particle at the commencement of the motion, or in the state of equilibrium [2128xii]. Substituting [2140, 2141] in [2142], it becomes as in [2143].

† (1585) Putting $a=1$ in [1486′, 1491], we get the value of V, corresponding to the stratum [2145′], which is represented in [2135″] by $\alpha V'$; hence

[2145a] $$\alpha\,V'=4\,\alpha\,\pi\,.\left\{Y^{(0)}+\tfrac{1}{3}\,r\,.\,Y^{(1)}+\tfrac{1}{5}\,r^2\,.\,Y^{(2)}+\&c.\right\};$$

π being the ratio of the semi-circumference of a circle to its radius. If we put ρ for the mean density of the earth, we shall have* $g = \frac{4}{3}\pi\rho$, [2146′]
consequently $4\pi = \frac{3g}{\rho}$. [2147]

It follows, from the preceding article, that the part of $\alpha V'$ depending upon the action of the sun and moon, and also in general upon the action of any number of attracting bodies, may be developed in a series of the form,† [2147′]

$$\alpha U^{(0)} + \alpha U^{(2)} + \alpha U^{(3)} + \alpha U^{(4)} + \&c. ; \qquad [2148]$$

$U^{(i)}$ being a rational and integral function of μ, $\sqrt{1-\mu^2}.\sin.\varpi$, $\sqrt{1-\mu^2}.\cos.\varpi$, of the order i, which satisfies the following equation of partial differentials, [2148′]

$$0 = \left\{ \frac{d.\left\{(1-\mu^2).\left(\frac{dU^{(i)}}{d\mu}\right)\right\}}{d\mu} \right\} + \frac{\left(\frac{ddU^{(i)}}{d\varpi^2}\right)}{1-\mu^2} + i.(i+1).U^{(i)}. \qquad [2149]$$

This being premised; if we substitute these values of y and V' in the equation [2143], the comparison of the similar functions $U^{(i)}$ and $Y^{(i)}$, will give,‡

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r being the distance of the attracted particle from the centre of the earth; and as this particle is placed upon the surface of the earth, r must differ from a, or 1, by quantities of the order α; therefore by neglecting α^2, we may put $r=1$; then dividing by α, it becomes as in [2146]. [2145b]

* (1586) This is the same as [1430l], changing ρ' into ρ, and putting $r=1$; then dividing by $\frac{1}{3}\rho$, we get 4π [2147]; substituting this in V' [2146], we get

$$V' = \frac{3g}{\rho}.\{Y^{(0)} + \tfrac{1}{3}Y^{(1)} + \tfrac{1}{5}Y^{(2)} + \&c.\} = \Sigma.\frac{3g}{\rho}.\frac{Y^{(i)}}{2i+1}. \qquad [2146a]$$

We may observe, that in the formula [1430l], to which we have referred, the value of g is computed for a homogeneous sphere [2135$^{\text{vii}}$] of the density ρ' or ρ; but it is evident, from [470‴, &c.], that the result will be the same, if we suppose the sphere to be composed of concentrical strata, of variable densities, the mean density of the whole mass being ρ. [2146b]

† (1587) The forms assumed in [2148, 2149], for several bodies, is the same as [2134, 2135], for one attracting body; observing that r [2134] represents the distance of the body from the earth, which is not affected by the differentials in [2135]. [2148a]

‡ (1588) Using the sign of finite integrals in [2148], we shall get $\alpha.\Sigma U^{(i)}$ for that part of $\alpha V'$, or $\Sigma U^{(i)}$ for the corresponding part of V'. Adding this to the other

[2150] $$\left(\frac{dd\,Y^{(i)}}{dt^2}\right)+\frac{i\,.\,(i+1)\,.\,lg}{(2i+1)\,.\,\rho}\,.\,\{(2i+1)\,.\,\rho-3\}\,.\,Y^{(i)}=i\,.\,(i+1)\,.\,l\,.\,U^{(i)}.$$

Supposing for brevity,

[2151] $$\frac{i\,.\,(i+1)\,.\,lg}{(2i+1)\,.\,\rho}\,.\,\{(2i+1)\,.\,\rho-3\}=\lambda_i^2\,;$$

[2150a] part of V' [2146a], we obtain its complete value $V'=\Sigma\frac{3g}{\rho}\,.\,\frac{Y^{(i)}}{2i+1}+\Sigma U^{(i)}$; substituting this and $y=\Sigma\,Y^{(i)}$ [2144] in [2143], we have

[2150b] $$\Sigma\left(\frac{dd\,Y^{(i)}}{dt^2}\right)=\Sigma\,lg\,.\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\Sigma\,lg\,.\,\frac{\left(\frac{dd\,Y^{(i)}}{d\varpi^2}\right)}{1-\mu^2}$$

[2150c] $$-\Sigma\frac{3lg}{(2i+1)\,.\,\rho}\,.\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,Y^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}-\Sigma\frac{3lg}{(2i+1)\,.\,\rho}\,.\,\frac{\left(\frac{dd\,Y^{(i)}}{d\varpi^2}\right)}{1-\mu^2}$$

[2150d] $$-\Sigma\,l\,.\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{d\,U^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}-\Sigma\,l\,.\,\frac{\left(\frac{dd\,U^{(i)}}{d\varpi^2}\right)}{1-\mu^2}.$$

The terms in the three lines of the second member of this expression are easily reduced, by means of [2145, 2149]. For if we transpose the two first terms of [2145], then multiply by $-lg$, and prefix the sign Σ, we shall get, for the first line, marked [2150b], the
[2150e] equivalent expression $-\Sigma\,lg\,.\,i\,.\,(i+1)\,.\,Y^{(i)}$. Repeating the operation with the factor $\frac{3lg}{(2i+1)\,.\,\rho}$, instead of $-lg$, we get the second line [2150c] equal to

$$\Sigma\frac{3lg}{(2i+1)\,.\,\rho}\,.\,i\,.\,(i+1)\,.\,Y^{(i)}.$$

In like manner, by transposing the two first terms of [2149], multiplying by l, and prefixing
[2150f] the sign Σ, we obtain the third line [2150d] equal to $\Sigma\,l\,.\,i\,.\,(i+1)\,.\,U^{(i)}$. Hence, by substitution in [2150b—d], we have,

[2150g] $$\Sigma\left(\frac{dd\,Y^{(i)}}{dt^2}\right)=-\Sigma\,lg\,.\,i\,.(i+1)\,.\,Y^{(i)}+\Sigma\frac{3lg}{(2i+1)\,.\,\rho}\,.\,i\,.(i+1)\,.Y^{(i)}+\Sigma\,l\,.\,i\,.(i+1)\,.U^{(i)}.$$

Transposing the two first terms of the second member, after reducing them to the common denominator $(2i+1)\,.\,\rho$, we get the general term depending on the index i, as in [2150]; and as there can be only one form of development of this kind [1479'], the terms depending on i must be put separately equal to nothing, and by this means we shall obtain the equation [2150]; which, by using the symbol λ_i^2 [2151], may be put under the following form,

[2150h] $$0=\left(\frac{dd\,Y^{(i)}}{dt^2}\right)+\lambda_i^2\,.\,Y^{(i)}-i\,.\,(i+1)\,.\,l\,.\,U^{(i)}.$$

the preceding differential equation will give, by integration,*

$$Y^{(i)} = l \,.\, M^{(i)} \,.\, \sin.\lambda_i t + l \,.\, N^{(i)} \,.\, \cos.\lambda_i t$$
$$+ \frac{i.(i+1)}{\lambda_i} \,.\, l \,.\, \sin.\lambda_i t \,.\, \int U^{(i)} \,.\, dt \,.\, \cos.\lambda_i t$$
$$- \frac{i.(i+1)}{\lambda_i} \,.\, l \,.\, \cos.\lambda_i t \,.\, \int U^{(i)} \,.\, dt \,.\, \sin.\lambda_i t. \qquad [2152]$$

$M^{(i)}$ and $N^{(i)}$ being rational and integral functions of μ, $\sqrt{1-\mu^2}\,.\,\sin.\varpi$, and $\sqrt{1-\mu^2}\,.\,\cos.\varpi$, which satisfy the following equations of partial differentials, [2152′]

$$0 = \left\{ \frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{dM^{(i)}}{d\mu}\right)\right\}}{d\mu} \right\} + \frac{\left(\frac{ddM^{(i)}}{d\varpi^2}\right)}{1-\mu^2} + i\,.\,(i+1)\,.\,M^{(i)}; \qquad [2153]$$

$$0 = \left\{ \frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{dN^{(i)}}{d\mu}\right)\right\}}{d\mu} \right\} + \frac{\left(\frac{ddN^{(i)}}{d\varpi^2}\right)}{1-\mu^2} + i\,.\,(i+1)\,.\,N^{(i)}. \qquad [2154]$$

The differential equation in $Y^{(i)}$ [2150] gives, by supposing $i=0$, $\left(\frac{ddY^{(0)}}{dt^2}\right)=0$; hence we get,† [2154′]

$$Y^{(0)} = l \,.\, M^{(0)} \,.\, t + l \,.\, N^{(0)}; \qquad [2155]$$

the equation [2144],

$$y = Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c., \qquad [2156]$$

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* (1589) If in [865] we put $y=Y^{(i)}$, $a=\lambda_i$, $aQ=-i.(i+1)\,.\,l\,.\,U^{(i)}$, it becomes as in [2150*h*]. Making the same changes in the value of y [870], and putting for brevity $c=\lambda_i l\,.\,M^{(i)}$, $c'=\lambda_i l\,.\,N^{(i)}$, so that $M^{(i)}$, $N^{(i)}$, may be the arbitrary constant quantities, it becomes as in [2152]. This value of $Y^{(i)}$ ought to satisfy the equation [2145], and it is evident that the two terms of [2152] depending on $U^{(i)}$ must satisfy the similar equation [2149]. Moreover, the terms of [2152] depending on $M^{(i)}$, $N^{(i)}$, being connected with the different expressions $\sin.\lambda_i t$, $\cos.\lambda_i t$, must separately satisfy the equation [2145]; which requires that we should have the two equations [2153, 2154]; observing that $M^{(i)}$, $N^{(i)}$, must satisfy the same conditions as $Y^{(i)}$ [2144′, &c.]. [2150*i*] [2150*k*]

† (1590) Multiplying the equation [2154′] by dt, and integrating, we get, $\left(\frac{dY^{(0)}}{dt}\right)$ equal to the constant quantity $l\,.\,M^{(0)}$. Again multiplying by dt, integrating, and adding the constant quantity $l\,.\,N^{(0)}$, we get $Y^{(0)}$ [2155]. [2154*a*]

will therefore become,*

General expression of y.

$$
\begin{aligned}
y = l.M^{(0)}.t + l.N^{(0)} &+ l.M^{(1)}.\sin.\lambda_1 t + l.N^{(1)}.\cos.\lambda_1 t \\
&+ l.M^{(2)}.\sin.\lambda_2 t + l.N^{(2)}.\cos.\lambda_2 t \\
&\quad \cdots\cdots\cdots \\
&+ l.M^{(i)}.\sin.\lambda_i t + l.N^{(i)}.\cos.\lambda_i t \\
&+ \&c. \\
&+ \frac{6l}{\lambda_2}.\sin.\lambda_2 t.\textstyle\int U^{(2)}.dt.\cos.\lambda_2 t \\
&- \frac{6l}{\lambda_2}.\cos.\lambda_2 t.\textstyle\int U^{(2)}.dt.\sin.\lambda_2 t \\
&\quad \cdots\cdots\cdots \\
&+ \frac{i.(i+1).l}{\lambda}.\sin.\lambda_i t.\textstyle\int U^{(i)}.dt.\cos.\lambda_i t \\
&- \frac{i.(i+1).l}{\lambda_i}.\cos.\lambda_i t.\textstyle\int U^{(i)}.dt.\sin.\lambda_i t \\
&+ \&c.
\end{aligned}
\qquad [2157]
$$

[2157′] *We can determine the functions* $N^{(0)}$, $N^{(1)}$, $N^{(2)}$, &c., by means of the initial figure of the fluid; and the functions $M^{(0)}$, $M^{(1)}$, $M^{(2)}$, &c., by means of the initial velocity; therefore the preceding expression of y will include all the figures, and all the velocities, which the fluid can possibly have; consequently the form of y is as general as can be desired.

[2157″] If the quantity $M^{(0)}$ be finite, the value of y will increase without any limit, and the equilibrium will not be stable, whatever may be the ratio of the density of the fluid to that of the sphere which it covers. But it is easy
[2157‴] to prove that the quantities $M^{(0)}$ and $N^{(0)}$ vanish, because the mass of
[2157⁗] the fluid is constant. This condition gives $\int y.d\mu.d\varpi = 0$,† the integral

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* (1591) Substituting in [2156] the value of $Y^{(0)}$ [2155], and those of $Y^{(1)}$, $Y^{(2)}$, &c., deduced from [2152], by putting successively $i=1$, $i=2$, &c., we get [2157];
[2157a] observing that $U^{(1)}$ vanishes from [2148], or $U^{(1)}=0$, and it will therefore vanish from [2157].

† (1592) If $R_{\prime}$ be the value of the radius of the spheroid, corresponding to μ, ϖ, when
[2157b] the fluid is in equilibrium, and R' its value when in motion, we shall have $R' = R_{\prime} + \alpha y$

being taken from $\mu=-1$ to $\mu=1$, and from $\varpi=0$ to $\varpi=2\pi$. Now we have in general, by § 12 of the third book [1476],

$$\int Y^{(i)} \cdot U^{(i')} \cdot d\mu \cdot d\varpi = 0,$$ [2158]

when i and i' are different integral numbers; therefore we shall have,*

$$\int y \cdot d\mu \cdot d\varpi = 4\pi \cdot Y^{(0)} = 4\pi \cdot l \cdot \{M^{(0)} \cdot t + N^{(0)}\}.$$ [2159]

Putting this quantity equal to nothing, we shall get $M^{(0)}=0$ and $N^{(0)}=0$. [2160] $M^{(0)}$, $N^{(0)}$, vanish.

Hence it follows, that the stability of the equilibrium of the fluid depends on the signs of the quantities λ_1^2, λ_2^2, &c. For if one of these quantities, as λ_i^2, be negative, the sine and cosine of the angle $\lambda_i t$ will be exponential quantities;† and they become arcs of a circle, if $\lambda_i^2=0$.‡ In these two [2160′] [2160″]

[2128xiv]. Substituting this in the second expression of the mass M of the spheroid [1467a], it becomes, by neglecting α^2,

$$M = \tfrac{1}{3}\int R'^3 \cdot d\mu \cdot d\varpi = \tfrac{1}{3} \cdot \int R_i^3 \cdot d\mu \cdot d\varpi + \alpha \cdot \int R_i^2 \cdot y \cdot d\mu \cdot d\varpi.$$

In the state of equilibrium, when $y=0$, this becomes $M=\frac{1}{3}\cdot\int R_i^3 . d\mu . d\varpi$. The difference of these two expressions, divided by α, is $0=\int R_i^2 . y . d\mu . d\varpi$. Supposing now the earth to be spherical, [2135vi], and its radius $R_i=1$ [2145′], the preceding equation will become as in [2157″″]. [2157c]

* (1593) Substituting the value of y [2156] in the equation $\int y . d\mu . d\varpi = 0$ [2157″″], and reducing as in note 1010, page 105, it becomes $4\pi \cdot Y^{(0)}=0$, or $Y^{(0)}=0$; hence $l \cdot \{M^{(0)} \cdot t + N^{(0)}\}=0$ [2155]; and as l is finite, we must have, for all values of t, $M^{(0)} \cdot t + N^{(0)} = 0$. Now when $t=0$, this becomes $N^{(0)}=0$; subtracting this from the preceding, we get $M^{(0)} \cdot t = 0$; which cannot exist for all values of t, except we have $M^{(0)}=0$, as in [2160]. [2158a]

† (1594) This is proved in note 179, Vol. I, page 187. [2159a]

‡ (1595) If we put $\lambda_i^2=0$ in [2150h], it becomes,

$$0=\left(\frac{dd\,Y^{(i)}}{dt^2}\right) - i \cdot (i+1) \cdot l \cdot U^{(i)};$$

multiplying this by dt, and integrating, we get, $\left(\frac{d\,Y^{(i)}}{dt}\right) = i \cdot (i+1) \cdot l \cdot U^{(i)} . t + B$, [2160a]

B being a constant quantity added to complete the integral. Again multiplying by dt, integrating, and adding the constant quantity A, we obtain

$$Y^{(i)} = \tfrac{1}{2} i \cdot (i+1) \cdot l \cdot U^{(i)} \cdot t^2 + Bt + A,$$ [2160b]

depending on the arc of a circle t, which may increase indefinitely.

cases, the value of y ceases to be periodical; and it is necessary that it should be periodical, for the stability of the equilibrium. Now we have [2151]

[2160‴] $$\lambda_i^2 = \frac{i.(i+1).lg}{(2i+1).\rho}.\{(2i+1).\rho-3\},$$

[2161] and this quantity cannot be positive, except in the case where* $\rho > \frac{3}{2i+1}$; i being an integral positive number, equal to unity, or greater than unity. *It is therefore necessary, for the stability of the equilibrium, that this condition should be satisfied, for all the values of* i; *but this cannot take place except* $\rho > 1$:
[2162] *or in other words, the density of the nucleus* [2146′] *must exceed that of the fluid. This is therefore the general condition of the stability of the equilibrium, and if it be satisfied, it will render the equilibrium stable, whatever be the*
[2162′] *primitive impulse of the fluid; but if it be not satisfied, the stability of the equilibrium will depend on the nature of that impulse.*

If the density of the nucleus exceed that of the fluid, the equilibrium of the fluid will be stable.

[2162″] For example, if the primitive impulse be such, that the centre of gravity of the fluid coincide with that of the nucleus which it covers; and at the same instant the motion of these centres, relatively to each other, be
[2163] nothing; then $Y^{(1)}$ and $\left(\frac{dY^{(1)}}{dt}\right)$ will be nothing, at the first instant; since this coincidence depends wholly upon the value of $Y^{(1)}$, as is shown
[2163′] in § 31 of the third book;† therefore this value will be nothing at every

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* (1596) l, g, ρ, are positive; and if we wish to have λ_i^2 [2151] positive, we must make
[2161a] the factor $(2i+1).\rho-3$ positive, or $\rho > \frac{3}{2i+1}$; but $M^{(0)}=0$, $N^{(0)}=0$, [2160]; therefore the least value of i which occurs in [2157] is $i=1$. The values of $\frac{3}{2i+1}$, corresponding to $i=1$, $i=2$, $i=3$, &c., are 1, $\frac{3}{5}$, $\frac{3}{7}$, &c.; so that
[2161b] the greatest of all these values is 1; and if ρ exceed 1, the equilibrium will be stable.

† (1597) In the time dt, after the first impulse, the function $Y^{(1)}$ becomes

[2163a] $$Y^{(1)} + \left(\frac{dY^{(1)}}{dt}\right).dt;$$

and in both these cases, the centres of gravity of the fluid and nucleus are supposed to coincide [2163]; therefore, by the formula [1745], both these expressions [2163a] must
[2163b] be equal to nothing, and their difference, divided by dt, will give $\left(\frac{dY^{(1)}}{dt}\right)=0$, as in [2163].

instant,* consequently the centre of gravity of the fluid will always coincide [2163′]
with that of the nucleus. In this case the stability of the equilibrium depends on the sign of λ_2^2, and to render this quantity positive, it is only necessary to have† $\rho > \frac{3}{5}$. [2164]

The value of y gives immediately those of u and v; for the equation

$$\left(\frac{d\,d\,u}{d\,t^2}\right) = g\cdot\left(\frac{d\,y}{d\,\mu}\right)\cdot\sqrt{1-\mu^2} - \left(\frac{d\,V'}{d\,\mu}\right)\cdot\sqrt{1-\mu^2}, \qquad [2165]$$

[2138], gives,‡

$$\left(\frac{d\,d\,u}{d\,t^2}\right) = \Sigma\,.\sqrt{1-\mu^2}\,.\left\{g\,.\left(1-\frac{3}{(2\,i+1)\,.\,\rho}\right).\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)-\left(\frac{d\,U^{(i)}}{d\,\mu}\right)\right\}; \qquad [2166]$$

the sign Σ of finite integrals includes all integral positive values of i, also [2166′]
$i=0$. But we have, by what precedes,§

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* (1598) Putting $i=1$, and $U^{(1)}=0$ [2157*a*], in [2152], we get,

$$Y^{(1)} = l\,.\,M^{(1)}\,.\,\sin.\,\lambda_1\,t + l\,.\,N^{(1)}\,.\,\cos.\,\lambda_1\,t\,; \qquad [2164a]$$

whose differential, relative to $d\,t$, gives

$$\left(\frac{d\,Y^{(1)}}{d\,t}\right) = l\,.\,\lambda_1\,M^{(1)}\,.\,\cos.\,\lambda_1\,t - l\,.\,\lambda_1\,N^{(1)}\,.\,\sin.\,\lambda\ t.$$

At the moment of impulse, when $t=0$, both these expressions become nothing [2163]; hence $0=l\,.\,N^{(1)}$, $0=l\,.\,\lambda_1\,M^{(1)}$; consequently $N^{(1)}=0$, $M^{(1)}=0$. [2164b]
Substituting these in the general value of $Y^{(1)}$ [2164*a*], it becomes $Y^{(1)}=0$, whatever be the value of t.

† (1599) We have $Y^{(0)}=0$ [2158*a*], $Y^{(1)}=0$ [2164*b*]; therefore the least value of i in [2152, 2157] is $i=2$, and the expression $\rho > \frac{3}{2\,i+1}$ [2161*a*] [2165a]
becomes $\rho > \frac{3}{5}$.

‡ (1600) Substituting y, V', [2150*a*], in [2165], we get [2166]. [2166a]

§ (1601) Transposing the first and third terms of [2150], then dividing by $i\,.\,(i+1)\,.\,l$, we get [2167]. Taking the differential of [2167] relatively to μ, multiplying by $\frac{\sqrt{(1-\mu^2)}}{d\,\mu}$, then prefixing the sign Σ, we obtain, for the first member, the same expression as in the [2167a]
second member of [2166], representing the value of $\left(\frac{d\,d\,u}{d\,t^2}\right)$; and the second member of the product is as in [2168].

[2167] $$g \cdot \left\{ 1 - \frac{3}{(2i+1) \cdot \rho} \right\} \cdot Y^{(i)} - U^{(i)} = - \frac{1}{i \cdot (i+1) \cdot l} \cdot \left(\frac{d\,d\,Y^{(i)}}{d\,t^2} \right);$$

therefore,

[2168] $$\frac{d\,d\,u}{d\,t^2} = - \Sigma \cdot \frac{\sqrt{1-\mu^2}}{i \cdot (i+1) \cdot l} \cdot \left\{ \frac{d \cdot \left(\frac{d\,d\,Y^{(i)}}{d\,t^2} \right)}{d\,\mu} \right\};$$

hence we deduce,*

[2169] $$u = G + H \cdot t - \Sigma \cdot \frac{\sqrt{1-\mu^2}}{i \cdot (i+1) \cdot l} \cdot \left(\frac{d\,Y^{(i)}}{d\,\mu} \right),$$

[2169′] G and H being arbitrary functions of μ and ϖ. In the same manner, we shall find,†

[2170] $$v = K + L \cdot t + \Sigma \cdot \frac{1}{i \cdot (i+1) \cdot l \cdot (1-\mu^2)} \cdot \left(\frac{d\,Y^{(i)}}{d\,\varpi} \right),$$

[2170′] K and L being functions of μ and ϖ, depending on the functions G and H. For if in the equation [2136],

[2171] $$y = l \cdot \left\{ \frac{d.(u.\sqrt{1-\mu^2})}{d\,\mu} \right\} - l \cdot \left(\frac{d\,v}{d\,\varpi} \right),$$

we substitute the preceding values of y, u, v, we shall find, by comparing
[2171′] separately the terms multiplied by t, and those which are independent of it,

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* (1602) Multiplying [2168] by $d\,t$, and integrating, adding the constant quantity H

[2169a] to complete the integral, we get $\frac{d\,u}{d\,t} = H - \Sigma \cdot \frac{\sqrt{(1-\mu^2)}}{i \cdot (i+1) \cdot l} \cdot \left(\frac{d \cdot \left(\frac{d\,Y^{(i)}}{d\,t} \right)}{d\,\mu} \right)$. Again

multiplying by $d\,t$, integrating, and adding the constant quantity G, we get u [2169]; observing that G, H, are functions of μ, ϖ, which are considered constant, in the preceding integrations.

† (1603) The value of v may be deduced from [2139], by integration, in the same manner as [2169] was deduced from [2138]; or we may derive the one from the other.
[2170a] For if we change, in [2138], u into v, $d\,\mu$ into $d\,\varpi$, and $\sqrt{(1-\mu^2)}$ into $-\frac{1}{1-\mu^2}$, it will become as in [2139]. The same changes being made in [2169], putting also K, L, for the arbitrary constant quantities, instead of G, H, it becomes as in [2170].

$$0=\left(\frac{d\,.\,G\,.\sqrt{1-\mu^2}}{d\,\mu}\right)-\left(\frac{d\,K}{d\,\varpi}\right);^*$$ [2172]

$$0=\left(\frac{d\,.\,H\,.\sqrt{1-\mu^2}}{d\,\mu}\right)-\left(\frac{d\,L}{d\,\varpi}\right);$$ [2173]

so that by means of the values $u=G+H\,.\,t$, and $v=K+L\,.\,t$, the [2173′]
surface of the fluid will always remain spherical.† To comprehend the motions of the fluid, in this hypothesis, we shall suppose that there is a small rotatory motion, of the order α, about the axis of the spheroid. The [2173″]
spherical figure of the fluid will be altered only by a quantity of the order α^2, since the centrifugal force will be of the same order α^2.‡ In this case, we shall have $u=0$, and $v=qt$,§ q being a coefficient independent of μ [2173‴]

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* (1604) Substituting y, u, v, [2150a, 2169, 2170], in [2136], transposing $\frac{\Sigma\,Y^{(i)}}{l}$ to the second member, connecting the terms under the sign Σ, also those multiplied by t, we get,

$$0=\left\{\left(\frac{d\,.\{G\,.\sqrt{(1-\mu^2)}\}}{d\,\mu}\right)-\left(\frac{d\,K}{d\,\varpi}\right)\right\}+t\,.\left\{\left(\frac{d\,.\{H\,.\sqrt{(1-\mu^2)}\}}{d\,\mu}\right)-\left(\frac{d\,L}{d\,\varpi}\right)\right\}$$
$$+\Sigma\,.\left\{\begin{array}{l}\frac{2\,\mu}{i\,.(i+1)\,.\,l}\,.\left(\frac{d\,Y^{(i)}}{d\,\mu}\right)-\frac{(1-\mu^2)}{i\,.(i+1)\,.\,l}\,.\left(\frac{d\,d\,Y^{(i)}}{d\,\mu^2}\right)\\ -\frac{1}{i\,.(i+1)\,.\,l\,.(1-\mu^2)}\,.\left(\frac{d\,d\,Y^{(i)}}{d\,\varpi^2}\right)-\frac{Y^{(i)}}{l}\end{array}\right\}.$$ [2171a]

The term under the sign Σ, is the same as the function [2145], divided by $-i\,.(i+1)\,.\,l$, as is evident by developing the first term; it therefore vanishes from [2171a]. To make the term multiplied by t vanish, we must put its coefficient equal to nothing, as in [2173]; the remaining term then becomes as in [2172].

† (1605) Substituting the values of u, v [2173′] in $\frac{y}{l}$ [2136], it becomes as in the first line of [2171a], which vanishes by means of the equations [2172, 2173], making $y=0$; consequently the radius of the surface, $1+\alpha\,y$ [2145′], becomes a sphere, [2172a]
whose radius is unity.

‡ (1606) The centrifugal force [54′] is of the order of the square of the velocity α^2; and by [1648] $\alpha\,y$ is of the order of the centrifugal force $\alpha\,\varphi$. Therefore in the present [2173a]
case $\alpha\,y$ is of the order α^2, and if we neglect terms of this order, the surface will become spherical.

§ (1607) The polar distance of a particle, and its longitude, in the state of equilibrium, are represented by θ, ϖ; which in the disturbed state become $\theta+\alpha\,u$, and $\varpi+\alpha\,v$, [2175a]

and ϖ. But we may conceive the fluid to turn about any other axis; and these motions being supposed to be very small, the fluid, when agitated by
[2173''''] any number of similar motions, will always preserve its spherical figure, except in quantities of the second order. All these motions are included in the formulas,*

[2174] $$u = G + H.t, \qquad v = K + L.t;$$

$G, H, K, L,$ being functions of μ and ϖ, which have to each other the preceding relations [2172, 2173]. These motions are not prejudicial to the stability
[2174'] of the equilibrium;† moreover they must quickly be destroyed by the friction, and the resistances of every kind, which the fluid suffers.

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respectively, [2128^{xv}]. If there be no motion in the direction of the meridian, as in the hypothesis [2173''], αu will be nothing, and $u = 0$. An angular motion αq, about the
[2175b] axis, will increase the longitude of the particle, in the time t, by the quantity $\alpha q t$; hence $\alpha v = \alpha q t$, or $v = q t$ [2173''']. In the original work it is given erroneously $v = q t \cdot \sqrt{(1 - \mu^2)}$.

* (1608) The rotation about the *first* axis, during the time t_1, gives values of αu, αv,
[2175c] which we shall call $\alpha u = \alpha p_1 t_1 = 0$, $\alpha v = \alpha q_1 t_1$. If there be also a revolution about a *second* axis, during the time t_2, which would cause the same particle to increase its polar distance by the quantity $\alpha u_2 = \alpha p_2 t_2$, and its longitude by $\alpha v_2 = \alpha q_2 t_2$; a revolution about a *third* axis, during the time t_3, which would increase the polar distance by $\alpha u_3 = \alpha p_3 t_3$, and the longitude by $\alpha v_3 = \alpha q_3 t_3$, &c.; then the sum of all these values αu_1, αu_2, αu_3, &c., αv_1, αv_2, αv_3, &c., will represent the complete values of αu and αv respectively; hence

[2175d] $$u = p_1 t_1 + p_2 t_2 + p_3 t_3 + \&c., \qquad v = q_1 t_1 + q_2 t_2 + q_3 t_3 + \&c.$$

If these motions commence at different times, we may put $t_1 = t$, $t_2 = e_2 + t$, $t_3 = e_3 + t$, &c. Substituting these in [2175d], we get,

$$u = (p_2 e_2 + p_3 e_3 + \&c.) + (p_1 + p_2 + p_3 + \&c.) \cdot t;$$

$$v = (q_2 e_2 + q_3 e_3 + \&c.) + (q_1 + q_2 + q_3 + \&c.) \cdot t;$$

which are of the same form as in [2174].

† (1609) The increments of polar distance and longitude [2174] are so adapted to each
[2175e] other, as to keep the fluid on the same level, and make $y = 0$ [2172a]. Consequently the motions of the fluid, corresponding to the values of u, v, [2174], do not affect the stability of the figure of the fluid; or in other words, it will always retain its spherical form, neglecting quantities of the order α^2.

3. *We shall now consider the case of nature, in which the spheroid covered by the sea has a rotatory motion.* The equation [2129] is transformed into the following [2136b], [2174'']

$$y = \left(\frac{d \cdot \gamma u \cdot \sqrt{1-\mu^2}}{d\mu}\right) - \left(\frac{d \cdot \gamma v}{d\varpi}\right); \qquad (A)$$ [2175]

The equation [2130] gives the two following,*

Fundamental equations for all cases.

$$\left(\frac{ddu}{dt^2}\right) - 2n \cdot \left(\frac{dv}{dt}\right) \cdot \mu \cdot \sqrt{1-\mu^2} = g \cdot \left(\frac{dy}{d\mu}\right) \cdot \sqrt{1-\mu^2} - \left(\frac{dV'}{d\mu}\right) \cdot \sqrt{1-\mu^2}$$ [2176]

$$\left(\frac{ddv}{dt^2}\right) + 2n \cdot \left(\frac{du}{dt}\right) \cdot \frac{\mu}{\sqrt{1-\mu^2}} = -\frac{g \cdot \left(\frac{dy}{d\varpi}\right)}{1-\mu^2} + \frac{\left(\frac{dV'}{d\varpi}\right)}{1-\mu^2}; \qquad (B)$$ [2176']

The general integration of these equations is very difficult: we shall limit ourselves here to a very extensive case, namely that in which γ *is a function of* μ *without* ϖ, and we shall put,† [2177]

Assumed values of y, u, v.

$$y = a \cdot \cos. (it + s\varpi + \varepsilon);$$ [2178]

$$u = b \cdot \cos. (it + s\varpi + \varepsilon);$$ [2178']

$$v = c \cdot \sin. (it + s\varpi + \varepsilon);$$ [2178'']

$$y - \frac{V'}{g} = a' \cdot \cos. (it + s\varpi + \varepsilon) = y' \; [2324];$$ [2178''']

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* (1611) The equations [2176, 2176'], are found in the same manner as [2138, 2139]; the only difference being in the terms depending upon n. The first member of [2138] is the coefficient of $d\theta$ [2130], when $n = 0$; and if we retain the term of n in that coefficient, it will become as in [2176]. The formula [2139] contains, for its first member, the coefficient of $d\varpi$ [2130], divided by $\sin.^2\theta$, or $1-\mu^2$, supposing $n=0$; and if we retain the term containing n, and divide it, in the same manner, we shall obtain the first member of [2176']. [2176a]

† (1612) The reasons for assuming these forms of y, u, v, V', will appear from the following considerations. The expression y [2144] is composed of a series of functions, similar to those in [1528a—e, &c.]; any one of which is composed of terms depending on $\sin. s\varpi$, $\cos. s\varpi$, or as it may be generally expressed $\cos. (s\varpi + \varepsilon')$, multiplied by an integral function of μ, $\sqrt{(1-\mu^2)}$, represented by M; and by quantities independent of μ, ϖ, represented by A; so that $y = \Sigma A M \cdot \cos. (s\varpi + \varepsilon')$, s being an integral number. [2177a] The expression A may contain the rotatory motion of the earth nt, and the mean motion of

[2178''''] a, b, c, a', being rational functions of μ and $\sqrt{1-\mu^2}$ [2183f], and s an integral number [2177a]. Substituting these values in the equations [2175—2176'], we shall get,*

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the attracting body $m\,t$, on which its declination v and right ascension ψ depend, as in [2133, 669, &c.]; by which means the time t will enter into the expression of A, under the

[2177b] form of the sine or cosine of a multiple of t, or as it may be expressed generally, $\cos.(i\,t+\varepsilon'')$; i being a quantity depending on n, m. Substituting this value of A in y [2177a], and reducing by means of [20] Int., it will become of the form [2178].

[2177c] If for brevity we put $\mu'=\gamma\,.\,\sqrt{(1-\mu^2)}$, μ' will be a function of μ independent of ϖ [2177]. Substituting this in [2175], and developing, we get

[2177d]
$$y=u\,.\left(\frac{d\mu'}{d\mu}\right)+\mu'.\left(\frac{du}{d\mu}\right)-\gamma\,.\left(\frac{dv}{d\varpi}\right).$$

Now $\left(\frac{d\mu'}{d\mu}\right)$, μ', γ, being independent of ϖ, they cannot either of them contain the quantity $\cos.(i\,t+s\,\varpi+\varepsilon)$, which will therefore be found only in the quantities u, $\left(\frac{du}{d\mu}\right)$, $\left(\frac{dv}{d\varpi}\right)$; and as the term of y depending on the angle $(it+s\varpi+\varepsilon)$, in the first member, must be destroyed by the corresponding terms of the second member, it is necessary that we should have,

[2177e]
$$u=b\,.\cos.(it+s\varpi+\varepsilon),\quad \left(\frac{du}{d\mu}\right)=b'.\cos.(it+s\varpi+\varepsilon),\quad \left(\frac{dv}{d\varpi}\right)=b''.\cos.(it+s\varpi+\varepsilon);$$

b, b', b'', being supposed independent of ϖ. This value of u is of the same form as [2178']. Multiplying the second of the equations by $d\mu$, and integrating relatively to $d\mu$, putting

[2177f] $\int b'\,.\,d\mu=b$, it becomes identical with the preceding. The third of the equations [2177e] being multiplied by $d\varpi$, and then integrated relatively to $d\varpi$, putting $b''=s\,c$, gives v [2178'']. Substituting the values of u, v, [2178', 2178''], in [2176'], multiplying by $-\frac{(1-\mu^2)}{g}\,.\,d\varpi$, integrating relatively to $d\varpi$, and using the value of a' deduced from

[2177g] [2179''], we get $a'\,.\cos.(i\,t+s\,\varpi+\varepsilon)=y-\frac{V'}{g}$, as in [2178'''], a' being independent of ϖ. Hence we see that the forms assumed in [2178—2178'''], agree with the proposed equations [2175—2176'].

* (1614) Substituting the values of y, u, v, [2178—2178''] in [2175], and dividing by $\cos.(i\,t+s\,\varpi+\varepsilon)$, we get [2179]. The partial differential of [2178'''] relative to μ, being multiplied by $g\,.\sqrt{(1-\mu^2)}$, gives,

[2179a]
$$g\,.\left(\frac{dy}{d\mu}\right).\sqrt{(1-\mu^2)}-\left(\frac{dV'}{d\mu}\right).\sqrt{(1-\mu^2)}=g\,.\left(\frac{da'}{d\mu}\right).\sqrt{(1-\mu^2)}\,.\cos.(it+s\varpi+\varepsilon);$$

substituting this in the second member of [2176]; also the differentials of u, v, [2178', 2178''],

$$a = \left(\frac{d \,.\, (\gamma b \,.\, \sqrt{1-\mu^2})}{d\mu}\right) - s\gamma c ;$$ [2179]

$$i^2 \,.\, b + 2nic \,.\, \mu \,.\, \sqrt{1-\mu^2} = -g \,.\, \left(\frac{da'}{d\mu}\right) \,.\, \sqrt{1-\mu^2} ;$$ [2179′]

$$i^2 \,.\, c + \frac{2nib \,.\, \mu}{\sqrt{1-\mu^2}} = -\frac{gsa'}{1-\mu^2}.$$ [2179″]

These two last equations give,*

$$b = \frac{\frac{2ngs}{i} \,.\, \mu a' - g \,.\, \left(\frac{da'}{d\mu}\right) \,.\, (1-\mu^2)}{(i^2 - 4n^2\mu^2) \,.\, \sqrt{1-\mu^2}} ;$$ [2180]

General expressions of b, c.

$$c = \frac{\frac{2ng}{i} \,.\, \left(\frac{da'}{d\mu}\right) \,.\, \mu \,.\, (1-\mu^2) - gsa'}{(i^2 - 4n^2\mu^2) \,.\, (1-\mu^2)}.$$ [2181]

Substituting these values of b and c in [2179], and putting for brevity,

$$z = \frac{\gamma}{i^2 - 4n^2\mu^2},$$ [2182]

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in the first member; then dividing by $-\cos.(it + s\varpi + \varepsilon)$, we get [2179′]. In like manner, the partial differential of [2178‴] relative to ϖ, being multiplied by $-\frac{g}{1-\mu^2}$, gives,

$$-\frac{g \,.\, \left(\frac{dy}{d\varpi}\right)}{1-\mu^2} + \frac{\left(\frac{dV'}{d\varpi}\right)}{1-\mu^2} = \frac{g}{1-\mu^2} \,.\, sa' \,.\, \sin.(it + s\varpi + \varepsilon) ;$$ [2179b]

substituting this in the second member of [2176′]; and the differentials of u, v, [2178′, 2178″], in the first member; then dividing by $-\sin.(it + s\varpi + \varepsilon)$, we get [2179″].

* (1615) Multiplying [2179′] by $\sqrt{(1-\mu^2)}$, also [2179″] by $-\frac{2n}{i} \,.\, \mu \,.\, (1-\mu^2)$, and adding the products, c vanishes, and we get,

$$(i^2 - 4n^2\mu^2) \,.\, (1-\mu^2)^{\frac{1}{2}} \,.\, b = \frac{2ngs}{i} \,.\, \mu a' - g \,.\, \left(\frac{da'}{d\mu}\right) \,.\, (1-\mu^2) ;$$ [2180a]

dividing this by the coefficient of b, we obtain [2180]. Again multiplying [2179′] by $-\frac{2n}{i} \,.\, \mu \,.\, (1-\mu^2)^{\frac{1}{2}}$, also [2179″] by $(1-\mu^2)$, and adding the products, b vanishes, and we have $(i^2 - 4n^2\mu^2) \,.\, (1-\mu^2) \,.\, c = \frac{2ng}{i} \,.\, \left(\frac{da'}{d\mu}\right) \,.\, \mu \,.\, (1-\mu^2) - gsa'$. Dividing [2180b] this by the coefficient of c, we get [2181].

we shall have,*

General differential equation between a, a', γ.

$$a = g \cdot d \cdot \left\{ \frac{z \cdot \left\{ \frac{2ns}{i} \cdot \mu a' - \left(\frac{da'}{d\mu}\right) \cdot (1-\mu^2) \right\}}{d\mu} \right\} \qquad (4)$$

[2183]
$$+ \frac{2ngs \cdot \mu z}{i \cdot (1-\mu^2)} \cdot \left\{ \frac{2ns}{i} \cdot \mu a' - \left(\frac{da'}{d\mu}\right) \cdot (1-\mu^2) \right\} + \frac{s^2 \cdot gz \cdot a' \cdot (i^2 - 4n^2\mu^2)}{i^2 \cdot (1-\mu^2)};$$

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* (1616) The value of a [2183] is easily deduced from [2179]; since the first and second *terms* of the value [2179], produce the first and second *lines* of the value [2183]. For if we multiply [2182] by $b \cdot (i^2 - 4n^2\mu^2) \cdot (1-\mu^2)^{\frac{1}{2}}$, using b [2180], we get, successively,

[2183a]
$$\begin{aligned}\gamma b \cdot (1-\mu^2)^{\frac{1}{2}} &= z \cdot \{b \cdot (i^2 - 4n^2\mu^2) \cdot (1-\mu^2)^{\frac{1}{2}}\} \\ &= g \cdot z \cdot \left\{ \frac{2ns}{i} \cdot \mu a' - \left(\frac{da'}{d\mu}\right) \cdot (1-\mu^2) \right\};\end{aligned}$$

taking its partial differential relatively to $d\mu$, and dividing by $d\mu$, we obtain the first term
[2183b] of [2179], as in the first line of [2183]. Substituting the value of $\gamma = z \cdot (i^2 - 4n^2\mu^2)$ [2182], in the second term of [2179], it becomes, by using [2181],

[2183c]
$$-s\gamma c = -sz \cdot (i^2 - 4n^2\mu^2) \cdot c = \frac{\frac{-2ngs.z}{i} \cdot \left(\frac{da'}{d\mu}\right) \cdot \mu \cdot (1-\mu^2) + s^2 \cdot gz \cdot a'}{1-\mu^2};$$

in which the term, multiplied by $\left(\frac{da'}{d\mu}\right)$, is the same as in the second line of [2183]. The other term of the preceding expression is $\frac{s^2 gz \cdot a'}{1-\mu^2}$, or $\frac{s^2 i^2 \cdot gz \cdot a'}{i^2 \cdot (1-\mu^2)}$, which is the same as the term containing i^2 in the numerator of the second line of [2183]. Lastly, the two remaining terms of the second line of [2183], are

[2183d]
$$\frac{2ngs \cdot \mu z}{i \cdot (1-\mu^2)} \cdot \frac{2ns}{i} \cdot \mu a' - \frac{s^2 \cdot gz \cdot a' \cdot 4n^2\mu^2}{i^2 \cdot (1-\mu^2)},$$

which mutually destroy each other; therefore the value of a is given correctly in [2183]. Before quitting this subject, we may remark, that y, V' [2150a] are composed of terms $Y^{(i)}$, $U^{(i)}$, which are rational and integral functions of μ, $(1-\mu^2)^{\frac{1}{2}} \cdot \sin.\varpi$, $(1-\mu^2)^{\frac{1}{2}} \cdot \cos.\varpi$,
[2183e] [2144′, 2148′]; therefore $y - \frac{V'}{g}$, or $a' \cdot \cos.(it + s\varpi + \varepsilon)$ [2178‴], must be a rational and integral function of the same quantities, and s an integral number [2177a]; hence, as a' does not contain ϖ [2177g], it must be a rational and integral function of μ,
[2183f] $(1-\mu^2)^{\frac{1}{2}}$; and its substitution in [2180, 2181, 2183] gives b, c, a, in terms of μ, which may be developed in rational and integral functions of μ, $(1-\mu^2)^{\frac{1}{2}}$, as in [1528a, &c.], or [1532a, &c.].

We shall here observe, that if $\frac{2ns}{i} \cdot \mu a' - \left(\frac{d a'}{d\mu}\right) \cdot (1-\mu^2)$ be divisible by $i^2 - 4n^2\mu^2$, the second member of this equation will not have the function $i^2 - 4n^2\mu^2$ in its denominator. [2183']

The equation [2183] includes what we have demonstrated in the preceding article, concerning the case of $n=0$, and γ equal to a constant quantity l; for then we have $z = \frac{l}{i^2}$ [2182], which changes the equation [2183] into the following,* [2183'']

$$i^2 a = lg \cdot \left\{ - \frac{d \cdot \left\{ (1-\mu^2) \cdot \left(\frac{d a'}{d\mu}\right) \right\}}{d\mu} + \frac{s^2 \cdot a'}{1-\mu^2} \right\}. \qquad (5) \qquad [2184]$$

Supposing that $a \cdot \cos.(it + s\varpi)$, when taken for $Y^{(f)}$, satisfies the following equation of partial differentials,† [2184']

$$0 = \left\{ \frac{d \cdot \left\{ (1-\mu^2) \cdot \left(\frac{d Y^{(f)}}{d\mu}\right) \right\}}{d\mu} \right\} + \frac{\left(\frac{dd Y^{(f)}}{d\varpi^2}\right)}{1-\mu^2} + f \cdot (f+1) \cdot Y^{(f)}; \qquad [2185]$$

then the part of $a' \cdot \cos.(it + s\varpi)$, arising from the attraction of an aqueous stratum, whose internal radius is 1, and external radius $1 + \alpha y$, will be, by the preceding article, $-\frac{4\pi}{(2f+1) \cdot g} \cdot a \cdot \cos.(it + s\varpi)$, or $-\frac{3a}{(2f+1) \cdot \rho} \cdot \cos.(it + s\varpi)$; because $g = \frac{4}{3}\pi\rho$ [2147]. Therefore, [2185'] [2185'']

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* (1617) Putting $n=0$ in [2183], we have

$$a = g \cdot \left\{ \frac{d \cdot \left\{ -z \cdot \left(\frac{d a'}{d\mu}\right) \cdot (1-\mu^2) \right\}}{d\mu} \right\} + \frac{s^2 \cdot g z \cdot a' \cdot i^2}{i^2 \cdot (1-\mu^2)}. \qquad [2184a]$$

Substituting $z = \frac{l}{i^2}$ [2183''], and multiplying by i^2, we get [2184].

† (1618) Changing i into f in [2145], it becomes as in [2185]; the corresponding term of V' [2146] is $\frac{4\pi}{2f+1} \cdot Y^{(f)}$; and that of $y - \frac{V'}{g}$ is

$$Y^{(f)} - \frac{4\pi}{(2f+1) \cdot g} \cdot Y^{(f)} = \left(1 - \frac{4\pi}{(2f+1) \cdot g}\right) \cdot Y^{(f)} = \left(1 - \frac{3}{(2f+1) \cdot \rho}\right) \cdot Y^{(f)} \quad [2185'']; \qquad [2185a]$$

substituting $Y^{(f)} = a \cdot \cos.(it + s\varpi)$ [2184'], it becomes [2185b]

$$y - \frac{V'}{g} = \left(1 - \frac{3}{(2f+1) \cdot \rho}\right) \cdot a \cdot \cos.(it + s\varpi). \qquad [2185c]$$

Comparing this with [2178'''], supposing $\varepsilon = 0$, we get [2186].

by supposing the part of a' corresponding to the action of the heavenly bodies to be nothing, we shall have,

[2186]
$$a' = \left\{ 1 - \frac{3}{(2f+1)\cdot\rho} \right\} \cdot a\,;$$

Now from the equation, in partial differentials of $Y^{(f)}$, we get,*

[2187]
$$0 = \frac{d\cdot\left\{ (1-\mu^2)\cdot\left(\frac{d\,a'}{d\,\mu}\right)\right\}}{d\,\mu} - \frac{s^2\cdot a'}{1-\mu^2} + f\cdot(f+1)\cdot a'\,;$$

therefore the equation [2184] will give,†

[2188]
$$i^2 = f\cdot(f+1)\cdot lg\cdot\left\{ 1 - \frac{3}{(2f+1)\cdot\rho} \right\}.$$

The numbers s and f being arbitrary, it is evident that we shall obtain the
[2188'] part of y which is independent of the action of the heavenly bodies, by connecting together all the expressions of $a\,.\cos.(it+s\varpi)$, corresponding to the different values that may be given to these numbers.

To obtain the part of y, depending upon the action of the heavenly
[2188''] bodies, we shall put $e\,.\cos.(it+s\varpi)$ for a term of the expression of V', relative to this action; so that when it is substituted for $Y^{(f)}$, in the preceding equation of partial differentials in $Y^{(f)}$ [2185], it will be satisfied; we shall then have,‡

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* (1619) Multiplying $Y^{(f)}$ [2135b], by $1-\frac{3}{(2f+1)\cdot\rho}$, and substituting a' [2186],
[2187a] it becomes $a'\,.\cos.(it+s\varpi)$. This must evidently satisfy [2185], when taken for $Y^{(f)}$, because the factor by which we have multiplied $Y^{(f)}$ is constant. This substitution being made, and the resulting equation divided by $\cos.(it+s\varpi)$, we get [2187].

† (1620) Multiplying [2187] by $\frac{lg}{a}$, and [2184] by $\frac{1}{a}$, then adding the products,
[2188a] we get $i^2 = f\cdot(f+1)\cdot\frac{lg}{a}\cdot a' = f\cdot(f+1)\cdot lg\cdot\left(1-\frac{3}{(2f+1)\cdot\rho}\right)$ [2186], as in the formula [2188].

‡ (1621) In the expression of $y - \frac{V'}{g}$ [2185c], the attractions of the heavenly bodies are neglected. If we suppose these attractions to increase V' by the quantity

$$a' = a \cdot \left(1 - \frac{3}{(2f+1) \cdot \rho}\right) - \frac{e}{g} ;$$ [2189]

therefore the equation [2184] will give,*

$$i^2 \cdot a = lg \cdot f \cdot (f+1) \cdot \left(1 - \frac{3}{(2f+1) \cdot \rho}\right) \cdot a - f \cdot (f+1) \cdot le ;$$ [2190]

consequently,

$$a = \frac{f \cdot (f+1) \cdot le}{lg \cdot f \cdot (f+1) \cdot \left(1 - \frac{3}{(2f+1) \cdot \rho}\right) - i^2} ;$$ [2191]

We shall thus obtain the part of y depending on the action of the heavenly bodies; and it is evident that these results coincide with those of the preceding article.†

$$e \cdot \cos. (it + s\varpi) \qquad [2188'']$$

the complete value of $y - \frac{V'}{g}$ will become

$$y - \frac{V'}{g} = \left\{ a \cdot \left(1 - \frac{3}{(2f+1) \cdot \rho}\right) - \frac{e}{g} \right\} \cdot \cos. (it + s\varpi).$$ [2189a]

Comparing this with [2178'''], we get a' [2189].

* (1622) Substituting a' [2189] in $i^2 a = f \cdot (f+1) \cdot lg \cdot a'$, [2188a], we get [2190a]
[2190]. Transposing the first and third terms, and dividing by the coefficient of a, we obtain [2191].

† (1623) To prove this coincidence, we shall observe that the general value of y [2157], is deduced from $Y^{(i)}$ [2152]; in which the terms containing $U^{(i)}$ depend on the action of [2191a]
the heavenly bodies [2148]; and if, for a moment, we neglect this part, and change i into f, and λ_f [2151] into i, to conform to the present notation, we shall get, from [2152],

$$Y^{(f)} = l \cdot M^{(f)} \cdot \sin. it + l \cdot N^{(f)} \cdot \cos. it.$$ [2191b]

Now from [2152', 1528a—c], $M^{(f)}$, $N^{(f)}$, are functions of the forms $A \cdot \sin. s\varpi$, $B \cdot \cos. s\varpi$, s being an integral number, as has been already seen [2177a, &c.]. Substituting these in the preceding value of $Y^{(f)}$, it will produce terms depending on the [2191c]
sines and cosines of the sum and difference of the angles it, $s\varpi$, which may be expressed under one general form $a \cdot \cos. (it + s\varpi + \varepsilon)$; being the same as in [2188'], neglecting [2191d]
for brevity the constant quantity ε, and changing, as above, λ_i^2 [2151] into i^2 [2188].

We shall now take into consideration the terms of [2152] depending on $U^{(i)}$, arising

[2191′] 4. *In the general case, where n does not vanish, and the depth of the sea is variable, the integration of the equation* [2183] *surpasses the power of analysis; but to determine the oscillations of the ocean, it is not necessary to integrate it generally, since it is sufficient merely to satisfy it.* For it is evident, that the part of the oscillations, which depends on the primitive state of the sea, must [2191″] have quickly disappeared, by the resistances of different kinds, which the waters of the ocean suffer in their motions; so that if it were not for the action of the sun and moon, the sea would long since have assumed a [2191‴] permanent state of equilibrium. The action of these two bodies continually disturbs it, and it is sufficient to ascertain the oscillations which depend on this action.

[2191⁗] If the distance of any heavenly body L, from the centre of the earth, be represented by r; the part of $\alpha V'$, relative to the action of this body upon [2191ᵛ] a fluid particle, developed according to the powers of $\frac{1}{r}$, neglecting the fourth power, will be, by § 1,*

[2191e] from the attractions of the heavenly bodies; and if, in the two terms of this expression, we change, as above, i into f, it will become, by putting for brevity λ for λ_f,

[2191f] $$Y^{(f)} = \frac{f.(f+1)}{\lambda}.l.\sin.\lambda t.\textstyle\int U^{(i)}.dt.\cos.\lambda t - \frac{f.(f+1)}{\lambda}.l.\cos.\lambda t.\int U^{(i)}.dt.\sin.\lambda t;$$

[2191g] which are the same as the terms of [870] depending on Q, changing α into $f.(f+1).l$, also a into λ, Q into $-U^{(i)}$; and if $Q = -U^{(i)} = K.\cos.(mt+\varepsilon)$ [870′], it will produce in $Y^{(f)}$ the term $Y^{(f)} = \frac{f.(f+1).l.K}{m^2-\lambda^2}.\cos.(mt+\varepsilon)$. If we now put $K=-e$, $\varepsilon = s\varpi$, $m=i$, the part of $Y^{(f)}$ depending on $U^{(i)} = e.\cos.(it+s\varpi)$

[2191h] [2188″] will be $Y^{(f)} = \frac{f.(f+1).le}{\lambda^2-i^2}.\cos.(it+s\varpi)$. Substituting the value of λ_f^2, or λ^2 [2151], in which i is changed into f, to conform to the present notation, the coefficient of $\cos.(it+s\varpi)$ becomes like the value of a [2191], making this part of

[2191i] $$Y^{(f)} = a.\cos.(it+s\varpi),$$

as in [2184′, 2191]; hence the formula [2152] produces the same results as in [2184′—2191].

* (1624) It follows from [2133′], that the part of $\alpha V'$ depending on the action of the body L, is found by subtracting the function [2133] from [2132]; and if we use the symbol δ [2133c], it becomes

$$\alpha V' = \frac{3L}{2r^3} \cdot \{[\cos.\theta \,.\, \sin.\mathrm{v} + \sin.\theta \,.\, \cos.\mathrm{v} \,.\, \cos.(nt + \varpi - \psi)]^2 - \tfrac{1}{3}\}\,; \qquad [2192]$$

or*

Parts of $\alpha V'$ producing the different oscillations.

$$\alpha V' = \frac{L}{4r^3} \cdot \{\sin.^2\mathrm{v} - \tfrac{1}{2} \,.\, \cos.^2\mathrm{v}\} \,.\, \{1 + 3 \,.\, \cos.2\theta\} \qquad [\text{First Oscillation.}] \qquad [2193]$$

$$+ \frac{3L}{r^3} \,.\, \sin.\theta \,.\, \cos.\theta \,.\, \sin.\mathrm{v} \,.\, \cos.\mathrm{v} \,.\, \cos.(nt + \varpi - \psi) \qquad [\text{Second Oscillation.}] \qquad [2194]$$

$$+ \frac{3L}{4r^3} \,.\, \sin^2\theta \,.\, \cos.^2\mathrm{v} \,.\, \cos.2 \,.\, (nt + \varpi - \psi). \qquad [\text{Third Oscillation.}] \qquad [2195]$$

$$\alpha V' = \frac{L}{(r^2 - 2r\delta + 1)^{\frac{1}{2}}} - \frac{L}{r} - \frac{L}{r^2} \,.\, \delta = \frac{L}{r} \,.\, \left(1 - \frac{2\delta}{r} + \frac{1}{r^2}\right)^{-\frac{1}{2}} - \frac{L}{r} - \frac{L}{r^2} \,.\, \delta$$

$$= \frac{L}{r} \,.\, \left(1 + \frac{\delta}{r} - \frac{1}{2r^2} + \tfrac{3}{2} \,.\, \frac{\delta^2}{r^2} + \&c.\right) - \frac{L}{r} - \frac{L}{r^2} \,.\, \delta = \frac{3L}{2r^3} \,.\, (\delta^2 - \tfrac{1}{3}), \qquad [2192a]$$

neglecting terms of the order $\frac{L}{r^4}$. Resubstituting δ [2133c], it becomes as in [2192].

* (1625) Developing the first term of [2192], we find that the part depending on the first power of $\cos.(nt + \varpi - \psi)$, is equal to the expression [2194]; and if in the term depending on the square of this cosine, we substitute

$$\cos.^2(nt + \varpi - \psi) = \tfrac{1}{2} + \tfrac{1}{2} \,.\, \cos.2 \,.\, (nt + \varpi - \psi), \qquad [2192b]$$

we shall find the part depending on $\cos.2 \,.\, (nt + \varpi - \psi)$, as in [2195]. The remaining terms of [2192], independent of the angle $nt + \varpi - \psi$, are as in the first of the following expressions, which is reduced to the form [2193] by putting $\cos.^2\theta = \frac{1}{2} + \frac{1}{2} \,.\, \cos.2\theta$, $\sin.^2\theta = \frac{1}{2} - \frac{1}{2} \,.\, \cos.2\theta$, and $-\frac{1}{3} = -\frac{1}{3} \,.\, (\sin.^2\mathrm{v} + \cos.^2\mathrm{v})$.

$$\frac{3L}{2r^3} \,.\, \{\cos.^2\theta \,.\, \sin.^2\mathrm{v} + \tfrac{1}{2} \,.\, \sin.^2\theta \,.\, \cos.^2\mathrm{v} - \tfrac{1}{3}\}$$

$$= \frac{3L}{2r^3} \,.\, \{(\tfrac{1}{2} + \tfrac{1}{2} \,.\, \cos.2\theta) \,.\, \sin.^2\mathrm{v} + \tfrac{1}{2} \,.\, (\tfrac{1}{2} - \tfrac{1}{2} \,.\, \cos.2\theta) \,.\, \cos.^2\mathrm{v} - \tfrac{1}{3} \,.\, (\sin.^2\mathrm{v} + \cos.^2\mathrm{v})\} \qquad [2192b']$$

$$= \frac{3L}{2r^3} \,.\, \{\sin.^2\mathrm{v} \,.\, (\tfrac{1}{2} + \tfrac{1}{2} \,.\, \cos.2\theta - \tfrac{1}{3}) + \tfrac{1}{2} \,.\, \cos.^2\mathrm{v} \,.\, (\tfrac{1}{2} - \tfrac{1}{2} \,.\, \cos.2\theta - \tfrac{2}{3})\}$$

$$= \frac{3L}{2r^3} \,.\, \{\sin.^2\mathrm{v} \,.\, (\tfrac{1}{6} + \tfrac{1}{2} \,.\, \cos.2\theta) - \tfrac{1}{2} \,.\, \cos.^2\mathrm{v} \,.\, (\tfrac{1}{6} + \tfrac{1}{2} \,.\, \cos.2\theta)\}$$

$$= \frac{L}{4r^3} \,.\, \{\sin.^2\mathrm{v} \,.\, (1 + 3 \,.\, \cos.2\theta) - \tfrac{1}{2} \,.\, \cos.^2\mathrm{v} \,.\, (1 + 3 \,.\, \cos.2\theta)\}$$

$$= \frac{L}{4r^3} \,.\, \{\sin.^2\mathrm{v} - \tfrac{1}{2} \,.\, \cos.^2\mathrm{v}\} \,.\, \{1 + 3 \,.\, \cos.2\theta\}. \qquad [2192c]$$

[2195′] *As the quantities* r, v, ψ, *vary with extreme slowness, in comparison with the*
Oscillations; *rotatory motion of the earth; the three preceding terms produce three different*
[2195″] *species of oscillations. The periods of the oscillations of the first kind are very*
First kind; *long; they are independent of the rotatory motion of the earth, and depend*
[2195‴] *wholly upon the motion of the body* L *in its orbit. The periods of the*
Second kind; *oscillations of the second species depend chiefly on the rotatory motion of the*
[2195⁗] *earth* nt; *their duration is nearly one day. Lastly, the periods of the*
Third kind. *oscillations of the third kind depend chiefly on the angle* $2nt$; *they are*
[2195v] *completed in about half a day.* The equation [2183] of the preceding article is a linear differential equation; hence it follows, that these three species of
[2195vi] oscillations can exist together, without being confounded with each other; therefore we may consider them separately.*

ON OSCILLATIONS OF THE FIRST KIND.

[2195vii] 5. *We shall suppose, in these researches, that the spheroid which is covered by the sea is an ellipsoid of revolution.* This is the most natural and simple hypothesis that can be adopted. In this case, the expression of γ is of
Depth of the sea. the form,†

[2196] $$\gamma = l \,.\, (1 - q\,\mu^2) = \text{the depth of the sea};$$

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* (1626) The quantities y, u, v, V', occur in [2175—2176′] only in the first power; and if each of them, instead of being expressed by one term, depending on the same angle
[2195a] $it + s\varpi + \varepsilon$, as in [2178—2178‴], were composed of the sum of several such terms, corresponding to different angles, we should obtain, for each of these angles, an equation similar to [2183]; by which the coefficients a, b, c, a', corresponding to any one of these
[2195b] angles, could be determined, independently of the values relating to the other angles; as is evident, from the common principles of linear equations. Therefore each of these angles may
[2195c] be taken into consideration separately from the others, in the same manner as if the oscillations were wholly independent of each other.

† (1627) The radius of the solid spheroid of revolution may be expressed by the quantity $a' . (1 - \alpha h' . \mu^2)$; and that of the surrounding fluid, in its state of equilibrium, independently
[2196a] of the attractions of the heavenly bodies, by $a'' . (1 - \alpha h'' . \mu^2)$ [1795]. The difference
[2196b] of these two expressions is equal to γ, the depth of the sea; which, by putting $a'' - a' = l$, $\alpha a'' h'' - \alpha a' h' = lq$, becomes $\gamma = l - lq . \mu^2$, as in [2196]. Substituting this in
[2196c] [2182], we get [2197]; and as this does not contain ϖ, it will be comprised in the case which we have taken into consideration in [2177—2191].

and we shall have,

$$z = \frac{l \cdot (1 - q \mu^2)}{i^2 - 4 n^2 \mu^2}. \qquad [2197]$$

We shall resume the equation [2183]; and as the oscillations of the first kind do not depend on the angle ϖ, we must put $s=0$, in this equation.* [2197′]
We shall then suppose

Value of a.

$$a = P^{(0)} + P^{(2)} + P^{(4)} + P^{(6)} \ldots\ldots + P^{(2f)}; \qquad [2198]$$

$P^{(2)}$, $P^{(4)}$, &c., being functions of μ^2 [2197e], which satisfy the following equation of partial differentials, whatever be the value of f, [2198′]

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* (1628) The function [2193] upon which the oscillations of the first kind depend, does not contain ϖ; therefore we must put $s=0$, in [2178, &c.]. The form of this function may be changed, by means of [34] Int., which gives,

$$1 + 3 \cdot \cos. 2\theta = 1 + 3 \cdot (2 \cdot \cos.^2\theta - 1) = 6 \cdot (\cos.^2\theta - \tfrac{1}{3}) = 6 \cdot (\mu^2 - \tfrac{1}{3}) \quad [2128^{xii}]. \qquad [2197a]$$

Substituting this in [2193], it becomes $\frac{3L}{2r^3} \cdot (\sin.^2 v - \frac{1}{2} \cdot \cos.^2 v) \cdot (\mu^2 - \frac{1}{3})$; and as this [2197b]
does not contain ϖ, its effect must be the same, for all particles of the fluid, in the same latitude, whatever be the longitude; consequently the height αy of the particle, above its level, arising from this cause, will depend on *its latitude only*; or, in other words, y must be a function of [2197c]
μ, independent of ϖ. Moreover, the expressions of the disturbing force [2197b], and the depth of the sea [2196], do not vary by changing the sign of μ or v; therefore a particle of the fluid, whose latitude is *south*, and its sine $-\mu$, is acted upon in exactly the same manner, as if the latitude were *north*, and its sine μ. Hence it follows, that y must be a function of the *even* powers of μ, or a function of μ^2; for if it should contain any *uneven* power [2197d]
of μ, the sign would change in south latitudes, and the values of y would become unequal; therefore a [2178] must be a function of μ^2. In this case, all the terms depending on ϖ [2197e]
must vanish from the expressions [1528a—e, &c.]; which will be reduced to their first terms, depending on $B_0^{(0)}$, $B_1^{(0)}$, $B_2^{(0)}$, $B_3^{(0)}$, &c.; and by this means the radical $(1-\mu^2)^{\frac{1}{2}}$, and its powers, will also disappear. Hence y, or a, will become a function of μ^2; without containing any term depending on the radical $(1-\mu^2)^{\frac{1}{2}}$, or the variable quantity ϖ. If we develop this expression of a, in a series of the form [2198], the terms $P^{(1)}$, $P^{(3)}$, $P^{(5)}$, [2197f]
&c., will vanish, because they contain only the uneven powers of μ; as is evident by the inspection of the terms depending on $B_1^{(0)}$, $B_3^{(0)}$, $B_5^{(0)}$, &c., in the formulas [1528a—e, &c.]; and we may observe, that the greatest exponent of μ, in a function of the form [2198], terminated by $P^{(2f)}$, is equal to $2f$, as is evident from the same formulas [2197g]
[1528a—e, &c.].

[2199] $$0=\frac{d.\left\{(1-\mu^2).\left(\frac{d\,P^{(2f)}}{d\mu}\right)\right\}}{d\mu}+2f.(2f+1).P^{(2f)}.*$$

[2199′] The part of a', corresponding to y, and to the aqueous stratum whose internal radius is unity, and external radius is $1+\alpha y$, will be, by what precedes,†

Value of a'.

[2200] $$\left(1-\frac{3}{\rho}\right).P^{(0)}+\left(1-\frac{3}{5\rho}\right).P^{(2)}+\left(1-\frac{3}{9\rho}\right).P^{(4)}\ldots\ldots+\left(1-\frac{3}{(4f+1).\rho}\right).P^{(2f)}.$$

[2200′] The part of a' corresponding to the action of the heavenly bodies, produces only quantities of the form $P^{(2)}$; for the function $1+3.\cos.2\theta$, by
[2200″] which it is multiplied in the preceding article [2193], is equal to $6.(\mu^2-\frac{1}{3})$ [2197a], and it is evident that this last function is of the form $P^{(2)}$ [1528c]. This being premised, if we substitute the values of a and a', in the equation [2183], and determine the arbitrary terms of $P^{(0)}$, $P^{(2)}$, &c., so that the function $\left(\frac{d\,a'}{d\mu}\right).(1-\mu^2)$ may be divisible by $i^2-4n^2\mu^2$, which
[2201] gives one equation of condition [2201d] between these arbitrary terms;‡

[2199a] * (1629) Changing Y [2185] into P, and f into $2f$, we get [2199]; observing that $P^{(2f)}$ being independent of ϖ [2197′, &c.], its partial differential, relatively to ϖ, vanishes.

† (1630) Putting, as in the last note, $2f$ for f, in [2186], we get,

$$a'=\left\{1-\frac{3}{(4f+1).\rho}\right\}.a;$$

and if we suppose the term $P'^{(2f)}$, in a', to correspond to $P^{(2f)}$, in a, we shall have,

[2200a] $$P'^{(2f)}=\left\{1-\frac{3}{(4f+1).\rho}\right\}.P^{(2f)}.$$

If we now put successively $f=0$, $f=1$, $f=2$, &c., we shall get,

[2200b] $$P'^{(0)}=\left\{1-\frac{3}{\rho}\right\}.P^{(0)},\qquad P'^{(2)}=\left\{1-\frac{3}{5\rho}\right\}.P^{(2)},\qquad \&c.;$$

whose sum gives a' [2200].

‡ (1631) Putting $s=0$ [2197′], in [2183], it becomes,

$$a=g.d.\left\{\frac{-z.\left(\frac{d\,a'}{d\mu}\right).(1-\mu^2)}{d\mu}\right\};$$

then the most elevated power of μ^2, in each member of this equation, will be μ^{2f}; and by comparing the coefficients of the several powers of μ^2, we [2201′]
shall obtain $f+1$ equations of condition; which, with the preceding, [2201], will form $f+2$ equations of condition. But the arbitrary terms [2201″]

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and by substituting z [2197], we get,

$$a = g \cdot d \cdot \left\{ \frac{-l \cdot (1 - q\mu^2) \cdot \left(\frac{\left(\frac{d a'}{d\mu}\right) \cdot (1-\mu^2)}{i^2 - 4n^2\mu^2} \right)}{d\mu} \right\}. \qquad [2201a]$$

Assuming now the most general values of $P^{(0)}$, $P^{(2)}, \ldots\ldots P^{(2f)}$, independent of ϖ, as they are given by formulas [1528a, &c.], and substituting them in a, a', [2198, 2200], they will become of the following forms, when arranged according to the powers of μ,

$$\begin{aligned} a &= G^{(0)} \cdot \mu^{2f} + G^{(2)} \cdot \mu^{2f-2} \ldots\ldots + G^{(2f-2)} \cdot \mu^2 + G^{(2f)}, \\ a' &= A^{(0)} \cdot \mu^{2f} + A^{(2)} \cdot \mu^{2f-2} \ldots\ldots + A^{(2f-2)} \cdot \mu^2 + A^{(2f)}. \end{aligned} \qquad [2201b]$$

The differential of this last, being taken relatively to μ, and then multiplied by the term $\frac{1-\mu^2}{d\mu}$, produces, for $\left(\frac{d a'}{d\mu}\right) \cdot (1-\mu^2)$, a function of the form,

$$\mu \cdot \{B^{(0)} \cdot \mu^{2f} + B^{(2)} \cdot \mu^{2f-2} + B^{(4)} \cdot \mu^{2f-4} \ldots\ldots + B^{(2f-2)} \cdot \mu^2 + B^{(2f)}\}. \qquad [2201c]$$

Dividing this by $i^2 - 4n^2\mu^2$, or rather by $-4n^2\mu^2 + i^2$, so as to produce a series descending according to the powers of μ, and continuing the division till we obtain the term μ, it will become of the form

$$\mu \cdot \left\{ C^{(0)} \cdot \mu^{2f-2} + C^{(2)} \cdot \mu^{2f-4} \ldots\ldots + C^{(2f-4)} \cdot \mu^2 + C^{(2f-2)} + \frac{C}{-4n^2\mu^2 + i^2} \right\}; \qquad [2201c']$$

and as this quantity is supposed to be divisible by $-4n^2\mu^2 + i^2$ [2201], we must have $C = 0$, which is the equation of condition mentioned in [2201]. Multiplying [2201c'] by [2201d]
$-l \cdot (1 - q\mu^2)$, or $l \cdot (q\mu^2 - 1)$, we get a series of the form

$$D^{(0)} \cdot \mu^{2f+1} + D^{(2)} \cdot \mu^{2f-1} \ldots\ldots + D^{(2f)} \cdot \mu.$$

Its differential, taken relatively to μ, and multiplied by $\frac{g}{d\mu}$, gives the second member of [2201a], under the form,

$$a = E^{(0)} \cdot \mu^{2f} + E^{(2)} \cdot \mu^{2f-2} \ldots\ldots + E^{(2f-2)} \cdot \mu^2 + E^{(2f)}. \qquad [2201e]$$

Comparing this with a [2201b], we get $f+1$ equations

$$G^{(0)} = E^{(0)}, \quad G^{(2)} = E^{(2)}, \quad G^{(4)} = E^{(4)}, \ldots\ldots G^{(2f)} = E^{(2f)}.$$

These, being connected with $C=0$, [2201d], make $f+2$ equations, as in [2201″]. [2201f]

[2201‴] of the function $P^{(0)}+P^{(2)}+P^{(4)}+\&c.$, amount to $f+1$;* connecting these with the indeterminate quantity q, we shall have $f+2$ indeterminate quantities, by means of which we can satisfy the preceding equations of condition; and thence also the equation [2183], for a determinate law of the depth of the sea. We shall obtain this law, by observing that if we put
[2202] $Q\,.\,\mu^{2f}$, for the term of a having the most elevated power of μ, the coefficient of μ^{2f-1}, in the function $-\left(\frac{d\,a'}{d\,\mu}\right).(1-\mu^2)$, divided by $i^2-4\,n^2\,\mu^2$, will be,†

[2203]
$$-f.\left\{1-\frac{3}{(4f+1).\rho}\right\}.\frac{Q}{2\,n^2}.$$

[2203′] Hence it follows, that the coefficient of μ^{2f}, in the second member of the equation [2183], will be,

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* (1632) The terms independent of ϖ, in $Y^{(0)}$, $Y^{(2)}$, &c., [1528a, &c.], are
[2201g] multiplied by the arbitrary constant quantities $B_0^{(0)}$, $B_2^{(0)}$, &c. In like manner, each of the functions $P^{(0)}$, $P^{(2)}$, $P^{(4)}$, ... $P^{(2f)}$, [2198], is multiplied by an arbitrary constant quantity. The number of these arbitrary terms is evidently $f+1$, as in [2201‴].
[2201h] Connecting this with q [2196], we obtain $f+2$ arbitrary quantities, being equal in number to the equations of condition [2201f].

† (1633) The terms $P^{(2f)}$ and $\left(1-\frac{3}{(4f+1).\rho}\right).P^{(2f)}$ in the values of a, a',
[2202a] [2198, 2200] must evidently contain the most elevated power of μ [2197g]. These terms are to each other as 1 to $1-\frac{3}{(4f+1).\rho}$; and as $Q.\mu^{2f}$ [2202] is the term of a, having
[2202b] the most elevated power of μ, the similar term of a' will be $\left(1-\frac{3}{(4f+1).\rho}\right).Q.\mu^{2f}$. Its differential relative to μ, being divided by $-d\,\mu$, gives the corresponding term of $-\left(\frac{d\,a'}{d\,\mu}\right)$, equal to $-2f.\left(1-\frac{3}{(4f+1).\rho}\right).Q.\mu^{2f-1}$. Again, if we develop $\frac{1-\mu^2}{i^2-4\,n^2\,\mu^2}$, or $\frac{\mu^2-1}{4\,n^2\,\mu^2-i^2}$, in a series descending according to the powers of μ,
[2202c] as in [2201c'], it will be of the form $\frac{1}{4n^2}+H.\mu^{-2}+H'.\mu^{-4}+\&c.$ This being multiplied by the preceding term of $-\left(\frac{d\,a'}{d\,\mu}\right)$, retaining only the part having the greatest exponent,
[2202d] produces the expression $-2f.\left(1-\frac{3}{(4f+1).\rho}\right).Q.\mu^{2f-1}.\frac{1}{4n^2}$, as in [2203].

$$f.(2f+1).\left\{1-\frac{3}{(4f+1).\rho}\right\}.\frac{lg.q.Q}{2n^2}.^{*}$$ [2204]

Putting this equal to Q, the coefficient of μ^{2f} in the first member, we shall have,

Value of q.

$$q=\frac{2n^2}{f.(2f+1).\left\{1-\frac{3}{(4f+1).\rho}\right\}.lg}.$$ [2205]

Therefore if we suppose the depth of the sea to be equal to l, decreased by the product of $l\mu^2$ into this value of q, we shall have, by the preceding [2205′] analysis, the oscillations of the first kind.

The ratio of the centrifugal force to the gravity at the equator, is [2205″] $\frac{n^2}{g}=\frac{1}{289}$ [1594a]. If we take for f a rather large number, as twelve or [2205‴] fourteen, the coefficient of μ^2 will be so small that it may be neglected, and then the depth of the sea will be nearly constant.† We shall thus obtain, [2205⁗]

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* (1634) If we multiply [2202d] by $gl.(1-q\mu^2)$ and retain only the power of μ having the greatest exponent, it will become $f.\left\{1-\frac{3}{(4f+1).\rho}\right\}.\frac{lg.q.Q}{2n^2}.\mu^{2f+1}$; whose differential relative to μ, being divided by $d\mu$, gives the corresponding term of the value [2205a] of a [2201a], as in [2204]. Putting this term equal to the assumed value $Q.\mu^{2f}$ [2202b], and dividing by the coefficient of q, we obtain [2205]. Substituting this in γ [2196], we get the depth of the sea, corresponding to these oscillations,

$$\gamma=l-\frac{2n^2\mu^2}{f.(2f+1).\left\{1-\frac{3}{(4f+1).\rho}\right\}.g}.$$ [2205b]

† (1635) Putting $f=14$, in [2205], and multiplying by l, we get

$$lq=\frac{2n^2}{14.29.\left\{1-\frac{3}{57.\rho}\right\}.g}.$$ [2205c]

Now by [2162] $\rho>1$, hence $\frac{1}{1-\frac{3}{57.\rho}}<\frac{1}{1-\frac{3}{57}}$, or $\frac{19}{18}$. Substituting this, and $\frac{n^2}{g}=\frac{1}{289}$ [2205″], we get $lq<\frac{2}{14.29}.\frac{19}{18}.\frac{1}{289}$, or $lq<\frac{1}{55579}$; the polar semi-axis of the earth being unity [2128ix]. Now this semi-axis is 6356677 metres [2035b]; hence $lq<\frac{6356677^m}{55579}$, or $lq<115$ metres; so that the greatest variation [2205d]

very nearly, the oscillations of the sea, in the case where the depth is everywhere the same.

[2205v] 6. *The value of* c [2181] *is very great in oscillations of the first kind, because of the divisor* i, *which affects several of the terms.* If we develop the part of the action of the moon [2193],

In oscillations of the first kind, c is very great.

[2206] $$\alpha V' = \frac{L}{4\,r^3} \cdot \{\sin.^2 v - \tfrac{1}{2} \cdot \cos.^2 v\} \cdot (1 + 3 \cdot \cos. 2\theta),$$

which produces the oscillations of the first kind, in sines and cosines of angles, increasing in proportion to the time ; and denote by

[2207] $$\alpha k \cdot (1 + 3 \cdot \cos. 2\theta) \cdot \cos. (i t + A),$$

any term of this development of $\alpha V'$; we shall find that k is multiplied
[2207′] by the tangent of the inclination of the lunar orbit to the ecliptic,* in the

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of the depth of the sea, in the hypothesis [2196, 2205‴], is less than 115 metres. This is very small in comparison with the general depth of the sea ; therefore we may consider this depth as being nearly constant, upon the principles here assumed.

* (1636) In the annexed figure, $CIGF$ represents an arc of the celestial equator; $NCED$
[2206a] the ecliptic; NM the lunar orbit; C the first point of Aries; N the moon's ascending node; M the moon's place at any time; the arc ME is perpendicular to the ecliptic CD; and the arcs EG, MHI, perpendicular to the equator. Then if we put, for brevity,

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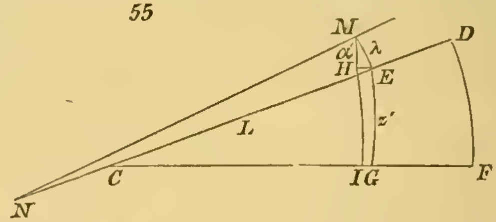

[2206b] $GE = IH = z'$, $HM = \alpha'$, $CE = L$, $ME = \lambda$, angle $ECG = C$, $MNE = N$,

we shall have the moon's declination $v = MI = MH + IH = z' + \alpha'$; and if we neglect α'^2, we shall find, as in [60, 61] Int.,

[2206c] $$\sin. v = \sin. z' + \alpha' \cdot \cos. z', \qquad \cos. v = \cos. z' - \alpha' \cdot \sin. z',$$

[2206d] $$\sin.^2 v - \tfrac{1}{2} \cdot \cos.^2 v = \sin.^2 z' - \tfrac{1}{2} \cdot \cos.^2 z' + 3\alpha' \cdot \sin. z' \cdot \cos. z'.$$

Now by construction we have nearly, angle $MEH = GEC$; hence if we consider MEH as a small rectangular triangle, we shall find very nearly $MH = ME \cdot \sin. MEH$, or $\alpha' = \lambda \cdot \sin. GEC = \lambda \cdot \frac{\cos. C}{\cos. z'}$ [1345^32], consequently $\alpha' \cdot \cos. z' = \lambda \cdot \cos. C$. In the triangle CGE we have $\sin. GE = \sin. CE \cdot \sin. ECG$; or in symbols,

terms where it represents the mean motion of the nodes of the lunar orbit; [2207″]
*but on account of the smallness of i, this term is very considerable, and is the greatest of all which occur in the expression of c.**

We must however make here an important observation. The resistance [2208]
which the waters of the ocean suffer must diminish considerably the oscillations of this kind, and render them of little importance. To prove this, we shall suppose the resistance to be proportional to the velocity, and [2208′]

$$\sin. z' = \sin. L \,.\, \sin. C\,;$$

multiplying this by the preceding expression of $\alpha.\cos.z'$, substituting the result in [2206d], and putting $3 \sin. C \,.\, \cos. C = c'$, we get

$$\sin.^2 \mathrm{v} - \tfrac{1}{2} \,.\, \cos.^2 \mathrm{v} = \sin.^2 z' - \tfrac{1}{2} \,.\, \cos.^2 z' + c' \,.\, \lambda \,.\, \sin. L. \qquad [2206e]$$

In the rectangular triangle NEM, we have $\tan. ME = \mathrm{tang}. MNE \,.\, \sin. NE$ [1345[31]], or nearly $\lambda = \mathrm{tang}. N \,.\, \sin. NE$; multiplying this by $c' \,.\, \sin. L$, or $c' \,.\, \sin. CE$, and reducing the product $c' \,.\, \sin. NE \,.\, \sin. CE$ [17] Int., it produces the term

$$\tfrac{1}{2} c' \,.\, \mathrm{tang}. N \,.\, \cos. (NE - CE) = \tfrac{1}{2} c' \,.\, \mathrm{tang}. N \,.\, \cos. NC.$$

Substituting this term of the factor [2206e] in [2206], and putting also

$$\alpha k = \frac{L}{4 r^3} \,.\, \tfrac{1}{2} c' \,.\, \mathrm{tang}. N, \qquad [2206f]$$

we obtain a term of [2206] of the form $\alpha k.(1+3.\cos.2\theta).\cos.NC$; in which $NC = it + A$ [2206g]
[2207, 2207″] is the distance of the node from the first point of Aries, A being that distance when $t = 0$; moreover the factor k is multiplied by the tangent of the inclination of the lunar orbit to the ecliptic, as in [2207′].

* (1637) If we put $s = 0$ [2197′], in a, b, [2183, 2180], all the terms, having the divisor i, will vanish; but c [2181] will become, by noticing only such terms,

$$c = \frac{\dfrac{2ng}{i} \,.\, \left(\dfrac{da'}{d\mu}\right) \,.\, \mu}{i^2 - 4n^2\mu^2}\,; \qquad [2207a]$$

and this might be great, on account of the smallness of i, if the fluid suffered no resistance. For in the time t, while the earth makes one revolution about its axis, we have $nt = 360^d$, [2128[xvi]]; and during this time, the moon's node describes the arc $it = 3^m\, 10^s$ nearly, [2207″], as is evident from the inspection of any lunar tables. The ratio of these two quantities is $\frac{n}{i} = 6800$ nearly; and though this is multiplied by the tangent of the [2207b]
inclination of the lunar orbit to the ecliptic, 0,09 nearly, it still remains quite large, and exceeds 600; in consequence of which it becomes necessary to examine into the effect of [2207c]
the terms having the divisor i.

[2208''] we shall put β for the coefficient of this resistance. The two equations deduced from the expression [2130] will then be,*

Equations to find u, v, with a resisting fluid.

[2209] $$\left(\frac{d\,d\,u}{d\,t^2}\right)-2n.\left(\frac{d\,v}{d\,t}\right).\mu.\sqrt{1-\mu^2}+\beta.\left(\frac{d\,u}{d\,t}\right)=g.\left(\frac{d\,y}{d\,\mu}\right).\sqrt{1-\mu^2}-\left(\frac{d\,V'}{d\,\mu}\right).\sqrt{1-\mu^2};$$

[2209'] $$\left(\frac{d\,d\,v}{d\,t^2}\right)+\frac{2\,n\,.\,\mu\,.\left(\frac{d\,u}{d\,t}\right)}{\sqrt{1-\mu^2}}+\beta.\left(\frac{d\,v}{d\,t}\right)=-\frac{g\,.\left(\frac{d\,y}{d\,\varpi}\right)}{1-\mu^2}+\frac{\left(\frac{d\,V'}{d\,\varpi}\right)}{1-\mu^2};$$

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* (1638) The quantities θ, ϖ, corresponding to the state of equilibrium, become $\theta+\alpha u$,
[2209a] $\varpi+\alpha\,v$, [2128xv], in the state of motion, at the end of the time t; $\alpha\,u$ and $\alpha\,v$ being the relative motions of the fluid in latitude and longitude [347''']. The differentials of these quantities, relative to t, being divided by $d\,t$, give the relative velocities of the particle, in
[2209b] latitude $\alpha.\left(\frac{d\,u}{d\,t}\right)$, and in longitude $\alpha.\left(\frac{d\,v}{d\,t}\right)$. This last expression must be multiplied by the sine of the polar distance θ, to reduce it to the parallel of latitude; so that the velocity in the direction of the meridian $a\,A$, in the figure page 346, is $\alpha.\left(\frac{d\,u}{d\,t}\right)$, and in the
[2209c] direction of the parallel of latitude $A\,B$, is $\alpha.\left(\frac{d\,v}{d\,t}\right).\sin.\theta$. Now if we suppose the resistance to be represented by the product of β [2208''] by the velocity, it may be resolved into two forces, in the directions $a\,A$, $A\,B$, by multiplying the relative velocities [2209b]
[2209d] by $-\beta$; the negative sign being prefixed, because the resistance tends to decrease these
[2209d'] velocities; hence they become respectively $-\alpha\,\beta.\left(\frac{d\,u}{d\,t}\right)$, $-\alpha\beta.\left(\frac{d\,v}{d\,t}\right).\sin.\theta$. These are to be multiplied by the elements of their directions, $a\,A$, $A\,B$, or $d\theta$, $d\varpi.\sin.\theta$, to produce the corresponding parts of $\alpha\,\mathrm{d}\,V'$ [2130']; which are $-\alpha\beta.\left(\frac{d\,u}{d\,t}\right).d\,\theta$, $-\alpha\,\beta.\left(\frac{d\,v}{d\,t}\right).\sin.^2\theta\,.\,d\,\varpi$; consequently the part of $\mathrm{d}\,V'$ arising from this source, is
[2209e] $\mathrm{d}\,V'=-\beta.\left(\frac{d\,u}{d\,t}\right).d\,\theta-\beta.\left(\frac{d\,v}{d\,t}\right).\sin.^2\theta\,.\,d\,\varpi$. Substituting this in formula [2130], and transposing the terms to the first member, the coefficients of $d\theta$, $d\varpi$, will be augmented
[2209f] by the terms $\beta.\left(\frac{d\,u}{d\,t}\right)$, $\beta.\left(\frac{d\,v}{d\,t}\right).\sin.^2\theta$; which is the same thing as to change
[2209g] $\left(\frac{d\,d\,u}{d\,t^2}\right)$ into $\left(\frac{d\,d\,u}{d\,t^2}\right)+\beta.\left(\frac{d\,u}{d\,t}\right)$, and $\left(\frac{d\,d\,v}{d\,t^2}\right)$ into $\left(\frac{d\,d\,v}{d\,t^2}\right)+\beta.\left(\frac{d\,v}{d\,t}\right)$, as is evident by inspection. This produces the same changes in [2176, 2176']; by which means they become as in [2209, 2209']. The equation [2129] does not contain V', therefore it remains unaltered.

since it is evident that the resistance must add to the two first members of these equations, the terms $\beta.\left(\frac{du}{dt}\right)$ and $\beta.\left(\frac{dv}{dt}\right)$. The equation [2129] will remain unaltered. [2209″]

We shall here consider only the terms depending on the angle it, in which the coefficient i is very small, and much less than β. In this case $\left(\frac{ddu}{dt^2}\right)$ [2209‴] and $\left(\frac{ddv}{dt^2}\right)$ may be neglected in comparison with $\beta.\left(\frac{du}{dt}\right)$ and $\beta.\left(\frac{dv}{dt}\right)$;* and as these terms are independent of the angle ϖ, the equation [2209′] will give,† [2209⁗]

Oscillations depending on angles it, in which i is very small.

$$\frac{2n\mu.\left(\frac{du}{dt}\right)}{\sqrt{1-\mu^2}}+\beta.\left(\frac{dv}{dt}\right)=0\,; \qquad [2210]$$

and [2209] will become,‡

* (1639) Putting $s=0$ [2197′] in [2178′, 2178″], we get

$$u=b.\cos.(it+\varepsilon), \qquad v=c.\sin.(it+\varepsilon)\,;$$

whose differentials are,

$$\left(\frac{du}{dt}\right)=-bi.\sin.(it+\varepsilon), \qquad \left(\frac{ddu}{dt^2}\right)=-bi^2.\cos.(it+\varepsilon),$$

$$\left(\frac{dv}{dt}\right)=ci.\cos.(it+\varepsilon), \qquad \left(\frac{ddv}{dt^2}\right)=-ci^2.\sin.(it+\varepsilon), \qquad [2210a]$$

Substituting these in [2209, 2209′], we shall find that the terms depending on ddu, ddv, have the factor i^2; while those of du, dv, have the factor ni, or βi. These last, being divided by the first, produce terms of the order $\frac{n}{i}$, or $\frac{\beta}{i}$; and as these quantities [2210b] are very large [2207b, 2209‴], it follows that the terms depending on du, dv, must be much greater than those arising from ddu, ddv; therefore these last may be neglected. [2210c]

† (1640) Since ϖ does not enter in oscillations of the first kind [2197′], the partial differentials of y, V', [2209′], relative to ϖ, vanish; and as ddv is also neglected [2210c], [2211a] the equation [2209′] becomes as in [2210].

‡ (1641) Multiplying [2210] by $\frac{2n}{\beta}.\mu.\sqrt{(1-\mu^2)}$, and adding the product to [2209], [2212a] neglecting ddu, as in [2210c], it becomes as in [2211].

[2211] $$\left(\frac{\beta^2+4n^2\mu^2}{\beta}\right)\cdot\left(\frac{du}{dt}\right)=g\cdot\left(\frac{dy}{d\mu}\right)\cdot\sqrt{1-\mu^2}-\left(\frac{dV'}{d\mu}\right)\cdot\sqrt{1-\mu^2}; \qquad (f)$$

This equation must be combined with the following,*

[2212] $$y=\left(\frac{d\cdot(\gamma u\cdot\sqrt{1-\mu^2})}{d\mu}\right).$$

[2212'] If we neglect quantities of the order i, the equation [2211] will give,†

[2213] $$0=g\cdot\left(\frac{dy}{d\mu}\right)-\left(\frac{dV'}{d\mu}\right);$$

or

[2214] $$gy-V'=0;$$

therefore,‡

[2215] $$u=\frac{\int V'\cdot d\mu}{g\gamma\cdot\sqrt{1-\mu^2}}.$$

[2215'] This value of u, being substituted in the equation [2211], will give a more correct value of $gy-V'$; but this first approximation is sufficiently accurate.

[2212b] * (1642) The equation [2212] is the same as [2136b], neglecting the term depending on ϖ, as in [2211a].

† (1643) Neglecting quantities of the order i as in [2212'], the quantity $\left(\frac{du}{dt}\right)$ [2210a]

[2213a] will vanish from the first member of [2211]; and then dividing by $\sqrt{(1-\mu^2)}$, we shall

[2213b] get the equation [2213]. Multiplying this by $d\mu$, and integrating, we get $gy-V'=0$; observing that $y=0$ when $V'=0$ [2128xiv]; because when the disturbing forces V'

[2213c] vanish, the fluid is supposed to be in its situation of equilibrium; therefore it is not necessary to add any constant quantity to this integral.

‡ (1644) From [2214] we get $y=\frac{V'}{g}$; substituting this in [2212], multiplying by

[2214a] $d\mu$, and integrating, we find $\frac{\int V'.d\mu}{g}=\gamma u.\sqrt{(1-\mu^2)}$. Dividing this by $\gamma.\sqrt{(1-\mu^2)}$, we obtain u [2215]. If this value of u be substituted in [2211], we shall obtain an equation from which we can find, by approximation, a more correct value of u; but this is unnecessary, because the corrections thus introduced are of the order i.

The value of V', relative to the action of the body L, is of the form $k \,.\, (1+3 \,.\, \cos. 2\,\theta) \,.\, \cos. (i\,t+A)$ [2207], or $6\,k \,.\, (\mu^2-\tfrac{1}{3}) \,.\, \cos. (it+A)$ [2216]
[2197*a*]. If $Q \,.\, (\mu^2-\tfrac{1}{3}) \,.\, \cos. (i\,t+A)$ be the corresponding part of y; [2216′]
the similar part of V', arising from the action of a stratum of the fluid, whose internal radius is 1, and external radius $1+\alpha\,y$, will be*

$$\frac{3\,g}{5\,\rho} \,.\, Q \,.\, (\mu^2-\tfrac{1}{3}) \,.\, \cos. (i\,t+A)\,; \qquad [2217]$$

therefore the equation $g\,y - V' = 0$ [2214], will give,†

$$0 = \left(1-\frac{3}{5\,\rho}\right) \,.\, Q \,.\, (\mu^2-\tfrac{1}{3}) - \frac{6\,k}{g} \,.\, (\mu^2-\tfrac{1}{3})\,; \qquad [2218]$$

hence we deduce

$$Q = \frac{6\,k}{g \,.\, \left(1-\frac{3}{5\,\rho}\right)}\,; \qquad [2219]$$

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* (1646) Substituting in [2178] the values $s=0$ [2197′], $\varepsilon = A$, also a [2198]; we get $y = \{P^{(0)}+P^{(2)}+P^{(4)}+\&c.\} \,.\, \cos. (i\,t+A)$. The term of V' [2216] [2217*a*]
depending on the action of the body L, has the factor $\mu^2-\tfrac{1}{3}$, which is of the form $Y^{(2)}$ [1528*c*]; and if, in this last expression, we change $Y^{(2)}$ into $P^{(2)}$, $B_2^{(0)}$ into Q, we shall have $P^{(2)} = Q \,.\, (\mu^2-\tfrac{1}{3})$. The part of y [2217*a*] corresponding, is

$$y = Q \,.\, (\mu^2-\tfrac{1}{3}) \,.\, \cos. (i\,t+A). \qquad [2217b]$$

Substituting this in [2178‴], and also the similar part of a', depending on the action of the stratum [2199′], which is $a' = \left(1-\frac{3}{5\,\rho}\right) \,.\, Q \,.\, (\mu^2-\tfrac{1}{3})$ [2200]; we get the corresponding part of $\frac{V'}{g}$, namely $\frac{V'}{g} = y - a' \,.\, \cos. (it+A) = \frac{3}{5\rho} \,.\, Q \,.\, (\mu^2-\tfrac{1}{3}) \,.\, \cos. (i\,t+A)$, [2217*c*]
as in [2217].

† (1647) Adding the two parts of V' [2216, 2217], we get

$$V' = \left\{\frac{3g}{5\rho} \,.\, Q + 6\,k\right\} \,.\, (\mu^2-\tfrac{1}{3}) \,.\, \cos. (i\,t+A).$$

Substituting this, and y [2217*b*], in $g\,y - V' = 0$ [2217], then dividing by

$$g \,.\, \cos. (i\,t+A), \qquad [2218a]$$

we get [2218]. Again dividing by the coefficient of Q, we obtain [2219]; substituting this in y [2216′], then multiplying by α, it becomes as in [2220].

consequently

[2220] $$\alpha y = \frac{6\alpha k \, . \, (\mu^2 - \frac{1}{3}) \, . \, \cos.(i t + A)}{g \, . \left(1 - \frac{3}{5\rho}\right)}.$$

The sum of all the terms, $\alpha k \, . \, \cos.(i t + A)$ [2207, 2206], is represented

[2220′] by $\alpha k \, . \, \cos.(i t + A) = \frac{L}{4 r^3} \, . \, \{\sin.^2 \mathrm{v} - \frac{1}{2} \, . \, \cos.^2 \mathrm{v}\}$; therefore *the whole value of* αy, *corresponding to oscillations of the first kind, arising from the action of the body* L, *will be*,*

Whole value of αy, in oscillations of the first kind.

[2221] $$\alpha y = \frac{L \, . \, \{\sin.^2 \mathrm{v} - \frac{1}{2} \, . \, \cos.^2 \mathrm{v}\} \, . \, \{1 + 3 \, . \, \cos. 2\theta\}}{4 r^3 \, . \, g \, . \left(1 - \frac{3}{5\rho}\right)}.$$

This value is that which takes place, if v and r are rigorously constant, and the fluid, at each instant, assumes the form corresponding to the state of equilibrium. For $\left(\frac{du}{dt}\right)$ and $\left(\frac{dv}{dt}\right)$ being nothing in the case of

[2221′] equilibrium, the differential equations in u and v become† $0 = g y - V'$.

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* (1649) The part of $\alpha V'$ [2216] depending on the attraction of the body L, being

[2221a] divided by the *constant* quantity $g \, . \left(1 - \frac{3}{5\rho}\right)$, produces the corresponding part of αy [2220]; and all the terms of $\alpha V'$, depending on oscillations of the first kind, depend on

[2221b] the function [2193]; therefore if we divide this function by the same quantity $g \, . \left(1 - \frac{3}{5\rho}\right)$, we shall obtain the complete value of αy [2221], depending on these oscillations.

† (1650) When the fluid is in equilibrium, the velocity of a particle is nothing; consequently

[2221c] the expressions of the velocities [2209b], $\alpha \, . \left(\frac{du}{dt}\right)$, $\alpha \, . \left(\frac{dv}{dt}\right)$, vanish; therefore $\left(\frac{du}{dt}\right)$, $\left(\frac{dv}{dt}\right)$, and their differentials $\left(\frac{ddu}{dt^2}\right)$, $\left(\frac{ddv}{dt^2}\right)$, become nothing. Substituting these in [2209, 2209′], we find that the first members of these expressions vanish; and the second members, divided respectively by $(1 - \mu^2)^{\frac{1}{2}}$, $-(1 - \mu^2)^{-1}$, become

[2221d] $$0 = g \, . \left(\frac{dy}{d\mu}\right) - \left(\frac{dV'}{d\mu}\right), \qquad 0 = g \, . \left(\frac{dy}{d\varpi}\right) - \left(\frac{dV'}{d\varpi}\right).$$

Multiplying these by $d\mu$, $d\varpi$, respectively, and adding the products, we get

$$g \, . \left\{ \left(\frac{dy}{d\mu}\right) . \, d\mu + \left(\frac{dy}{d\varpi}\right) . \, d\varpi \right\} - \left\{ \left(\frac{dV'}{d\mu}\right) . \, d\mu + \left(\frac{dV'}{d\varpi}\right) . \, d\varpi \right\} = 0,$$

Therefore, when the variations of r *and* v *are very small, we may determine the oscillations of the first kind, as if the fluid assumed, at each instant, its situation of equilibrium under the action of the body which attracts it.* The error of this supposition becomes less, as the velocity of the attracting body decreases, therefore it is insensible for the sun. It might be sensible for the moon, because of the rapidity of the motion of this luminary in its orbit; but as the oscillations of the first kind are by observation very small, we may, even in computing the action of the moon, make use of the preceding value of αy. [2221''] [2221'''] [2221'''']

Although we have obtained these results, by supposing the resistance to be proportional to the velocity; it is evident that they exist, whatever be the law of the resistance. In general we may adopt them without sensible error, whenever the fluid, which is disturbed in its equilibrium, returns to this state, by means of* [2221v]

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or $g \cdot \mathrm{d}y - \mathrm{d}V' = 0$; θ, ϖ, being considered as the variable quantities [2130']. The integral of this last expression is $gy - V' = 0$, as in [2221']; no constant quantity being added, for the reasons stated in [2213c]. [2221e]

* (1651) The velocity of a particle of the fluid, in the *direction* of the meridian, is $\alpha \cdot \left(\frac{du}{dt}\right)$ [2209b], and in the *direction* of the parallel, $\alpha \cdot \left(\frac{dv}{dt}\right) \cdot \sin.\theta$ [2209c]. The whole horizontal velocity, which we shall call αW, is the square root of the sum of the squares of these quantities, or $\alpha W = \alpha \cdot \left\{ \left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2 \cdot \sin.^2\theta \right\}^{\frac{1}{2}}$. We shall now suppose the resistance to be proportional to a function of αW, represented by $\alpha W \cdot \beta$; β being a function of W. This resistance, in the direction of the motion of the moving particle, may be resolved into two others; the one in the direction of the meridian, represented by $-\alpha \cdot \left(\frac{du}{dt}\right) \cdot \beta$; the other in the direction of the parallel of latitude, $-\alpha \cdot \left(\frac{dv}{dt}\right) \cdot \beta \cdot \sin.\theta$; as is evident from the usual rule for the resolution of a force. These expressions of the resistances being of the same form as in [2209d'], we may deduce from them the same value of $\mathrm{d}V'$, as in [2209e], and the same additional terms $\beta \cdot \left(\frac{du}{dt}\right)$, $\beta \cdot \left(\frac{dv}{dt}\right)$, to the first members of the equations [2209, 2209']; therefore the equations [2210, 2211] retain the same form; β being a function of W, instead of being constant as in [2208'']. Then neglecting $\left(\frac{du}{dt}\right)$ [2211], because it is of the order i, we obtain $gy - V' = 0$, as in [2213b]; which is therefore the same for all laws of resistance. [2221f] [2221g] [2221h] [2221i] [2221k]

*the resistances it suffers, in a less time than that which is required to complete the revolution of the attracting body.**

ON OSCILLATIONS OF THE SECOND KIND.

The part of $\alpha V'$ depending on the action of the body L.

7. The part of the action of the body L, which produces these oscillations, is equal to [2194],

[2222]
$$\frac{3L}{r^3}\,.\,\sin.\,\text{v}\,.\,\cos.\,\text{v}\,.\,\sin.\,\theta\,.\,\cos.\,\theta\,.\,\cos.\,(nt+\varpi-\psi).$$

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* (1652) When the moon has any particular declination, represented by v, it produces

[2221*l*] a corresponding oscillation of the first kind; and if this be not destroyed by the resistance, in one revolution of the moon, it will be augmented by the renewed action of the moon, when it has completed the revolution, and again attained to nearly the same declination. In this case the successive oscillations will interfere with each other; and we cannot then suppose the fluid to take the form, corresponding to the equilibrium under the forces which act upon

[2221*m*] it. On the contrary, if the resistance nearly destroys one oscillation, before a different one commences, we may suppose the fluid to assume nearly the form of equilibrium corresponding

[2221*n*] to the forces acting upon it; hence we may obtain, as in [2221*e*], $gy-V'=0$, and consequently αy, as in [2220].

(1653) We shall now make a few remarks on the method of calculation used in formulas [2196—2221]. When a value of a' is found, which satisfies [2183], we may substitute it in

[2221*o*] [2180, 2181], and we shall get values of a, b, c; and then, from [2178—2178'''], the values of y, u, v, V', which will satisfy the equations [2175—2176']. Having obtained these, it will be unnecessary to seek for other values, depending on the same angle $it+\varepsilon$. For if we suppose the quantities a', a, b, c, y, u, v, V', to be increased by the terms $a'_{,}$, $a_{,}$, $b_{,}$, $c_{,}$,

[2221*p*] $y_{,}$, $u_{,}$, $v_{,}$, $V'_{,}$, respectively; and that the new values $a'+a'_{,}$, $b+b_{,}$, &c., when substituted for a', a, &c., in [2183], and in the similar equations [2176—2181], will satisfy them; we shall find that the difference between these two expressions of the equation [2183], is an equation of exactly the same form, in which a, a', are changed into $a_{,}$, $a'_{,}$, respectively, or in other words, an accent is marked below the letters. The same results are obtained in the similar equations [2176—2181]; but the function $V'_{,}$, which takes the place of V',

[2221*q*] in [2176, 2176'], does not contain the function $U^{(i)}$ [2148]; because the value of V' [2150*a*], being subtracted from the similar value of

$$V'+V'_{,}=\Sigma\,\frac{3g}{\rho}\,.\,\frac{Y^{(i)}+Y^{(i)}_{,}}{2i+1}+\Sigma\,U^{(i)},$$

[2221*r*] gives $V'_{,}=\Sigma\,\frac{3g}{\rho}\,.\,\frac{Y^{(i)}_{,}}{2i+1}$, which is independent of $U^{(i)}$ [2148] arising from the action

The development of this function, in sines and cosines of angles proportional to the time, gives a series of terms, of the form

$$\alpha k . \mu . \sqrt{1-\mu^2} . \cos. (i t + \varpi - A) ;$$ [2223]

in which i differs but very little from n; because the motion of the attracting body is very small in comparison with the rotatory motion of the earth.*
We shall now resume the equation [2183], supposing $s=1$, and z as [2224]
in [2197],

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of the body L. Hence the values of $u_{,}$, $v_{,}$, $y_{,}$, &c., must depend on the same equations, as if this body did not act on the fluid; and then the oscillations corresponding to the angle $i t+\varepsilon$, which depends on the motions of this body, ought evidently to vanish. Similar remarks may be made relative to oscillations of the second and third kinds, in which [2221s]
$s=1$, $s=2$.

* (1654) Putting $n't$ for the *mean* motion of the body L, its true place in the orbit will be represented by the second of the equations [669], its distance r by the first of those [2223a]
equations, and its latitude by [679]; n being changed into n', in all of them, to conform to the present notation. All the terms of these series depend on quantities of the form $\genfrac{}{}{0pt}{}{\sin.}{\cos.} s' n' t$; s' being a whole number. The true place of the body in its orbit, being [2223b]
reduced to the ecliptic, is of a similar form [675]. Having the longitude, and the latitude, of the moon, expressed in such series, the declination v, which is deduced from this longitude and latitude, must also be of a similar form; consequently the factor $\frac{3L}{r^3} . \sin. \mathrm{v} . \cos. \mathrm{v}$ [2223c]
[2222] contains terms of the form $\genfrac{}{}{0pt}{}{\sin.}{\cos.} s' n' t$, or as it may be generally expressed,

$$\cos. (s' n' t + A').$$

Factors of this kind, being multiplied by the term $\cos. (n t + \varpi - \psi)$, which occurs in [2223d]
[2222], produce terms of the form $\alpha k . \cos. \{(n \pm s' n') . t + \varpi - \psi \pm A'\}$ [20] Int.;
and by putting $n \pm s' n' = i$, $-\psi \pm A' = -A$, it becomes $\alpha k . \cos. (i t + \varpi - A)$. [2223d']
Now n' is very small, in comparison with n; since $n' = \frac{1}{27} . n$ for the moon, and
$n' = \frac{1}{366} . n$ for the sun; and in the chief terms of their attractions, s' does not exceed one [2223e]
or two; therefore i must be nearly equal to n. Lastly, since

$$\frac{3L}{r^3} . \sin. \mathrm{v} . \cos. \mathrm{v} . \cos. (n t + \varpi - \psi)$$

is composed of terms of the form $\alpha k . \cos. (it + \varpi - A)$, we shall find, in multiplying by $\sin. \theta . \cos. \theta = \mu . \sqrt{(1-\mu^2)}$, that $\frac{3L}{r^3} . \sin. \mathrm{v} . \cos. \mathrm{v} . \sin. \theta . \cos. \theta$ is composed of terms of the form $\alpha k . \mu . \sqrt{(1-\mu^2)} . \cos. (i t + \varpi - A)$, as in [2223]. This angle $(i t + \varpi - A)$, being compared with [2178'''], depending on V', gives $s=1$, $\varepsilon = -A$. [2223f]

Value of z.

[2225] $$z = \frac{l \cdot (1 - q\mu^2)}{i^2 - 4n^2\mu^2};$$

Assumed values of a.

we shall also suppose that a is expressed by the series,*

[2226] $$a = \mu \cdot \sqrt{1-\mu^2} \cdot \{P^{(0)} + P^{(2)} + P^{(4)} \ldots\ldots + P^{(2f-2)}\};$$

$P^{(0)}$, $P^{(2)}$, &c., being such functions of μ^2, that if we put

[2227] $$Y^{(2f)} = \mu \cdot \sqrt{1-\mu^2} \cdot P^{(2f-2)},$$

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* (1655) If we follow the same method of calculation as the author, we shall find, that this *assumed* value of a gives the corresponding value of a' [2231]; and that these expressions

[2225a] of a, a', satisfy the equation [2183], supposing the value of q to be as in [2238]. But if it were required to know why the value of a was assumed of the form in [2226], we might state the reasons for it in the following manner, according to the usual principles of development of linear equations. The function [2222], on which the oscillations of the

[2225b] second kind depend, contains the factor $\sin.\theta \cdot \cos.\theta$, or $\mu \cdot \sqrt{(1-\mu^2)}$, which *vanishes* when $\mu=0$, or when $\sqrt{(1-\mu^2)}=0$; but in cases where the disturbing force vanishes, it is natural to suppose, that the effect resulting from this force also vanishes; and to produce

[2225c] this effect in αy, y, or a, [2178], we may suppose *a to have the same factor* $\mu \cdot \sqrt{(1-\mu^2)}$, as in [2226]. This factor of y may also be supposed to be common to every term $Y^{(2f)}$,

[2225d] into which y is developed [2144, &c.], by which means it becomes as in [2227]; observing that terms of the form $Y^{(2f+1)}$ must be rejected, because no such terms, depending on $\sin.\varpi$, $\cos.\varpi$, have the factor $\mu \cdot \sqrt{(1-\mu^2)}$, as is evident by the inspection of the

[2225e] formulas [1528a—e, &c]; hence we must put $Y^{(1)}=0$, $Y^{(3)}=0$, &c., in [2144]. The terms depending on $\sin.\varpi$, $\cos.\varpi$, in $Y^{(2)}$, $Y^{(4)}$, &c., [1528c, e, &c.], are

[2225f] of the form $Y^{(2f)} = C \cdot \mu \cdot \sqrt{(1-\mu^2)} \cdot \{A' \cdot \sin.\varpi + B' \cdot \cos.\varpi\}$, A', B', being independent of μ; and C a function of μ^2 and constant quantities. The values of C, corresponding

[2225g] to $Y^{(2)}$, $Y^{(4)}$, $Y^{(6)}$, &c., being 1, $\mu^2 - \frac{3}{7}$, $\mu^4 - \frac{10}{11} \cdot \mu^2 + \frac{5}{33}$, &c., as is evident from [1528c, e, 1528]. This value of $Y^{(2f)}$ may be made to depend on an angle of the

[2225h] same form as in [2223], by putting $A' = -h \cdot \sin.(it - A)$, $B' = h \cdot \cos.(it - A)$; whence

$$A' \cdot \sin.\varpi + B' \cdot \cos.\varpi = h \cdot \{-\sin.\varpi \cdot \sin.(it - A) + \cos.\varpi \cdot \cos.(it - A)\}$$

[2225i] $$= h \cdot \cos.(it + \varpi - A);$$

by which means the function [2225f] becomes $Y^{(2f)} = Ch \cdot \mu \cdot \sqrt{(1-\mu^2)} \cdot \cos.(it + \varpi - A)$, Ch being a function of μ^2 and constant quantities; and if we represent it by $P^{(2f-2)}$, we

[2225k] shall get $Y^{(2f)} = \mu \cdot \sqrt{(1-\mu^2)} \cdot P^{(2f-2)} \cdot \cos.(it + \varpi - A)$; which produces, for y [2144], the same function as would be obtained by the substitution of a [2226], $Y^{(2f)}$ [2227], in

[2225l] [2178]; observing that the quantity we have named $Y^{(2f)}$ in this note, is equal to that in [2227] multiplied by $\cos.(it + \varpi - A)$.

we shall have, whatever be the value of f,*

Differential equation in $Y^{(2f)}$.

$$0=\left\{\frac{d\cdot\left\{(1-\mu^2)\cdot\left(\frac{d\,Y^{(2f)}}{d\mu}\right)\right\}}{d\mu}\right\}-\frac{Y^{(2f)}}{1-\mu^2}+2f.(2f+1).Y^{(2f)}. \qquad [2228]$$

The part of a' relative to the action of an aqueous stratum, whose internal radius is unity, and external radius $1+\alpha y$, will be, by what precedes,† [2228′]

$$-\frac{3\mu.\sqrt{1-\mu^2}}{\rho}.\left\{\tfrac{1}{5}.P^{(0)}+\tfrac{1}{9}.P^{(2)}+\tfrac{1}{13}.P^{(4)}\ldots\ldots+\frac{1}{4f+1}.P^{(2f-2)}\right\}. \qquad [2229]$$

The part of a', relative to the action of the body L,‡ is $-\frac{k}{g}.\mu.\sqrt{1-\mu^2}$; [2230]

therefore we shall have,§

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* (1657) The second differential of $Y^{(2f)}$ [2225k], relative to ϖ, gives

$$\left(\frac{d\,d\,Y^{(2f)}}{d\varpi^2}\right)=-\mu.\sqrt{(1-\mu^2)}.P^{(2f-2)}.\cos.(i\,t+\varpi-A)=-Y^{(2f)}; \qquad [2228a]$$

$P^{(2f-2)}$ being independent of ϖ [2225k]. Substituting this, and $i=2f$, in [2145], we get [2228].

† (1658) The part of V' [2146a], depending on the term $Y^{(2f)}$ of this aqueous stratum, is $\frac{3g}{(4f+1).\rho}.Y^{(2f)}$. Substituting the value of $Y^{(2f)}$ [2225k], we get the corresponding [2228b] term of $-\frac{V'}{g}=-\mu.\sqrt{(1-\mu^2)}.\frac{3}{(4f+1).\rho}.P^{(2f-2)}.\cos.(i\,t+\varpi-A)$; hence the [2228$c$] corresponding term of a' [2178‴] is $-\mu.\sqrt{(1-\mu^2)}.\frac{3}{(4f+1).\rho}.P^{(2f-2)}$, as in [2229].

‡ (1659) This part of $\alpha V'$ is given in [2194], or [2222], and is supposed to be composed of a series of terms of the form $\alpha k.\mu.\sqrt{(1-\mu^2)}.\cos.(it+\varpi-A)$ [2223]. [2229a] Dividing this by $-\alpha g$, we get the corresponding part of $-\frac{V'}{g}$ [2178‴], equal to $-\frac{k}{g}.\mu.\sqrt{(1-\mu^2)}.\cos.(i\,t+\varpi-A)$; and this produces in a' [2178‴] the term [2229b] $-\frac{k}{g}.\mu.\sqrt{(1-\mu^2)}$, as in [2230].

§ (1660) Substituting in [2178‴], the two parts of $-\frac{V'}{g}$ [2228c, 2229b], also y [2178, 2226], and dividing by $\cos.(i\,t+\varpi+\varepsilon)$, we get a' [2231]; observing that [2230a] $s=1$ [2223f], and $\varepsilon=-A$, in the case now under consideration.

Value of a'.

[2231] $$a' = \mu.\sqrt{1-\mu^2}.\left\{\begin{array}{l}\left(1-\frac{3}{5\rho}\right).P^{(0)}+\left(1-\frac{3}{9\rho}\right).P^{(2)}\,.....+\left(1-\frac{3}{(4f+1).\rho}\right).P^{(2f-2)} \\ -\frac{k}{g}\end{array}\right\}.$$

We shall suppose the indeterminate constant quantities, by which the functions $P^{(0)}$, $P^{(2)}$, &c., are multiplied, to be such that the function

[2231′] $$\frac{2n}{i}.\mu a' - \left(\frac{da'}{d\mu}\right).(1-\mu^2)$$

may be divisible by $i^2 - 4n^2\mu^2$; which requires but one equation of condition,* between these indeterminate quantities; then the second member of [2183] will have no denominator. For by considering, in the second member, only the terms which have $\sqrt{1-\mu^2}$ for a divisor, and supposing

[2232] $a' = F\mu.\sqrt{1-\mu^2}$; F being a rational and integral function of μ^2; the three parts of this second member will become,†

[2233] $$-\left(\frac{2n}{i}+1\right).\frac{gz.\mu^3 F}{\sqrt{1-\mu^2}} + \frac{2n}{i}.\left(\frac{2n}{i}+1\right).\frac{gz.\mu^3 F}{\sqrt{1-\mu^2}} + \frac{(i^2-4n^2\mu^2).gz.\mu F}{i^2.\sqrt{1-\mu^2}},$$

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* (1661) This is proved in like manner as in [2201a—d], where the equation of

[2230b] condition is $C=0$.

† (1662) Putting for brevity

[2231a] $$W = \frac{2n}{i}.\mu a' - \left(\frac{da'}{d\mu}\right).(1-\mu^2);$$

$$F = \left(1-\frac{3}{5\rho}\right).P^{(0)}+\left(1-\frac{3}{9\rho}\right).P^{(2)}\,.....+\left(1-\frac{3}{(4f+1).\rho}\right).P^{(2f-2)}-\frac{k}{g};$$

[2231b] we get, from [2231], $a' = F.\mu.(1-\mu^2)^{\frac{1}{2}}$ [2232]; and from [2183, 2224],

[2231c] $$a = g.\left(\frac{d.(zW)}{d\mu}\right) + \frac{2ng.\mu z}{i.(1-\mu^2)}.W + \frac{gz.a'.(i^2-4n^2\mu^2)}{i^2.(1-\mu^2)}.$$

Substituting in W the value of a' [2231b], and its differential

[2231d] $$\left(\frac{da'}{d\mu}\right) = -\frac{F\mu^2}{(1-\mu^2)^{\frac{1}{2}}} + (1-\mu^2)^{\frac{1}{2}}.\left(\frac{d.(F\mu)}{d\mu}\right), \qquad \text{we get}$$

[2231e] $$W = \left(\frac{2n}{i}+1\right).\mu^2.(1-\mu^2)^{\frac{1}{2}}.F - (1-\mu^2)^{\frac{3}{2}}.\left(\frac{d.(F\mu)}{d\mu}\right).$$

This value, and that of a' [2231b] are to be substituted in [2231c], and it will then contain, in its second member, before reduction, terms having the factors $(1-\mu^2)^{-\frac{1}{2}}$, $(1-\mu^2)^{\frac{1}{2}}$, $(1-\mu^2)^{\frac{3}{2}}$; but we shall find, by connecting together the terms having the factor $(1-\mu^2)^{-\frac{1}{2}}$,

or $gz.\mu.F.\sqrt{1-\mu^2}$; hence it follows, that the second member has not [2234]
$\sqrt{1-\mu^2}$ in its denominator. Therefore, by substituting the values of a and a' in that equation, and dividing it by $\mu.\sqrt{1-\mu^2}$, then comparing the similar powers of μ, we shall get f equations of condition.* These, being [2235]

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that they may be reduced to the second form $(1-\mu^2)^{\frac{1}{2}}$. To prove this, we shall notice [2231f]
only those terms of the second member [2231c], which have the factor $(1-\mu^2)^{-\frac{1}{2}}$. Now if we develop the first of these terms, it becomes,

$$g.\left(\frac{d.(zW)}{d\mu}\right)=gW.\left(\frac{dz}{d\mu}\right)+gz.\left(\frac{dW}{d\mu}\right);$$

and as z [2225] does not contain $(1-\mu^2)$, the first term of the second member will not [2231f′]
contain $(1-\mu^2)^{-\frac{1}{2}}$, as is evident from [2231e]; but the second term $gz.\left(\frac{dW}{d\mu}\right)$ will produce the first term of [2233], by means of the differential of the factor, $(1-\mu^2)^{\frac{1}{2}}$, which occurs in the first term of W [2231c]. Again, by the substitution of W [2231e] in the second term of [2231c], it produces the second term of [2233], having the factor $(1-\mu^2)^{-\frac{1}{2}}$. Lastly, by the substitution of a' [2231b], in the third term of [2231c], we get the third term of [2233]; and these three terms are all that occur in the second member of [2231c] with [2231g]
this factor $(1-\mu^2)^{-\frac{1}{2}}$. The function [2233] has the common factor $\frac{gz.\mu F}{i^2.(1-\mu^2)^{\frac{1}{2}}}$, and [2231h]
if we divide all the terms by this factor, and then reduce the quotient to its most simple terms, we shall get successively

$$-(2ni+i^2).\mu^2+2n.(2n+i).\mu^2+(i^2-4n^2\mu^2)=-i^2\mu^2+i^2=i^2.(1-\mu^2); \qquad [2231i]$$

multiplying this by the preceding factor, we get $gz.\mu F.(1-\mu^2)^{\frac{1}{2}}$ for the value of the function [2233], as in [2234]. Moreover, each term of the second member of [2231c] contains the factor μ, as is evident by the inspection of the values of F, a', W, [2231a, b, c], and z [2225]; therefore the second member becomes divisible by $\mu.(1-\mu^2)^{\frac{1}{2}}$. Hence, after the substitution of a, a', [2226, 2231], in [2231c], it becomes divisible by $\mu.(1-\mu^2)^{\frac{1}{2}}$, [2231k]
and the radical $(1-\mu^2)^{\frac{1}{2}}$ disappears from the expression.

* (1663) Assuming in [2226] the most general expressions of $P^{(0)}$, $P^{(2)}$, $P^{(4)}$, &c., given by [1528, 1528a, &c.], and arranging them according to the powers of μ^2, as we have [2233a]
already done in a similar case in [2201b], we shall get the value of a [2234a]. Substituting this in [2231], we get the value of a' [2234b];

$$a=\mu.(1-\mu^2)^{\frac{1}{2}}.\{G^{(0)}.\mu^{2f-2}+G^{(2)}.\mu^{2f-4}\ldots\ldots+G^{(2f-4)}.\mu^2+G^{(2f-2)}\}; \qquad [2234a]$$

$$a'=\mu.(1-\mu^2)^{\frac{1}{2}}.\{A^{(0)}.\mu^{2f-2}+A^{(2)}.\mu^{2f-4}\ldots\ldots+A^{(2f-4)}.\mu^2+A^{(2f-2)}\}; \qquad [2234b]$$

$A^{(0)}$, $A^{(2)}$, &c., being dependent on the f arbitrary quantities $G^{(0)}$, $G^{(2)}$, ... $G^{(2f-2)}$. Substituting these values of a, a', in [2183], or rather in [2231c] reduced as in [2231k],

connected with the preceding, will form $f+1$ equations of condition, which are to be satisfied; and as the number of the indeterminate quantities,
[2235′] including q, is $f+1$; we shall have as many indeterminate quantities as equations.

[2235″] To obtain the value of q, we shall put $Q\,.\,\mu^{2f-1}\,.\,\sqrt{1-\mu^2}$ for the term of a depending upon the highest power of μ; the similar term of a' will be,*

[2236] $$Q\,.\left(1-\frac{3}{(4f+1)\,.\,\rho}\right).\,\mu^{2f-1}\,.\,\sqrt{1-\mu^2}\,;$$

which gives,*

[2237] $$\frac{lg\,.\,q}{2\,n^2}\,.\left(2f^2+f+\frac{n}{i}\right).\,Q\,.\left(1-\frac{3}{(4f+1)\,.\,\rho}\right).\,\mu^{2f-1}\,.\,\sqrt{1-\mu^2},$$

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and putting the coefficients of the same power of μ in each member equal to each other, we shall obtain f equations of condition. Connecting these with $C=0$ [2230b], we have
[2234c] $f+1$ equations to determine the $f+1$ arbitrary constant quantities

$$q, \quad G^{(0)}, \quad G^{(2)}, \ldots\ldots G^{(2f-2)}, \qquad \text{as in [2235′].}$$

* (1664) The greatest exponent of μ, in [2226, 2231], must evidently arise from $P^{(2f-2)}$. This term of a [2226] is multiplied by $\mu\,.\,(1-\mu^2)^{\frac{1}{2}}$, and the corresponding
[2235a] term of a', [2231] is multiplied by $\mu\,.\,(1-\mu^2)^{\frac{1}{2}}\,.\left\{1-\frac{3}{(4f+1)\,.\,\rho}\right\}$; consequently this part of a' can be deduced from a, by multiplying by $1-\frac{3}{(4f+1)\,.\,\rho}$. In the same way, we may derive [2236] from [2235″]

† (1665) Proceeding in the same manner as in [2202a—d], retaining only the most elevated powers of μ, in a, a'; and putting, for brevity

[2236a] $$Q'=\left(1-\frac{3}{(4f+1)\,.\,\rho}\right).\,Q\,; \qquad R=\frac{lg\,.\,q}{2\,n^2}\,.\,Q'\,.\,(1-\mu^2)^{\frac{1}{2}}\,.\,\mu^{2f-1}\,;$$

[2236b] we get, as in [2235″, 2236], $a=Q\,.\,\mu^{2f-1}\,.\,(1-\mu^2)^{\frac{1}{2}}$, $a'=Q'\,.\,\mu^{2f-1}\,.\,(1-\mu^2)^{\frac{1}{2}}$; hence,

$$\left(\frac{d\,a'}{d\mu}\right)=Q'\,.\,(2f-1)\,.\,\mu^{2f-2}\,.\,(1-\mu^2)^{\frac{1}{2}}-Q'\,.\,\mu^{2f}\,.\,(1-\mu^2)^{-\frac{1}{2}}$$

[2236c] $$=Q'\,.\,(1-\mu^2)^{-\frac{1}{2}}\,.\{(2f-1)\,.\,\mu^{2f-2}-2f\,.\,\mu^{2f}\}.$$

Substituting these in W [2231a], and retaining only the highest power of μ, we get,

[2236d] $$W=Q'\,.\,(1-\mu^2)^{\frac{1}{2}}\,.\,\mu^{2f}\,.\left\{\frac{2n}{i}+2f\right\}.$$

for the similar term of the second member of the equation [2183]; by putting it equal to the corresponding term of a, we shall get,

$$q = \frac{2\,n^2}{l g \cdot \left(1 - \frac{3}{(4f+1)\cdot\rho}\right)\cdot\left(2f^2+f+\frac{n}{i}\right)}; \qquad [2238]$$

Value of q.

Multiplying this by g, and by the value of z [2225], reduced, as in [2202c], to the form $z = \frac{l\cdot(q\mu^2-1)}{4\,n^2\mu^2-i^2} = \frac{l\,q}{4n^2} + H_{,}\cdot\mu^{-2} + \&c.$; retaining only the highest power of μ, we get [2236e]
successively, by using the symbol R [2236a],

$$g\,z\,.\,W = Q'\cdot(1-\mu^2)^{\frac{1}{2}}\cdot\mu^{2f}\cdot\frac{l\,q\cdot g}{2\,n^2}\cdot\left(\frac{n}{i}+f\right) \qquad [2236f]$$

$$= R\,.\,\mu\,.\left(\frac{n}{i}+f\right). \qquad [2236g]$$

The differential of [2236f], relative to μ, being reduced by putting

$$-\frac{1}{1-\mu^2} = \frac{1}{\mu^2-1} = \mu^{-2}+\mu^{-4}+\&c., \qquad [2236h]$$

and then retaining only the highest power of μ, using also R [2236a], gives

$$g\cdot\left(\frac{d\,.(z\,W)}{d\mu}\right) = Q'\cdot(1-\mu^2)^{\frac{1}{2}}\cdot\mu^{2f-1}\cdot\frac{l\,g\,.\,q}{2\,n^2}\cdot\left\{2f\cdot\left(\frac{n}{i}+f\right) - \frac{\mu^2}{1-\mu^2}\cdot\left(\frac{n}{i}+f\right)\right\}$$

$$= Q'\cdot(1-\mu^2)^{\frac{1}{2}}\cdot\mu^{2f-1}\cdot\frac{l\,g\,.\,q}{2\,n^2}\cdot\left\{2f^2+\frac{2\,n\,.f}{i}+\frac{n}{i}+f\right\}$$

$$= R\,.\left\{2f^2+\frac{2\,n\,.f}{i}+\frac{n}{i}+f\right\}. \qquad [2236i]$$

Substituting the value of $\frac{1}{1-\mu^2}$ [2236h], in the two last terms of [2231c], and retaining only the highest powers of μ, we get,

$$a = g\cdot\left(\frac{d\,.(z\,W)}{d\mu}\right) - \frac{2\,n}{i}\cdot\mu^{-1}\cdot g\,z\,.\,W + \frac{4n^2}{i^2}\cdot g\,z\,.\,a'. \qquad [2236k]$$

Substituting, in the two last terms of the second member, the values [2236g, a, b, e], we obtain,

$$-\frac{2\,n}{i}\cdot\mu^{-1}\cdot g\,z\,.\,W = R\,.\left\{-\frac{2n^2}{i^2}-\frac{2\,n\,.f}{i}\right\}, \qquad [2236l]$$

$$+\frac{4n^2}{i^2}\cdot g\,z\,.\,a' = \frac{4n^2}{i^2}\cdot g\cdot\frac{l\,q}{4n^2}\cdot Q'\cdot\mu^{2f-1}\cdot(1-\mu^2)^{\frac{1}{2}} = \frac{2n^2}{i^2}\cdot R\,. \qquad [2236m]$$

Adding together the three equations [2236i, l, m], the first member of the sum becomes like

therefore the depth of the sea being supposed equal to*

Depth of the sea.

[2239]
$$l - \frac{2n^2\mu^2}{g\cdot\left(1-\frac{3}{(4f+1)\cdot\rho}\right)\cdot\left(2f^2+f+\frac{n}{i}\right)},$$

we can determine, by the preceding analysis, the oscillations of the second kind.

[2239′] This law of the depth depends on the value of i, therefore it is not the same for all the terms in which the action of the attracting body is developed; now this identity is indispensable for the admission of any particular law of the depth. *But we may observe, that as* i *differs but very little from* n, *we*

[2240] *may suppose* $\frac{n}{i}=1$ [2223e], *and then the preceding law of the depth of the sea becomes independent of* i. This is also very nearly equal to that we have found, in the preceding article, for the oscillations of the first kind,

[2241] if f be sufficiently large to permit us to neglect $\frac{n}{i}$ in comparison with $2f^2+f$.†

Consideration of the case in which n and i are equal.

[2241′] 8. *The consideration of* i *being nearly equal to* n, *leads to a very simple and remarkable expression of* a, *which furnishes an explanation of one of the*

[2242] *principal phenomena of the tides.* If we put $i=n$, we shall have [2225]

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the second member of [2236k], and the second member of the sum, rejecting the terms depending on $\pm\frac{2n^2}{i^2}$, $\pm\frac{2n\cdot f}{i}$, which mutually destroy each other, becomes

$$R\cdot\left(2f^2+f+\frac{n}{i}\right).$$

[2236n] Resubstituting the value of R [2236a], we get [2237], for the corresponding term of a [2236k]. Putting this equal to the assumed value [2235″], and then dividing by the coefficient of q, we get [2238].

[2239a] * (1666) Substituting [2238] in [2196], we get the depth of the sea, as in [2239].

[2240a] † (1668) If $\frac{n}{i}$ be so small, in comparison with $2f^2+f$, that we may neglect it; the expression [2238] will become identical with [2205], which corresponds to the first oscillation.

Value of z.

$$z = \frac{l \cdot (1 - q\,\mu^2)}{n^2 \cdot (1 - 4\,\mu^2)}. \qquad [2243]$$

We shall now suppose, in the equation [2183], that $s = 1$, $i = n$, and* $a = Q \cdot \mu \cdot \sqrt{1-\mu^2}$, Q being a coefficient independent of μ; we shall then have, by what precedes,† Value of a. [2244]

Value of a'.

$$a' = \left\{ \left(1 - \frac{3}{5\,\rho}\right) \cdot Q - \frac{k}{g} \right\} \cdot \mu \cdot \sqrt{1-\mu^2}\,; \qquad [2245]$$

which gives,‡

$$\frac{2\,n}{i} \cdot \mu\,a' - \left(\frac{d\,a'}{d\,\mu}\right) \cdot (1 - \mu^2) = - \frac{(1 - 4\mu^2)}{\mu} \cdot a'. \qquad [2246]$$

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* (1670) We have found, in [$2223d'$], $i = n \pm s'n'$; and if we suppose, as in [2242], $i = n$, it will be the same as to neglect n' in comparison with n; and this is equivalent to the supposition that the attracting body does not change its place during the time nt of one oscillation, in which case v, r, [2222], remain constant; consequently the factor k, deduced from [2222], may be considered as constant, in the part of [2241a]

$$\alpha\,V' = \alpha\,k \cdot \mu \cdot \sqrt{(1-\mu^2)} \cdot \cos.\,(i\,t + \varpi - A) \qquad [2223],$$

on which the functions of the second kind depend. This part of $\alpha\,V'$ contains the factor $\mu . \sqrt{(1-\mu^2)}$, which must produce, in the equation [2178'''], a term having the same factor; and this must be balanced by terms of the similar forms in y, a', a, [2178''', 2178]. [2241b] Assuming therefore, for a, the value [2244], we get, as in the next note, a' [2245]. These values must satisfy [2183], and by this condition we shall hereafter obtain the value of Q [2249].

† (1671) Comparing the values of a [2244, 2226], we get $P^{(0)} = Q$, $P^{(2)} = 0$, $P^{(4)} = 0$, &c. Substituting these in [2231], we get a' [2245]. [2245a]

‡ (1672) Putting for brevity $Q' = \left(1 - \frac{3}{5\,\rho}\right) \cdot Q - \frac{k}{g}$, we get $a' = Q' . \mu . \sqrt{(1-\mu^2)}$ [2245]. Its differential gives,

$$\left(\frac{d\,a'}{d\,\mu}\right) = Q' \cdot \frac{1 - 2\mu^2}{\sqrt{(1-\mu^2)}} = \{ Q' \cdot \mu \cdot \sqrt{(1-\mu^2)} \} \cdot \frac{1 - 2\mu^2}{\mu . (1-\mu^2)} = a' \cdot \frac{1 - 2\mu^2}{\mu . (1-\mu^2)}. \qquad [2246a]$$

Substituting this, and $i = n$ [2242], in the first member of [2246], it becomes,

$$2\,\mu\,a' - a' \cdot \frac{(1 - 2\,\mu^2)}{\mu} = - a' \cdot \frac{(1 - 4\,\mu^2)}{\mu},$$

as in [2246]. This is the quantity named W [$2231a$]; and if we multiply it by z [2243], we shall get,

$$z\,W = - \frac{l}{n^2} \cdot \frac{(1 - q\,\mu^2)}{\mu} \cdot a' = - \frac{l}{n^2} \cdot (\mu^{-1} - q\,\mu) \cdot a'. \qquad [2246b]$$

The second member of the equation [2183] is by this means reduced to the

[2247] term* $\frac{2lg \cdot qa'}{n^2}$. Putting this quantity equal to the first member, or the supposed value of a, we shall find,†

[2248] $$Q = \frac{2lgq}{n^2} \cdot \left(1 - \frac{3}{5\rho}\right) \cdot Q - \frac{2lq}{n^2} \cdot k;$$

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* (1673) The partial differential of the last expression of zW [2246b], being multiplied by g, and reduced by means of [2246a], gives successively,

$$g \cdot \left(\frac{d.zW}{d\mu}\right) = \frac{lg}{n^2} \cdot (\mu^{-2} + q) \cdot a' - \frac{lg}{n^2} \cdot (\mu^{-1} - q\mu) \cdot \left(\frac{da'}{d\mu}\right)$$

$$= \frac{lga'}{n^2} \cdot \left\{ \mu^{-2} + q - (\mu^{-1} - q\mu) \cdot \frac{1-2\mu^2}{\mu.(1-\mu^2)} \right\} = \frac{lga'}{n^2} \cdot \left\{ \mu^{-2} + q - (\mu^{-2} - q) \cdot \frac{1-2\mu^2}{1-\mu^2} \right\}$$

[2247a] $$= \frac{lga'}{n^2} \cdot \left\{ \mu^{-2} \cdot \frac{\mu^2}{1-\mu^2} + q \cdot \frac{2-3\mu^2}{1-\mu^2} \right\} = \frac{lga'}{n^2} \cdot \frac{(1+2q-3q\mu^2)}{1-\mu^2}.$$

The same value of zW [2246b] gives [2247b], by putting $i=n$ [2242]; moreover, from z [2243], we get [2247c],

[2247b] $$\frac{2ng\mu}{i.(1-\mu^2)} \cdot zW = -\frac{2g\mu}{1-\mu^2} \cdot \frac{l}{n^2} \cdot (\mu^{-1} - q\mu) \cdot a' = \frac{lga'}{n^2} \cdot \frac{(-2+2q\mu^2)}{1-\mu^2};$$

[2247c] $$\frac{gza'.(i^2 - 4n^2\mu^2)}{i^2 \cdot (1-\mu^2)} = \frac{lga'}{n^2} \cdot \frac{(1-q\mu^2)}{1-\mu^2}.$$

Adding together the expressions [2247a, b, c], the first member of the sum becomes as in the second member of [2231c], and the second member of the sum, reduced by neglecting

[2247d] the terms which destroy each other, becomes $\frac{lga'}{n^2} \cdot \frac{2q-2q\mu^2}{1-\mu^2} = \frac{lga'}{n^2} \cdot 2q$, as in [2247].

† (1674) Substituting in [2231c] the value of its second member [2247], and that of its first member [2244], we get

[2247e] $$Q \cdot \mu \cdot \sqrt{(1-\mu^2)} = \frac{2lgq}{n^2} \cdot a' = \frac{2lgq}{n^2} \cdot \left\{ \left(1 - \frac{3}{5\rho}\right) \cdot Q - \frac{k}{g} \right\} \cdot \mu \cdot \sqrt{(1-\mu^2)} \qquad [2245].$$

Dividing by $\mu . \sqrt{(1-\mu^2)}$, we obtain [2248]; from which we easily deduce the value of Q [2249]. Hence the assumed value of a [2244] becomes,

[2247f] $$a = \frac{2lqk.\mu.\sqrt{(1-\mu^2)}}{2lgq \cdot \left(1 - \frac{3}{5\rho}\right) - n^2} = \frac{2lq.k.\sin.\theta.\cos.\theta}{2lgq \cdot \left(1 - \frac{3}{5\rho}\right) - n^2} \qquad [2128^{xii}].$$

hence we deduce,

$$Q = \frac{2\,l\,q\,.\,k}{2\,l\,g\,q\,.\left(1-\frac{3}{5\rho}\right)-n^2}. \qquad [2249]$$

Therefore the part of αy, corresponding to the term

$$\alpha k\,.\,\sin.\theta\,.\,\cos.\theta\,.\,\cos.(i\,t+\varpi-A) \qquad [2223], \qquad [2250]$$

will be,*

$$\frac{2\,l\,q\,.\sin.\theta\,.\cos.\theta}{2\,l\,g\,q\,.\left(1-\frac{3}{5\rho}\right)-n^2}\,.\,\alpha k\,.\,\cos.(i\,t+\varpi-A). \qquad [2251]$$

But the sum of the terms $\alpha k\,.\,\cos.(i\,t+\varpi-A)$ [2223] is, by what precedes, the development of the function

$$\frac{3L}{r^3}\,.\,\sin.\mathrm{v}\,.\,\cos.\mathrm{v}\,.\,\cos.(n\,t+\varpi-\psi)\,; \qquad [2252]$$

therefore we shall have, for the whole part of αy, relative to oscillations of the second kind,†

Value of αy, depending on oscillations of the second kind.

$$\frac{\frac{6L}{r^3}\,.\,l\,q\,.\,\sin.\theta\,.\,\cos.\theta\,.\,\sin.\mathrm{v}\,.\,\cos.\mathrm{v}\,.\,\cos.(n\,t+\varpi-\psi)}{2\,l\,g\,q\,.\left(1-\frac{3}{5\rho}\right)-n^2}\,; \qquad [2253]$$

and this value is general, whatever q may be; that is, whatever may be the law of the depth of the sea; provided the spheroid which it covers is an ellipsoid of revolution [2196a]. [2253′]

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* (1675) Substituting a [2247f] in [2178], multiplying it by α, and putting $s=1$, $\varepsilon=-A$, [2230a], we get for αy the expression [2251], depending on the term [2223] $\alpha k\,.\,\sin.\theta\,.\,\cos.\theta\,.\,\cos.(i\,t+\varpi-A)$; and the former may be derived from the latter, by multiplying it by $\dfrac{2\,l\,q}{2\,l\,g\,q\,.\left(1-\frac{3}{5\rho}\right)-n^2}$. [2251$a$]

† (1676) The function [2252], multiplied by $\sin.\theta\,.\,\cos.\theta$, represents the part of the disturbing force $\alpha V'$ [2222] producing the oscillations of the second kind. Therefore if we multiply [2252] by $\sin.\theta\,.\,\cos.\theta$, and by the factor [2251a], we shall obtain the [2252a] corresponding value of αy [2253], as is evident from the last note.

The difference of the two tides of the same day, depends on oscillations of the second kind. For when the attracting body passes the upper meridian of the particle, we have $nt+\varpi-\psi=0$ [2131a]; and when it passes the
[2254] lower meridian, $nt+\varpi-\psi=200°$ [2131b]; therefore the excess of the tide in the first instance, above that in the second, is*

Difference of the two tides of the same day.

[2255]
$$\frac{\frac{12\,L}{r^3}\,.\,lq\,.\,\sin.\theta\,.\,\cos.\theta\,.\,\sin.v\,.\,\cos.v}{2\,lgq\,.\left(1-\frac{3}{5\,\rho}\right)-n^2}.$$

The observations made in our ports, prove that this difference is very small;
[2256] *which requires that lq should be very small, in comparison with $\frac{n^2}{g}$.*† Hence

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* (1677) If we put L' for the coefficient of $\cos.(nt+\varpi-\psi)$ [2253], we shall
[2253a] have $\alpha y=L'.\cos.(nt+\varpi-\psi)$. Now when the attracting body is on the meridian, above the horizon, we have $nt+\varpi-\psi=0$ [2131a]; and then $\alpha y=L'$. When
[2253b] the body is on the meridian, below the horizon, we have $nt+\varpi-\psi=200°$ [2131b]; and then $\alpha y=-L'$. The difference of these two expressions is $2L'$, as in [2255].

† (1678) If we divide the numerator and denominator of [2255] by lgq, it becomes,

[2255a]
$$\frac{\frac{12\,L}{r^3 g}\,.\,\sin.\theta\,.\,\cos.\theta\,.\,\sin.v\,.\,\cos.v}{2\,.\left(1-\frac{3}{5\,\rho}\right)-\frac{n^2}{g}\,.\,\frac{1}{lq}}.$$

The numerator of this is of the same order as the ratio of four times the disturbing force $\alpha V'$ [2222], to gravity g; and as this is of considerable magnitude, the expression [2255a], or
[2255b] [2255], cannot become small, unless the denominator $2\,.\left(1-\frac{3}{5\,\rho}\right)-\frac{n^2}{g}\,.\,\frac{1}{lq}$ is large; which requires that $\frac{n^2}{g}\,.\,\frac{1}{lq}$ should be a very large quantity; therefore in general q must be very small, in comparison with $\frac{n^2}{g}$; and then the denominator of [2255a] will be *negative*, if lq be *positive*; but *positive*, if lq be *negative*; $\frac{n^2}{g}$ being a positive quantity, representing the ratio of the centrifugal force, to gravity [1594a, &c.]. If the place of observation be on the same side of the equator as the attracting body, $\sin.\theta$ and
[2255c] $\sin.v$ will have the same sign; the numerator of [2255a] will be positive, and the whole of that expression will have a *different* sign from lq. Now it appears from observation, that this sign of [2255] is positive; therefore lq must be *negative*, and equal to $-lq'$; q' being a
[2255d] positive quantity; and then the depth of the sea [2196] becomes $l+lq'.\mu^2=l+lq'.\sin.^2\text{lat.}$, which is greatest at the poles.

this supposition gives a very simple explanation of the phenomenon. In this case the denominator of the preceding fraction is negative; and if, as observations seem to indicate, the superior tide exceeds the inferior, [2256′] lq will be a negative quantity, and the sea a little deeper at the poles than at the equator. But this inference is restricted by the hypothesis [2196], that the fluid is regularly spread over an ellipsoid of revolution, which is not [2256″] the case of nature.

If we substitute the value of Q in the expression of a', we shall get,*

$$a' = \frac{\frac{n^2k}{g}\cdot\mu\cdot\sqrt{1-\mu^2}}{2lgq\cdot\left(1-\frac{3}{5\rho}\right)-n^2}. \qquad [2257]$$

Substituting this in the expressions of b and c, § 3, we get, by supposing [2257′] $s=1$, and $i=n$,†

$$b = \frac{-k}{2lgq\cdot\left(1-\frac{3}{5\rho}\right)-n^2}; \qquad [2258]$$

$$c = \frac{k\mu}{\left\{2lgq\cdot\left(1-\frac{3}{5\rho}\right)-n^2\right\}\cdot\sqrt{1-\mu^2}}. \qquad [2259]$$

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* (1679) Substituting Q [2249] in [2245], and then reducing the terms to a common denominator, we get [2257].

† (1680) Putting $s=1$, in [2180]; then substituting, in the numerator, the value [2246], we get the first of the following expressions of b, which, by putting $i=n$ [2223], and substituting a' [2257], becomes as in [2258];

$$b = -\frac{g\cdot(1-4\mu^2)\cdot a'}{(i^2-4n^2\mu^2)\cdot\mu\cdot\sqrt{(1-\mu^2)}} = -\frac{ga'}{n^2\cdot\mu\cdot\sqrt{(1-\mu^2)}} = \frac{-k}{2lgq\cdot\left(1-\frac{3}{5\rho}\right)-n^2}; \qquad [2258a]$$

making the same substitutions, $s=1$, $i=n$, in [2181], then using the values of $\left(\frac{da'}{d\mu}\right)$ [2246a], we get successively,

$$c = \frac{2g\cdot\left(\frac{da'}{d\mu}\right)\cdot\mu\cdot(1-\mu^2)-ga'}{n^2\cdot(1-4\mu^2)\cdot(1-\mu^2)} = \frac{2ga'\cdot(1-2\mu^2)-ga'}{n^2\cdot(1-4\mu^2)\cdot(1-\mu^2)}$$

$$= \frac{ga'\cdot(1-4\mu^2)}{n^2\cdot(1-4\mu^2)\cdot(1-\mu^2)} = \frac{ga'}{n^2\cdot(1-\mu^2)}; \qquad [2258b]$$

substituting a' [2257], we obtain [2259].

[2259′] Hence it follows, that the part of αu, relative to oscillations of the second kind, is*

[2260]
$$\alpha u = -\frac{\frac{3L}{r^3} \cdot \sin. \text{v} \cdot \cos. \text{v} \cdot \cos. (nt + \varpi - \psi)}{2lgq \cdot \left(1 - \frac{3}{5\rho}\right) - n^2};$$

αu, and αv, relative to oscillations of the second kind.

and the part of αv, relative to the same oscillations, is†

[2261]
$$\alpha v = \frac{\frac{3L}{r^3} \cdot \frac{\cos. \theta}{\sin. \theta} \cdot \sin. \text{v} \cdot \cos. \text{v} \cdot \sin. (nt + \varpi - \psi)}{2lgq \cdot \left(1 - \frac{3}{5\rho}\right) - n^2}.$$

ON OSCILLATIONS OF THE THIRD KIND.

The part of $\alpha V'$ depending on the action of the body L.

9. The part of the action of the body L, which produces these oscillations, is, by [2195], equal to

[2262]
$$\frac{3L}{4r^3} \cdot \sin.^2 \theta \cdot \cos.^2 \text{v} \cdot \cos. 2 \cdot (nt + \varpi - \psi).$$

[2260a] * (1681) Putting $s = 1$, $\varepsilon = -A$, in [2178′], multiplying by α, and substituting [2258], we get $\alpha u = -\frac{1}{2lgq \cdot \left(1 - \frac{3}{5\rho}\right) - n^2} \cdot \alpha k \cdot \cos. (it + \varpi - A)$. Comparing this with [2223], we find, that if this part of the disturbing force,

$$\alpha k \cdot \mu \cdot \surd(1 - \mu^2) \cdot \cos. (it + \varpi - A),$$

be divided by $-\left\{2lgq \cdot \left(1 - \frac{3}{5\rho}\right) - n^2\right\} \cdot \mu \cdot \surd(1 - \mu^2)$, it will produce the corresponding part of αu; hence if the whole of the disturbing force $\alpha V'$ [2222] be

[2260b] divided by the same quantity, or by $-\left\{2lgq \cdot \left(1 - \frac{3}{5\rho}\right) - n^2\right\} \cdot \sin. \theta \cdot \cos. \theta$, it will produce the whole value of αu [2260].

† (1682) Multiplying [2178″] by α, and substituting [2259, 2260a], we get,

[2261a]
$$\alpha v = \frac{1}{2lgq \cdot \left(1 - \frac{3}{5\rho}\right) - n^2} \cdot \frac{\mu}{\surd(1 - \mu^2)} \cdot \alpha k \cdot \sin. (it + \varpi - A).$$

The development of this function, in cosines of angles, increasing in proportion to the time, gives a series of terms, of the form

$$\alpha k \,.\, \sin.^2 \theta \,.\, \cos.(i t + 2 \varpi - A) \,; \qquad [2262']$$

i differing but little from $2n$.*

We shall now resume the equation [2183], supposing $s=2$,† and z as in [2197, 2225], [2263]

$$z = \frac{l\,.\,(1-q\,\mu^2)}{i^2 - 4\,n^2\,\mu^2}\,; \qquad [2264]$$

we shall also suppose a to be expressed by the following series,‡ Value of a.

$$a = (1-\mu^2)\,.\,\{P^{(0)} + P^{(2)} + P^{(4)} \,.....\, + P^{(2f-2)}\}\,; \qquad [2265]$$

Comparing this with [2223], it appears that if the part of the disturbing force

$$a\,k\,.\,\mu\,.\,\sqrt{(1-\mu^2)}\,.\,\cos.(i t + \varpi - A),$$

be divided by $\left\{ 2 l g q \,.\, \left(1 - \frac{3}{5\rho}\right) - n^2 \right\} . (1-\mu^2) \,.\, \text{tang.}(i t + \varpi - A)$, it will produce the corresponding term of αv. Therefore if we divide the whole of the disturbing force $\alpha V'$ [2222], by the same quantity, [2261b]

$$\left\{ 2 l g q \,.\, \left(1 - \frac{3}{5\rho}\right) - n^2 \right\} . \sin.^2\theta \,.\, \text{tang.}(i t + \varpi - A),$$

we shall obtain the whole value of αv [2261]; i being nearly equal to n [2257'].

* (1683) This may be proved in a manner similar to that used in [2223a—f]. [2262a]

† (1684) The coefficient of ϖ, in [2195], or [2262], is to be put equal to s, to conform to [2178, &c.]; hence $s=2$. [2263a]

‡ (1685) This *assumed* value of a gives a', as in [2270], and these expressions satisfy the equation [2183], using the value of q [2278]. If we wish to know why this form was assumed, we may proceed as in note 1655, page 568, and we shall find, as in [2225c], that a ought to have the same factor $\sin.^2\theta$, or $1-\mu^2$, as [2262]; so that if we suppose, as in [2144], $y = Y^{(0)} + Y^{(1)} + Y^{(2)} + Y^{(3)} \,.....\, + Y^{(2f)}$. we must put [2263b]

$$Y^{(2f)} = (1-\mu^2)\,.\,P^{(2f-2)}\,.\,\cos.(i t + 2\varpi - A)\,; \qquad [2263c]$$

as is evident, from the method of investigation of the value of $Y^{(2f)}$ [2227]. If we now seek, in [1528a—d, &c.], the terms depending on $\sin.2\varpi$, $\cos.2\varpi$, which are also multiplied by $(1-\mu^2)$; and put $A_n^{(2)}.\sin.2\varpi + B_n^{(2)}.\cos.2\varpi$ under the form $A^{(n)}.\cos.(it+2\varpi-A)$, as in [2225i], we shall find, as in [2225c—g], [2263d]

[2265′] $P^{(0)}$, $P^{(2)}$, $P^{(4)}$, &c., being rational and integral functions of μ^2, of such
[2266] forms, that by putting $Y^{(2f)}=(1-\mu^2).P^{(2f-2)}$, we may have,*

[2267] $$0=\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dY^{(2f)}}{d\mu}\right)\right\}}{d\mu}\right\}-\frac{4\,Y^{(2f)}}{1-\mu^2}+2f.(2f+1).Y^{(2f)}.$$

[2267′] The part of a', corresponding to the action of an aqueous stratum, whose internal radius is unity, and external radius $1+\alpha y$, will be, by what precedes,†

[2268] $$-\frac{3.(1-\mu^2)}{\rho}.\left\{\tfrac{1}{5}.P^{(0)}+\tfrac{1}{9}.P^{(2)}+\tfrac{1}{13}.P^{(4)}.....+\frac{1}{4f+1}.P^{(2f-2)}\right\}.$$

[2269] The part of a', corresponding to the action of the body L, is‡ $-\frac{k}{g}.(1-\mu^2)$;

[2263e] $$Y^{(0)}=0,\qquad Y^{(1)}=0,\qquad Y^{(2)}=(1-\mu^2).A^{(2)}.\cos.(it+2\varpi-A),$$
$$Y^{(3)}=0,\qquad Y^{(4)}=(1-\mu^2).(\mu^2-\tfrac{1}{7}).A^{(4)}.\cos.(it+2\varpi-A),\qquad \&c.$$

If we now put $A^{(2)}=P^{(0)}$, $A^{(4)}.(\mu^2-\frac{1}{7})=P^{(2)}$, &c., the value of y will
[2263f] become $y=(1-\mu^2).\{P^{(0)}+P^{(2)}.....+P^{(2f-2)}\}.\cos.(it+2\varpi-A)$. Comparing this with [2178], we get a [2265].

* (1686) The assumed value of $Y^{(2f)}$ [2263e] gives

$$\left(\frac{ddY^{(2f)}}{d\varpi^2}\right)=-4(1-\mu^2).P^{(2f-2)}.\cos.(it+2\varpi-A)=-4\,Y^{(2f)};$$

Substituting this, and $i=2f$, in the equation [2145], it becomes as in [2267]; which
[2266a] would also be satisfied, if we were to reject the factor $\cos.(it+2\varpi-A)$, and put simply $Y^{(2f)}=(1-\mu^2).P^{(2f-2)}$, as in [2266].

† (1687) The part of V' [2146a], depending on the attraction of this stratum, being divided by $-g$, gives, by using [2263e],

$$-\frac{V'}{g}=-\frac{3}{\rho}.\Sigma\frac{Y^{(i)}}{2i+1}=-\frac{3}{\rho}.\Sigma\frac{Y^{(2f)}}{4f+1}=-\frac{3.(1-\mu^2)}{\rho}.\Sigma\frac{P^{(2f-2)}}{4f+1}.\cos.(it+2\varpi-A).$$

[2266b] This produces, in a' [2178‴], the terms $-\frac{3.(1-\mu^2)}{\rho}.\Sigma\frac{P^{(2f-2)}}{4f+1}$, beginning with $f=1$ and $P^{(0)}$, as in [2268].

‡ (1688) The part of $\alpha V'$, depending on the attracting body, is composed of a series of terms of the form

[2269a] $$\alpha k.\sin.^2\theta.\cos.(it+2\varpi-A)\quad[2262'],\quad\text{or}\quad \alpha k.(1-\mu^2)\cos.(it+2\varpi-A);$$

therefore we shall have,*

Value of a'.

$$a'=(1-\mu^2)\cdot\left\{\begin{array}{l}\left(1-\frac{3}{5\rho}\right).P^{(0)}+\left(1-\frac{3}{9\rho}\right).P^{(2)}\ldots\ldots+\left(1-\frac{3}{(4f+1).\rho}\right).P^{(2f-2)}\\ -\frac{k}{g}\end{array}\right\}. \qquad [2270]$$

We shall suppose the indeterminate constant quantities, by which the functions $P^{(0)}$, $P^{(2)}$, &c., are multiplied, to be such that the function

$$\frac{4n}{i}\cdot\mu a'-\left(\frac{d a'}{d\mu}\right)\cdot(1-\mu^2) \qquad [2271]$$

may be divisible by $(i^2-4n^2\mu^2)$, which requires only one equation of condition between these indeterminate quantities.† Then the second member of the equation [2183] will have no denominator; moreover it will be divisible by $1-\mu^2$, like the first member. For by supposing‡ $a'=(1-\mu^2).F$, [2272]
and considering only the terms not divisible by $1-\mu^2$, the three parts of this second member will become,§

$$-\left(\frac{2n+i}{i}\right).\mu^2.4gz.F+\left(\frac{2n+i}{i}\right).\mu^2.\frac{8n}{i}.gz.F+\left(\frac{i^2-4n^2\mu^2}{i^2}\right).4gz.F; \qquad [2273]$$

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which must produce, in $-\frac{V'}{g}$, terms of the form $-\frac{k}{g}.(1-\mu^2).\cos.(it+2\varpi-A)$; and in a' [2178'''], terms of the form [2269].

* (1689) Substituting a [2265] in [2178], and the resulting value of y, with those of $-\frac{V'}{g}$ [2268, 2269], in [2178''], we shall get, for a' the value [2270]. [2270a]

† (1690) This may be proved in a similar manner to that in [2201d], where the equation of condition is $C=0$. [2271a]

‡ (1691) Putting

$$F=\left\{\left(1-\frac{3}{5\rho}\right).P^{(0)}-\frac{k}{g}+\left(1-\frac{3}{9\rho}\right).P^{(2)}\ldots\ldots+\left(1-\frac{4}{(4f+1).\rho}\right).P^{(2f-2)}\right\}, \qquad [2272a]$$

in [2270], it becomes as in [2272].

§ (1692) Putting $W'=\frac{4n}{i}.\mu a'-\left(\frac{da'}{d\mu}\right).(1-\mu^2)$, and $s=2$, in [2183], it [2273a]
becomes,

[2274] or* $4.(1-\mu^2).gz.F$; consequently their sum is divisible by $1-\mu^2$. Substituting the preceding values of a, a', in the equation [2183], then dividing it by $1-\mu^2$, and comparing the coefficients of the powers of μ, we shall get f equations. These, being connected with the preceding [2271a], will form a system of $f+1$ equations of condition, which must be satisfied.†
[2274'] Now the number of indeterminate quantities, including q, is also $f+1$; therefore we shall have as many indeterminate quantities as equations.

[2275] To obtain the value of q, we shall put $Q.\mu^{2f-2}$ for the term having the greatest exponent of μ in $P^{(2f-2)}$; the corresponding term of a' is

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[2273a'] $$a=g.\left(\frac{d.(zW')}{d\mu}\right)+\frac{4ng.\mu z}{i.(1-\mu^2)}.W'+\frac{4gza'.(i^2-4n^2\mu^2)}{i^2.(1-\mu^2)}$$

[2273b] $$=g.W'.\left(\frac{dz}{d\mu}\right)+gz.\left(\frac{dW'}{d\mu}\right)+\frac{4ng.\mu z}{i.(1-\mu^2)}.W'+\frac{4gza'.(i^2-4n^2\mu^2)}{i^2.(1-\mu^2)}.$$

Substituting a' [2272] in W' [2273a], reducing, and then taking its differential, we get,

$$W'=\frac{4n}{i}.\mu(1-\mu^2).F+2\mu.F.(1-\mu^2)-(1-\mu^2)^2.\left(\frac{dF}{d\mu}\right)$$

[2273c] $$=\left(\frac{2n+i}{i}\right).2\mu.(1-\mu^2).F-(1-\mu^2)^2.\left(\frac{dF}{d\mu}\right);$$

[2273d] $$\left(\frac{dW'}{d\mu}\right)=\left(\frac{2n+i}{i}\right).2.(1-\mu^2).F-\left(\frac{2n+i}{i}\right).4\mu^2.F+\left(\frac{4n+6i}{i}\right).\mu.(1-\mu^2).\left(\frac{dF}{d\mu}\right)$$
$$-(1-\mu^2)^2.\left(\frac{ddF}{d\mu^2}\right).$$

If we retain only those terms of [2273b] which have not the factor $(1-\mu^2)$, we shall find, by substituting [2273c, d], that the first term of [2273b] may be neglected; the second
[2273e] term produces the first of [2273]; the third term produces the second of [2273]; and the fourth term of [2273b], by the substitution of a' [2272], produces the last term of [2273].

* (1693) If we put the expression [2273] equal to $4gz.F.G$, we shall find, by successive reductions,

$$G=-\left(\frac{2n+i}{i}\right).\mu^2+\left(\frac{2n+i}{i}\right).\mu^2.\frac{2n}{i}+\frac{i^2-4n^2\mu^2}{i^2}$$

[2274a] $$=1-\mu^2.\left\{\frac{2n+i}{i}-\frac{(2n+i)}{i}.\frac{2n}{i}+\frac{4n^2}{i^2}\right\}=1-\mu^2, \qquad \text{as in [2274].}$$

† (1694) The demonstration of this is made as in [2234a—c], putting $(1-\mu^2)$ for
[2274a] the factor of a, a', [2234a, b], instead of $\mu.\surd(1-\mu^2)$. Observing that the greatest index in [2198] is $2f$, and in [2265] is $2f-2$, so that f is decreased by unity in this last expression; and for the same reason $f+2$ [2201f] becomes $f+1$, as in [2274'].

$$(1-\mu^2)\,.\,\mu^{2f-2}\,.\,Q\,.\left(1-\frac{3}{(4f+1).\rho}\right);^{*}$$ [2276]

the corresponding term of the second member of the equation [2183] is†

$$\frac{lg\,.\,q}{2\,n^2}\,.\left(2f^2+f+\frac{2\,n}{i}\right).\left(1-\frac{3}{(4f+1)\,.\,\rho}\right).\,Q\,.\,(1-\mu^2)\,.\,\mu^{2f-2}.$$ [2277]

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* (1695) The greatest exponent of μ, in the value of a [2265], must evidently arise from the term $P^{(2f-2)}\,.\,(1-\mu^2)$; and that of a', from the term

$$(1-\mu^2)\,.\left(1-\frac{3}{(4f+1)\,.\,\rho}\right).\,P^{(2f-2)}, \qquad [2270],$$ [2275a]

which are to each other as 1 to $1-\frac{3}{(4f+1)\,.\,\rho}$. Multiplying this last expression, by the assumed value of the coefficient of $P^{(2f-2)}\,.\,(1-\mu^2)$, namely $(1-\mu^2)\,.\,\mu^{2f-2}\,.\,Q$, we get the corresponding term of a' [2276].

† (1696) Proceeding in the same manner as in note 1665, page 572, noticing the highest power of μ, we shall put,

$$Q'=\left\{1-\frac{3}{(4f+1)\,.\,\rho}\right\}.\,Q, \qquad R=\frac{lgq}{2n^2}\,.\,Q'.\,\mu^{2f};$$ [2276a]

and then from [2275, 2265, 2276], we have

$$a=Q\,.\,\mu^{2f-2}\,.\,(1-\mu^2), \qquad a'=Q'\,.\,\mu^{2f-2}\,.\,(1-\mu^2).$$ [2276b]

If we retain, in the following calculations, only the highest power of μ, we may put $a=-Q\,.\,\mu^{2f}$, $a'=-Q'\,.\,\mu^{2f}$; hence $\left(\frac{d\,a'}{d\,\mu}\right)=-2f\,.\,Q'\,.\,\mu^{2f-1}$. Substituting [2276c] these in [2273a], we get,

$$W'=-\frac{4\,n}{i}\,.\,Q'\,.\,\mu^{2f+1}-2f\,.\,Q'\,.\,\mu^{2f+1}=Q'\,.\,\mu^{2f+1}\,.\left(-\frac{4\,n}{i}-2f\right).$$ [2276d]

Multiplying this by g, and by the value of z [2236e], we obtain the first expression [2276e]; whose differential is given in [2276f]. These formulas are reduced by using R [2276a],

$$g\,z\,.\,W'=Q'\,.\,\mu^{2f+1}\,.\,\frac{l\,g\,q}{4\,n^2}\,.\left(-\frac{4\,n}{i}-2f\right)=\tfrac{1}{2}R\,.\,\mu\,.\left(-\frac{4\,n}{i}-2f\right);$$ [2276e]

$$g\,.\left(\frac{d.(z\,W')}{d\,\mu}\right)=Q'\,.\,\mu^{2f}\,.\,\frac{l\,g\,q}{4\,n^2}\,.\left(-4f^2-2f-\frac{4\,n}{i}-\frac{8\,n\,f}{i}\right)$$
$$=R\,.\left(-2f^2-f-\frac{2\,n}{i}-\frac{4\,n\,f}{i}\right).$$ [2276f]

Substituting the value of $(1-\mu^2)^{-1}$ [2236h], in the two last terms of [2273a'], we get,

$$a=g\,.\left(\frac{d.(z\,W')}{d\,\mu}\right)-\frac{4\,n}{i}\,.\,\mu^{-1}\,.\,g\,z\,.\,W'+\frac{16\,n^2}{i^2}\,.\,g\,z\,.\,a'.$$ [2276g]

Putting this equal to the corresponding term of the first member,

$$Q\,.\,(1-\mu^2)\,.\,\mu^{2f-2},$$

we shall get,*

Value of q.

[2278]
$$q=\frac{2n^2}{lg.\left(1-\frac{3}{(4f+1)\,.\,\rho}\right).\left(2f^2+f+\frac{2n}{i}\right)}.$$

Hence, if we suppose the depth of the sea [2196, 2278] equal to

Depth of the sea.

[2279]
$$l-\frac{2n^2\mu^2}{g\,.\left(1-\frac{3}{(4f+1)\,.\,\rho}\right).\left(2f^2+f+\frac{2n}{i}\right)},$$

we can determine, by the preceding analysis, the oscillations of the third kind. This expression is different for the different values which i may have; *but we may observe, that* i [2262'] *being nearly equal to* $2n$, *we can*
[2279'] *suppose* $\frac{2n}{i}=1$; *and then we shall have the same expression, for the depth*

Now by means of the values [2276e, c, 2236e], we find,

[2276h]
$$-\frac{4n}{i}\,.\,\mu^{-1}\,.\,gz\,.\,W'=R\,.\left(\frac{8n^2}{i^2}+\frac{4nf}{i}\right);$$

[2276i]
$$+\frac{16n^2}{i^2}\,.\,gz\,.\,a'=\frac{16n^2}{i^2}\,.\,g\,.\,\frac{lq}{4n^2}\,.\,(-Q'\,.\,\mu^{2f})=R\,.\left(-\frac{8n^2}{i^2}\right).$$

Adding together the equations [2276f, h, i], and neglecting the terms depending on $\pm\frac{4nf}{i}$, $\pm\frac{8n^2}{i^2}$, which mutually destroy each other; substituting also, instead of the first member, its value a [2276g], we get,

[2276k]
$$a=R\,.\left(-2f^2-f-\frac{2n}{i}\right)=-\frac{lgq}{2n^2}\,.\,Q'\,.\,\mu^{2f}.\left(2f^2+f+\frac{2n}{i}\right).$$

Now it has been proved, in [2271—2274], that the second member of [2183], or the value of a, is divisible by $1-\mu^2$; therefore the preceding term of a, depending on μ^{2f}, must arise from a factor of the form $-(1-\mu^2)\,.\,\mu^{2f-2}$. Substituting this for μ^{2f}, in [2276k],
[2276l] it becomes $a=\frac{lgq}{2n^2}\,.\,Q'\,.\,(1-\mu^2)\,.\,\mu^{2f-2}\,.\left(2f^2+f+\frac{2n}{i}\right)$, as in [2277], using Q' [2276a].

* (1697) The term of $P^{(2f-2)}$ containing the highest power of μ, is $Q\,.\,\mu^{2f-2}$ [2275]; the corresponding term of a [2265] is $a=(1-\mu^2)\,.\,Q\,.\,\mu^{2f-2}$. Putting this
[2278a] equal to its value [2277], and dividing by the coefficient of q, we get [2278]. Hence the depth of the sea [2196] becomes as in [2279].

of the sea, as we have found in § 7,* *relative to the oscillations of the second kind.* The coincidence of these two expressions is evidently necessary for the admission of this law of the depth. We may even make this depth coincide with that found in § 5, for oscillations of the first kind, supposing f to be so great, that we may neglect unity, in comparison with $2f^2+f$;† [2280]
but we have observed, in § 6, that the resistance suffered by the sea, in its motion, renders the oscillations of the first kind independent of the depth of the sea ;‡ so that it is only necessary to consider those laws of the depth, in [2280′]
which we can determine the oscillations of the second and the third kinds.

10. *We have remarked, in* § 8, *that to satisfy the observations, we must suppose the depth of the sea to be nearly constant* ;§ *we shall now determine the* [2280″]
oscillations of the third kind in this hypothesis. We shall also suppose that r, ψ, *and* v, *vary so slowly, in comparison with the variations of the angle* $2nt$, *that we may consider them as constant quantities. We shall also neglect*

Tides when the depth of the sea is constant.

the fraction‖ $\frac{1}{\rho}$, *which expresses the ratio of the density of the sea, to the* [2280‴]

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* (1698) In [2239] we have $\frac{n}{i}=1$ nearly [2240]; and in [2279], $\frac{2n}{i}=1$ nearly [2279′]; hence these expressions are nearly identical. [2279a]

† (1699) Neglecting $\frac{2n}{i}=1$ [2279′], in the factor $2f^2+f+\frac{2n}{i}$ [2279]; and $\frac{n}{i}=1$ [2240], in the factor $2f^2+f+\frac{n}{i}$ [2239]; they become $2f^2+f$, or $f.(2f+1)$, as in [2205b]; by which means the three expressions of the depth of the sea [2205b, 2239, 2279] become identical. [2280a]

‡ (1700) This follows from [2221], where αy is given, independent of l, q, on which the depth of the sea [2196] depends. [2280b]

§ (1701) It is shown in [2256], from observation, that lq is very small; therefore the depth of the sea [2196] is nearly constant. [2280c]

‖ (1702) The density of water being taken for unity, the mean density of the earth will be $\rho=5$, nearly; this being the value assumed by the author [10642], in conformity to the calculations of Dr. Hutton, from the observations of the attraction of the mountain Schehallien; and from the experiments of Mr. Cavendish on the attraction of small spherical [2280d]
bodies. Hence $\frac{3}{5\rho}=\frac{3}{25}$, $\frac{3}{9\rho}=\frac{3}{45}$, &c., ... $\frac{3}{(4f+1).\rho}=\frac{3}{5.(4f+1)}$; all of which [2280e]

mean density of the earth; which ratio is very small, as appears by the observations made on the attractions of mountains. This being premised, and then putting*

[2281] $$y = a \cdot \cos. (2nt + 2\varpi - 2\psi),$$

we shall get,†

[2282] $$\alpha a' = \alpha a - \frac{3L}{4r^3 . g} \cdot (1 - \mu^2) \cdot \cos.^2 \text{v} ;$$

[2283] therefore the equation [2183] will become, by observing that‡ $z = \frac{l}{4n^2 . (1-\mu^2)}$

[2284] $$\frac{4n^2}{lg} . \alpha a . (1-\mu^2)^2 = -\alpha . \left(\frac{dda}{d\mu^2}\right) . (1-\mu^2)^2 + (6+2\mu^2) . \alpha a - \frac{6L}{r^3 . g} \cdot (1-\mu^2) \cdot \cos.^2 \text{v}. \ \S$$

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are small, in comparison with the term unity of the series [2270]; and if we neglect them, on account of their smallness, we shall have, by using a [2265],

[2280f] $$a' = (1-\mu^2) . \left\{ P^{(0)} + P^{(2)} \ldots\ldots + P^{(2f-2)} - \frac{k}{g} \right\} = a - \frac{k}{g} . (1-\mu^2).$$

[2281a] * (1703) The quantities r, ψ, v, being supposed constant; the development of the force [2262], made as in [2223a—f], will contain simply the angle $2nt + 2\varpi - 2\psi$; because
[2281b] the terms $s'n'$, A', [2223c—d], will not be introduced by the quantities r, ψ, v, which are supposed to be constant. Therefore, instead of the value of y [2178], we may assume that, given in [2281], depending on the same angle $2 . (nt + \varpi - \psi)$, which occurs in [2262].

[2282a] † (1704) Multiplying [2280f] by α, we get $\alpha a' = \alpha a - \frac{1}{g} . \alpha k . (1 - \mu^2)$. Now as r, ψ, v, are supposed to be constant [2281a], we may, instead of the series of terms [2262], take only one term $\alpha k . \sin.^2 \theta . \cos. 2 . (nt + \varpi - \psi)$. Putting this equal to the expression [2262], and then dividing by $\sin.^2 \theta . \cos. 2 . (nt + \varpi - \psi)$, we get $\alpha k = \frac{3L}{4r^3} . \cos.^2 \text{v}$. Substituting this in [2282a], we obtain [2282].

‡ (1705) This value of z is the same as that used in [2197, 2225, 2264], putting
[2283a] $i = 2n$, and $q = 0$, in order to render the depth of the sea [2196] constant, and equal to l.

§ (1706) The differential of z [2283], gives $\left(\frac{dz}{d\mu}\right) = \frac{2l\mu}{4n^2 . (1-\mu^2)^2}$. Substituting
[2284a] these, and $i = 2n$, in [2273b], multiplying by $\frac{4n^2}{lg} . \alpha . (1-\mu^2)^2$, and reducing, we get,

We may put this equation under a more simple form, by making Value of x.

$$1-\mu^2=x^2\,[=\cos.^2\,\text{lat.}],$$ [2285]

and supposing dx constant; by which means we shall have,* Equation to find a.

$$x^2.(1-x^2).\alpha.\left(\frac{dda}{dx^2}\right)-x.\alpha.\left(\frac{da}{dx}\right)-2\alpha a.\left\{4-x^2-\frac{2n^2}{lg}.x^4\right\}+\frac{6L}{r^3g}.x^2.\cos.^2\text{v}=0.$$ [2286]

$$\frac{4n^2}{lg}.\alpha a.(1-\mu^2)^2$$

$$=\frac{4n^2}{lg}.\alpha.(1-\mu^2)^2.\left\{\frac{2lg\mu.W'}{4n^2.(1-\mu^2)^2}+\frac{lg}{4n^2.(1-\mu^2)}.\left(\frac{dW'}{d\mu}\right)+\frac{2lg\mu.W'}{4n^2.(1-\mu^2)^2}+\frac{4lga'}{4n^2.(1-\mu^2)}\right\}$$ [2284b]

$$=2\alpha W'.\mu+\alpha.\left(\frac{dW'}{d\mu}\right).(1-\mu^2)+2\alpha W'.\mu+4\alpha a'.(1-\mu^2)$$

$$=4\alpha W'.\mu+\alpha.\left(\frac{dW'}{d\mu}\right).(1-\mu^2)+4\alpha a'.(1-\mu^2).$$ [2284c]

Now $i=2n$, makes W' [2273a] become $W'=2\mu a'-\left(\frac{da'}{d\mu}\right).(1-\mu^2)$; hence $\left(\frac{dW'}{d\mu}\right)=2a'+4\mu.\left(\frac{da'}{d\mu}\right)-\left(\frac{dda'}{d\mu^2}\right).(1-\mu^2)$. Moreover, the second differential of $\alpha a'$ [2282], taken relatively to μ, and divided by $d\mu^2$, considering g, L, r, v, as constant, gives $\alpha.\left(\frac{dda'}{d\mu^2}\right)=\alpha.\left(\frac{dda}{d\mu^2}\right)+\frac{6L}{4r^3g}.\cos.^2\text{v}$. Substituting the values of W', $\left(\frac{dW'}{d\mu}\right)$, [2284d], in [2284c]; reducing, using $\alpha a'$ [2282], and its differential [2284e], we finally obtain [2284f]; and by connecting the terms of this last expression, depending on L, it becomes as in [2284], [2284d] [2284e]

$$\frac{4n^2}{lg}.\alpha a.(1-\mu^2)^2$$

$$=4\alpha\mu.\left\{2\mu a'-\left(\frac{da'}{d\mu}\right).(1-\mu^2)\right\}+\alpha.(1-\mu^2).\left\{2a'+4\mu.\left(\frac{da'}{d\mu}\right)-\left(\frac{dda'}{d\mu^2}\right).(1-\mu^2)\right\}$$ [2284f]

$$+4\alpha a'.(1-\mu^2)$$

$$=\alpha a'.\{6+2\mu^2\}-\alpha.\left(\frac{dda'}{d\mu^2}\right).(1-\mu^2)^2$$

$$=\left\{\alpha a-\frac{3L}{4r^3g}.(1-\mu^2).\cos.^2\text{v}\right\}.(6+2\mu^2)-\left\{\alpha.\left(\frac{dda}{d\mu^2}\right)+\frac{6L}{4r^3g}.\cos.^2\text{v}\right\}.(1-\mu^2)^2.$$ [2284g]

* (1707) Supposing a in the first place to be a function of μ, and then a function of x; taking the partial differentials as in [462, &c.], we get,

$$\left(\frac{da}{d\mu}\right)=\left(\frac{da}{dx}\right).\left(\frac{dx}{d\mu}\right);\qquad \left(\frac{dda}{d\mu^2}\right)=\left(\frac{dda}{dx^2}\right).\left(\frac{dx}{d\mu}\right)^2+\left(\frac{da}{dx}\right).\left(\frac{ddx}{d\mu^2}\right).$$ [2284h]

To satisfy this equation, we shall put,*

[2287] $$\alpha a = A^{(1)} . x^2 + A^{(2)} . x^4 + A^{(3)} . x^6 + \&c.$$

This value being substituted in the preceding differential equation, will give, in the first place, by comparing the coefficients of x^2,

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Also from $x = (1-\mu^2)^{\frac{1}{2}}$ [2285], we obtain

[2286a] $$\left(\frac{dx}{d\mu}\right) = -\frac{\mu}{(1-\mu^2)^{\frac{1}{2}}}; \qquad \left(\frac{ddx}{d\mu^2}\right) = -\frac{1}{(1-\mu^2)^{\frac{3}{2}}};$$

hence $\left(\frac{dda}{d\mu^2}\right) = \left(\frac{dda}{dx^2}\right) . \frac{\mu^2}{1-\mu^2} - \left(\frac{da}{dx}\right) . \frac{1}{(1-\mu^2)^{\frac{3}{2}}}$; and by substitution in [2234], it becomes

$$\frac{4n^2}{lg} . \alpha a . (1-\mu^2)^2 = -\alpha . \left(\frac{dda}{dx^2}\right) . \mu^2 . (1-\mu^2) + \alpha . (1-\mu^2)^{\frac{1}{2}} . \left(\frac{da}{dx}\right) + (6+2\mu^2) . \alpha a$$
$$- \frac{6L}{r^3 g} . (1-\mu^2) . \cos.^2 v.$$

[2286b] Substituting $(1-\mu^2)^{\frac{1}{2}} = x$, $\mu^2 = 1 - x^2$, it becomes,

[2286c] $$\frac{4n^2}{lg} . \alpha a . x^4 = -\alpha . \left(\frac{dda}{dx^2}\right) . (1-x^2) . x^2 + \alpha x . \left(\frac{da}{dx}\right) + (8-2x^2) . \alpha a - \frac{6L}{r^3 g} . x^2 . \cos.^2 v;$$

which is easily reduced to the form [2286], by transposing all the terms to the first member.

* (1708) This agrees with the form assumed in [2265]. For $P^{(0)}$, $P^{(2)}$, &c., being functions of μ^2 [2266]; if we put $\mu^2 = 1 - x^2$, the expression $P^{(0)} + P^{(2)} \ldots\ldots + P^{(2f-2)}$,
[2287a] may be put equal to $(A^{(1)} + A^{(2)} . x^2 + A^{(3)} . x^4 + \&c.)$. Multiplying this by $1-\mu^2$, or x^2, we get the corresponding part of a [2265], which is here called αa. If we use the characteristic
[2287b] of finite integrals Σ, the expression [2287] may be put under the form $\alpha a = \Sigma A^{(f)} . x^{2f}$, which gives $\alpha . \left(\frac{da}{dx}\right) = \Sigma 2f . A^{(f)} . x^{2f-1}$, $\alpha . \left(\frac{dda}{dx^2}\right) = \Sigma (4f^2 - 2f) A^{(f)} . x^{2f-2}$; hence [2286] becomes

[2287c] $$0 = \Sigma \left\{ \begin{array}{l} x^2 . (1-x^2) . (4f^2-2f) . A^{(f)} . x^{2f-2} - x . 2f . A^{(f)} . x^{2f-1} \\ +2 . \left(-4 + x^2 + \frac{2n^2}{lg} . x^4\right) . A^{(f)} . x^{2f} \end{array} \right\} + \frac{6L}{r^3 g} . x^2 . \cos.^2 v;$$

and by arranging the terms, between the braces, according to the powers of x^2, we obtain,

[2287d] $$0 = \Sigma \left\{ \begin{array}{l} \frac{4n^2}{lg} . A^{(f)} . x^{2f+4} - (2f-2) . (2f+1) . A^{(f)} . x^{2f+2} \\ \qquad + 4 . (f-2) . (f+1) . A^{(f)} . x^{2f} \end{array} \right\} + \frac{6L}{r^3 g} . x^2 . \cos.^2 v;$$

in which f includes the positive integral numbers 1, 2, 3, 4, &c.

$$A^{(1)} = \frac{3L}{4r^3g} \cdot \cos.^2 v.^*$$ [2288]

The comparison of the coefficients of x^4 will give the identical equation $0=0$;† lastly, the comparison of the coefficients of x^{2f+4}, f being equal to, or greater than, unity, will give,‡ [2288′]

$$0 = A^{(f+2)} \cdot (2f^2+6f) - A^{(f+1)} \cdot (2f^2+3f) + \frac{2n^2}{lg} \cdot A^{(f)}.$$ [2289]

By means of this equation, we may find the values of $A^{(3)}$, $A^{(4)}$, &c., when $A^{(1)}$ and $A^{(2)}$ are known. We may put this equation under the following form,§

$$\frac{A^{(f+1)}}{A^{(f)}} = \frac{\frac{2n^2}{lg}}{2f^2+3f-(2f^2+6f) \cdot \frac{A^{(f+2)}}{A^{(f+1)}}};$$ [2290]

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* (1709) If we take the least value of f, namely $f=1$, it will produce in [2287d], under the sign Σ, a term depending upon x^2. This being connected with the term without the sign Σ, having the same factor x^2, must be put equal to nothing; hence

$$-8A^{(1)} \cdot x^2 + \frac{6L}{r^3g} \cdot x^2 \cdot \cos.^2 v = 0;$$ [2288a]

from which we easily obtain [2288].

† (1710) Putting $f=1$, and then $f=2$, in [2287d], and retaining only the terms having the coefficient x^4, we shall find that the first of these values of f produces the expression $-(2f-2) \cdot (2f+1) \cdot A^{(1)} \cdot x^4 = 0$; and the second $4 \cdot (f-2) \cdot (f-1) \cdot A^{(2)} \cdot x^4 = 0$; [2288$b$] so that the coefficient of x^4 produces the identical equation $0=0$ [2288′].

‡ (1711) To obtain the term of [2287d] multiplied by x^{2f+4}, we must, in the second term, change f into $f+1$; and in the third, change f into $f+2$. Putting the resulting [2289a] expression equal to nothing, and dividing by $2x^{2f+4}$, we shall get [2289].

§ (1712) If we transpose the two first terms of [2289], we may put it under the following form,

$$A^{(f+1)} \cdot \left\{ (2f^2+3f) - (2f^2+6f) \cdot \frac{A^{(f+2)}}{A^{(f+1)}} \right\} = \frac{2n^2}{lg} \cdot A^{(f)}.$$ [2290a]

Dividing this by $A^{(f)}$, and by the coefficient of $A^{(f+1)}$, we get [2290].

hence we find,*

[2291] $$\frac{A^{(f+1)}}{A^{(f)}}=\cfrac{\frac{2n^2}{lg}}{2f^2+3f-\cfrac{\frac{4n^2}{lg}\cdot(f^2+3f)}{2\cdot(f+1)^2+3\cdot(f+1)-\cfrac{\frac{4n^2}{lg}\cdot\{(f+1)^2+3\cdot(f+1)\}}{2\cdot(f+2)^2+3\cdot(f+2)-\&c.}}};$$

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* (1713) The equation [2290] must be satisfied for all integral values of f; and if we write $f+1$ for f, we shall get,

[2290b] $$\frac{A^{(f+2)}}{A^{(f+1)}}=\frac{\frac{2n^2}{lg}}{2\cdot(f+1)^2+3\cdot(f+1)-2\cdot\{(f+1)^2+3\cdot(f+1)\}\cdot\frac{A^{(f+3)}}{A^{(f+2)}}}.$$

Substituting this in the second member of [2290], we obtain,

[2290c] $$\frac{A^{(f+1)}}{A^{(f)}}=\cfrac{\frac{2n^2}{lg}}{2f^2+3f-\cfrac{\frac{4n^2}{lg}\cdot(f^2+3f)}{2\cdot(f+1)^2+3\cdot(f+1)-2\cdot\{(f+1)^2+3\cdot(f+1)\}\cdot\frac{A^{(f+3)}}{A^{(f+2)}}}}.$$

Now changing f into $f+2$, we shall have an expression of $\frac{A^{(f+3)}}{A^{(f+2)}}$ in terms of $\frac{A^{(f+5)}}{A^{(f+4)}}$; and by substituting this in [2290c], we shall obtain a *second* value of $\frac{A^{(f+1)}}{A^{(f)}}$, expressed in terms of $\frac{A^{(f+5)}}{A^{(f+4)}}$. Changing in this, f into $f+4$, we get the value of
[2290d] $\frac{A^{(f+5)}}{A^{(f+4)}}$, in terms of $\frac{A^{(f+9)}}{A^{(f+8)}}$; and by substituting it in the second value of $\frac{A^{(f+1)}}{A^{(f)}}$, we obtain a *third* value of that quantity. Proceeding in this way, we finally obtain the formula [2291]; which, by putting $f=1$, and multiplying by $A^{(1)}$, gives [2292].

[2290e] An expression of the form [2291] is called a *continued fraction*, which may be generally expressed by

Continued fraction.

[2290f] $$B=a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\&c.}}}};$$

and, by supposing $f=1$, it becomes,

$$A^{(2)}=\cfrac{\frac{2n^2}{lg}.A^{(1)}}{2.1^2+3.1-\cfrac{\frac{4n^2}{lg}.(1^2+3.1)}{2.2^2+3.2-\cfrac{\frac{4n^2}{lg}.(2^2+3.2)}{2.3^2+3.3-\cfrac{\frac{4n^2}{lg}.(3^2+3.3)}{2.4^2+3.4-\&c.}}}}. \qquad [2292]$$

the series being continued infinitely; or broken off after any number of terms, as in the following examples, whose values are derived from each other, by the usual operations of arithmetic, commencing the calculation with the last term,

$$B=\cfrac{1}{4+\cfrac{1}{5}}=\frac{5}{20+1}=\frac{5}{21}; \qquad B=\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5}}}=\cfrac{1}{3+\frac{5}{21}}=\frac{21}{63+5}=\frac{21}{68};$$

$$B=\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5}}}}=\cfrac{1}{2+\frac{21}{68}}=\frac{68}{136+21}=\frac{68}{157}. \qquad [2290g]$$

It has been lately proposed to write a fraction of the form [2290f] in the following manner, which takes up less space;

Abridged form of continued fractions.

$$B=a+\frac{\alpha}{b}+\frac{\beta}{c}+\frac{\gamma}{d}+\frac{\delta}{e}+\&c. \qquad [2290g']$$

If we compute successively 1, 2, 3, 4, &c. terms of the series [2290f], we may reduce them to the form of common fractions, as in [2290h—l, &c.]; observing that when the fraction is less than unity, we shall have $a=0$.

$$a \qquad =\frac{a}{1}; \qquad [2290h]$$

$$a+\frac{\alpha}{b} \qquad =\frac{ab+\alpha}{b}; \qquad [2290i]$$

$$a+\frac{\alpha}{b}+\frac{\beta}{c} \qquad =\frac{abc+\beta a+\alpha c}{bc+\beta}; \qquad [2290k]$$

$$a+\frac{\alpha}{b}+\frac{\beta}{c}+\frac{\gamma}{d}=\frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}; \qquad [2290l]$$

&c.

Hence we may find $A^{(2)}$ by means of $A^{(1)}$. The ratio of the centrifugal
[2292′] force to the gravity at the equator is expressed by $\frac{n^2}{g} = \frac{1}{289}$ [1594a].

Each of the fractions contained in the second members of the series [2290h, &c.], may be derived, from the two immediately preceding it, in the following manner.

Place these fractions in their natural order, in the horizontal line [2290o], the first term of
[2290m] the series being $\frac{1}{0}$; and above them the quantities a, b, c, d, &c., as *upper indices*, in the line [2290n]; also below them respectively, the quantities α, β, γ, δ, &c., [2290p], as *lower indices*;

Approximate value of B.

[2290n] a $\quad b$ $\quad c$ $\quad d$ $\quad e$ [Upper index.]

[2290o] $$\frac{1}{0}; \quad \frac{a}{1}; \quad \frac{ab+\alpha}{b}; \quad \frac{abc+\beta a+\alpha c}{bc+\beta}; \quad \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}.$$

[2290p] α $\quad \beta$ $\quad \gamma$ $\quad \delta$ $\quad \varepsilon$ [Lower index.]

Then if we compare the successive fractions [2290o] together, we shall easily find, by induction, that each fraction can be deduced from the *two* immediately preceding it, by the following general rule.

[2290q] *Multiply the last numerator by the index placed over it, and the numerator which precedes this by the index below it; the sum of these products, noticing the signs, is the numerator of the next term of the series. In like manner we must multiply the last denominator by the index placed over it, and the denominator which precedes this by the index below it; the sum of these products is the new denominator.*

General rule for the summation of continued fractions.

A common fraction, expressed in large numbers, may be reduced to a continued fraction,
[2290r] by the usual method of finding the greatest common measure of these two numbers; dividing the greater by the less, and so on, always dividing the last divisor by the last remainder. The several quotients are the denominators b, c, d, &c.; and the numerators
[2290s] are all equal to unity. Thus the fraction $B = \frac{68}{157}$ [2290g] may be successively reduced to the following forms,

[2290t] $$B = \tfrac{68}{157} = \tfrac{1}{2} + \tfrac{21}{68} = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{5}{21} = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5}.$$

When a fraction is reduced to a series of this kind, in which the numerators are all equal to unity, and the denominators positive, the series of terms, commencing with the first, approximates towards the true value of the fraction B; so that the sum of any *even*
[2290u] number of terms will be *less* than the true value of B, and the sum of any *odd* number of terms will be *greater* than the true value of B; until we arrive at the last term of the
[2290v] series, which gives the accurate value of B, when the series terminates. Thus in the preceding example, by taking successively one, two, three and four terms, the series becomes,

[2290w] $$\tfrac{1}{2} = \tfrac{1}{2}; \qquad \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{3}{7}; \qquad \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{13}{30}; \qquad \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} = \tfrac{68}{157};$$

Therefore by supposing successively $\frac{2\,n^2}{lg} = 20,\quad \frac{2\,n^2}{lg} = 5,\quad \frac{2\,n^2}{lg} = \frac{5}{2};$ [2293]

the corresponding depths of the sea will be* $\frac{1}{2890},\quad \frac{1}{722,5},\quad \frac{1}{361,25};$ the radius of the earth being taken for unity. This being supposed, we [2293'] shall find by the preceding analysis, that the corresponding values of αa will be,†

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which are alternately greater and less than the true value, as is easily seen, by the manner in which these terms are successively formed. In this way, the ratio of the diameter of a circle to its circumference is expressed by

$$\frac{100000000000}{314159265335} = \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \&c.\,; \qquad [2290x]$$

from which we easily deduce the well known values $\frac{1}{3}$, $\frac{7}{22}$, $\frac{113}{355}$, &c.; which are [2290y] alternately greater and less than the true ratio. We shall not enter into any further explanation of the properties of these fractions, which have been treated of by several authors, particularly by La Grange, in his addition to the Elements of Algebra by Euler.

* (1714) If we put $c = \frac{2n^2}{lg} = \frac{2}{l} \cdot \frac{1}{289}$ [2292'], we shall get $l = \frac{2}{289 \,.\, c}$. [2292a]
Substituting the values of c [2293], namely $c = 20$, $c = 5$, $c = \frac{5}{2}$; we obtain, for the constant depth of the sea l, the values respectively corresponding, as in [2293].

† (1715) Dividing $A^{(2)}$ [2292] by $A^{(1)}$, and putting $\frac{2n^2}{lg} = 20$, we get the following expression [2293a], in which we have used the symbol B, representing the infinite series [2292b] [2293b]. This separation of the series into two parts is made for the convenience of calculation; since the series B converges very fast, and its value being found, and substituted in [2293a], it becomes very easy to compute the value of the fraction, as in [2293f]. We have altered the form of writing these fractions, according to the method proposed in [2292c] [2290g'] In this series, the second difference of the numerators, beginning with the second, is 80; the second difference of the denominators 4.

$$\frac{A^{(2)}}{A^{(1)}} = \frac{20}{5} - \frac{160}{14} - \frac{400}{27} - \frac{720}{44} - \frac{1120}{65} - \frac{1600}{90} - \frac{2160}{119} - \frac{2800}{152} - \frac{3520}{189} - \frac{4320}{B}\,; \qquad [2293a]$$

$$B = 230 - \frac{5200}{275} - \frac{6160}{324} - \frac{7200}{377} - \frac{8320}{434} - \frac{9520}{495} - \frac{10800}{560} - \&c. \qquad [2293b]$$

The value B, computed as in [2290o], forms a series of fractions, as in the following table; in which the first line contains the upper indices; the second, the successive values of B; [2293c] the third, the lower indices; the fourth, the successive values of B in decimals.

[2294] $$\alpha a = \frac{3L}{4r^3.g} \, . \, x^2 . \cos.^2 \text{v} \, . \left\{ \begin{array}{l} 1,0000 \quad + 20,1862 \, . \, x^2 + 10,1164 \, . \, x^4 \\ - 13,1047 \, . \, x^6 - 15,4488 \, . \, x^8 - \; 7,4581 \, . \, x^{10} \\ - \; 2,1975 \, . \, x^{12} - \; 0,4501 \, . \, x^{14} - \; 0,0687 \, . \, x^{16} \\ - \; 0,0082 \, . \, x^{18} - \; 0,0008 \, . \, x^{20} - \; 0,0001 \, . \, x^{22} \end{array} \right\} ;$$

[2293d]

230,	275,	324,	377,	434,	495,	560,	&c.
$\frac{1}{0}$,	$\frac{230}{1}$,	$\frac{58050}{275}$,	$\frac{17391400}{82940}$,	$\frac{6138597800}{29288380}$,	$\frac{2519454997200}{12021096120}$,	$\frac{1198690772558000}{5671617201800}$,	&c.
—5200,	—6160,	—7200,	—8320,	—9520,	—10800,	—12160,	&c.
∞,	230,	211,1,	209,68,	209,591,	209,586,	B=209,5859,	&c.

[2293e] Hence it appears that the series [2293b], approximates very fast; and that, by taking only seven terms, we get very nearly $B = 209,5859$.

If we now substitute this value of B in [2293a], we may obtain, by a similar computation, a series of fractions, approximating successively towards the value of $\frac{A^{(2)}}{A^{(1)}}$. But as the numbers thus produced are quite large, it was thought best to use a different process, and begin with the tenth, computing the terms in an inverted order, as in the adjoined table, which needs no explanation. From this we find that

$189 - \frac{4320}{209,5859}$	=	168,38793
$152 - \frac{3520}{168,38793}$	=	131,095889
$119 - \frac{2800}{131,095889}$	=	97,64159
$90 - \frac{2160}{97,64159}$	=	67,87828
$65 - \frac{1600}{67,87828}$	=	41,42839
$44 - \frac{1120}{41,42839}$	=	16,96540
$27 - \frac{720}{16,96540}$	=	—15,43932
$14 + \frac{400}{15,43932}$	=	39,907875
$5 - \frac{160}{39,907875}$	=	0,99076
$\frac{20}{0,99076}$	=	20,1864
	=	$\frac{A^{(2)}}{A^{(1)}}$

[2293f] $$\frac{A^{(2)}}{A^{(1)}} = 20,1864, \quad \text{or} \quad A^{(2)} = 20,1864 \, . \, A^{(1)}.$$

Now if we divide [2289] by $2f^2 + 6f$, and put $\frac{2n^2}{lg} = 20$, we shall get,

[2293g] $$A^{(f+2)} = \frac{2f+3}{2f+6} \, . \, A^{(f+1)} - \frac{10}{f \cdot (f+3)} \, . \, A^{(f)}.$$

Putting now successively $f = 1, \ 2, \ 3,$ &c., we may obtain $A^{(3)}, \ A^{(4)}, \ A^{(5)}$, &c., in terms of the preceding quantities; and by substituting their values in terms of $A^{(1)}$, they may all be made to depend on $A^{(1)}$. Thus

$$A^{(3)} = \tfrac{5}{8} \, . \, A^{(2)} - \tfrac{5}{2} \, . \, A^{(1)} = 10,1165 \, . \, A^{(1)}, \qquad A^{(4)} = \tfrac{7}{10} \, . \, A^{(3)} - \tfrac{5}{5} \, . \, A^{(2)} = -13,1048 \, . \, A^{(1)},$$

$$A^{(5)} = \tfrac{9}{12} . A^{(4)} - \tfrac{5}{9} . A^{(3)} = -15,4489 . A^{(1)}, \qquad A^{(6)} = \tfrac{11}{14} \, . \, A^{(5)} - \tfrac{5}{14} \, . \, A^{(4)} = - \; 7,4581 \, . \, A^{(1)},$$

[2293h] $$A^{(7)} = \tfrac{13}{16} . A^{(6)} - \tfrac{5}{20} . A^{(5)} = -2,1975 . A^{(1)}, \qquad A^{(8)} = \tfrac{15}{18} \, . \, A^{(7)} - \tfrac{5}{27} \, . \, A^{(6)} = - \; 0,4501 \, . \, A^{(1)},$$

$$A^{(9)} = \tfrac{17}{20} . A^{(8)} - \tfrac{5}{35} . A^{(7)} = -0,0687 . A^{(1)}, \qquad A^{(10)} = \tfrac{19}{22} \, . \, A^{(9)} - \tfrac{5}{44} \, . \, A^{(8)} = - \; 0,0082 \, . \, A^{(1)},$$

$$A^{(11)} = \tfrac{21}{24} . A^{(10)} - \tfrac{5}{54} . A^{(9)} = -0,0008 . A^{(1)}, \qquad A^{(12)} = \tfrac{23}{26} \, . \, A^{(11)} - \tfrac{5}{65} \, . \, A^{(10)} = - \; 0,0001 \, . \, A^{(1)}.$$

Substituting $A^{(1)}$ [2288] in these expressions of $A^{(2)}, \ A^{(3)}, \ A^{(4)}$, &c.; and then the resulting values in the formula [2287]; we shall obtain αa [2294] nearly.

$$\alpha a = \frac{3L}{4r^3.g} . x^2 . \cos.^2 \text{v} . \left\{ \begin{matrix} 1{,}0000 \quad + 6{,}1960 . x^2 + 3{,}2474 . x^4 \\ + 0{,}7238 . x^6 + 0{,}0919 . x^8 + 0{,}0076 . x^{10} \\ + 0{,}0004 . x^{12} \end{matrix} \right\} ;*$$ Values of αa, in terms of $x=$ cos.lat. [2295]

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* (1716) Putting $\frac{2n^2}{lg} = 5$, in [2292], and then dividing by $A^{(1)}$, we get, by separating the series into two parts, as in the preceding note,

$$\frac{A^{(2)}}{A^{(1)}} = \tfrac{5}{5} - \tfrac{40}{14} - \tfrac{100}{27} - \tfrac{180}{44} - \tfrac{280}{B} ; \qquad B = 65 - \tfrac{400}{90} - \tfrac{540}{119} - \tfrac{700}{152} - \tfrac{880}{189} - \tfrac{1080}{230} - \&c.;$$ [2295a]

The second differences of the numerators are 20, and those of the denominators 4. We then obtain the value of B, by the method [2290n—q], as in the following table : [2295b]

65,	90,	119,	152,	189,	230,	&c.
$\frac{1}{0}$,	$\frac{65}{1}$,	$\frac{5450}{90}$,	$\frac{613450}{10170}$,	$\frac{89429400}{1482840}$,	$\frac{16362320600}{271307160}$,	&c.
—400,	—540,	—700,	—880,	—1080,	—1300,	&c.
∞,	65,	60,5,	60,319,	60,3095,	60,3092 $= B$,	&c.

[2295c]

Having found $B = 60{,}3092$, we must substitute it in $\frac{A^{(2)}}{A^{(1)}}$ [2295a], and we shall get $\frac{A^{(2)}}{A^{(1)}} = 0 + \frac{5}{5} - \frac{40}{14} - \frac{100}{27} - \frac{180}{44} - \frac{280}{60{,}3092}$. The value of this series may be found by an inverted operation, as in [2293f] ; but as the numbers are not very large, we may find it by the usual method [2290n—q], as follows,

0,	5,	14,	27,	44,	60,3092.	
$\frac{1}{0}$,	$\frac{0}{1}$,	$\frac{5}{5}$,	$\frac{70}{30}$,	$\frac{1390}{310}$,	$\frac{48560}{8240}$,	$\frac{2539414}{410148}$.
5,	—40,	—100,	—180,	—280.		
∞,	0,	1,	$2\frac{1}{3}$,	4,4,	5,8,	$6{,}1914 = \frac{A^{(2)}}{A^{(1)}}$,

[2295d]

Hence $A^{(2)} = 6{,}1914 . A^{(1)}$. This differs a little from La Place's value, $6{,}1960 . A^{(1)}$ [2295e] [2295] ; as it appears from the coefficient of the second term of the series between the braces. Now putting $\frac{2n^2}{lg} = 5$, in [2289], and dividing by $2f^2 + 6f$, we obtain

$$A^{(f+2)} = \frac{2f+3}{2f+6} . A^{(f+1)} - \frac{5}{2f.(f+3)} . A^{(f)}.$$ [2295f]

Substituting successively $f=1$, $f=2$, &c., we obtain $A^{(3)}$, $A^{(4)}$, &c., in terms of the preceding quantities, and then their values in terms of $A^{(1)}$, namely,

$$A^{(3)} = \tfrac{5}{8} . A^{(2)} - \tfrac{5}{8} . A^{(1)} = 3{,}2446 . A^{(1)}, \qquad A^{(4)} = \tfrac{7}{10} . A^{(3)} - \tfrac{5}{20} . A^{(2)} = 0{,}7234 . A^{(1)},$$
$$A^{(5)} = \tfrac{9}{12} . A^{(4)} - \tfrac{5}{36} . A^{(3)} = 0{,}0919 . A^{(1)}, \qquad A^{(6)} = \tfrac{11}{14} . A^{(5)} - \tfrac{5}{56} . A^{(4)} = 0{,}0076 . A^{(1)},$$ [2295g]
$$A^{(7)} = \tfrac{13}{16} . A^{(6)} - \tfrac{5}{80} . A^{(5)} = 0{,}0004 . A^{(1)}.$$

[2296] $$\alpha a = \frac{3L}{4r^3 . g} . x^2 . \cos.^2 \mathrm{v} . \left\{ \begin{array}{l} 1{,}0000 + 0{,}7504 . x^2 + 0{,}1566 . x^4 \\ \qquad + 0{,}01574 . x^6 + 0{,}0009 . x^8 \end{array} \right\} .^*$$

11. *We shall now collect together the expressions of the different oscillations of the sea.* Those of the first kind [2221], neglecting the density of the sea, in comparison with that of the earth, are†

[2297] $$\frac{L}{4r^3 . g} . \{ \sin.^2 \mathrm{v} - \tfrac{1}{2} . \cos.^2 \mathrm{v} \} . \{ 1 + 3 . \cos. 2\theta \}.$$

Substituting, in each of these, the value of $A^{(1)}$ [2288], and then these last values in [2287], we obtain the expression of αa corresponding to this case, as in [2295] nearly. The difference arises from the above error in $A^{(2)}$.

* (1717) Putting $\frac{2n^2}{lg} = \frac{5}{2}$, in [2292], and dividing by $A^{(1)}$, we obtain, for this case,

[2296a] $$\frac{A^{(2)}}{A^{(1)}} = 0 + \frac{2,5}{5} - \frac{20}{14} - \frac{50}{27} - \frac{90}{44} - \frac{140}{65} - \frac{200}{90} - \frac{270}{119} - \frac{350}{152} - \&c.$$

As this series evidently approximates very fast, we may compute it, without any separation, by the method [2290n—q],

[2296b] $$\begin{array}{ccccccccc} 0, & 5, & 14, & 27, & 44, & 65, & 90, & 119, & \&c. \\ \frac{1}{0}, & \frac{0}{1}, & \frac{2,5}{5}, & \frac{35}{50}, & \frac{820}{1100}, & \frac{32930}{43900}, & \frac{2025650}{2699500}, & \frac{175722500}{234175000}, & \&c. \\ 2{,}5, & -20, & -50, & -90, & -140, & -200, & -270, & -350, & \&c. \\ \infty, & 0, & 0{,}5, & 0{,}7, & 0{,}74, & 0{,}7501, & 0{,}75038, & 0{,}7503897 = \frac{A^{(2)}}{A^{(1)}}, & \&c. \end{array}$$

[2296c] Hence we have nearly $A^{(2)} = 0{,}7504 . A^{(1)}$; Now putting $\frac{2n^2}{lg} = \frac{5}{2}$, in [2289], and dividing it by $2f^2 + 6f$, we get $A^{(f+2)} = \frac{2f+3}{2f+6} . A^{(f+1)} - \frac{5}{4f.(f+3)} . A^{(f)}$. Hence by putting $f = 1, 2, 3$, &c., and proceeding as in the last notes, we find,

[2296d] $$A^{(3)} = \tfrac{5}{8} . A^{(2)} - \tfrac{5}{16} . A^{(1)} = 0{,}1565 . A^{(1)}, \qquad A^{(4)} = \tfrac{7}{10} . A^{(3)} - \tfrac{5}{40} . A^{(2)} = 0{,}01575 . A^{(1)},$$
$$A^{(5)} = \tfrac{9}{12} . A^{(4)} - \tfrac{5}{72} . A^{(3)} = 0{,}0009 . A^{(1)}.$$

Substituting these in [2287], and then using the value of $A^{(1)}$ [2288], we obtain for αa the expression [2296] nearly.

[2297a] † (1718) If we neglect $\frac{3}{5\rho} = \frac{3}{25}$ [2280e], in comparison with unity, the expression of the oscillations of the first kind [2221], becomes as in [2297].

We have seen that the oscillations of the second kind are nothing when the depth of the sea is everywhere the same.* Lastly, the oscillations of the third kind are expressed by $\alpha a.\cos.\{2nt+2\varpi-2\psi\}$.† The sum of these oscillations is the whole value of αy; therefore we shall have, [2297′]

$$\alpha y=\frac{L}{4r^3.g}.\{\sin.^2\mathrm{v}-\tfrac{1}{2}.\cos.^2\mathrm{v}\}.(1+3.\cos.2\theta)+\alpha a.\cos.(2nt+2\varpi-2\psi). \qquad [2298]$$

If we suppose L to be the sun, r its mean distance from the earth, mt its mean sideral motion, we shall have, by the theory of central forces,‡ [2299]

$$\frac{3L}{4r^3.g}=\frac{3n^2}{4g}.\frac{m^2}{n^2}=\frac{3}{4.289.(366{,}26)^2}. \qquad [2300]$$

This quantity is a fraction of the radius of the earth, which we have taken for unity [2193]; and if we multiply it by the number of metres contained in that radius, we shall have, [2300′]

$$\frac{3L}{4r^3.g}=0^{\mathrm{met.}},12316\,; \qquad [2301]$$

and we must vary this quantity, in proportion to the cube of the mean distance of the sun from the earth, to its actual distance.

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* (1719) If the depth of the sea be constant, and equal to l, in all latitudes, q [2196] will be nothing; and then the expression of the oscillations of the second kind [2253] will vanish. [2297b]

† (1720) The assumed value of y [2281] gives $\alpha y=\alpha a.\cos.(2nt+2\varpi-2\psi)$ for the oscillations of the third kind. Adding this to the other oscillations [2297, &c.], we get the complete value [2298]. [2298a]

‡ (1721) The sideral motion of the earth being mt, its angular velocity is m, therefore its actual velocity is proportional to mr; the square of this, divided by the radius r, gives the centrifugal force m^2r [54′]. This is equal to the attractive force of the sun $\frac{L}{r^2}$; hence $L=m^2r^3$. Substituting this in the first member of [2300], we get the second member, and then the third, by using $\frac{n^2}{g}=\frac{1}{289}$ [1594a], and $\frac{m}{n}=\frac{1}{366{,}26}$ [1591a]. This numerical value is expressed in parts of the radius taken as unity; and if we multiply it by the mean radius in metres $6366193^{\mathrm{met.}}$ [2035b], it becomes as in [2301]. [2300a] [2300b] [2300c]

[2301′] *If we put* e *for the ratio of the mass of the moon, divided by the cube of its mean distance from the earth, to the mass of the sun, divided by the cube of its mean distance; we shall have, for the moon,**

Value of e.

[2302] $$\frac{3L}{4r^3.g} = e\,.\,0^{\text{met.}},12316\,;$$

[2302′] and we must vary this quantity, in proportion to the cube of the mean distance of the moon, to its actual distance.

Hence it follows, that if we put v', ψ', for the declination and right
[2302″] ascension of the moon, we shall have, by means of the combined action of
[2303] the sun and moon, when the depth of the sea is equal to $\frac{1}{10}\cdot\frac{n^2}{g}$, or $\frac{1}{2890}$ of the radius of the earth,†

[2304] $$\alpha y = 0^{\text{met.}},12316.\left\{\frac{1+3.\cos.2\theta}{3}\right\}.\left\{\sin.^2v - \tfrac{1}{2}.\cos.^2v + e.\sin.^2v' - \tfrac{1}{2}e.\cos.^2v'\right\}$$

Tides in a sea of the depth of $1\frac{1}{3}$ miles nearly.

$$+\,0^{\text{met.}},12316.\left(\begin{array}{ll} 1,0000 & +20,1862.x^2 \\ +10,1164.x^4 & -13,1047.x^6 \\ -15,4488.x^8 & -\ 7,4581.x^{10} \\ -\ 2,1975.x^{12} & -\ 0,4501.x^{14} \\ -\ 0,0687.x^{16} & -\ 0,0082.x^{18} \\ -\ 0,0008.x^{20} & -\ 0,0001.x^{22} \end{array}\right).x^2.\left\{\begin{array}{c} \cos.^2v.\cos.(2nt+2\varpi-2\psi) \\ +e.\cos.^2v'.\cos.(2nt+2\varpi-2\psi') \end{array}\right\}.$$

[2304′] We shall hereafter find that $e=3$ [2706], in the mean distances of the sun and moon; therefore if we suppose these two bodies to be at these

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* (1721a) If we accent the letters L, r, ψ, v, to make them correspond to the moon, as in [2426]; the expression of the function in the first member of [2301] will become, for the
[2302a] moon, $\frac{3L'}{4r'^3.g}$; and if we put, as in [2301′], $\frac{L'}{r'^3} = e.\frac{L}{r^3}$, we shall get, by using [2301], $\frac{3L'}{4r'^3.g} = e.\frac{3L}{4r^3.g} = e.0^{\text{met.}},12316$, as in [2302].

† (1722) Multiplying the second member of [2298] by e, and changing v, ψ, into v', ψ', we obtain the part of αy depending on the moon [2301′]. Adding this to the part
[2303a] depending on the sun [2298]; then substituting the value of αa [2294], and that of $\frac{3L}{4r^3.g}$ [2301]; we get the formula [2304].

distances; and also in opposition or conjunction, in the plane of the equator; the high and low water will correspond to the cases in which the angle [2304″] $2nt + 2\varpi - 2\psi$ is nothing, or equal to $200°$.* Hence we find $7^{\mathrm{met.}},34$, [2305] for the difference between the high and the low water, under the equator, where $x = 1$.† *But by a remarkable singularity, the low water takes place* Remarkable result. *when the two bodies are in the meridian, and the high water when they are in the horizon; so that the tide subsides, at the equator, under the body which* [2305′] *attracts it.* In proceeding from the equator, towards the poles, we find that

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* (1723) We shall put, for brevity

$$X = 1{,}0000 + 20{,}1862\,.\,x^2 + 10{,}1164\,.\,x^4 - 13{,}1047\,.\,x^6 - 15{,}4488\,.\,x^8 - 7{,}4581\,.\,x^{10} - 2{,}1975\,.\,x^{12} - \&c. \qquad [2304a]$$

Now when the luminaries are in the equator, we have $\mathrm{v} = 0$, $\mathrm{v}' = 0$; $\sin.\mathrm{v} = 0$, $\cos.\mathrm{v} = 1$; $\sin.\mathrm{v}' = 0$, $\cos.\mathrm{v}' = 1$; also $e = 3$. Substituting these in [2304], [2304b] we get,

$$\alpha y = -0^{\mathrm{met.}},12316 \times \tfrac{2}{3}\,.\,(1 + 3\,.\,\cos.2\,\theta) + 0^{\mathrm{met.}},12316\,.\,X x^2\,.\,\{\cos.2\,.\,(nt + \varpi - \psi) + 3\,.\,\cos.2\,.\,(nt + \varpi - \psi')\}. \qquad [2304c]$$

When the sun is in the meridian *above* the horizon, we have $nt + \varpi - \psi = 0$; when in the meridian *below* the horizon, $nt + \varpi - \psi = 200°$ [2131*a*, *b*], and in both cases, [2304*d*] $\cos.2\,.\,(nt + \varpi - \psi) = 1$. In like manner, when the moon is in the meridian, above or below the horizon, we have $\cos.2\,.\,(nt + \varpi - \psi') = 1$. Therefore the term multiplied [2304*e*] by X [2304*c*], must be a maximum, when these bodies are in conjunction, or in opposition, on the meridian; and the value of αy [2304*c*] will then become

$$\alpha y = 0^{\mathrm{met.}},12316\,.\,\{-\tfrac{2}{3} - 2\,.\,\cos.2\,\theta + 4\,X\,.\,x^2\}. \qquad [2304f]$$

The luminaries being still on the equator, as in [2304*b*], if we suppose them to be in the horizon, or in other words $100°$ distant from the meridian, we shall have $(nt + \varpi - \psi)$ and $(nt + \varpi - \psi')$ equal to $100°$, or $300°$;

$$\cos.2\,.\,(nt + \varpi - \psi) = \cos.2\,.\,(nt + \varpi - \psi') = -1\,;$$
$$\alpha y = 0^{\mathrm{met.}},12316\,.\,\{-\tfrac{2}{3} - 2\,.\,\cos.2\,\theta - 4\,X x^2\}. \qquad [2304g]$$

Moreover, it is evident that the expressions [2304*f*, *g*] correspond to the maximum and [2304*h*] minimum values of αy, or to the times of high and low water, neglecting the variations of v, v', ψ, ψ'.

† (1723*a*) At the equator we have $\theta = 100°$ [2128$^{\mathrm{xi}}$], $\cos.\theta = 0$, $\cos.2\,\theta = -1$, [2305*a*] $x^2 = 1 - \mu^2 = 1$ [2285], $X = -38{,}7370 + 31{,}3026 = -7{,}4344$ [2304*a*]. Hence when the bodies are in the *meridian*, the expression [2304*f*] becomes

$$\alpha y = 0^{\mathrm{met.}},12316\,.\,\{\tfrac{4}{3} - 4 \times 7{,}4344\} = -3^{\mathrm{met.}},50\,;$$

in any place near the eighteenth degree of latitude, either north or south,
[2305''] the difference between the high and the low water is nothing.* Hence it follows, that in all the zone, included between the two parallels of latitude
[2305'''] of eighteen degrees, the low water takes place when the bodies pass the meridian, and beyond these parallels, the high water takes place at the same instant.

[2305''''] In the case of $l = \frac{4}{10} \cdot \frac{n^2}{g}$, or when the depth of the sea is equal to $\frac{1}{722,5}$, we find,†

[2305b] and as this is *negative*, the water is *depressed below* its level. But when the bodies are in the *horizon*, the expression [2304g] becomes

$$a\,y = 0^{\text{met.}},12316 \,.\, \{\tfrac{4}{3} + 4 \times 7,4344\} = 3^{\text{met.}},83\,;$$

[2305c] and as this is *positive*, the water is *elevated above* its level. The sum of $3^{\text{met.}},50$, $3^{\text{met.}},83$, is $7^{\text{met.}},33$, as in [2305] nearly.

* (1724) If we neglect the variations of v, v', ψ, ψ', in the interval between the high and low water; putting also for brevity A for the first line of [2304], $B = \cos.^2 v + e \,.\, \cos.^2 v'$, and X as in [2304a]; we shall find, by proceeding as in the last note, that at the
[2305d] time of high water, $a\,y = A + 0^{\text{met.}},12316 \,.\, X x^2 \,.\, B$ [2304], and at the time of low water, $a\,y = A - 0^{\text{met.}},12316 \,.\, X x^2 \,.\, B$. The difference of these two expressions $0^{\text{met.}},12316 \,.\, 2\,X x^2 \,.\, B$, represents the rise of the tide, which becomes nothing when $X = 0$. A slight inspection of the formula [2304a] shows that x^2 must be nearly equal to
[2305e] unity; and if we put $x^2 = 1 - z$, and neglect the third and higher powers of z, we shall find $0 = -7,43 + 115 \,.\, z - 241 \,.\, z^2$; hence we obtain $z = 0,08$, and
[2305f] $x = \sqrt{(1-z)} = 0,96 = \cos.\, 18^\circ$ nearly as in [2305'']. If we put X [2304a] under the following form,

[2305g]
$$X = 1,0000 + 20,1862 \,.\, x^2 + 10,1164 \,.\, x^4 - x^6 \,.\, \{13,1047 + 15,4488 \,.\, x^2 + 7,4581 \,.\, x^4 + 2,1975 \,.\, x^6 + \&c.\}\,;$$

it will be evident that any value of x less than 0,96 [2305f], will render the negative terms
[2305h] less than the positive; therefore the value of X will be *positive*, between the latitudes of 18° and 100°, and the high water will be at the time of passing the meridian; the contrary takes place between the equator and 18°, as in [2305'''].

† (1725) The formula [2306] is computed in the same manner as [2304], using $a\,a$ [2295]. The quantity corresponding to [2304a] becomes, in this case,

[2306a]
$$X = 1,0000 + 6,1960 \,.\, x^2 + 3,2474 \,.\, x^4 + 0,7238 \,.\, x^6 + 0,0919 \,.\, x^8 + 0,0076 \,.\, x^{10} + 0,0004 \,.\, x^{12}\,;$$

$$\alpha y = 0^{\text{met.}},12316 \,.\, \left\{ \frac{1+3.\cos.2\theta}{3} \right\} . \{\sin.^2 v - \tfrac{1}{2}.\cos.^2 v + e.\sin.^2 v' - \tfrac{1}{2} e.\cos.^2 v'\}$$ [2306]

$$+0^{\text{met.}},12316 \,.\, \left\{ \begin{array}{l} 1,0000 \quad +6,1960.x^2 \\ +3,2474.x^4 + 0,7238.x^6 \\ +0,0919.x^8 + 0,0076.x^{10} \\ +0,0004.x^{12} \end{array} \right\} .x^2. \left\{ \begin{array}{c} \cos.^2 v.\cos.(2nt+2\varpi-2\psi) \\ +e.\cos.^2 v'.\cos.(2nt+2\varpi-2\psi') \end{array} \right\};$$

Tides in a sea of the depth of $5\frac{1}{4}$ miles nearly.

and we shall have, in the above hypothesis, $11^{\text{met.}},05$, for the difference between high and low water, under the equator; but in this case, the time of high water is everywhere the same as that of the passage of the body over the meridian. [2306′]

Lastly, in the case of $l = \frac{8}{10} \cdot \frac{n^2}{g}$, or where the depth of the sea is double of the preceding [2305′′′′′], we find,* [2306″]

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and by using the same values of v, v′, e, [2304b], the expression [2306] becomes similar to [2304c],

$$\begin{aligned} \alpha y = & -0^{\text{met.}},12316 \times \tfrac{2}{3}.(1+3.\cos.2\theta) \\ & + 0^{\text{met.}},12316 \,.\, X x^2.\{\cos.2.(nt+\varepsilon-\psi) + 3.\cos.2.(nt+\varepsilon-\psi')\}; \end{aligned}$$ [2306b]

and the values corresponding to [2304f, g], are

$$\alpha y = 0^{\text{met.}},12316.\{-\tfrac{2}{3} - 2.\cos.2\theta + 4 X x^2\}, \qquad \text{and}$$
$$\alpha y = 0^{\text{met.}},12316.\{-\tfrac{2}{3} - 2.\cos.2\theta - 4 X x^2\};$$

whose difference is $0^{\text{met.}},98528 \,.\, X x^2$. This expresses the excess of the height of the tide, when the bodies are in conjunction or in opposition, on the meridian, above the height when both these bodies are in the horizon of a plane situated upon the equator. Now the value of X [2306a] is always positive, therefore the expression [2306c] is positive; consequently the high tide takes place when the bodies are in the meridian. At the equator, where $x=1$, we have $X = 11,2671$ [2306a], hence the expression [2306c] becomes $11^{\text{met.}},1$ as in [2306′] nearly. [2306c] [2306d] [2306e]

* (1726) The formula [2307] is found like [2304, 2304a], using αa [2296] instead of [2294]. In this case the quantity corresponding to X [2304a] becomes

$$X = 1,0000 + 0,7504.x^2 + 0,1566.x^4 + 0,01574.x^6 + 0,0009.x^8;$$

and when $x=1$, we have $X = 1,9236$. Substituting this in [2306c], which represents the excess of the tides, when the bodies are in the meridian, above the corresponding low water, it becomes $0^{\text{met.}},98528 \,.\, X x^2 = 1^{\text{met.}},90$, as in [2307]. [2307$a$]

[2307] $$\alpha y = 0^{\text{met.}},12316 \cdot \left\{ \frac{1+3.\cos.2\theta}{3} \right\} \cdot \{ \sin.^2 \mathrm{v} - \tfrac{1}{2} \cdot \cos.^2 \mathrm{v} + e \cdot \sin.^2 \mathrm{v}' - \tfrac{1}{2} e \cdot \cos.^2 \mathrm{v}' \}$$

Tides in a sea of the depth of 11 miles nearly.

$$+ 0^{\text{met.}},12316 \cdot \left\{ \begin{array}{c} 1,0000 + 0,7504 \cdot x^2 \\ + 0,1566 \cdot x^4 + 0,01574 \cdot x^6 \\ + 0,0009 \cdot x^8 + \&c. \end{array} \right\} \cdot x^2 \cdot \left\{ \begin{array}{c} \cos.^2 \mathrm{v} \cdot \cos.(2nt + 2\varpi - 2\psi) \\ + e \cdot \cos.^2 \mathrm{v}' . \cos.(2nt + 2\varpi - 2\psi') \end{array} \right\};$$

[2307′] and in the same hypothesis as above, we shall have $1^{\text{met.}},90$ for the difference of the heights of the tide, at the equator.

If we increase the depth of the sea, the value of αy decreases; but there is a limit in this decrease, and the value of αa is quickly reduced to

[2308] $\frac{3L}{4r^3.g} \cdot x^2 \cdot \cos.^2 \mathrm{v}$; we then find $0^{\text{met.}},98528$, for the difference of the

[2309] heights of the tide, under the equator,* when the two bodies are in conjunction in the plane of the equator; therefore this quantity is the limit of this difference of the tides.

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* (1727) When $\frac{2n^2}{lg}$ is small in comparison with f, the formula [2291] will be

[2309a] reduced to its first term, nearly $\frac{A^{(f+1)}}{A^{(f)}} = \frac{\frac{2n^2}{lg}}{2f^2+3f}$; therefore $A^{(f+1)}$ must be very small in comparison with $A^{(f)}$; the coefficients $A^{(1)}$, $A^{(2)}$, $A^{(3)}$, &c., must then rapidly decrease, so that we may neglect all but the first term $A^{(1)}$, and we shall have $\alpha a = A^{(1)} \cdot x^2 = \frac{3L}{4r^3.g} \cdot x^2 \cdot \cos.^2 \mathrm{v}$ [2287, 2288], as in [2308]. Substituting this in

[2309b] [2281], we get $\alpha y = \frac{3L}{4r^3.g} \cdot \cos.^2 \mathrm{v} . x^2 . \cos.2 . (nt + \varpi - \psi)$; being the part depending on the sun. If we multiply this by e, and change v, ψ, into v', ψ', we shall have, as in [2303a], the part depending on the moon. The sum of these two expressions is the whole value of αy; and by using $\frac{3L}{4r^3.g}$ [2301], we get

[2309c] $$\alpha y = 0^{\text{met.}},12316 . x^2 . \{ \cos.^2 \mathrm{v} \cdot \cos.2 \cdot (nt + \varpi - \psi) + e . \cos.^2 \mathrm{v}' . \cos.2 \cdot (nt + \varpi - \psi') \}.$$

This may be deduced from [2306b], by neglecting the terms free from X, and then putting $X = 1$; by which means the difference of the heights of the tide [2306c] becomes $0,^{\text{met.}},98528 . x^2$; or simply $0^{\text{met.}},98528$; because $x=1$ under the equator [2285]. This quantity is evidently the least value, or limit of the difference of the heights of the tide; and it will be shown in [2319e], that it corresponds to the hypothesis assumed in [2309′].

12. *The limit we have assigned, corresponds to the case where the sea assumes at each instant the figure of equilibrium corresponding to the forces which act on it.* In this hypothesis, the value of αy may be very easily determined, whatever be the law of the depth and density of the sea. For it is the same as to suppose, in the equation [2130], that the motions of the attracting body, and the rotation of the earth, are so slow, that we may neglect the quantities* [2309'] [2309'']

Tides where the sea assumes the figure of equilibrium corresponding to the forces acting on it.

$$\left(\frac{ddu}{dt^2}\right), \qquad n.\left(\frac{du}{dt}\right), \qquad \left(\frac{ddv}{dt^2}\right), \qquad n.\left(\frac{dv}{dt}\right); \qquad [2310]$$

and then this equation gives, by integration,

$$\alpha y = \frac{\alpha V'}{g}. \qquad [2311]$$

The part of $\alpha V'$, depending on the action of the body L, is, by [2193—2195], equal to† [2311']

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* (1728) If the attracting body be at rest, and the earth have no rotatory motion, the fluid will assume its permanent figure of equilibrium; and then we shall have, as in [2221c], $\frac{du}{dt}=0$, $\frac{dv}{dt}=0$, $\frac{ddu}{dt^2}=0$, $\frac{ddv}{dt^2}=0$. Substituting these in [2130], it becomes [2310a] $0=-g.\mathrm{d}y+\mathrm{d}V'$; its integral, found as in [2221e], is $0=-gy+V'$; hence $\alpha y=\frac{\alpha V'}{g}$. Now if the motion of the sun or moon and the rotatory motion of the earth be very small, so that the fluid at every instant can be supposed to assume the figure corresponding to its permanent state, we may consider the quantities [2310] to be so small that they may be neglected in [2130], and we shall then obtain the same results as in [2310b]. [2310b] [2310c]

† (1729) This expression is the same as [2193—2195]; and since

$$1+3.\cos.2\theta=6.(\mu^2-\tfrac{1}{3}) \quad [2197a], \qquad \sin.\theta.\cos.\theta=\mu.(1-\mu^2)^{\frac{1}{2}}, \qquad \sin.^2\theta=1-\mu^2,$$

it becomes,

$$\frac{6L}{4r^3}.(\sin.^2\mathrm{v}-\tfrac{1}{2}.\cos.^2\mathrm{v}).(\mu^2-\tfrac{1}{3})+\frac{3L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}.\{\mu.(1-\mu^2)^{\frac{1}{2}}.\cos.(nt+\varpi-\psi)\}$$
$$+\frac{3L}{4r^3}.\cos.^2\mathrm{v}.\{(1-\mu^2).\cos.(2nt+2\varpi-2\psi). \qquad [2312a]$$

If this be multiplied by a constant but indeterminate quantity Q, putting $B_2^{(0)}$, $B_2^{(1)}$, $B_2^{(2)}$, for the coefficients of $\mu^2-\frac{1}{3}$, $\mu.(1-\mu^2)^{\frac{1}{2}}.\cos.(nt+\varpi-\psi)$, $(1-\mu^2).\cos.2.(nt+\varpi-\psi)$ respectively, it will become of the form of $Y^{(2)}$ [1528c]; ϖ being changed into $nt+\varpi-\psi$. This expression of $Y^{(2)}$ satisfies the equation [2313], which is the same as [2145], when $i=2$. [2312b]

$$\frac{L}{4r^3}\cdot\{\sin.^2 v-\tfrac{1}{2}\cdot\cos.^2 v\}\cdot(1+3\cdot\cos.2\theta)$$

[2312] $$+\frac{3L}{r^3}\cdot\sin.v\cdot\cos.v\cdot\sin.\theta\cdot\cos.\theta\cdot\cos.(nt+\varpi-\psi)$$

$$+\frac{3L}{4r^3}\cdot\cos.^2 v\cdot\sin.^2\theta\cdot\cos.(2nt+2\varpi-2\psi).$$

We shall suppose the corresponding part of αy to be equal to this quantity,
[2312′] multiplied by the indeterminate quantity Q; the product being of the form $Y^{(2)}$, or such that when taken for $Y^{(2)}$ it will satisfy this equation of partial differentials [2145]

[2313] $$0=\left\{\frac{d\cdot\left\{(1-\mu^2)\cdot\left(\frac{dY^{(2)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddY^{(2)}}{d\varpi^2}\right)}{1-\mu^2}+6\cdot Y^{(2)}.$$

Then the part of $\alpha V'$, corresponding to the action of the fluid stratum, whose internal radius is unity, and external radius $1+\alpha y$, will be
[2314] $\frac{4}{5}\pi\cdot Y^{(2)}$ [2146, 2314d], or $\frac{3}{5\rho}\cdot g\cdot Y^{(2)}$ [2147]; therefore the equation
[2315] $\alpha g y=\alpha V'$ [2311] will give*

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* (1730) The part of $\alpha V'$, arising from the attraction of the aqueous stratum, whose internal radius is 1, and external $1+\alpha y$, is

[2314a] $$\frac{3g}{\rho}\cdot\left\{\alpha\cdot Y^{(0)}+\frac{\alpha}{3}\cdot Y^{(1)}+\frac{\alpha}{5}\cdot Y^{(2)}+\&c.\right\}\qquad[2146a].$$

Adding this to the part of $\alpha V'$ [2312] depending on the body L, which for brevity we shall call $Y'^{(2)}$, we shall get the whole value of $\alpha V'$, namely,

[2314b] $$\alpha V'=Y'^{(2)}+\frac{3g}{\rho}\cdot\{\alpha Y^{(0)}+\tfrac{1}{3}\cdot\alpha Y^{(1)}+\tfrac{1}{5}\cdot\alpha Y^{(2)}+\&c.\}.$$

Substituting this, and y [2144], in [2311], we obtain

$$\alpha\cdot\{Y^{(0)}+Y^{(1)}+Y^{(2)}+Y^{(3)}+\&c.\}$$

[2314c] $$=\frac{3}{\rho}\cdot\alpha Y^{(0)}+\frac{3}{3\rho}\cdot\alpha Y^{(1)}+\left(\frac{3}{5\rho}\cdot\alpha Y^{(2)}+\frac{1}{g}\cdot Y'^{(2)}\right)+\frac{3}{7\rho}\cdot\alpha Y^{(3)}+\&c.$$

Now from the nature of the functions $Y^{(i)}$, the parts of each member of this equation, depending on the same value of i, ought to destroy each other; which gives, when i differs from 2, $Y^{(i)}=\frac{3}{(2i+1)\cdot\rho}\cdot Y^{(i)}$, or $\left\{1-\frac{3}{(2i+1)\cdot\rho}\right\}\cdot Y^{(i)}=0$; hence
[2314c'] generally $Y^{(i)}=0$. The terms depending on $i=2$, give

Expressions of the tide caused by the sun.

$$\alpha y = \frac{L}{4r^3.g.\left(1-\frac{3}{5\rho}\right)}.\{\sin.^2 v - \tfrac{1}{2}.\cos.^2 v\}.(1+3.\cos.2\theta)$$

$$+\frac{3L}{r^3.g.\left(1-\frac{3}{5\rho}\right)}.\sin.v.\cos.v.\sin.\theta.\cos.\theta.\cos.(nt+\varpi-\psi)$$ [2316]

$$+\frac{3L}{4r^3.g.\left(1-\frac{3}{5\rho}\right)}.\cos.^2 v.\sin.^2\theta.\cos.(2nt+2\varpi-2\psi).$$

In the hypothesis under consideration, if the sun and moon are in conjunction with the same declination; then the excess of the midday high tide above the following low water, will be,* [2316′]

$$\frac{3L}{2r^3.g.\left(1-\frac{3}{5\rho}\right)}.(1+e).\sin.^2\theta.\cos.^2 v.\{1+2.\mathrm{tang}.v.\cot.\theta\};$$ [2317]

$$\alpha Y^{(2)} = \frac{3}{5\rho}.\alpha Y^{(2)} + \frac{1}{g}.Y'^{(2)}, \qquad \text{or} \qquad \alpha Y^{(2)} = \frac{Y'^{(2)}}{\left(1-\frac{3}{5\rho}\right).g}.$$ [2314d]

In this case the expression [2144] becomes $y = Y^{(2)}$, hence

$$\alpha y = \alpha Y^{(2)} = \frac{Y'^{(2)}}{\left(1-\frac{3}{5\rho}\right).g};$$ [2314e]

and by substituting the value of $Y'^{(2)}$ [2312], it becomes as in [2316].

* (1730a) We shall put, for brevity, A equal to the first line of the second member of [2316]; B equal to the coefficient of $\cos.(nt+\varpi-\psi)$, in the second line; and C equal to the coefficient of $\cos.(2nt+2\varpi-2\psi)$, in the third line; and we shall get

$$\alpha y = A + B.\cos.(nt+\varpi-\psi) + C.\cos.2.(nt+\varpi-\psi).$$ [2316a]

When the sun and moon are in *conjunction*, with the same declination, the values of A, B, C, for the moon, will be equal to those corresponding to the sun, multiplied by e [2301′]; therefore to obtain the whole of αy, we must multiply [2316a] by $1+e$; hence

$$\alpha y = (1+e).\{A + B.\cos.(nt+\varpi-\psi) + C.\cos.2.(nt+\varpi-\psi)\}.$$ [2316b]

But $nt+\varpi-\psi=0$ [2131a], when these bodies are in the meridian; and when they have passed 100° from the meridian, $nt+\varpi-\psi=100^{\circ}$. The former value gives the midday elevation $(1+e).(A+B+C)$; the latter the following low tide, nearly $(1+e).(A-C)$; subtracting this from the preceding, we get the excess of the midday [2316c]

and the excess of the midnight tide above the same low water, will be nearly,*

[2318] $$\frac{3L}{2r^3 \cdot g \cdot \left(1-\frac{3}{5\rho}\right)} \cdot (1+e) \cdot \sin.^2\theta \cdot \cos.^2 \mathrm{v} \cdot \{1-2 \cdot \mathrm{tang.\,v} \cdot \cot.\theta\}.$$

These two quantities [2317, 2318] will therefore be to each other in the
[2318'] ratio of $1+2 \cdot \mathrm{tang.\,v} \cdot \cot.\theta$ to $1-2 \cdot \mathrm{tang.\,v} \cdot \cot.\theta$; hence, for Brest, where $\theta = 46^\circ\, 26'$ nearly, if the two luminaries have 23° of north declination, the two quantities will be in the ratio of 1,7953 to 0,2047;† that
[2319] is, the first will be about eight times the second. According to observations,

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tide above the following low water, $(1+e) \cdot (B+2C)$; which, by substituting the values of B, C, becomes

[2316d] $$\frac{3L}{2r^3 \cdot g \cdot \left(1-\frac{3}{5\rho}\right)} \cdot (1+e) \cdot \sin.^2\theta \cdot \cos.^2 \mathrm{v} \cdot \left\{1+2 \cdot \frac{\sin.\mathrm{v}}{\cos.\mathrm{v}} \cdot \frac{\cos.\theta}{\sin.\theta}\right\};$$

and this is easily reduced to the form [2317].

* (1731) When the bodies are in the meridian, below the horizon, we have the quantity $nt+\varpi-\psi=200^\circ$ [2131b]; and then [2316b] becomes $(1+e) \cdot (A-B+C)$, from which subtracting the value of αy corresponding to the preceding low tide, $(1+e) \cdot (A-C)$ nearly [2316c], it becomes $(1+e) \cdot (2C-B)$, for the difference between the midnight tide and the preceding low water nearly; and by substituting the values of C, B, it becomes

[2318a] $$\frac{3L}{2r^3 \cdot g \cdot \left(1-\frac{3}{5\rho}\right)} \cdot (1+e) \cdot \sin.^2\theta \cdot \cos.^2 \mathrm{v} \cdot \left\{1-2 \cdot \frac{\sin.\mathrm{v}}{\cos.\mathrm{v}} \cdot \frac{\cos.\theta}{\sin.\theta}\right\};$$

which is easily reduced to the form of [2318]; and then the expressions [2317, 2318] are as $1+2 \cdot \mathrm{tang.\,v} \cdot \cot.\theta$ to $1-2 \cdot \mathrm{tang.\,v} \cdot \cot.\theta$. It may be observed, that the preceding value $(A-C) \cdot (1+e)$ is not exactly the value of αy at the time of *low water;* this is to be found by determining the angle $nt+\varpi-\psi$ which makes [2316b] a *minimum*, which differs a little from the preceding.

[2319a] † (1732) This ratio is as 1,85 to 0,15 nearly, instead of 1,7953 to 0,2047. The author, in computing the value of $1 \pm 2 \cdot \mathrm{tang.\,v} \cdot \cot.\theta$, seems to have used the sine of $\mathrm{v}=23^\circ$ instead of its tangent.

When the sun and moon are in the equator, we have $\mathrm{v}=0$; and the expressions
[2319b] [2317, 2318] become equal to each other, as in the first member of [2319c]. This, by

these two quantities differ but little from each other, therefore the present hypothesis is very far from representing the observations in this particular, and *we see that it is indispensable, in the theory of the ebb and flow of the sea, to notice the rotatory motion of the earth, and the motions of the attracting bodies.* [2319′]

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neglecting $\frac{3}{5\rho}$ in comparison with 1, as in [2280e], then putting $e=3$ [2304′], and $\theta=100^{\circ}$, finally becomes as in [2319d], by using [2301],

$$\frac{3L}{2r^3.g.\left(1-\frac{3}{5\rho}\right)}.(1+e).\sin.^2\theta=\frac{3L}{2r^3.g}.4.\sin.^2\theta=\frac{3L}{4r^3.g}.8.\sin.^2\theta \qquad [2319c]$$

$$=0^{\text{met.}},12316\times 8.\sin.^2\theta=0^{\text{met.}},98528.\sin.^2\theta=0^{\text{met.}},98528. \qquad [2319d]$$

This expresses the rise of the tide, at a place situated in the equator, when the sun and moon are in conjunction in the equator; the fluid being supposed, as in [2309′], to assume at each instant the figure of equilibrium corresponding to the forces which act upon it. This quantity [2319e] $0^{\text{met.}},98528$ is the same as is mentioned in [2309].

CHAPTER II.

ON THE STABILITY OF THE EQUILIBRIUM OF THE SEA.

13. We *have seen, in* [2162, 2135^{vi}], *that if the earth have no rotatory*
[2319''] *motion, and the depth of the sea be constant, the equilibrium will be stable, provided the mean density of the earth exceed that of the sea. We shall now generalize this theorem, and shall show that it takes place, whatever be the depth of the sea, or the rotatory motion of the earth.*

General equations for all parts of the fluid.

We shall resume the general equations of the motions of the sea given in § 36 of the first book,*

[2320] $$0=\left(\frac{d.r^2 s}{dr}\right)+r^2.\left\{\left(\frac{dv}{d\varpi}\right)-\left(\frac{d.(u.\sqrt{1-\mu^2})}{d\mu}\right)\right\}; \qquad (5)$$

[2321] $$\left(\frac{ddu}{dt^2}\right)-2n\mu.\sqrt{1-\mu^2}.\left(\frac{dv}{dt}\right)=g.\left(\frac{dy'}{d\mu}\right).\sqrt{1-\mu^2};\dagger \qquad (6)$$

[2322] $$(1-\mu^2).\left(\frac{ddv}{dt^2}\right)+2n\mu.\sqrt{1-\mu^2}.\left(\frac{du}{dt}\right)=-g.\left(\frac{dy'}{d\varpi}\right). \qquad (7)$$

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* (1733) By development we find, $\left(\frac{d.(u.\sin.\theta)}{d\theta}\right).\frac{1}{\sin.\theta}=\left(\frac{du}{d\theta}\right)+\frac{u.\cos.\theta}{\sin.\theta}$. Substituting in the first member the values of $\sin.\theta=\sqrt{(1-\mu^2)}$, and $d\theta.\sin.\theta=-d\mu$,
[2320a] [2136a], we get $\left(\frac{du}{d\theta}\right)+\frac{u.\cos.\theta}{\sin.\theta}=-\left(\frac{d.(u.\sqrt{1-\mu^2})}{d\mu}\right)$; hence the expression [337] becomes as in [2320].

† (1734) The differential, or rather the variation of the assumed equation [2324], is
[2320b] $g.\delta y'=g.\delta y-\delta V'$. Comparing this with [341c—d], we get, for any part of the fluid,

These equations refer to any particle, either upon or below the surface of the sea; the co-ordinates of this particle being $\theta+\alpha u$, $\varpi+\alpha v$, *and* $r+\alpha s$; [2323]
$r+\alpha s$ being its distance from the centre of the earth; moreover

$$g y'=g y-V'. \qquad [2324]$$

If we integrate the equation [2320], from the surface of the spheroid which the fluid covers, to the surface of the sea, we shall find,*

Equation of continuity; second form.

$$r'^2 . s'-r_{\prime}^2 . s_{\prime}=\int r^2 d r . \left\{\left(\frac{d . (u . \sqrt{1-\mu^2})}{d \mu}\right)-\left(\frac{d v}{d \varpi}\right)\right\}; \qquad [2325]$$

r' and s' correspond to the surface of the sea, and $r_{\prime}$, $s_{\prime}$, to the surface of the spheroid. Putting γ for the depth of the sea, and supposing it to be [2326]
very small, we shall have $r'=r_{\prime}+\gamma$, which gives, [2327]

$$r^2 . \delta \theta . \left\{\left(\frac{d d u}{d t^2}\right)-2 n . \sin. \theta . \cos. \theta . \left(\frac{d v}{d t}\right)\right\}$$
$$+r^2 . \delta \varpi . \left\{\sin.^2 \theta . \left(\frac{d d v}{d t^2}\right)+2 n . \sin. \theta . \cos. \theta . \left(\frac{d u}{d t}\right)\right\}=-g . \delta y'; \qquad [2320c]$$

the terms depending on δr being neglected, in conformity with the remarks which precede the formula [341c]. Substituting for $\delta y'$ its value $\left(\frac{d y'}{d \theta}\right) . \delta \theta+\left(\frac{d y'}{d \varpi}\right) . \delta \varpi$, similar to [2137b]; and then putting, according to the usual principles [2138a], the coefficients of [2320d]
$\delta \theta$, in each member, equal to each other, we get the first equation [2320e]; and the coefficients of $\delta \varpi$ give, in like manner, the second equation [2320e],

$$r^2 . \left\{\left(\frac{d d u}{d t^2}\right)-2 n . \sin. \theta . \cos. \theta . \left(\frac{d v}{d t}\right)\right\}=-g . \left(\frac{d y'}{d \theta}\right);$$
$$r^2 . \left\{\sin.^2 \theta . \left(\frac{d d v}{d t^2}\right)+2 n . \sin. \theta . \cos. \theta . \left(\frac{d u}{d t}\right)\right\}=-g . \left(\frac{d y'}{d \varpi}\right). \qquad [2320e]$$

Now r^2 differs from unity, by terms of the order γ [2326, 2329]; therefore if we divide these two equations by r^2, neglecting terms of the order γg, as in [2130b], and substituting the values [2320a], we shall obtain the equations [2321, 2322]; observing that by this substitution we have $\left(\frac{d y'}{d \theta}\right)=-\left(\frac{d y'}{d \mu}\right) . \sin. \theta=-\left(\frac{d y'}{d \mu}\right) . \sqrt{(1-\mu^2)}$ [2320a]. [2320f]

* (1735) If the equation of continuity [2320] be multiplied by dr, and then integrated; commencing the integral at the bottom of the sea, where $r=r_{\prime}$, $s=s_{\prime}$; and terminating at the surface, where $r=r'$, $s=s'$; we shall get the second form of this equation, [2325a]
as in [2325]. This is reduced to the third form [2334], by the method given in [2326—2333].

[2328] $$r'^2 . s' - r_{,}^2 . s_{,} = r_{,}^2 . (s' - s_{,}) + 2\, r_{,} . \gamma s' + \gamma^2 s' .^*$$

[2329] Therefore, *as the mean radius of the earth is taken for unity*, we shall have nearly,

[2330] $$r'^2 . s' - r_{,}^2 . s_{,} = s' - s_{,} .$$

Now we have,†

[2331] $$\alpha s' = \alpha y + \alpha u' . \left(\frac{d r'}{d \theta}\right) + \alpha v' . \left(\frac{d r'}{d \varpi}\right);$$

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* (1736) Substituting r' [2327], in the first member of [2328], it becomes like the second member of this equation. Now the figure of the bottom of the sea being very
[2328a] different from that of its surface, the values $\alpha s'$, $\alpha s_{,}$ [2331, 2332] will generally vary so much from each other, that their difference $\alpha s' - \alpha s_{,}$ may be supposed of the same
[2328b] order as $\alpha s'$ or $\alpha s_{,}$; so that $s' - s_{,}$ may be considered of the same order as $s_{,}$; and the term of the second member of [2328], multiplied by $s' - s_{,}$, will generally be much greater than the two other terms, containing the very small factor γ [2326]. If we neglect these two terms, the equation [2328] will become $r'^2 . s' - r_{,}^2 . s_{,} = r_{,}^2 . (s' - s_{,})$; and
[2328c] as $r_{,}$ differs from unity [2329], by quantities of the order γ, we shall obtain the equation
[2328d] [2330], neglecting terms of the order $\gamma . (s' - s_{,})$.

† (1737) A particle of fluid at the bottom of the sea, having the co-ordinates $r_{,}$, $\theta_{,}$, $\varpi_{,}$,
[2331a] at the origin of the motion, will have the co-ordinates $r_{,} + \alpha s_{,}$, $\theta_{,} + \alpha u_{,}$, $\varpi_{,} + \alpha v_{,}$, [2323, 2326], at the expiration of the time t; so that the polar distance of the particle will increase by $\alpha u_{,}$, and its longitude by $\alpha v_{,}$. Now the surface of the spheroid, or the bottom of the sea, being supposed to be known, $r_{,}$ will be a function of $\theta_{,}$ and $\varpi_{,}$, which
[2331b] we shall represent by $r_{,} = \varphi(\theta_{,}, \varpi_{,})$. When $r_{,}$, $\theta_{,}$, $\varpi_{,}$, are increased by $\alpha s_{,}$, $\alpha u_{,}$, $\alpha v_{,}$, respectively, the radius becomes $r_{,} + \alpha s_{,} = \varphi(\theta_{,} + \alpha u_{,}, \varpi_{,} + \alpha v_{,})$. Developing the second member, in terms of $\alpha u_{,}$, $\alpha v_{,}$, [610], retaining only the first power of α, by which means we may change $d\theta_{,}$, $d\varpi_{,}$, into $d\theta$, $d\varpi$, in the denominators, we get,

[2331c] $$r_{,} + \alpha s_{,} = \varphi(\theta_{,}, \varpi_{,}) + \left(\frac{d . \varphi(\theta_{,}, \varpi_{,})}{d \theta}\right) . \alpha u_{,} + \left(\frac{d . \varphi(\theta_{,}, \varpi_{,})}{d \varpi}\right) . \alpha v_{,} .$$

Subtracting from this the value of $r_{,}$ [2331b], we get, as in [2332],

[2331d] $$\alpha s_{,} = \left(\frac{d . \varphi(\theta_{,}, \varpi_{,})}{d \theta}\right) . \alpha u_{,} + \left(\frac{d . \varphi(\theta_{,}, \varpi_{,})}{d \varpi}\right) . \alpha v_{,} = \left(\frac{d r_{,}}{d \theta}\right) . \alpha u_{,} + \left(\frac{d r_{,}}{d \varpi}\right) . \alpha v_{,} .$$

If in this we change $r_{,}$, $s_{,}$, $u_{,}$, $v_{,}$, into r', s', u', v', we shall obtain

[2331e] $$\alpha s' = \left(\frac{d r'}{d \theta}\right) . \alpha u' + \left(\frac{d r'}{d \varpi}\right) . \alpha v',$$

corresponding to the *surface of equilibrium* of the fluid; or, in other words, the part

in which u' and v' correspond to the surface of the sea. In like manner we find,

$$\alpha s_{,} = \alpha u_{,} \cdot \left(\frac{d r_{,}}{d \theta}\right) + \alpha v_{,} \cdot \left(\frac{d r_{,}}{d \varpi}\right), \qquad [2332]$$

$u_{,}$ and $v_{,}$ being the values corresponding to the surface of the spheroid; therefore we shall have,*

$$s' - s_{,} = y - u' \cdot \left(\frac{d r'}{d \mu}\right) \cdot \sqrt{1-\mu^2} + u_{,} \cdot \left(\frac{d r_{,}}{d \mu}\right) \cdot \sqrt{1-\mu^2} + v' \cdot \left(\frac{d r'}{d \varpi}\right) - v_{,} \cdot \left(\frac{d r_{,}}{d \varpi}\right); \qquad [2333]$$

so that we shall find nearly†

Equation of continuity; third form.

$$y = u' \cdot \left(\frac{d r'}{d \mu}\right) \cdot \sqrt{1-\mu^2} - u_{,} \cdot \left(\frac{d r_{,}}{d \mu}\right) \cdot \sqrt{1-\mu^2} - v' \cdot \left(\frac{d r'}{d \varpi}\right) + v_{,} \cdot \left(\frac{d r_{,}}{d \varpi}\right)$$
$$+ \int d r \cdot \left\{ \left(\frac{d \cdot (u \cdot \sqrt{1-\mu^2})}{d \mu}\right) - \left(\frac{d v}{d \varpi}\right) \right\}. \qquad (8) \qquad [2334]$$

This equation is not restricted, like [2129], *or* [347, 342vii], *to the condition that u and v must be the same, for all the particles situated on the same radius;* [2334′] and it is evident, that by satisfying this condition, the two equations will agree.‡

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depending on the supposition that the particle changes its place upon the surface of equilibrium, without rising above it. To this we must add αy [2128xiv], which represents the elevation of the particle *above its surface of equilibrium*. The sum represents $\alpha s'$ [2331].

* (1738) Subtracting [2332] from [2331], and dividing by α, we obtain

$$s' - s_{,} = y + u' \cdot \left(\frac{d r'}{d \theta}\right) - u_{,} \cdot \left(\frac{d r_{,}}{d \theta}\right) + v' \cdot \left(\frac{d r'}{d \varpi}\right) - v_{,} \cdot \left(\frac{d r_{,}}{d \varpi}\right); \qquad [2333a]$$

substituting $\left(\frac{d r'}{d \theta}\right) = -\left(\frac{d r'}{d \mu}\right) \cdot \sqrt{(1-\mu^2)}$, $\left(\frac{d r_{,}}{d \theta}\right) = -\left(\frac{d r_{,}}{d \mu}\right) \cdot \sqrt{(1-\mu^2)}$ [2320f], [2333b]

we get [2333].

† (1740) The value of $s' - s_{,}$ [2330, 2325], is represented by the second member of [2325]; in which we may put $r^2 = 1$, neglecting terms of the order $\gamma \cdot (s' - s_{,})$ [2334a] [2328d]. Substituting this in [2333], and transposing all the terms of the second member except y, we get [2334].

‡ (1741) If we suppose u, v, to be the same, for all the particles situated upon the same [2334a′] radius, we shall have $u = u' = u_{,}$, $v = v' = v_{,}$. Hence

Now if we add the equation [2321], multiplied by $dr.d\mu.d\varpi.\left(\frac{du}{dt}\right)$, to

[2334″] the equation [2322] multiplied by $dr.d\mu.d\varpi.\left(\frac{dv}{dt}\right)$, we shall get, by integration,*

$$\int\int\int dr.d\mu.d\varpi.\left\{\left(\frac{du}{dt}\right).\left(\frac{ddu}{dt^2}\right)+\left(\frac{dv}{dt}\right).\left(\frac{ddv}{dt^2}\right).(1-\mu^2)\right\} \qquad (9)$$

[2335] $$=\int\int\int dr.d\mu.d\varpi.\left\{g.\left(\frac{dy'}{d\mu}\right).\left(\frac{du}{dt}\right).\sqrt{1-\mu^2}-g.\left(\frac{dy'}{d\varpi}\right).\left(\frac{dv}{dt}\right)\right\}.$$

[2334b] $$u'.\left(\frac{dr'}{d\mu}\right)-u_{,}.\left(\frac{dr_{,}}{d\mu}\right)=\ u.\left(\frac{d.(r'-r_{,})}{d\mu}\right)=u.\left(\frac{d\gamma}{d\mu}\right) \quad [2327],$$
$$-v'.\left(\frac{dr'}{d\varpi}\right)+v_{,}.\left(\frac{dr_{,}}{d\varpi}\right)=-v.\left(\frac{d.(r'-r_{,})}{d\varpi}\right)=-v.\left(\frac{d\gamma}{d\varpi}\right);$$

and the terms of the second member of [2334] independent of the sign $\int$, become

[2334c] $$u.\left(\frac{d\gamma}{d\mu}\right).\sqrt{(1-\mu^2)}-v.\left(\frac{d\gamma}{d\varpi}\right).$$

We shall put, for brevity, M equal to the coefficient of dr, in the quantity under the sign $\int$ [2334]; then all the terms of M being independent of r [2334d′], we shall get

$$\int dr.M=M.\int dr=M.(r'-r_{,})=M.\gamma=\gamma.\left(\frac{d.(u.\sqrt{1-\mu^2})}{d\mu}\right)-\gamma.\left(\frac{dv}{d\varpi}\right)$$

[2334d] $$=\gamma.\sqrt{(1-\mu^2)}.\left(\frac{du}{d\mu}\right)-\frac{\gamma u.\mu}{\sqrt{(1-\mu^2)}}-\gamma.\left(\frac{dv}{d\varpi}\right).$$

Connecting this with the part [2334c], we get, for the equation [2334], the expression [2334e]. Substituting in this the values similar to [2320a, f], we get [2334f], which, by successive reductions, becomes of the form [2334g], agreeing with [2129],

[2334e] $$y=u.\left(\frac{d\gamma}{d\mu}\right).\sqrt{(1-\mu^2)}-v.\left(\frac{d\gamma}{d\varpi}\right)+\gamma.\sqrt{(1-\mu^2)}.\left(\frac{du}{d\mu}\right)-\frac{\gamma u.\mu}{\sqrt{(1-\mu^2)}}-\gamma.\left(\frac{dv}{d\varpi}\right)$$

[2334f] $$=-u.\left(\frac{d\gamma}{d\theta}\right)-v.\left(\frac{d\gamma}{d\varpi}\right)-\gamma.\left(\frac{du}{d\theta}\right)-\frac{\gamma u.\cos.\theta}{\sin.\theta}-\gamma.\left(\frac{dv}{d\varpi}\right)$$

$$=-\left\{u.\left(\frac{d\gamma}{d\theta}\right)+\gamma.\left(\frac{du}{d\theta}\right)\right\}-\left\{v.\left(\frac{d\gamma}{d\varpi}\right)+\gamma.\left(\frac{dv}{d\varpi}\right)\right\}-\frac{\gamma u.\cos.\theta}{\sin.\theta}$$

[2334g] $$=-\left(\frac{d.(\gamma u)}{d\theta}\right)-\left(\frac{d.(\gamma v)}{d\varpi}\right)-\frac{\gamma u.\cos.\theta}{\sin.\theta}.$$

* (1742) The terms multiplied by n disappear from this sum; and by prefixing the signs

[2335a] of integration $\int\int\int$, it becomes as in [2335]. The limits of the integral are similar to those in [1433″, 2335′].

To make these integrals include the whole mass of the fluid, we must take them from $r=r_{\prime}$ to $r=r'$, from $\mu=-1$ to $\mu=1$, and from $\varpi=0$ [2335′]
to $\varpi=2\pi$. Integrating relatively to μ, and observing that y' is independent [2336]
of r,* we shall find,†

$$\int\int dr\,.\,d\mu\,.\left(\frac{du}{dt}\right).\left(\frac{dy'}{d\mu}\right).\sqrt{1-\mu^2}$$

$$=\int y'.\left(\frac{du}{dt}\right).dr.\sqrt{1-\mu^2}-\int y'.\left\{\frac{d.\int\left\{\left(\frac{du}{dt}\right).dr.\sqrt{1-\mu^2}\right\}}{d\mu}\right\}.d\mu+\text{const.} \qquad [2337]$$

At the two extremities of this integral, where $\mu=-1$ and $\mu=1$, we
have $y'.\left(\frac{du}{dt}\right).\sqrt{1-\mu^2}=0$;‡ therefore [2338]

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* (1743) This follows from the assumed value of $g.\delta y'$ [2320b], compared with [2336a]
[341d, &c.].

† (1744) Putting for brevity $W=\left(\frac{du}{dt}\right).dr.\sqrt{(1-\mu^2)}$, in the first member of [2337a]
[2337], then integrating by parts [1716a], relatively to μ, neglecting for a moment the sign of integration relatively to r, we get successively,

$$\int dr\,.\,d\mu\,.\left(\frac{du}{dt}\right).\left(\frac{dy'}{d\mu}\right).\sqrt{(1-\mu^2)}=\int d\mu\,.\left(\frac{dy'}{d\mu}\right).W=y'.W-\int y'.\left(\frac{dW}{d\mu}\right).d\mu. \qquad [2337b]$$

Now prefixing the sign of integration relative to dr, we get

$$\int\int dr\,.\,d\mu\,.\left(\frac{du}{dt}\right).\left(\frac{dy'}{d\mu}\right).\sqrt{(1-\mu^2)}=\int y'.W-\int\int y'.\left(\frac{dW}{d\mu}\right).d\mu\,; \qquad [2337c]$$

which is the same as [2337]; observing that the last term may be put under the form

$$\int y'.\left\{\int\left(\frac{dW}{d\mu}\right)\right\}.d\mu,$$

because y' is independent of r [2336], so that we may separate the part $\left\{\int\left(\frac{dW}{d\mu}\right)\right\}$ [2337d]
which depends on the sign of integration relative to r, from the factor y', which does not contain r.

‡ (1745) At the two limits of the integral relative to μ, namely $\mu=\pm 1$ [2335′], [2338a]
the expression of W [2337a] becomes $W=0$; therefore the term $y'.W$ free from the sign $\int$, vanishes from [2337b]; and for the same reason, the term $\int y'.W$ vanishes from [2337c]; or in other words, the expression [2339] must be put equal to nothing; observing that the author, in finding the integral [2337b], has added a constant quantity to the second member, putting $y'.W+\text{constant}$ instead of $y'.W$. Substituting [2339] [2338b]
in [2337], we get [2340].

[2339] $$0 = \int y' \cdot \left(\frac{du}{dt}\right) \cdot dr \cdot \sqrt{1-\mu^2} + \text{const.}$$

consequently

[2340] $$\int\int dr \cdot d\mu \cdot \left(\frac{du}{dt}\right) \cdot \left(\frac{dy'}{d\mu}\right) \cdot \sqrt{1-\mu^2} = -\int y' \cdot \left\{ \frac{d \cdot \int \left\{ \left(\frac{du}{dt}\right) \cdot dr \cdot \sqrt{1-\mu^2} \right\}}{d\mu} \right\} \cdot d\mu.$$

Now we have*

[2341] $$\left(\frac{d \cdot \left\{ \int \left(\frac{du}{dt}\right) \cdot dr \cdot \sqrt{1-\mu^2} \right\}}{d\mu} \right) = \int dr \cdot \left\{ \frac{d \cdot \left\{ \left(\frac{du}{dt}\right) \cdot \sqrt{1-\mu^2} \right\}}{d\mu} \right\} + \left(\frac{du'}{dt}\right) \cdot \left(\frac{dr'}{d\mu}\right) \cdot \sqrt{1-\mu^2} - \left(\frac{du_{,}}{dt}\right) \cdot \left(\frac{dr_{,}}{d\mu}\right) \cdot \sqrt{1-\mu^2};$$

[2340a] * (1747) Putting $\mathcal{N} = \left(\frac{du}{dt}\right) \cdot \sqrt{(1-\mu^2)}$, and $\int_{r_{,}}^{r'} \mathcal{N} \cdot dr = R$, we shall get, for the first member of [2341],

[2340b] $$\left\{ \frac{d \cdot \left\{ \int \left(\frac{du}{dt}\right) \cdot dr \cdot \sqrt{(1-\mu^2)} \right\}}{d\mu} \right\} = \left(\frac{d \cdot (\int \mathcal{N} \cdot dr)}{d\mu} \right) = \left(\frac{dR}{d\mu}\right);$$

which may be reduced to the same form as the second member of [2341]. For in finding

[2340c] $\left(\frac{dR}{d\mu}\right)$ from $R = \int_{r_{,}}^{r'} \mathcal{N} \cdot dr$, we must observe, that when μ is increased by $d\mu$, the quantity $\mathcal{N}$ is increased by $\left(\frac{d\mathcal{N}}{d\mu}\right) \cdot d\mu$, by which means we obtain, in $\left(\frac{dR}{d\mu}\right)$, the

[2340d] expression $\int_{r_{,}}^{r'} \left(\frac{d\mathcal{N}}{d\mu}\right) \cdot dr$, corresponding to the term under the sign of integration in the second member of [2341]. Moreover, we must observe, that although the variable quantity r is independent of μ, yet the limiting values of r, namely $r_{,}$, r', are functions of μ, ϖ; so that when μ is increased by $d\mu$, the limits of the integral $r_{,}$, r', in the value of R [2340c],

[2340e] will be changed into $r_{,} + \left(\frac{dr_{,}}{d\mu}\right) \cdot d\mu$, $r' + \left(\frac{dr'}{d\mu}\right) \cdot d\mu$, respectively; by which means the quantity R is increased, at the limit r', by the element $\mathcal{N}' \cdot \left(\frac{dr'}{d\mu}\right) \cdot d\mu$, as in formulas

[2340f] [1430a, &c.]; and is decreased, at the limit $r_{,}$, by the element $\mathcal{N}_{,} \cdot \left(\frac{dr_{,}}{d\mu}\right) \cdot dt$; $\mathcal{N}'$, $\mathcal{N}_{,}$, being the values of $\mathcal{N}$ corresponding to these limits respectively. Hence the part of

[2340g] $\left(\frac{dR}{d\mu}\right)$ corresponding to these two elements is $\mathcal{N}' \cdot \left(\frac{dr'}{d\mu}\right) - \mathcal{N}_{,} \cdot \left(\frac{dr_{,}}{d\mu}\right)$, as in the second line of the second member of [2341].

therefore*

$$\int\!\!\int\!\!\int dr\,.\,d\mu\,.\,d\varpi\,.\left(\frac{du}{dt}\right).\left(\frac{dy'}{d\mu}\right).\sqrt{1-\mu^2}$$

$$=\int\!\!\int y'\,.\,d\mu\,.\,d\varpi\,.\left\{\begin{array}{c}\left(\frac{du_{\prime}}{dt}\right).\left(\frac{dr_{\prime}}{d\mu}\right).\sqrt{1-\mu^2}-\left(\frac{du'}{dt}\right).\left(\frac{dr'}{d\mu}\right).\sqrt{1-\mu^2}\\ -\int dr\,.\left\{\frac{d\,.\left\{\left(\frac{du}{dt}\right).\sqrt{1-\mu^2}\right\}}{d\mu}\right\}\end{array}\right\}. \qquad [2342]$$

Again, by integrating relatively to ϖ, we have,†

$$\int\!\!\int dr.d\varpi.\left(\frac{dv}{dt}\right).\left(\frac{dy'}{d\varpi}\right)=\int y'.\left(\frac{dv}{dt}\right).dr-\int y'.\left\{\frac{d.\left\{\int\left(\frac{dv}{dt}\right).dr\right\}}{d\varpi}\right\}.d\varpi+\text{const.} \qquad [2343]$$

At the two extremities of the integral, where $\varpi=0$ and $\varpi=2\varpi$, [2335′], the value of $y'.\left(\frac{dv}{dt}\right)$ is the same, since it refers to the same particle; therefore we have,‡

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* (1748) Multiplying [2340] by $d\varpi$, substituting in the second member of the product the value of the function [2341], then annexing the sign of integration relative to ϖ, we get [2342]. [2341a]

† (1749) Putting for brevity $P=dr\,.\left(\frac{dv}{dt}\right)$, in the first expression [2343b]; then integrating by parts, relatively to ϖ, we get successively, [2343a]

$$\int dr\,.\,d\varpi\,.\left(\frac{dv}{dt}\right).\left(\frac{dy'}{d\varpi}\right)=\int d\varpi\,.\,P\,.\left(\frac{dy'}{d\varpi}\right)=P\,.\,y'-\int y'\,.\left(\frac{dP}{d\varpi}\right).\,d\varpi. \qquad [2343b]$$

Integrating relatively to r, we get, by observing that y' is independent of r [2336],

$$\int\!\!\int dr\,.\,d\varpi\,.\left(\frac{dv}{dt}\right).\left(\frac{dy'}{d\varpi}\right)=\int P\,.\,y'-\int\!\!\int y'\,.\left(\frac{dP}{d\varpi}\right).\,d\varpi. \qquad [2343c]$$

This expression is the same as in [2343], because the last term may be put under the form $\int y'\,.\left\{\int\left(\frac{dP}{d\varpi}\right)\right\}.\,d\varpi$, in like manner as in [2337d].

‡ (1750) At the two limits of the integral relative to ϖ, namely $\varpi=0$, $\varpi=2\pi$, the expression $P\,.\,y'$ [2343b] has the same value, and it will therefore vanish from this definite integral. For the same reason, $\int P\,.\,y'$ will vanish from [2343c]; or it must be put equal to nothing, as in [2344], where it is connected with the constant term added to complete the integral. Substituting [2344] in [2343], we obtain [2345]. [2344a]

[2344] $$\int y' . \left(\frac{dv}{dt}\right) . dr + \text{constant} = 0 ;$$

hence

[2345] $$\int\int dr . d\varpi . \left(\frac{dv}{dt}\right) . \left(\frac{dy'}{d\varpi}\right) = -\int y' . \left(\frac{d . \left\{\int \left(\frac{dv}{dt}\right) . dr\right\}}{d\varpi}\right) . d\varpi .$$

Now we have*

[2346] $$\left(\frac{d . \left\{\int \left(\frac{dv}{dt}\right) . dr\right\}}{d\varpi}\right) = \int dr . \left(\frac{ddv}{d\varpi\, dt}\right) + \left(\frac{dv'}{dt}\right) . \left(\frac{dr'}{d\varpi}\right) - \left(\frac{dv_{\prime}}{dt}\right) . \left(\frac{dr_{\prime}}{d\varpi}\right) .$$

therefore†

[2347] $$\int\int dr . d\varpi . \left(\frac{dv}{dt}\right) . \left(\frac{dy'}{d\varpi}\right) = -\int y' . d\varpi . \left\{ \left(\frac{dv'}{dt}\right) . \left(\frac{dr'}{d\varpi}\right) - \left(\frac{dv_{\prime}}{dt}\right) . \left(\frac{dr_{\prime}}{d\varpi}\right) + \int dr . \left(\frac{ddv}{d\varpi\, dt}\right) \right\} ;$$

consequently‡

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* (1751) This expression is obtained in the same manner as [2341] is found in
[2346a] [2340a—g], putting $N = \left(\frac{dv}{dt}\right)$, $\int_{r_{\prime}}^{r'} N . dr = R$; and taking the partial differentials relatively to ϖ, instead of μ; by which means the part under the sign of integration,
[2346b] corresponding to [2340d], becomes $\int_{r_{\prime}}^{r'} \left(\frac{dN}{d\varpi}\right) . d\varpi$, as in the first term of the second member of [2346]. The terms corresponding to the limits [2340g] become

[2346c] $$N' . \left(\frac{dr'}{d\varpi}\right) - N_{\prime} . \left(\frac{dr_{\prime}}{d\varpi}\right),$$

as in the second and third terms of the second member of [2346]. We may also derive the integral [2346] from [2341], by changing $\left(\frac{du}{dt}\right) . \sqrt{(1-\mu^2)}$ into $\left(\frac{dv}{dt}\right)$, and $d\mu$ into $d\varpi$.

† (1752) Multiplying [2346] by $-y' . d\varpi$, and then prefixing the sign of integration
[2347a] relatively to ϖ, we get the value of $-\int y' . \left(\frac{d . \left\{\int \left(\frac{dv}{dt}\right) . dr\right\}}{d\varpi}\right) . d\varpi$. Substituting this in [2345], we obtain [2347].

‡ (1753) The value of the first term of the first member of [2348] is obtained by multiplying the expression [2342] by g. This produces the first, second and fifth terms of

$$\int\!\!\int\!\!\int dr\,.\,d\mu\,.\,d\varpi\,.\left\{g\,.\left(\frac{du}{dt}\right).\left(\frac{dy'}{d\mu}\right).\sqrt{1-\mu^2}-g\,.\left(\frac{dv}{dt}\right).\left(\frac{dy'}{d\varpi}\right)\right\}$$

$$=\int\!\!\int gy'.d\mu.d\varpi.\left\{\begin{array}{l}\left(\frac{du_{,}}{dt}\right).\left(\frac{dr_{,}}{d\mu}\right).\sqrt{1-\mu^2}-\left(\frac{du'}{dt}\right).\left(\frac{dr'}{d\mu}\right).\sqrt{1-\mu^2}+\left(\frac{dv'}{dt}\right).\left(\frac{dr'}{d\varpi}\right)\\ -\left(\frac{dv_{,}}{dt}\right).\left(\frac{dr_{,}}{d\varpi}\right)-\int dr.\left\{\left(\frac{d.\left\{\left(\frac{du}{dt}\right).\sqrt{1-\mu^2}\right\}}{d\mu}\right)-\left(\frac{ddv}{d\varpi\,dt}\right)\right\}\end{array}\right\}. \qquad [2348]$$

The second member of this equation is reduced, by means of the equation [2334], to the term $-\int\!\!\int gy'.\left(\frac{dy}{dt}\right).d\mu.d\varpi$; therefore the equation [2349]
[2335] becomes*

$$\int\!\!\int\!\!\int dr\,.\,d\mu\,.\,d\varpi\,.\left\{\left(\frac{du}{dt}\right).\left(\frac{ddu}{dt^2}\right)+\left(\frac{dv}{dt}\right).\left(\frac{ddv}{dt^2}\right).(1-\mu^2)\right\}$$

$$=-\int\!\!\int d\mu\,.\,d\varpi\,.\,g\,y'.\left(\frac{dy}{dt}\right). \qquad (10) \qquad [2350]$$

We shall here neglect the attraction of the heavenly bodies, and shall consider only the mutual action of the particles of the sea, and the terrestrial [2350']

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the second member of [2348]. The remaining terms are deduced from the second term of [2348a] the first member, by multiplying [2347] by $-g.d\mu$, and prefixing the sign of integration relatively to μ.

* (1754) If we take the partial differential of [2334], relatively to t; in which case r, μ, ϖ, which depend on the primitive state of the fluid, must be considered as constant; and u, u', $u_{,}$, v, v', $v_{,}$, variable; we shall get, by neglecting the terms of the same order as in [2334a],

$$\left(\frac{dy}{dt}\right)=\left(\frac{du'}{dt}\right).\left(\frac{dr'}{d\mu}\right).\sqrt{(1-\mu^2)}-\left(\frac{du_{,}}{dt}\right).\left(\frac{dr_{,}}{d\mu}\right).\sqrt{(1-\mu^2)}-\left(\frac{dv'}{dt}\right).\left(\frac{dr'}{d\varpi}\right)$$
$$+\left(\frac{dv_{,}}{dt}\right).\left(\frac{dr_{,}}{d\varpi}\right)+\int dr.\left\{\left(\frac{d.\left\{\left(\frac{du}{dt}\right).\sqrt{(1-\mu^2)}\right\}}{d\mu}\right)-\left(\frac{ddv}{d\varpi\,dt}\right)\right\}. \qquad [2349a]$$

Multiplying this by $-gy'.d\mu.d\varpi$, and prefixing the double sign of integration, the second member becomes the same as the second of [2348]; which is therefore equal to $-\int\!\!\int gy'.\left(\frac{dy}{dt}\right).d\mu.d\varpi$ [2349], representing the value of the first member of [2348]. Substituting this in the second member of [2335], we get [2350].

[2350″] spheroid. The value of V' then arises from the attraction of an aqueous stratum, whose internal radius is r', and external radius $r'+\alpha y$; r' being
[2350‴] nearly equal to unity. We have seen, in [2144, 2146], that if we suppose

[2351] $$y = Y^{(0)} + Y^{(1)} + Y^{(2)} + Y^{(3)} + Y^{(4)} + \&c.,$$

we shall have [2146a],

[2352] $$V' = \frac{3g}{\rho} \cdot \{ Y^{(0)} + \tfrac{1}{3} \cdot Y^{(1)} + \tfrac{1}{5} \cdot Y^{(2)} + \tfrac{1}{7} \cdot Y^{(3)} + \&c. \}.$$

Now we have generally [1476],

[2353] $$\int\int Y^{(i)} \cdot Y^{(i')} \cdot d\mu \cdot d\varpi = 0,$$

when i and i' are different numbers; therefore we shall have,*

$$\int\int y' \cdot \left(\frac{dy}{dt}\right) \cdot d\mu \cdot d\varpi$$

[2354] $$= \int\int d\mu \cdot d\varpi \cdot \left\{ \begin{array}{l} \left(1-\frac{3}{\rho}\right) . Y^{(0)} . \left(\frac{d Y^{(0)}}{dt}\right) + \left(1-\frac{3}{3\rho}\right) . Y^{(1)} . \left(\frac{d Y^{(1)}}{dt}\right) \\ + \left(1-\frac{3}{5\rho}\right) . Y^{(2)} . \left(\frac{d Y^{(2)}}{dt}\right) + \left(1-\frac{3}{7\rho}\right) . Y^{(3)} . \left(\frac{d Y^{(3)}}{dt}\right) + \&c. \end{array} \right\}.$$

[2354a] * (1755) Substituting $y = \Sigma\, Y^{(i)}$ [2351], $V' = g \cdot \Sigma \frac{3 \cdot Y^{(i)}}{(2i+1) \cdot \rho}$ [2352], in
[2354b] [2324], then dividing by g, we get, $y' = \Sigma \left(1 - \frac{3}{(2i+1) \cdot \rho}\right) \cdot Y^{(i)}$. Moreover, the differential of the preceding value of y, taken relatively to dt, gives the expression $\left(\frac{dy}{dt}\right) = \Sigma \left(\frac{d Y^{(i)}}{dt}\right)$. Substituting these in the first member of [2354], we obtain,

[2354c] $$\int\int y' \cdot \left(\frac{dy}{dt}\right) \cdot d\mu \cdot d\varpi = \int\int d\mu \cdot d\varpi \cdot \left\{ \Sigma \left(1 - \frac{3}{(2i+1) \cdot \rho}\right) \cdot Y^{(i)} \times \Sigma \left(\frac{d Y^{(i)}}{dt}\right) \right\}$$

[2354d] $$= \int\int d\mu . d\varpi . \Sigma \left\{ \left(1 - \frac{3}{(2i+1) \cdot \rho}\right) \cdot Y^{(i)} \cdot \left(\frac{d Y^{(i)}}{dt}\right) \right\};$$

the last expression being deduced from that which immediately precedes it, by observing that if we put $\left(\frac{d Y^{(i)}}{dt}\right) = Y'^{(i)}$, the value of $Y'^{(i)}$ will be subjected to the same differential equation as $Y^{(i)}$ [2145]; and then, by means of [2353], we may neglect all products of the form $Y^{(i)} \cdot Y'^{(i')}$, in which i differs from i'; and may retain, in the product
[2354e] $\Sigma\, Y^{(i)} \times \Sigma\, Y'^{(i)}$, only the term $\Sigma\, Y^{(i)} \cdot Y'^{(i)}$, as in [2354d], or in [2354].

By § 2 we have $Y^{(0)}=0$;* therefore the equation [2350] becomes, by integrating relatively to the time t,† [2354′]

$$\int\!\!\int\!\!\int dr\,.\,d\mu\,.\,d\varpi\,.\left\{\left(\frac{du}{dt}\right)^2+\left(\frac{dv}{dt}\right)^2.(1-\mu^2)\right\} \qquad (11)$$

$$=M-g.\int\!\!\int d\mu.d\varpi.\left\{\left(1-\frac{1}{\rho}\right).Y^{(1)2}+\left(1-\frac{3}{5\rho}\right).Y^{(2)2}+\left(1-\frac{3}{7\rho}\right).Y^{(3)2}+\&c.\right\}; \qquad [2355]$$

M being an arbitrary constant quantity. It is evident that the first member of this equation, [multiplied by α^2], expresses very nearly the living force of the fluid mass, noticing only the relative velocity of its particles upon the terrestrial spheroid.‡ [2355′]

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* (1756) The mass of the fluid being constant, we have, by formulas [2157′′′′, 2159], $4\pi\,.\,Y^{(0)}=0$, or $Y^{(0)}=0$. [2354f]

† (1757) Multiplying [2354] by $-g$, substituting the product in [2350], and then putting $Y^{(0)}=0$ [2354′], we get,

$$\int\!\!\int\!\!\int dr\,.\,d\mu\,.\,d\varpi\,.\left\{\left(\frac{du}{dt}\right).\left(\frac{ddu}{dt^2}\right)+\left(\frac{dv}{dt}\right).\left(\frac{ddv}{dt^2}\right).(1-\mu^2)\right\}$$

$$=-g\,.\int\!\!\int d\mu\,.\,d\varpi\,.\left\{\left(1-\frac{1}{\rho}\right).\,Y^{(1)}\,.\left(\frac{dY^{(1)}}{dt}\right)+\left(1-\frac{3}{5\rho}\right).\,Y^{(2)}\,.\left(\frac{dY^{(2)}}{dt}\right)+\&c.\right\}. \qquad [2355a]$$

Multiplying this by 2, and integrating relatively to t, we get [2355]; observing that

$$\int 2\,.\left(\frac{du}{dt}\right).\left(\frac{ddu}{dt^2}\right)=\left(\frac{du}{dt}\right)^2; \qquad \int 2\,.\left(\frac{dv}{dt}\right).\left(\frac{ddv}{dt^2}\right)=\left(\frac{dv}{dt}\right)^2;$$

$$\int 2\,.\,Y^{(i)}\,.\left(\frac{dY^{(i)}}{dt}\right)=Y^{(i)2}. \qquad [2355b]$$

‡ (1758) The polar co-ordinates θ, ϖ, r, of a particle of the fluid at the commencement of the motion, become, at the end of the time t, $\theta+\alpha u$, $nt+\varpi+\alpha v$, $r+\alpha s$, [2355c] [323ᵛ]. The increments of these quantities, in the time dt, are represented by their differentials, αdu, $ndt+\alpha dv$, αds, respectively; θ, ϖ, r, n, being given quantities for each particle, which do not vary with the time t. If we notice only the relative motion of the particles, we must neglect the quantity ndt, depending on the rotatory motion, which is common to the whole body; then the increments become αdu, αdv, αds. [2355d] From the two first of these terms we get, for the square of the relative velocity of the particle, $\alpha^2\,.\,W^2=\alpha^2\,.\left(\frac{du}{dt}\right)^2+\alpha^2\,.\left(\frac{dv}{dt}\right)^2.(1-\mu^2)$ [2221g], upon the surface of a [2355e] sphere, whose radius is nearly equal to unity [2350′′′]; and if we neglect the square of αs, [2355f]

[2355''] M is a constant quantity, depending on the initial state of the motion of the sea, and is independent of the time t; it is very small, when we suppose the primitive motion to be small.*

If ρ be greater than unity, the function

[2356] $$-g \cdot \iint d\mu \cdot d\varpi \cdot \left\{ \left(1-\frac{1}{\rho}\right) \cdot Y^{(1)2} + \left(1-\frac{3}{5\rho}\right) \cdot Y^{(2)2} + \&c. \right\}$$

will always be negative;† it will be less than M, since the first member of the preceding equation [2355] is necessarily positive.‡ Therefore $Y^{(1)}$,
[2356'] $Y^{(2)}$, &c., cannot contain increasing exponential quantities, or arcs of a

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on account of its smallness, this value of $\alpha^2 . W^2$ may be taken as the square of the relative velocity of the particle. Now the mass dM' of a particle of the fluid, whose
[2355g] density is unity, is represented very nearly by $dM' = dr \cdot d\mu \cdot d\varpi$, as is evident from [1431d, 2350''']. Hence the first member of [2355], multiplied by α^2, is represented by $\alpha^2 . \iiint dM' . W^2$; and if we take the integrals within the limits [2335'], it represents the
[2355h] whole relative living force of the fluid [144, &c.]. The factor α^2 was accidentally omitted in [2355'] by the author.

* (1759) M, being independent of the time, must remain the same for all values of t;
[2356a] therefore it must depend on the original forces, impressed on the fluid at the commencement of the motion, when $t=0$. If the primitive motion be very small, the value of y, and also those of $Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}$, &c. [2351], together with the living force of the fluid, will then be small; therefore the first member of [2355], which is proportional to the
[2356b] living force, will be small; consequently *all the terms* of the value of M, deduced from [2355], will also be small.

† (1760) If $\rho > 1$, all the terms $\left(1-\frac{1}{\rho}\right)$, $\left(1-\frac{3}{5\rho}\right)$, &c., will be *positive*. Moreover, $Y^{(1)2}$, $Y^{(2)2}$, &c., being squares, will be *positive;* consequently each of the terms $\iint\left(1-\frac{1}{\rho}\right) . Y^{(1)2} . d\mu . d\varpi$, $\iint\left(1-\frac{3}{5\rho}\right) . Y^{(2)2} . d\mu . d\varpi$, &c., taken
[2356c] within the prescribed limits, as well as the sum of all of them, will be *positive*. This sum, multiplied by the *negative* quantity $-g$, will produce a *negative* quantity, as in [2356].

‡ (1761) This first member is necessarily positive; because if we multiply it by the positive quantity α^2, the result will represent the sum of the products, formed by multiplying
[2356d] each particle of the fluid by the square of its relative velocity [2355e]; and all the *terms* of this *sum* are evidently *positive*.

circle.* Hence it follows, that *the equilibrium of the sea is stable, if its density be less than the mean density of the earth.* [2356″]

Equilibrium of the sea is stable if its density be less than that of the earth.

14. *If the density of the sea exceed the mean density of the earth, its figure ceases to be stable in many cases.* We have seen, in [2160—2163], that if [2356‴] the earth have no rotatory motion, the depth of the sea be constant, and ρ be [2356⁗] less than unity; we may impart such a motion to the fluid, that the equation of its surface will contain the time, under the form of increasing exponential quantities; which is contrary to the stability of the equilibrium. The same result is obtained, in cases where the earth is supposed to have a rotatory motion; and the spheroid covered by the sea is a *solid of revolution*, [2356ᵛ] whatever be the law of the depth of the sea.

We shall resume the equations [2355], in which we shall suppose that the values of u and v are nearly the same, for all the particles situated upon the [2357] same radius. *We shall suppose, at the origin of the motion, that*

$$y=h\mu, \qquad \left(\frac{dy}{dt}\right)=0, \qquad \left(\frac{du}{dt}\right)=0, \qquad \left(\frac{dv}{dt}\right)=0. \qquad [2358]$$

Example of simple oscillations, which may be unstable, if the density of the sea exceed the mean density of the earth.

The fluid being then left to the action of its gravity, and to the attraction of its particles, must assume a motion, composed of an infinite number of simple oscillations; so that we shall have, by their union,†

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* (1762) If $Y^{(1)}$, $Y^{(2)}$, &c., contain any terms of the form

$$\alpha\, t \,.\, \cos.\,(\alpha\, t+b), \qquad \alpha\, e^{at},$$

they may become very great, in the course of time, even when they are multiplied by a very small coefficient α; and the value of M, deduced from [2355], may become very great; [2357a] which is contrary to what is supposed in [2355″].

† (1764) The nucleus, or solid part of the earth, is supposed, in [2356ᵛ], to be a solid of revolution; also at the commencement of motion, $y=h\mu$ [2358], h being a constant quantity, independent of μ, ϖ. Now this expression of y being independent of ϖ, the form [2359a] of the surface of the fluid must also be a figure of revolution, when $t=0$; and it is evident, as there are no external forces acting upon the fluid, that it must continue to be a figure of revolution, at the expiration of the time t; consequently the general expression of y must be [2359b] independent of ϖ. In this case the formulas [2178, &c.] will become

$$y=a\,.\cos.\,(it+\varepsilon), \qquad u=b\,.\,\cos.\,(it+\varepsilon), \qquad v=c\,.\,\sin.\,(it+\varepsilon)\,; \quad \text{or rather}$$

$$y=\Sigma a\,.\cos.\,(it+\varepsilon), \qquad u=\Sigma b\,.\,\cos.\,(it+\varepsilon), \qquad v=\Sigma c\,.\,\sin.\,(it+\varepsilon)\,; \qquad [2359c]$$

[2359] $$y = a \cdot \cos.(it+\varepsilon) + a_1 \cdot \cos.(i_1 t+\varepsilon_1) + a_2 \cdot \cos.(i_2 t+\varepsilon_2) + \&c.\,;$$

[2359′] a, a_1, a_2, &c., being functions of μ. The constant quantities i, i_1, i_2, &c., ε, ε_1, ε_2, &c., must be so dependent upon each other, that at the origin of the motion, when $t=0$, we may have,*

[2360] $$a \cdot \cos.\varepsilon + a_1 \cdot \cos.\varepsilon_1 + a_2 \cdot \cos.\varepsilon_2 + \&c. = h\mu\,;$$

[2361] $$ai \cdot \sin.\varepsilon + a_1 i_1 \cdot \sin.\varepsilon_1 + a_2 i_2 \cdot \sin.\varepsilon_2 + \&c. = 0.$$

The corresponding values of $\left(\frac{du}{dt}\right)$ and of $\left(\frac{dv}{dt}\right)$ are, by § 3, of the form†

[2362] $$-ib \cdot \sin.(it+\varepsilon) - i_1 b_1 \cdot \sin.(i_1 t+\varepsilon_1) - \&c.\,;$$

[2363] $$ic \cdot \cos.(it+\varepsilon) + i_1 c_1 \cdot \cos.(i_1 t+\varepsilon_1) + \&c.\,;$$

and they must become nothing when $t=0$. This being premised, if we substitute the preceding values in the equation [2355], it will become, by integrating, relatively to r,

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[2359d] the coefficients a, b, c, being independent of ϖ [2359a]; and they are also nearly independent of r, by the hypothesis assumed in [2357]; hence a, b, c, may be considered as functions of μ only, as in [2359′].

* (1765) Putting $t=0$ in [2359], and then $y=h\mu$ [2358], we get [2360]. Taking the differential of y [2359], relatively to t, we get

[2360a] $$\left(\frac{dy}{dt}\right) = -ai \cdot \sin.(it+\varepsilon) - a_1 i_1 \cdot \sin.(i_1 t+\varepsilon_1) - \&c.\,;$$

Substituting in this $t=0$, and the value of $\left(\frac{dy}{dt}\right)=0$ [2358]; then changing the signs of all the terms, we get [2361].

† (1766) The values u, v, [2359c] give

[2361a] $$\left(\frac{du}{dt}\right) = -\Sigma ib \cdot \sin.(it+\varepsilon), \qquad \left(\frac{dv}{dt}\right) = \Sigma ic \cdot \cos.(it+\varepsilon),$$

as in [2362, 2363]. Putting $t=0$, and then $\left(\frac{du}{dt}\right)=0$, $\left(\frac{dv}{dt}\right)=0$, [2358], we get

[2361b] $$0 = \Sigma ib \cdot \sin.\varepsilon, \qquad 0 = \Sigma ic \cdot \cos.\varepsilon.$$

$$\int\int \tfrac{1}{2}\gamma.d\mu.d\varpi.\Sigma i^2.\{c^2.(1-\mu^2)+b^2\}+\int\int\tfrac{1}{2}\gamma.d\mu.d\varpi.\Sigma i^2.\{c^2.(1-\mu^2)-b^2\}.\cos.(2it+2\varepsilon)$$
$$+\int\int\gamma.d\mu.d\varpi.\Sigma i i_1.\{b b_1+c c_1.(1-\mu^2)\}.\cos.(it-i_1t+\varepsilon-\varepsilon_1)$$
$$+\int\int\gamma.d\mu.d\varpi.\Sigma i i_1.\{c c_1.(1-\mu^2)-b b_1\}.\cos.(it+i_1t+\varepsilon+\varepsilon_1)$$
$$=M-g.\int\int\tfrac{1}{2}d\mu.d\varpi.\Sigma\left\{\left(1-\frac{1}{\rho}\right).P^{(1)2}+\left(1-\frac{3}{5\rho}\right).P^{(2)2}+\&c.\right\} \quad (12)$$ [2364]
$$-g.\int\int\tfrac{1}{2}d\mu.d\varpi.\Sigma\left\{\left(1-\frac{1}{\rho}\right).P^{(1)2}+\left(1-\frac{3}{5\rho}\right).P^{(2)2}+\&c.\right\}.\cos.(2it+2\varepsilon)$$
$$-g.\int\int d\mu.d\varpi.\Sigma\left\{\left(1-\frac{1}{\rho}\right).P^{(1)}.P_1^{(1)}+\left(1-\frac{3}{5\rho}\right).P^{(2)}.P_1^{(2)}+\&c.\right\}.\left\{\begin{matrix}\cos.(it-i_1t+\varepsilon-\varepsilon_1)\\+\cos.(it+i_1t+\varepsilon+\varepsilon_1)\end{matrix}\right\}.*$$

The characteristic Σ of finite integrals includes all the values i, i_1, &c. The quantities $P^{(1)}$, $P^{(2)}$, &c., are the coefficients of $\cos.(it+\varepsilon)$ in $Y^{(1)}$, [2365]

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* (1767) Putting, as in [2365],

$$Y^{(i)}=P^{(i)}.\cos.(it+\varepsilon)+P_1^{(i)}.\cos.(i_1t+\varepsilon_1)+P_2^{(i)}.\cos.(i_2t+\varepsilon_2)+\&c.\,;$$ [2364a]

then taking the square of this expression, and substituting

$$\cos.(i_nt+\varepsilon_n)^2=\tfrac{1}{2}+\tfrac{1}{2}.\cos.2.(i_nt+\varepsilon_n), \qquad \text{also}$$ [2364b]
$$2.\cos.(i_nt+\varepsilon_n).\cos.(i_mt+\varepsilon_m)=\cos.(i_nt-i_mt+\varepsilon_n-\varepsilon_m)+\cos.(i_nt+i_mt+\varepsilon_n+\varepsilon_m),$$

[6, 20] Int.; we shall have, by using the characteristic Σ, as in [2365],

$$Y^{(i)2}=\tfrac{1}{2}.\Sigma P^{(i)2}+\tfrac{1}{2}.\Sigma P^{(i)2}.\cos.2.(it+\varepsilon)$$
$$+\Sigma P^{(i)}.P_1^{(i)}.\{\cos.(it-i_1t+\varepsilon-\varepsilon_1)+\cos.(it+i_1t+\varepsilon+\varepsilon_1)\}.$$ [2364c]

Deducing from this the values of $Y^{(1)2}$, $Y^{(2)2}$, $Y^{(3)2}$, &c., and substituting them in the second member of [2355], we obtain the second member of [2364]. The values of $\left(\frac{du}{dt}\right)$, $\left(\frac{dv}{dt}\right)$, [2361a], being squared and reduced as above, by means of the formulas [1, 17, 20] Int., become

$$\left(\frac{du}{dt}\right)^2=\tfrac{1}{2}.\Sigma i^2.b^2-\tfrac{1}{2}.\Sigma i^2.b^2.\cos.2.(it+\varepsilon)+\Sigma i i_1.b b_1.\cos.(it-i_1t+\varepsilon-\varepsilon_1)$$
$$-\Sigma i i_1.b b_1.\cos.(it+i_1t+\varepsilon+\varepsilon_1)\,;$$
[2364d]
$$\left(\frac{dv}{dt}\right)^2=\tfrac{1}{2}.\Sigma i^2.c^2+\tfrac{1}{2}.\Sigma i^2.c^2.\cos.2.(it+\varepsilon)+\Sigma i i_1.c c_1.\cos.(it-i_1t+\varepsilon-\varepsilon_1)$$
$$+\Sigma i i_1.c c_1.\cos.(it+i_1t+\varepsilon+\varepsilon_1).$$

$Y^{(2)}$, &c.; $P_1^{(1)}$, $P_1^{(2)}$, &c., are the coefficients of $\cos.(i_1 t+\varepsilon_1)$ in the same functions $Y^{(1)}$, $Y^{(2)}$, &c.; and so on for the rest. The comparison of the terms independent of t, in this equation, gives,

[2366] $$\int\!\!\int \tfrac{1}{2}\gamma\,.\,d\mu\,.\,d\varpi\,.\,\Sigma\, i^2\,.\,\{c^2\,.\,(1-\mu^2)+b^2\} = M - g\,.\int\!\!\int \tfrac{1}{2}d\mu.d\varpi.\Sigma\left\{\left(1-\frac{1}{\rho}\right).P^{(1)2}+\left(1-\frac{3}{3\rho}\right).P^{(2)2}+\&c.\right\}; \qquad (13)$$

then we have

[2367] $$\int\!\!\int \tfrac{1}{2}\gamma\,.\,d\mu\,.\,d\varpi\,.\,i^2\,.\,\{c^2\,.\,(1-\mu^2)-b^2\} = -g\,.\int\!\!\int \tfrac{1}{2}d\mu\,.\,d\varpi\,.\left\{\left(1-\frac{1}{\rho}\right).P^{(1)2}+\left(1-\frac{3}{5\rho}\right).P^{(2)2}+\&c.\right\}; \qquad (14)$$

for the fluid can have each of the simple oscillations, relative to the coefficients i, i_1, &c., separately; since by substituting, in the equations [2175—2176′],
[2367′] the preceding values of y, $\left(\frac{du}{dt}\right)$, $\left(\frac{dv}{dt}\right)$, the terms depending on $\sin.(it+\varepsilon)$ and $\cos.(it+\varepsilon)$ must separately destroy each other. Now by considering only the oscillations relative to the angle $(it+\varepsilon)$, and supposing all the terms relative to the other angles to be nothing, the
[2367″] equation [2364] will give the equation [2367], by comparing the coefficients of $\cos.2.(it+\varepsilon)$. Hence, by adding together all the equations similar to [2367], we shall obtain,*

Now we have supposed, in [2357], that u, v, have the same values, for all particles situated on the part of the radius $r'-r_{,}$; therefore u, v, are independent of r; and by integrating the first member of [2355] relatively to r, and substituting $\int_{r_{,}}^{r'} dr = r'-r_{,}=\gamma$, we shall get, for that first member, the expression

[2364e] $$\int\!\!\int \gamma\,.\,d\mu\,.\,d\varpi\,.\left\{\left(\frac{du}{dt}\right)^2+\left(\frac{dv}{dt}\right)^2.(1-\mu^2)\right\}.$$

This is easily reduced to the form of the first member of [2364], by using the values [2364d]. Putting the terms independent of t equal to each other, we get [2366]. The terms depending on $\cos.2.(it+\varepsilon)$, being put equal to each other, produce the equation [2367].

* (1768) In the same manner as [2367] was found, by putting the coefficients of $\cos.2.(it+\varepsilon)$ equal to each other, we shall get, by putting the coefficients of the terms $\cos.2.(i_1 t+\varepsilon_1)$, $\cos.2.(i_2 t+\varepsilon_2)$, &c., equal to each other,

$$\iint \tfrac{1}{2}\gamma \,.\, d\mu \,.\, d\varpi \,.\, \Sigma\, i^2 \{c^2 \,.\, (1-\mu^2) - b^2\}$$
$$= -g \,.\, \iint \tfrac{1}{2} d\mu \,.\, d\varpi \,.\, \Sigma \left\{ \left(1-\frac{1}{\rho}\right) . P^{(1)2} + \left(1-\frac{3}{5\rho}\right) . P^{(2)2} + \&c. \right\}. \qquad [2368]$$

If we subtract this from the equation [2366], we shall get,

$$\iint \gamma \,.\, d\mu \,.\, d\varpi \,.\, \Sigma\, i^2 \,.\, b^2 = M. \qquad [2369]$$

At the origin of the motion, we have by hypothesis [2358], $y = Y^{(1)} = h\mu$, $\left(\frac{du}{dt}\right) = 0$, $\left(\frac{dv}{dt}\right) = 0$; then the equation [2355] gives,*

$$0 = M - \tfrac{4}{3}\pi \,.\, g\, h^2 \,.\, \left(1-\frac{1}{\rho}\right). \qquad [2370]$$

therefore

$$\iint \tfrac{1}{2}\gamma \,.\, d\mu \,.\, d\varpi \,.\, i_1^2 \,.\, \{c_1^2 \,.\, (1-\mu^2) - b_1^2\}$$
$$= -g \,.\, \iint \tfrac{1}{2} d\mu \,.\, d\varpi \,.\, \left\{ \left(1-\frac{1}{\rho}\right) . P_1^{(1)2} + \left(1-\frac{3}{5\rho}\right) . P_1^{(2)2} + \&c. \right\};$$
$$\iint \tfrac{1}{2}\gamma \,.\, d\mu \,.\, d\varpi \,.\, i_2^2 \,.\, \{c_2^2 \,.\, (1-\mu^2) - b_2^2\} \qquad [2367a]$$
$$= -g \,.\, \iint \tfrac{1}{2} d\mu \,.\, d\varpi \,.\, \left\{ \left(1-\frac{1}{\rho}\right) . P_2^{(1)2} + \left(1-\frac{3}{5\rho}\right) . P_2^{(2)2} + \&c. \right\};$$

&c.

Adding the equations [2367, 2367a], we get [2368]; subtracting this from [2366], we obtain [2369].

* (1769) At the origin of the motion [2358], $y = h\mu = Y^{(1)}$ [1528b]. Comparing this with [2351], we get $Y^{(0)} = 0$, $Y^{(2)} = 0$, $Y^{(3)} = 0$, &c. Substituting these, and $\left(\frac{du}{dt}\right) = 0$, $\left(\frac{dv}{dt}\right) = 0$, [2358], in [2355], it will be reduced to this form,

$$0 = M - g \,.\, \iint d\mu \,.\, d\varpi \,.\, \left(1-\frac{1}{\rho}\right) . h^2\mu^2. \qquad [2370a]$$

Now by integrating as in [1483g, h, &c.], we get

$$\int_0^{2\pi} d\varpi = 2\pi; \qquad \int_{-1}^{1} \mu^2\, d\mu = \tfrac{2}{3}; \qquad [2370b]$$

substituting these in [2370a], we get the equation [2370], or

$$M = \tfrac{4}{3}\pi \,.\, g h^2 \,.\, \left(1-\frac{1}{\rho}\right); \qquad [2370c]$$

hence the expression [2369] becomes as in [2371].

[2371] $$\int\int\gamma\,.\,d\mu\,.\,d\varpi\,.\,\{i^2 b^2+i_1^2 b_1^2+\&c.\}=\tfrac{4}{3}\pi g\,.\left(1-\frac{1}{\rho}\right).\,h^2.$$

[2371'] *If* ρ *be less than unity, or in other words, if the density of the sea exceed the mean density of the earth, the second member of this equation will be negative; therefore the first member will also be negative. This is impossible, while*
[2371''] i^2, i_1^2, i_2^2, &c., *are positive;* therefore in this case, some one of these quantities will be negative; consequently the expression of* y *will contain exponential quantities, and the equilibrium will be unstable.*

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* (1770) If i^2, i_1^2, &c., b^2, b_1^2, &c., be *positive*, all the terms of the first member of [2371] will be positive, and the signs will not be changed by integration;
[2371a] consequently the first member will be *positive*. Now as this does not agree with the second member [2371'], one or more of the quantities i^2, i_1^2, &c., must be *negative*, and then cos. $(i\,t+\varepsilon)$, &c., may become exponential quantities, as is shown in note 179, page 187, Vol. I.

CHAPTER III.

ON THE MANNER OF NOTICING, IN THE THEORY OF THE EBB AND FLOW OF THE SEA, THE VARIOUS CIRCUMSTANCES IN EACH PORT, WHICH HAVE AN INFLUENCE ON THE TIDES.

15. We have supposed, in the first chapter, that the earth is a solid of revolution; and we have determined, in this hypothesis, the oscillations of the sea. *We shall now assume a more natural form, and shall suppose the* [2371‴] *earth to have any figure whatever.* In this case, on account of the resistance which the sea suffers, the oscillations of the first kind will be the same as we have found in § 6 [2221]. With respect to the inequalities of the [2371⁗] second and third kinds, the value of y will be formed of a series of sines and cosines of angles, proportional to the time t; and by naming one of these angles it, we shall have,*

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* (1771) The values of y, u, v, y', [2178—2178‴], corresponding to the case in which [2372a] γ is a function of μ [2177], may be reduced to the forms [2372—2375]. For if we put for brevity $\varpi' = \cos.(s\varpi+\varepsilon)$, $\varpi_{,} = \sin.(s\varpi+\varepsilon)$, we shall get, by means of [21, 23] Int., [2372b]

$$\cos.(it+s\varpi+\varepsilon) = \cos.(s\varpi+\varepsilon).\cos.it - \sin.(s\varpi+\varepsilon).\sin.it = \varpi'.\cos.it - \varpi_{,}.\sin.it;$$

$$\sin.(it+s\varpi+\varepsilon) = \cos.(s\varpi+\varepsilon).\sin.it + \sin.(s\varpi+\varepsilon).\cos.it = \varpi'.\sin.it + \varpi_{,}.\cos.it. \quad [2372c]$$

Substituting these in [2178—2178‴], and putting $F = a\varpi'$, $G = -a\varpi_{,}$, $F' = g\,a'\varpi'$, $G' = -g\,a'\varpi_{,}$, $H = b\,\varpi'$, $K = -b\,\varpi_{,}$, $P = c\,\varpi_{,}$, $Q = c\,\varpi'$; they become as in [2372d] [2372—2375] respectively. The quantities F, G, &c., [2372d], found in this manner, are evidently functions of μ, ϖ, [2178‴, 2372b], which are restricted to the form of γ assumed [2372e] in [2177]; but we may avoid this restriction, by supposing F, G, &c., to be any functions of μ, ϖ, of such forms as will satisfy the equations [2175—2176′]. By this means we [2372f] obtain the equations [2376—2381, &c.], as in the subsequent part of this article.

[2372] $$y = F.\cos.it + G.\sin.it;$$

[2373] $$gy - V' = F'.\cos.it + G'.\sin.it = gy';$$

[2374] $$u = H.\cos.it + K.\sin.it;$$

[2375] $$v = P.\cos.it + Q.\sin.it.$$

General expressions of y, y', u, v, V'.

These values, being substituted in the equations [2175—2176'], will give the six following equations, by comparing the coefficients of the sines and cosines of it.*

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* (1772) It is evident that the second members of the equations [2176, 2176'] are

[2376a] respectively $\left(\frac{d.(gy-V')}{d\mu}\right).\sqrt{(1-\mu^2)}$; $\frac{-\left(\frac{d.(gy-V')}{d\varpi}\right)}{1-\mu^2}$; which, by means of $gy - V'$ [2373], become respectively,

$$\left\{\left(\frac{dF'}{d\mu}\right).\cos.it + \left(\frac{dG'}{d\mu}\right).\sin.it\right\}.\sqrt{(1-\mu^2)}, \quad \text{and} \quad \frac{-\left(\frac{dF'}{d\varpi}\right).\cos.it - \left(\frac{dG'}{d\varpi}\right).\sin.it}{1-\mu^2}.$$

Taking the differentials of [2374, 2375], relatively to t, we get,

$$\left(\frac{du}{dt}\right) = -i.H.\sin.it + i.K.\cos.it; \qquad \left(\frac{ddu}{dt^2}\right) = -i^2.H.\cos.it - i^2.K.\sin it;$$

[2376b]

$$\left(\frac{dv}{dt}\right) = -i.P.\sin.it + i.Q.\cos.it; \qquad \left(\frac{ddv}{dt^2}\right) = -i^2.P.\cos.it - i^2.Q.\sin.it.$$

Substituting these in the first member of [2176], we obtain,

$$-i^2.H.\cos.it - i^2.K.\sin.it + 2ni.P.\mu.\sqrt{(1-\mu^2)}.\sin.it - 2ni.Q.\mu.\sqrt{(1-\mu^2)}.\cos.it$$

[2376c] $$= \left\{\left(\frac{dF'}{d\mu}\right).\cos.it + \left(\frac{dG'}{d\mu}\right).\sin.it\right\}.\sqrt{(1-\mu^2)}.$$

The coefficients of $-\cos.it$, being put equal in each member, give [2376]; and the coefficients of $-\sin.it$ give [2377]. The equation [2176'] becomes, by like substitutions,

$$-i^2.P.\cos.it - i^2.Q.\sin.it - \frac{2n\mu}{\sqrt{(1-\mu^2)}}.i.H.\sin.it + \frac{2n\mu}{\sqrt{(1-\mu^2)}}.i.K.\cos.it$$

[2376d] $$= \frac{-\left(\frac{dF'}{d\varpi}\right).\cos.it - \left(\frac{dG'}{d\varpi}\right).\sin.it}{1-\mu^2}.$$

Multiplying this by $-(1-\mu^2)$, and putting separately the coefficients of $\cos.it$, $\sin.it$, equal to each other in both members, we get [2378, 2379] respectively. Substituting the values of y, u, v, [2372, &c.], in [2175]; then from the coefficients of $\cos.it$, $\sin.it$, we obtain, in like manner, [2380, 2381].

$$i^2 . H + 2ni . \mu . \sqrt{1-\mu^2} . Q = -\left(\frac{dF'}{d\mu}\right) . \sqrt{1-\mu^2} ;$$ [2376]

$$i^2 . K - 2ni . \mu . \sqrt{1-\mu^2} . P = -\left(\frac{dG'}{d\mu}\right) . \sqrt{1-\mu^2} ;$$ [2377]

$$(1-\mu^2) . i^2 . P - 2ni . \mu . \sqrt{1-\mu^2} . K = \left(\frac{dF'}{d\varpi}\right) ;$$ [2378]

$$(1-\mu^2) . i^2 . Q + 2ni . \mu . \sqrt{1-\mu^2} . H = \left(\frac{dG'}{d\varpi}\right) ;$$ [2379]

$$F = \left(\frac{d . (\gamma H . \sqrt{1-\mu^2})}{d\mu}\right) - \left(\frac{d . (\gamma P)}{d\varpi}\right) ;$$ [2380]

$$G = \left(\frac{d . (\gamma K . \sqrt{1-\mu^2})}{d\mu}\right) - \left(\frac{d . (\gamma Q)}{d\varpi}\right) .$$ [2381]

Hence we easily obtain,*

Values of H, K, P, Q.

$$H = \frac{-\left(\frac{dF'}{d\mu}\right) . \sqrt{1-\mu^2} - \frac{\frac{2n}{i} . \mu . \left(\frac{dG'}{d\varpi}\right)}{\sqrt{1-\mu^2}}}{i^2 - 4n^2\mu^2} ;$$ [2382]

$$K = \frac{-\left(\frac{dG'}{d\mu}\right) . \sqrt{1-\mu^2} + \frac{\frac{2n}{i} . \mu . \left(\frac{dF'}{d\varpi}\right)}{\sqrt{1-\mu^2}}}{i^2 - 4n^2\mu^2} ;$$ [2383]

$$P = \frac{\left(\frac{dF'}{d\varpi}\right) - \frac{2n}{i} . \mu . (1-\mu^2) . \left(\frac{dG'}{d\mu}\right)}{(1-\mu^2) . (i^2 - 4n^2\mu^2)} ;$$ [2384]

$$Q = \frac{\left(\frac{dG'}{d\varpi}\right) + \frac{2n}{i} . \mu (1-\mu^2) . \left(\frac{dF'}{d\mu}\right)}{(1-\mu^2) . (i^2 - 4n^2\mu^2)} ;$$ [2385]

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* (1773) Multiplying [2379] by $-\frac{2n\mu}{i . \sqrt{(1-\mu^2)}}$, adding it to [2376], then dividing [2383a] the sum by $i^2 - 4n^2\mu^2$, we obtain [2382]. Multiplying [2378] by $\frac{2n\mu}{i . \sqrt{(1-\mu^2)}}$, adding it to [2377], then dividing by $i^2 - 4n^2\mu^2$, we obtain [2383]. Multiplying [2377] by $\frac{2n\mu . \sqrt{(1-\mu^2)}}{i}$, adding [2378], then dividing the sum by $(1-\mu^2) . (i^2 - 4n^2\mu^2)$, we

and then we find,*

[2386] $$F = -\frac{\gamma.(1-\mu^2).\left(\frac{d\,dF'}{d\mu^2}\right)}{i^2-4n^2\mu^2} - \frac{\gamma.\left(\frac{d\,dF'}{d\varpi^2}\right)}{(1-\mu^2).(i^2-4n^2\mu^2)}$$

Value of F.

$$-\left\{\frac{d.\left(\frac{\gamma.(1-\mu^2)}{i^2-4n^2\mu^2}\right)}{d\mu}\right\}.\left(\frac{dF'}{d\mu}\right) - \frac{\left(\frac{d\gamma}{d\varpi}\right).\left(\frac{dF'}{d\varpi}\right)}{(1-\mu^2).(i^2-4n^2\mu^2)}$$

$$+\frac{\frac{2n}{i}.\mu.\left(\frac{d\gamma}{d\varpi}\right)}{i^2-4n^2\mu^2}.\left(\frac{dG'}{d\mu}\right) - \frac{2n}{i}.\left\{\frac{d.\left(\frac{\gamma\mu}{i^2-4n^2\mu^2}\right)}{d\mu}\right\}.\left(\frac{dG'}{d\varpi}\right).$$

Changing in this equation F into G, F' into G', and the contrary;

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get [2384]. Lastly, multiplying [2376] by $-\frac{2n\mu.\surd(1-\mu^2)}{i}$, adding [2379], then dividing by $(1-\mu^2).(i^2-4n^2\mu^2)$, we obtain [2385].

* (1774) Substituting H, P, [2382, 2384], in [2380], we get [2386a]; from which we easily obtain [2386b], by merely separating the terms, and reducing the factors of some of the quantities.

[2386a] $$F = \left\{\frac{d.\left\{-\left(\frac{dF'}{d\mu}\right).\gamma.\frac{1-\mu^2}{i^2-4n^2\mu^2} - \frac{2n\gamma}{i}.\left(\frac{dG'}{d\varpi}\right).\frac{\mu}{i^2-4n^2\mu^2}\right\}}{d\mu}\right\}$$

$$-\left\{\frac{d.\left\{\frac{\gamma.\left(\frac{dF'}{d\varpi}\right) - \frac{2n\gamma}{i}.\mu.(1-\mu^2).\left(\frac{dG'}{d\mu}\right)}{(1-\mu^2).(i^2-4n^2\mu^2)}\right\}}{d\varpi}\right\}$$

[2386b] $$= -\left\{\frac{d.\left\{\left(\frac{\gamma.(1-\mu^2)}{i^2-4n^2\mu^2}\right).\left(\frac{dF'}{d\mu}\right)\right\}}{d\mu}\right\} - \frac{\left\{\frac{d.\left\{\gamma.\left(\frac{dF'}{d\varpi}\right)\right\}}{d\varpi}\right\}}{(1-\mu^2).(i^2-4n^2\mu^2)}$$

$$+\left\{\frac{\frac{2n}{i}.\mu.\left\{\frac{d.\left\{\gamma.\left(\frac{dG'}{d\mu}\right)\right\}}{d\varpi}\right\}}{(i^2-4n^2\mu^2)}\right\} - \frac{2n}{i}.\left\{\frac{d.\left\{\left(\frac{\gamma\mu}{i^2-4n^2\mu^2}\right).\left(\frac{dG'}{d\varpi}\right)\right\}}{d\mu}\right\}.$$

Developing [2386b], considering $\left(\frac{\gamma.(1-\mu^2)}{i^2-4n^2\mu^2}\right)$, $\left(\frac{\gamma\mu}{i^2-4n^2\mu^2}\right)$, as simple terms, we easily obtain [2386], by changing the order of the terms, and observing that the quantities depending on $\left(\frac{d\,dG'}{d\mu\,d\varpi}\right)$ mutually destroy each other.

also the signs of the terms multiplied by $\frac{2n}{i}$; we shall obtain another equation [2386′]
between G, G', F'. The combination of these two equations gives F and G.* Value of G.

We may, by means of these equations, determine generally the law of the depth of the sea, which would render the oscillations of the second kind [2386″] nothing, in all places upon the earth. For in these oscillations, i differs but very little from n [2223e]; therefore we may suppose $i=n$, in the [2386‴] preceding equations. Moreover, by hypothesis [2387a], we have $F=0$, $G=0$; hence the values of F' and G' will be, by § 7, [2387]

$$F'=M.\mu.\sqrt{1-\mu^2}.\cos.\varpi;\qquad G'=-M.\mu.\sqrt{1-\mu^2}.\sin.\varpi; \qquad [2388]$$

M being a function of t, independent of μ and ϖ.† Substituting these

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* (1775) Changing F' into G', G' into F', $\frac{2n}{i}$ into $-\frac{2n}{i}$, the value of [2386c] H [2382] becomes like K [2383], and P [2384] changes into Q [2385]; therefore F [2380] changes into G [2381]. Hence the value of G may be deduced from that of F [2386], by making the same changes [2386c] in [2386]. [2386d]

† (1776) If we suppose the oscillations of the second kind to vanish, as in [2386″], we shall have $y=0$ [2372] for all values of t; hence $0=F.\cos.it+G.\sin.it$. Putting successively $it=0$ and $it=100^\circ$, we get $F=0$, $G=0$. The part [2387a] of $\alpha V'$, on which the oscillations of the second kind depend, is given in [2222]. Substituting in this expression $\cos.\theta=\mu$, $\sin.\theta=\sqrt{(1-\mu^2)}$, and by [23] Int., $\cos.(nt+\varpi-\psi)=\cos.nt.\cos.(\varpi-\psi)-\sin.nt.\sin.(\varpi-\psi)$, it becomes [2387b]

$$\alpha V'=\frac{3L}{r^3}.\sin.v.\cos.v.\mu.\sqrt{(1-\mu^2)}.\{\cos.nt.\cos.(\varpi-\psi)-\sin.nt.\sin.(\varpi-\psi)\}. \qquad [2387c]$$

If we now put $\frac{3L}{r^3}.\sin.v.\cos.v=-\alpha M$, we shall get,

$$V'=-M.\mu.\sqrt{(1-\mu^2)}.\{\cos.nt.\cos.(\varpi-\psi)-\sin.nt.\sin.(\varpi-\psi)\}. \qquad [2387d]$$

Substituting this and $y=0$, also $i=n$, in [2373], we obtain

$$M.\mu.\sqrt{(1-\mu^2)}.\{\cos.(\varpi-\psi).\cos.nt-\sin.(\varpi-\psi).\sin.nt\}$$
$$=F'.\cos.nt+G'.\sin.nt. \qquad [2387e]$$

Comparing separately the coefficients of $\cos.nt$, $\sin.nt$, we find the values of F', G', [2388]; changing in [2388] ϖ into $\varpi-\psi=\varpi'$, which does not alter the result of the [2387f] calculation [2391].

values in the preceding equation between F, F', and G', we shall find,*

[2389] $$0 = \cos.\varpi \,.\left(\frac{d\gamma}{d\mu}\right).\sqrt{1-\mu^2} + \frac{\mu.\left(\frac{d\gamma}{d\varpi}\right).\sin.\varpi}{\sqrt{1-\mu^2}}.$$

The equation between G, G', and F', will give,

[2388a] * (1777) We might substitute immediately the values of F, F', G', in [2386]; but it is rather more simple to compute H, P, and then to substitute them in [2380]. The differentials of [2388, 2387f] give

[2388b] $$\left(\frac{dF'}{d\mu}\right) = M.\frac{1-2\mu^2}{\sqrt{(1-\mu^2)}}.\cos.\varpi'; \qquad \left(\frac{dF'}{d\varpi}\right) = -M.\mu.\sqrt{(1-\mu^2)}.\sin.\varpi';$$
$$\left(\frac{dG'}{d\mu}\right) = M.\frac{(-1+2\mu^2)}{\sqrt{(1-\mu^2)}}.\sin.\varpi'; \qquad \left(\frac{dG'}{d\varpi}\right) = -M.\mu.\sqrt{(1-\mu^2)}.\cos.\varpi'.$$

Substituting these, and $i=n$ [2386'''''], in H, P, [2382, 2384], we get,

[2388c] $$H = \frac{-M.(1-2\mu^2).\cos.\varpi'+2M.\mu^2.\cos.\varpi'}{n^2.(1-4\mu^2)} = -\frac{M.\cos.\varpi'.(1-4\mu^2)}{n^2.(1-4\mu^2)} = -\frac{M}{n^2}.\cos.\varpi';$$

$$P = \frac{-M.\mu.\sqrt{(1-\mu^2)}.\sin.\varpi'-2M.\mu.\sqrt{(1-\mu^2)}.(-1+2\mu^2).\sin.\varpi'}{(1-\mu^2).n^2.(1-4\mu^2)}$$

[2388d] $$= \frac{M.\mu.\sin.\varpi'}{n^2.\sqrt{(1-\mu^2)}}.\frac{\{-1-2.(-1+2\mu^2)\}}{(1-4\mu^2)} = \frac{M.\mu.\sin.\varpi'}{n^2.\sqrt{(1-\mu^2)}}.$$

Substituting these values of H, P, and $F=0$ [2387], in [2380], it becomes,

[2388e] $$F=0=-\frac{M}{n^2}.\left(\frac{d.[\gamma.\sqrt{(1-\mu^2)}.\cos.\varpi']}{d\mu}\right)-\frac{M}{n^2}.\left(\frac{d.\left(\frac{\gamma\mu}{\sqrt{(1-\mu^2)}}.\sin.\varpi'\right)}{d\varpi}\right).$$

Dividing this by $-\frac{M}{n^2}$, then developing the differentials, we get, by successive reductions, and neglecting the terms which mutually destroy each other,

$$0=\left(\frac{d.[\gamma.\sqrt{(1-\mu^2)}.\cos.\varpi']}{d\mu}\right)+\left(\frac{d.\left(\frac{\gamma\mu}{\sqrt{(1-\mu^2)}}.\sin.\varpi'\right)}{d\varpi}\right)$$

$$=\left(\frac{d\gamma}{d\mu}\right).\sqrt{(1-\mu^2)}.\cos.\varpi'-\frac{\gamma\mu}{\sqrt{(1-\mu^2)}}.\cos.\varpi'+\frac{\mu}{\sqrt{(1-\mu^2)}}.\left(\frac{d\gamma}{d\varpi}\right).\sin.\varpi'+\frac{\gamma\mu}{\sqrt{(1-\mu^2)}}.\cos.\varpi'$$

[2388f] $$=\left(\frac{d\gamma}{d\mu}\right).\sqrt{(1-\mu^2)}.\cos.\varpi'+\frac{\mu}{\sqrt{(1-\mu^2)}}.\left(\frac{d\gamma}{d\varpi}\right).\sin.\varpi' \qquad [2389]$$

$$0 = \sin.\varpi \,.\left(\frac{d\gamma}{d\mu}\right).\sqrt{1-\mu^2} - \frac{\mu\,.\left(\frac{d\gamma}{d\varpi}\right).\cos.\varpi}{\sqrt{1-\mu^2}}.^{*}$$ [2390]

Hence we deduce† $\left(\frac{d\gamma}{d\mu}\right)=0$, $\left(\frac{d\gamma}{d\varpi}\right)=0$; consequently γ is equal to [2391] a constant quantity; therefore *the oscillations of the second kind cannot vanish, for the whole earth, except in the single case where the depth of the sea is constant.* [2391′]

Oscillations of the second kind may vanish in all parts of the earth.

If the oscillations of the third kind vanish in every part of the earth, we shall have $F=0$, $G=0$, for these oscillations; also by § 9,‡ [2391″]

$$F' = N.(1-\mu^2).\cos.2\varpi\,; \qquad G' = -N.(1-\mu^2).\sin.2\varpi\,;$$ [2392]

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* (1778) If we change ϖ' into $\varpi'+100^{\circ}$, in [2388], F' will change into G', and G' into $-F'$. The same changes being made in H, P, [2382, 2384], they will become [2388g] respectively like K, Q, [2383, 2385]; and by this means F [2380] will change into G [2381]. Hence if we change ϖ' into $\varpi'+100^{\circ}$, in the expression of $F=0$ [2388e], we shall have the corresponding values of $G=0$ [2387]. From this we may deduce an expression similar to [2388f]; which can also be derived from [2388f], by writing $\varpi'+100^{\circ}$ for ϖ'. Hence we get the following expression, which is the same as [2390] with the signs changed; observing always to use ϖ' instead of ϖ, as in [2387f]:

$$0 = -\left(\frac{d\gamma}{d\mu}\right).\sqrt{(1-\mu^2)}\,.\sin.\varpi' + \frac{\mu}{\sqrt{(1-\mu^2)}}\,.\left(\frac{d\gamma}{d\varpi}\right).\cos.\varpi'.$$ [2388h]

† (1779) Multiplying [2388f] by $\frac{\cos.\varpi'}{\sqrt{(1-\mu^2)}}$, and [2388h] by $\frac{-\sin.\varpi'}{\sqrt{(1-\mu^2)}}$, adding the products, putting $\cos.^2\varpi'+\sin.^2\varpi'=1$, we obtain $\left(\frac{d\gamma}{d\mu}\right)=0$. Substituting this [2390a] in [2388f], dividing by $\frac{\mu}{\sqrt{(1-\mu^2)}}\,.\sin.\varpi'$, we get $\left(\frac{d\gamma}{d\varpi}\right)=0$. Now $\left(\frac{d\gamma}{d\mu}\right)=0$ shows that γ is independent of μ; and $\left(\frac{d\gamma}{d\varpi}\right)=0$, that γ is independent of ϖ; hence γ is independent of μ, ϖ, and must therefore be the same for all latitudes and longitudes. Consequently the oscillations of the second kind disappear, only when the depth of the sea [2390b] is the same for all places upon the earth [2128x].

‡ (1780) The oscillations of the third kind being by hypothesis equal to nothing, gives $y=0$, and $F=0$, $G=0$, as in [2387a]. The part of αV [2262] on which [2392a] the oscillations of the third kind depend, becomes, by putting $\sin.^2\theta=1-\mu^2$, $2nt=it$ [2393], $\varpi-\psi=\varpi'$ [2387f], $\frac{3L}{4r^3}.\cos.^2 v = -\alpha N$, [2392b]

[2393] N being a function of t, independent of μ and ϖ. We may also suppose that $i=2n$ very nearly [2262']. This being premised, the equation between F, F', G', [2386], will give,*

[2394] $$0=\frac{2\gamma\,.\cos.2\varpi}{1-\mu^2}+\mu\,.\left(\frac{d\gamma}{d\mu}\right).\cos.2\varpi+\frac{(1+\mu^2)\,.\left(\frac{d\gamma}{d\varpi}\right).\sin.2\varpi}{2\,.(1-\mu^2)};$$

$$\alpha V'=\frac{3L}{4r^3}.(1-\mu^2).\cos.^2\mathrm{v}.\cos.(2nt+2\varpi-2\psi)=\frac{3L}{4r^3}.(1-\mu^2).\cos.^2\mathrm{v}.\cos.(it+2\varpi')$$

[2392c] $$=-\alpha N.(1-\mu^2).\{\cos.it.\cos.2\varpi'-\sin.it.\sin.2\varpi'\}.$$

Dividing this by α, we get V'; which is to be substituted, with $y=0$ [2392a], in [2373], and we shall get,

[2392d] $$N.(1-\mu^2).\{\cos.it.\cos.2\varpi'-\sin.it.\sin.2\varpi'\}=F'.\cos.it+G'.\sin.it.$$

Putting the coefficients of $\cos.it$, $\sin.it$, separately equal to each other, in both members, we get F', G', [2392], ϖ being changed into ϖ', as in [2387f].

* (1781) If we put $2n=i$ [2393], and $F=0$ [2391''], in [2386]; then multiplying it by i^2, we get,

[2392e] $$i^2F=0=-\gamma.\left(\frac{ddF'}{d\mu^2}\right)-\frac{\gamma}{(1-\mu^2)^2}.\left(\frac{ddF'}{d\varpi^2}\right)-\left(\frac{d\gamma}{d\mu}\right).\left(\frac{dF'}{d\mu}\right)-\frac{1}{(1-\mu^2)^2}.\left(\frac{d\gamma}{d\varpi}\right).\left(\frac{dF'}{d\varpi}\right)$$
$$+\frac{\mu}{1-\mu^2}.\left(\frac{d\gamma}{d\varpi}\right).\left(\frac{dG'}{d\mu}\right)-\left(\frac{d.\left(\frac{\gamma\mu}{1-\mu^2}\right)}{d\mu}\right).\left(\frac{dG'}{d\varpi}\right).$$

Now the formulas [2392] give, by changing ϖ into ϖ' [2387f],

$$\left(\frac{dF'}{d\mu}\right)=-2N.\mu.\cos.2\varpi'; \qquad \left(\frac{ddF'}{d\mu^2}\right)=-2N.\cos.2\varpi';$$

[2392f] $$\left(\frac{dG'}{d\varpi}\right)=-2N.(1-\mu^2).\cos.2\varpi'; \qquad \left(\frac{dF'}{d\varpi}\right)=-2N.(1-\mu^2).\sin.2\varpi';$$

$$\left(\frac{ddF'}{d\varpi^2}\right)=-4N.(1-\mu^2).\cos.2\varpi'; \qquad \left(\frac{dG'}{d\mu}\right)=2N.\mu.\sin.2\varpi'.$$

Substituting these in [2392e], and connecting together the terms depending on $\sin.2\varpi'$, $\cos.2\varpi'$, we obtain

[2392g] $$i^2F=0=N.\sin.2\varpi'.\left\{\frac{2}{1-\mu^2}.\left(\frac{d\gamma}{d\varpi}\right)+\frac{2\mu^2}{1-\mu^2}.\left(\frac{d\gamma}{d\varpi}\right)\right\}$$
$$+N.\cos.2\varpi'.\left\{2\gamma+\frac{4\gamma}{1-\mu^2}+2\mu.\left(\frac{d\gamma}{d\mu}\right)+2.(1-\mu^2).\left(\frac{d.\left(\frac{\gamma\mu}{1-\mu^2}\right)}{d\mu}\right)\right\};$$

the equation between G, G', and F', will give,*

$$0=\frac{2\gamma.\sin.2\varpi}{1-\mu^2}+\mu.\left(\frac{d\gamma}{d\mu}\right).\sin.2\varpi-\frac{(1+\mu^2).\left(\frac{d\gamma}{d\varpi}\right).\cos.2\varpi}{2.(1-\mu^2)}. \quad [2395]$$

Hence we deduce† $\left(\frac{d\gamma}{d\varpi}\right)=0$, and [2396]

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in which the coefficient of $\mathcal{N}.\sin.2\varpi'$ is easily reduced to the form $\frac{2.(1+\mu^2)}{1-\mu^2}.\left(\frac{d\gamma}{d\varpi}\right)$. Substituting, in the coefficient of $\mathcal{N}.\cos.2\varpi'$, the development of

$$\left(\frac{d.\left(\frac{\gamma\mu}{1-\mu^2}\right)}{d\mu}\right)=\frac{\mu}{1-\mu^2}.\left(\frac{d\gamma}{d\mu}\right)+\frac{1+\mu^2}{(1-\mu^2)^2}.\gamma; \qquad \text{it becomes}$$

$$2\gamma+\frac{4\gamma}{1-\mu^2}+2\mu.\left(\frac{d\gamma}{d\mu}\right)+2.(1-\mu^2).\left\{\frac{\mu}{1-\mu^2}.\left(\frac{d\gamma}{d\mu}\right)+\frac{1+\mu^2}{(1-\mu^2)^2}.\gamma\right\}$$
$$=2\gamma.\left\{1+\frac{2}{1-\mu^2}+\frac{1+\mu^2}{1-\mu^2}\right\}+2.\left(\frac{d\gamma}{d\mu}\right).\{\mu+\mu\}=\frac{8\gamma}{1-\mu^2}+4\mu.\left(\frac{d\gamma}{d\mu}\right). \quad [2392g']$$

Hence [2392g] becomes

$$i^2F=0=\mathcal{N}.\sin.2\varpi'.\frac{2.(1+\mu^2)}{1-\mu^2}.\left(\frac{d\gamma}{d\varpi}\right)+\frac{8\gamma.\mathcal{N}}{1-\mu^2}.\cos.2\varpi'+4\mu.\mathcal{N}.\left(\frac{d\gamma}{d\mu}\right).\cos.2\varpi'. \quad [2392h]$$

Dividing this by $4\mathcal{N}$, we get $\frac{i^2F}{4\mathcal{N}}=0$, as in [2394], ϖ being changed into ϖ', as [2392i]
in [2387f].

* (1782) If we change 2ϖ into $2\varpi+100^\circ$, or $2\varpi'$ into $2\varpi'+100^\circ$, the expressions F', G', [2392] will become G', $-F'$, respectively; hence H [2382] changes into K [2383], and P [2384] into Q [2385]; therefore F [2380] will become [2394a]
G [2381]; so that by changing 2ϖ into $2\varpi+100^\circ$, F will become G. Therefore by making the same change in [2386], we shall get the value of G; and if G be put equal to nothing, we may deduce the value of $\frac{i^2G}{4\mathcal{N}}=0$, from that of $\frac{i^2F}{4\mathcal{N}}=0$ [2394, 2392i], [2394b]
by writing $2\varpi'+100^\circ$ for $2\varpi'$, or $2\varpi+100^\circ$ for 2ϖ; by this means, [2394] with its signs changed becomes as in [2395], ϖ' being written for ϖ, as in [2387f].

† (1783) Supposing, as in the preceding note, 2ϖ to be changed into $2\varpi'$, then multiplying [2394] by $\sin.2\varpi'$, also [2395] by $-\cos.2\varpi'$, and adding together the products, putting $\cos.^2\varpi'+\sin.^2\varpi'=1$, we get $\frac{(1+\mu^2).\left(\frac{d\gamma}{d\varpi}\right)}{1-\mu^2}=0$; hence $\left(\frac{d\gamma}{d\varpi}\right)=0$. [2396a]
Substituting this in [2395], and dividing by $\sin.2\varpi'$, we obtain [2397].

[2397] $$0=\frac{2\gamma}{1-\mu^2}+\mu\cdot\left(\frac{d\gamma}{d\mu}\right).$$

This last equation gives, by integration,*

[2398] $$\gamma=\frac{A\cdot(1-\mu^2)}{\mu^2};$$

[2398′] A being an arbitrary constant quantity. According to this value of γ, the depth of the sea would be infinite at the equator, where $\mu=0$ [2128xii]. This cannot be admitted; *therefore no admissible law of the depth of the sea can make the oscillations of the third kind vanish in all parts of the earth.*

[2398″] Oscillations of the third kind cannot vanish in all parts of the earth.

The rapidity of the angular rotatory motion of the earth, in comparison with the angular motions of the sun and moon, enables us to suppose $i=n$,
[2399] in the oscillations of the second kind; and $i=2n$, in the oscillations of the third kind. Moreover, it is evident from the preceding analysis, and from
[2299′] § 6, 8, and 9, that *if we suppose the quantities* L, v, ψ, r, *to refer to the sun, and mark them with one accent for the moon,* the elevation αy of a particle of the surface of the sea, above the surface of equilibrium, which would take place independently of the action of these two bodies, is nearly of the form,†

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* (1784) From $\left(\frac{d\gamma}{d\varpi}\right)=0$ [2396], we perceive that γ is independent of ϖ, and is therefore a function of μ only; hence [2397] may be written $0=\frac{2\gamma}{1-\mu^2}+\frac{\mu\,d\gamma}{d\mu}$.
[2398a] Multiplying this by $\frac{\mu\,d\mu}{1-\mu^2}$, we get $0=\frac{2\gamma\cdot\mu\,d\mu}{(1-\mu^2)^2}+\frac{\mu^2 d\gamma}{1-\mu^2}$. The integral of this is $A=\frac{\gamma\mu^2}{1-\mu^2}$, A being the arbitrary constant quantity. From this we easily find γ [2398].

† (1785) If we put $\cos.^2 v=1-\sin.^2 v$ in [2221], it becomes

$$\frac{-L\cdot(1-3\cdot\sin.^2 v)\cdot(1+3\cdot\cos.2\theta)}{8r^3 g\cdot\left(1-\frac{3}{5\rho}\right)};$$

which is the part of αy, depending on oscillations of the first kind, and arising from the force of the sun. Accenting the letters L, v, r, we obtain the part corresponding to the
[2400a] moon, $\frac{-L'\cdot(1-3\cdot\sin.^2 v')\cdot(1+3\cdot\cos.2\theta)}{8r'^3 g\cdot\left(1-\frac{3}{5\rho}\right)}$. The sum of these is the whole value of αy, depending on the oscillations of the first kind, being the same as in the first line of
[2400a′] the second member of [2400]. If we put $\cos.\theta=\mu$, $M=\frac{L}{r^3}\cdot\sin.v\cdot\cos.v$,

$$\alpha y = -\frac{(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)}.\left\{\frac{L}{r^3}.(1-3.\sin.^2 \mathrm{v})+\frac{L'}{r'^3}.(1-3.\sin.^2 \mathrm{v}')\right\}$$ [2400]

$$+A.\left\{\begin{array}{l}\frac{L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}.\cos.(nt+\varpi-\psi-\beta)\\ +\frac{L'}{r'^3}.\sin.\mathrm{v}'.\cos.\mathrm{v}'.\cos.(nt+\varpi-\psi'-\beta)\end{array}\right\}$$

Approximate value of the height of the tide.

$$+B.\left\{\begin{array}{l}\frac{L}{r^3}.\cos.^2\mathrm{v}.\cos.2.(nt+\varpi-\psi-\lambda)\\ +\frac{L'}{r'^3}.\cos.^2\mathrm{v}'.\cos.2.(nt+\varpi-\psi'-\lambda)\end{array}\right\};$$

$A=\dfrac{6lq.\mu.\sqrt{(1-\mu^2)}}{2lgq.\left(1-\frac{3}{5\rho}\right)-n^2}$, the expression of the elevation of the tide αy [2253], [2400b]
depending on the oscillations of the second kind upon an ellipsoid of revolution, becomes

$$\alpha y = A.\frac{L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}.\cos.(nt+\varpi-\psi) = A.M.\cos.(nt+\varpi-\psi),$$ [2400c]

A being, in this particular case, a function of μ, depending on the depth of the sea, &c. [2196]; and M a function, which is proportional to the disturbing force of the sun. In supposing $i=n$ [2399], the motion of the sun is neglected in comparison with the rotatory motion of the earth [2223e], so that M may be considered as very nearly constant, and to be [2400d] the same for all parts of the earth, independent of μ, ϖ. If this disturbing force M be increased in any ratio, the preceding value of αy [2400c] will be increased in the like proportion. The same method of reasoning may be applied to the general value of αy, deduced from [2372], which must be proportional to M; and if we put

$$\alpha F = A.M.\cos.(\varpi-\psi-\beta), \quad \alpha G = -A.M.\sin.(\varpi-\psi-\beta), \quad it = nt \quad [2399],$$ [2400e]

we shall get

$$\alpha y = A.M.\{\cos.nt.\cos.(\varpi-\psi-\beta)-\sin.nt.\sin.(\varpi-\psi-\beta)\}$$
$$= A.M.\cos.(nt+\varpi-\psi-\beta) = A.\frac{L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}.\cos.(nt+\varpi-\psi-\beta);$$ [2400f]

A, B, being like F, G, functions of μ, ϖ, [2372e], determined by means of the equations [2386, 2386′]; and in the present calculation, in which $i=n$, they may be considered as independent of L, r, v, ψ. Therefore we may obtain nearly the part of αy depending [2400g] on the lunar action, by changing in [2400f] L, r, v, ψ, into L', r', v′, ψ', respectively, without altering A, β; by which means it will become,

$$\alpha y = A.\frac{L'}{r'^3}.\sin.\mathrm{v}'.\cos.\mathrm{v}'.\cos.(nt+\varpi-\psi'-\beta).$$ [2400h]

[2400′] A, B, β, λ, being functions of μ, ϖ, depending on the law of the depth of the sea. This form is so general, that it includes a great variety of the phenomena of tides, which might take place in the different ports. It appears from § 7 and 9, that if the earth which forms the bottom of the [2400″] sea be a solid of revolution, the instant of the *maximum* or *minimum* of the oscillations of the second, or of the third kind, will be the same as that of the passage of the body which produces it over the meridian.* But we see, by the preceding formula, that in general, *when the depth of the sea is* [2400‴] *assumed in any arbitrary manner whatever, these times may be very different, and the hours of the tide may be very variable from one port to another.* This is conformable to observation.

Time of the tide varies with the depth of the sea.

In some ports the oscillations of the second kind may be insensible, [2400⁗] while in other ports we cannot perceive any oscillations of the third kind.† But according to the preceding formula, the *maxima* or *minima* of these oscillations will follow the times of passage of the respective bodies over the meridian, by the same interval [2400*k*, *l*], since the quantities β and λ are the same, for each of the two bodies. Now we shall hereafter see, that this result is contrary to observations ; so that the preceding formula, even when

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The sum of the two parts of αy [2400*f*, *h*], is nearly equal to the part of αy depending on oscillations of the second kind, as in the second and third lines of the second member of [2400].

In like manner we obtain the part of αy, depending on oscillations of the third kind, by [2400*i*] putting in αy [2372], $i = 2n$, and using the quantities B, λ, instead of A, β ; changing also $M = \frac{L}{r^3} \cdot \sin. v \cdot \cos. v$ [2222, 2400*a*′] into $M = \frac{L}{r^3} \cdot \cos.^2 v$ [2262], corresponding to the oscillations of the third kind, as in the fourth and fifth lines of [2400].

* (1786) This is evident from the expressions of αy [2400*c*, *i*, 2297′, 2294, &c.], using [2131*c*]. In the more general case treated of in [2400‴], the values of αy are as [2400*k*] in [2400*f*, &c., 2400] ; and it is evident that [2400*f*] is a maximum, when $nt + \varpi - \psi = \beta$, at which time the body is at the distance β from the upper meridian [2131*c*]. In like manner the part of αy [2400] depending on the angle $2 \cdot (nt + \varpi - \psi - \lambda)$ becomes [2400*l*] a maximum when $nt + \varpi - \psi = \lambda$, at the distance λ from the meridian.

† (1787) A, B, are functions of μ, ϖ, [2400′], and there may be places on the earth [2400*m*] where $A = 0$, then we shall have no oscillation of the *second* kind in [2400]. If $B = 0$, there will be no oscillation of the *third* kind in [2400].

taken in its greatest extent, will not fully satisfy all the observed phenomena. [2400v]
The irregularity of the depth of the ocean, the manner in which it is spread over the earth, the position and declivity of the shores, their connexions with the adjoining coasts, the currents, and the resistances which the waters suffer, cannot possibly be submitted to an accurate calculation, though these causes modify the oscillations of this great fluid mass. All we can do is to analyze the general phenomena which must result from the attractions of [2400vi] *the sun and moon, and to deduce from the observations such data as are indispensable, for completing in each port the theory of the ebb and flow of the tides. These data are the arbitrary quantities, depending on the extent of* [2400vii] *the surface of the sea, its depth, and the local circumstances of the port.*

Sources of irregularity in the tides.

16. *We shall now examine, in this more general point of view, the theory of the oscillations of the ocean, and shall see if it corresponds with the observations of the tides.* We may consider the sea as a system of an infinite [2400viii] number of particles, which act on each other by their pressure and mutual attraction. These particles are also acted upon by gravity and the attractive forces of the sun and moon. Without the action of these two last forces, [2400ix] the sea would long since have been in equilibrium; therefore the law of these forces must regulate the motions of the tide.

General method of computing the tides.

To obtain the attractive forces of the sun and moon, upon a particle of the surface of the sea, determined by the co-ordinates R, ϖ, θ; R being [2400x] the radius drawn from the centre of the earth to the particle; we shall put $\alpha V'$ for the following expression,*

$$\alpha V' = \frac{3L.R^2}{2r^3}.\{[\sin.v.\cos.\theta + \cos.v.\sin.\theta.\cos.(nt+\varpi-\psi)]^2 - \tfrac{1}{3}\}$$
$$+\frac{3L'.R^2}{2r'^3}.\{[\sin.v'.\cos.\theta + \cos.v'.\sin.\theta.\cos.(nt+\varpi-\psi')]^2 - \tfrac{1}{3}\}. \qquad [2401]$$

$\alpha V'$.

* (1789) In the terms [2132, 2133], from which [2192] is derived, the radius of the earth is put equal to unity; but all these expressions must be of the order $\frac{L}{r}$ [2132], and as L is [2401a] of the third degree in R, they must be of the second degree in R, r. This form is obtained by multiplying [2192] by R^2, considered as unity; and then it becomes as in the first line of the second member of [2401]. The second line is deduced from the first, by accenting the letters L, r, v; hence we obtain the part relative to the moon. The sum of these two

Then the sum of the lunar and solar forces, resolved in the direction of the
[2401′] radius of the earth, will be $\alpha.\left(\frac{dV'}{dR}\right)$, or $2\alpha V'$ [2401c], putting
Forces. $R=1$, after taking the differential. The sum of these forces, resolved in a direction perpendicular to that radius, and in the plane of the meridian of
[2401″] the particle, will be $\alpha.\left(\frac{dV'}{d\theta}\right)$. Lastly, the sum of the same forces, resolved in a direction perpendicular to the plane of the meridian, will be
[2401‴] $\frac{\alpha.\left(\frac{dV'}{d\varpi}\right)}{\sin.\theta}$. These expressions are very nearly accurate for the sun, on account of its great distance from the earth, which renders the terms
[2401⁗] multiplied by $\frac{L}{r^4}$ * almost insensible. They are less exact for the moon; but we have not been able to discover in the phenomena of the tides, anything
[2401ᵛ] depending on forces of the order $\frac{L'}{r'^4}$. Perhaps a greater number of more accurate observations than have hitherto been made, may render the effects of these forces sensible.

[2401ᵛⁱ] Tides produced by the sun, supposing it to move in the plane of the equator. *We shall in the first place consider only the action of the sun, and shall suppose it to move in the plane of the equator, with a uniform motion, and always at the same distance from the centre of the earth.* The three preceding forces will then become,†

[2401b] parts is the whole value of $\alpha V'$, from which we may deduce the attractions in the directions of the radius, meridian, and parallel of latitude, as in [1811l], from these we easily find the expressions [2401′—2401‴]; changing r into R, and neglecting the signs, which serve only to indicate the direction of the forces [1811m]. The differential of $\alpha V'$ [2401], relative to R, being divided by dR, and then putting $R=1$, gives, as in
[2401c] [2401′], $\alpha.\left(\frac{dV'}{dR}\right)=2\alpha V'$.

[2401d] * (1790) Terms of this order were neglected [2192a, &c.], in computing $\alpha V'$ [2192].

† (1791) If we notice only the part of $\alpha V'$ [2401] depending on the sun, putting $v=0$ [2401ᵛⁱ], and $\cos.^2(nt+\varpi-\psi)=\frac{1}{2}+\frac{1}{2}.\cos.2.(nt+\varpi-\psi)$, we shall get

$$\alpha V'=\frac{3L.R^2}{2r^3}.\{\sin.^2\theta.\cos.^2(nt+\varpi-\psi)-\tfrac{1}{3}\}$$

[2402a] $$=\frac{3L}{2r^3}.\{\tfrac{1}{2}.\sin.^2\theta-\tfrac{1}{3}+\tfrac{1}{2}.\sin.^2\theta.\cos.2.(nt+\varpi-\psi)\}.$$

Forces.

$$\frac{3L}{2r^3}\cdot\{\sin.^2\theta-\tfrac{2}{3}+\sin.^2\theta.\cos.(2nt+2\varpi-2\psi)\};$$ [Central force.] [2402]

$$\frac{3L}{2r^3}\cdot\sin.\theta.\cos.\theta.\{1+\cos.(2nt+2\varpi-2\psi)\};\qquad (A)$$ [Meridional force.] [2403]

$$-\frac{3L}{2r^3}\cdot\sin.\theta.\sin.(2nt+2\varpi-2\psi).$$ [Perpendicular to the meridian.] [2404]

By means of the constant forces

$$\frac{3L}{2r^3}\cdot\{\sin.^2\theta-\tfrac{2}{3}\}\ [2402],\ \text{and}\ \ \frac{3L}{2r^3}\cdot\sin.\theta.\cos.\theta\ [2403],$$ [2404']

the sea would finally settle in a state of equilibrium. Therefore these forces merely alter a little the permanent figure which the sea would assume by means of its rotatory motion. But *the three variable parts of the preceding forces must excite oscillations in the ocean, and we shall now proceed to determine the nature of these oscillations.*

These forces, after an interval of half a day, again have the same values. Now we may establish it as a general principle of dynamics, that *the state of a system of bodies, in which the primitive conditions of the motion have disappeared by the resistances it suffers, is periodical, like the forces which act on it.* Therefore the state of the ocean must become the same, at each interval of half a day; so that there must be an ebb and flow during that interval. [Principle of dynamics.] [2404'']

To show this, by a method of reasoning which may be applied in all similar cases, we shall suppose that at any time a, the height of the tide in any part was h, and that it again became of the same height, after the intervals $f^{(1)}$, $f^{(2)}$, $f^{(3)}\ldots\ldots f^{(i)}$, &c., counted from the time a. Then $a+f^{(i)}$ is the time when the height of the sea is h, after the number i of these intervals. If we suppose i to be very great, this time will not depend on the conditions of motion which existed at the time a considered as the instant of the origin of the motion. For all these conditions must have disappeared by the friction and resistance of every kind, which the sea [2404'''] [2404'''']

Hence the force in the direction of the radius R, represented by $2\alpha V'$ [2401'], is as in [2402]. Substituting the value of $\alpha V'$ [2402a] in the forces [2401'', 2401'''], they become respectively as in [2403, 2404].

suffers in these oscillations. Therefore *the motion of the sea finally becomes independent of these conditions, and depends only upon the forces which act on it; so that it is impossible to discover the primitive state of the sea from its present condition.*

We shall now suppose that, at the time denoted by $a +$ half a day, all
[2404v] the conditions of the motions of the sea are the same as at the time a. Then since the solar forces are the same, and vary in like manner in both cases; it is evident that in the second case, the successive intervals in which the height of the sea is h, commencing with the time $a +$ half a day, are,
[2404vi] as in the first case, $f^{(1)}$, $f^{(2)}$, $f^{(3)}$, &c.; so that at the time

$$a + f^{(i)} + \text{half a day},$$

[2404vii] the height of the tide is h. But as i is very great, and the actual situation of the sea is independent of everything that has any relation to the origin of its motion, it is evident that the time $a + f^{(i)} +$ half a day must correspond with one of the times when the height of the sea is h in the first case; therefore we must have

[2405] $$a + f^{(i)} + \text{half a day} = a + f^{(i+r)},$$

r being an integral number; therefore

[2406] $$f^{(i+r)} - f^{(i)} = \text{half a day}.$$

Hence it follows that the state of the sea must become the same after an interval of half a day.

It is probable, that if the whole ocean were disturbed, by any cause, the
[2406′] resistances it meets with would destroy the effect of this cause, in the interval of a few months; so that after that interval, the tides would resume their natural order. Hence we can judge of the little influence of the winds; for however violent they may be, they are only local, and affect merely the surface of the sea. Therefore by taking the mean of a great
[2406″] number of observations, continued during several years, the result will represent nearly the effect of the regular forces acting on the ocean.

Effect of the disturbing forces, soon destroyed by the resistance, &c.

We shall suppose a right line to be drawn, whose parts represent the time;
[2406‴] *and upon this, as the axis of the abscisses, a curve to be described, whose ordinates express the height of the tide;* the part of the curve corresponding to a portion of the absciss representing a half day will determine the entire

Curve representing the tides.

curve. For the whole will be formed by repeating this part *ad infinitum.* [2406‴]
Thus the interval between two consecutive high tides will be half a day, and the same takes place with two consecutive low tides.

17. *For the purpose of ascertaining this curve, we shall suppose that there is a second sun L, perfectly equal to the first, and moved in the same manner* [2406ᵛ]
in the plane of the equator, with this difference only, that it precedes the first in its orbit by the angle $n'T$; in which $n'=n-m$, and $m=\frac{d\psi}{dt}$. [2406ᵛⁱ]
We shall obtain the forces relative to this new sun, by changing ψ into $\psi+n'T$,* in the expression of the variable forces acting on the sea, which [2406ᵛⁱⁱ]
we have given in the last article [2402—2404]. These new forces being added to the others will produce the following,

$$\frac{3L}{2r^3}\cdot\sin.^2\theta\,.\{\cos.(2nt+2\varpi-2\psi)+\cos.(2nt+2\varpi-2\psi-2n'T)\}; \qquad [2407]$$

$$\frac{3L}{2r^3}\cdot\sin.\theta\,.\cos.\theta\,.\{\cos.(2nt+2\varpi-2\psi)+\cos.(2nt+2\varpi-2\psi-2n'T)\}; \qquad [2408]$$

$$-\frac{3L}{2r^3}\cdot\sin.\theta\,.\{\sin.(2nt+2\varpi-2\psi)+\sin.(2nt+2\varpi-2\psi-2n'T)\}. \qquad [2409]$$

If we put $L_{,}=2L\,.\cos.n'T$, these three forces will become† [2409′]

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* (1792) The variable part of the forces [2402—2404] are

$$\frac{3L}{2r^3}\cdot\sin.^2\theta\,.\cos.(2nt+2\varpi-2\psi), \qquad \frac{3L}{2r^3}\cdot\sin.\theta\,.\cos.\theta\,.\cos.(2nt+2\varpi-2\psi),$$
$$-\frac{3L}{2r^3}\cdot\sin.\theta\,.\sin.(2nt+2\varpi-2\psi)\,; \qquad [2407a]$$

the constant parts being neglected [2404′]. If another body L, whose right ascension is $\psi+n'T$ [2406ᵛⁱ], be added; the three corresponding forces will be found by changing ψ into $\psi+n'T$ in the preceding values. These new forces, being added to the preceding [2407a], will become as in [2407—2409].

† (1793) Putting for brevity $\Pi=2nt+2\varpi-2\psi$, we get, by using [20, 18] Int., [2408a]
and $L_{,}$ [2409′],

$$L_{,}\,.\cos.(\Pi-n'T)=2L\,.\cos.n'T\,.\cos.(\Pi-n'T)=L\,.\{\cos.\Pi+\cos.(\Pi-2n'T)\},$$
$$L_{,}\,.\sin.(\Pi-n'T)=2L\,.\cos.n'T\,.\sin.(\Pi-n'T)=L\,.\{\sin.\Pi+\sin.(\Pi-2n'T)\}. \qquad [2408b]$$

The first of these expressions being substituted in [2407, 2408], produces [2410, 2411] respectively; the second changes [2409] into [2412].

[2410] $$\frac{3L_{,}}{2r^3}\cdot\sin.^2\theta\cdot\cos.(2nt+2\varpi-2\psi-n'T);$$

[2411] $$\frac{3L_{,}}{2r^3}\cdot\sin.\theta\cdot\cos.\theta\cdot\cos.(2nt+2\varpi-2\psi-n'T);$$

[2412] $$-\frac{3L_{,}}{2r^3}\cdot\sin.\theta\cdot\sin.(2nt+2\varpi-2\psi-n'T).$$

These last forces produce a flow and ebb, similar to that which the body L
[2412'] would produce, if its mass were changed into $L_{,}$, and the time t were decreased by $\frac{1}{2}T$, in the forces of the preceding article.* Therefore if we
[2413] put y'' for the ordinate of the curve of the height of the tide, corresponding
[2414] to the absciss $t-\frac{1}{2}T$, we shall have $\frac{L_{,}\cdot y''}{L}$ for the height of the tide produced by the three preceding forces.

This height, by the nature of very small oscillations, is equal to the sum
[2414'] of the heights of the tide arising from the action of the two suns L. For we know that the whole motion of a system, acted upon by very small forces, is the sum of the partial motions, that each force would produce in the system if it acted separately. It is in this manner that small waves, made in a basin, are formed the one above the other, in the same manner as
[2414''] if they were separately excited upon the level surface of stagnant water. This evidently follows, from the circumstance, that very small oscillations are given, by *linear* differential equations, whose complete integrals are the

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* (1794) The body L is supposed to move uniformly in the plane of the equator [2401^{vi}], with the angular velocity m [2406^{vi}]; therefore if the time t be changed into
[2411a] $t-\frac{1}{2}T$, as in [2412'], the angle $2nt$ will become $2nt-nT$, and 2ψ will change into $2\psi-mT$ [2406^{vi}]. By this means the angle $2nt+2\varpi-2\psi$, which occurs in [2407a], becomes $2nt+2\varpi-2\psi-(n-m)\cdot T=2nt+2\varpi-2\psi-n'T=T'$, for brevity [2406^{vi}]; and the three forces [2407a] change into

[2411b] $$\frac{3L}{2r^3}\cdot\sin.^2\theta\cdot\cos.T'\,;\qquad \frac{3L}{2r^3}\cdot\sin.\theta\cdot\cos.\theta\cdot\cos.T'\,;\qquad -\frac{3L}{2r^3}\cdot\sin.\theta\cdot\sin.T'\,;$$

which, by writing $L_{,}$ for L, or multiplying by $\frac{L_{,}}{L}$, become like the forces [2410—2412], Now if we suppose that the forces [2411b] produce a tide denoted by y'', the forces
[2411c] [2410—2412] must cause a tide equal to $\frac{L_{,}}{L}\cdot y''$ [2414], because the ratio of these forces is as 1 to $\frac{L_{,}}{L}$.

sums of all the partial integrals which satisfy them.* Hence if y be the ordinate of the curve of the height of the tide corresponding to the time t, [2414‴]
and y' the ordinate corresponding to the time $t-T$,† $y+y'$ will be the sum of these heights; therefore we shall have,

Important equation.

$$y+y'=\frac{L_{,}.y''}{L}. \qquad (o)$$ [2415]

Now if we develop y' and y'', in series arranged according to the powers of T, we shall find, by neglecting the powers exceeding the second,‡

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* (1795) The equations [2320—2322] are linear relative to the required quantities s, u, v; and if we have several triplets of values s_1, u_1, v_1, s_2, u_2, v_2, s_3, u_3, v_3, &c., which satisfy these equations respectively; we may also put $s=a_1 s_1+a_2 s_2+a_3 s_3+$&c., [2412a]
$u=a_1 u_1+a_2 u_2+a^3 u_3+$&c., $v=a_1 v_1+a_2 v_2+a_3 v_3+$&c.; a_1, a_2, a_3, &c., being any arbitrary constant quantities; it being evident, from the linear form of the equations [2320—2322], that these last values of s, u, v, which include all these partial integrals, will [2412b]
also satisfy these equations, and may be considered as the general integrals.

† (1796) In the same manner as we proved, in [2411a, &c.], that by changing t into $t-\frac{1}{2}T$, the angle $2nt+2\varpi-2\psi$ becomes $2nt+2\varpi-2\psi-n'T$, we may prove that by changing t into $t-T$, the angle $2nt+2\varpi-2\psi$ becomes

$$2nt+2\varpi-2\psi-2n'T;$$ [2414a]

so that y being put for the tide corresponding to the time t, and y' for that corresponding to $t-T$, their sum $y+y'$ will represent the tide corresponding to the two suns, or in other words, to the forces [2407—2409]; and as these forces have been proved to be equivalent to those in [2410—2412] which produce the tide $\frac{L_{,}.y''}{L}$ [2414], it is evident, [2414b]
that these expressions of the tides must be equal to each other; hence we get [2415].

‡ (1797) Putting in the formula [617] $\alpha=-T$, $\varphi(t)=y$, $\varphi(t+\alpha)=y'$, we shall obtain [2416a]; and if we put $\alpha=-\frac{1}{2}T$, $\varphi(t)=y$, $\varphi(t+\alpha)=y''$, we shall get [2416b].

$$y'=y-T.\frac{dy}{dt}+\tfrac{1}{2}T^2.\frac{d^2y}{dt^2}-\tfrac{1}{6}T^3.\frac{d^3y}{dt^3}+\text{\&c.};$$ [2416a]

$$y''=y-\tfrac{1}{2}T.\frac{dy}{dt}+\tfrac{1}{8}T^2.\frac{d^2y}{dt^2}-\tfrac{1}{48}T^3.\frac{d^3y}{dt^3}+\text{\&c.}$$ [2416b]

If we neglect T^3 and higher powers, they become as in [2416]; and it will be shown in [2416g], that these terms are not wanted in finding [2418], which is the object of the present calculation.

[2416]
$$y' = y - T \cdot \frac{dy}{dt} + \tfrac{1}{2} T^2 \cdot \frac{ddy}{dt^2};$$
$$y'' = y - \tfrac{1}{2} T \cdot \frac{dy}{dt} + \tfrac{1}{8} T^2 \cdot \frac{ddy}{dt^2}.$$

Moreover, we have*

[2417]
$$L_{,} = 2L - L \cdot n'^2 T^2;$$

these values, being substituted in the equation [2415], give†

[2418]
$$\frac{d^2 y}{dt^2} = -4 n'^2 \cdot y.$$

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* (1798) Substituting in $L_{,}$ [2409′] the value of $\cos. n' T$ [44] Int., we get

$$L_{,} = 2L \cdot \{1 - \tfrac{1}{2} n'^2 T^2 + \tfrac{1}{24} n'^4 T^4 - \&c.\};$$

which, by neglecting T^4, becomes as in [2417]. The preceding value of $L_{,}$ gives

[2416c]
$$\frac{L_{,}}{L} = 2 - n'^2 T^2 + \tfrac{1}{12} n'^4 T^4 - \&c.$$

† (1799) Substituting in [2415] the values of y', y'', $\frac{L_{,}}{L}$, [2416a, b, c], we get [2416d].

[2416c′] Neglecting $2y - T \cdot \frac{dy}{dt}$, which occur in both members, and dividing by $\tfrac{1}{4} T^2$, we obtain [2416e], and by reduction [2416f].

$$2y - T \cdot \frac{dy}{dt} + \tfrac{1}{2} T^2 \cdot \frac{d^2y}{dt^2} - \tfrac{1}{6} T^3 \cdot \frac{d^3y}{dt^3} + \&c.$$

$$= \left\{ y - \tfrac{1}{2} T \cdot \frac{dy}{dt} + \tfrac{1}{8} T^2 \cdot \frac{d^2y}{dt^2} - \tfrac{1}{48} T^3 \cdot \frac{d^3y}{dt^3} + \&c. \right\} \cdot \{2 - n'^2 T^2 + \&c.\}$$

[2416d]
$$= 2y - T \cdot \frac{dy}{dt} + \tfrac{1}{4} T^2 \cdot \frac{d^2y}{dt^2} - \tfrac{1}{24} T^3 \cdot \frac{d^3y}{dt^3} - n'^2 T^2 \cdot y + \tfrac{1}{2} n'^2 T^3 \cdot \frac{dy}{dt} + \&c.;$$

[2416e]
$$2 \cdot \frac{d^2y}{dt^2} - \tfrac{4}{6} T \cdot \frac{d^3y}{dt^3} + \&c. = \frac{d^2y}{dt^2} - \tfrac{1}{6} T \cdot \frac{d^3y}{dt^3} - 4 n'^2 \cdot y + 2 n'^2 T \cdot \frac{dy}{dt} + \&c.;$$

[2416f]
$$\frac{d^2y}{dt^2} - \tfrac{1}{2} T \cdot \frac{d^3y}{dt^3} + \&c. = -4 n'^2 \cdot y + 2 n'^2 T \cdot \frac{dy}{dt} + \&c.$$

This must exist for all values of T, and when $T = 0$, it becomes as in [2418]. It may be observed, that the coefficients of T, T^2, &c. [2416f] produce an equation similar to [2418]. For if we subtract [2418] from [2416f], and then divide by $-\tfrac{1}{2} T$, retaining

[2416g] only the terms independent of T, we shall get $\frac{d^3y}{dt^3} = -4 n'^2 \cdot \frac{dy}{dt}$, which is the same as the differential of [2418], divided by dt.

Hence we deduce, by integration,*

$$y = \frac{B \cdot L}{r^3} \cdot \cos. (2nt + 2\varpi - 2\psi - 2\lambda);$$ [2419]

B and λ being two arbitrary constant quantities, of which the first depends on the magnitude of the total tide in the port, and the second on the hour [2419′] of the tide, or the time after the passage of the sun over the meridian.

This expression of y gives the law, by which the solar tide ebbs and flows. For if we suppose a vertical curve to be described, and represent by its [2419″] *circumference an interval of half a day, and by its diameter the total rise of the solar tide, or the difference of elevations between high and low water; we may take any arc of this circle, commencing at its lowest point, to express the intervals of time from low water; and then the versed sine of this arc will be the height of the tide corresponding to that time.*† [2420]

Law of the ebb and flow of the solar tide.

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* (1800) Putting $a = 2n'$ in [865′], it becomes as in [2418], and the second value of y [864a] is

$y = b \cdot \cos. (2n't + \varphi) = b \cdot \cos. (2nt - 2mt + \varphi) = b \cdot \cos. (2nt - 2\psi + \varphi)$ [2406^{vi}]. [2419a]

To render this symmetrical with the similar expressions in this book, we may change the arbitrary quantities b, φ, into $B \cdot \frac{L}{r^3}$, $2\varpi - 2\lambda$, respectively, and we shall get [2419].

A little reflection will make it evident that the quantities B, λ, must be nearly constant. For the height of the tide y must be nearly proportional to the disturbing forces [2402—2404], which vary as the ratio $\frac{L}{r^3}$; therefore B [2419] must be nearly constant, and depend on [2419b] the mean rise of the tide in the particular port under consideration. If the moon's angular velocity in her orbit were the same as that of the sun, the effect of these velocities on the values of B, λ, would be very nearly alike, so that they might be taken the same for both bodies. The effect of the greater velocity of the moon is taken into consideration in the [2419c] values of B, λ, [2439].

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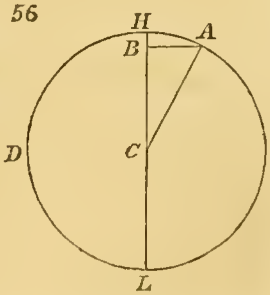

† (1801) Upon the vertical diameter $HL = 2B \cdot \frac{L}{r^3}$, and centre C, describe a circle $HALD$; make the angle $HCA = 2nt + 2\varpi - 2\psi - 2\lambda$, then draw the line AB, perpendicular to HL, and we shall have, from [2419],

$$y = CA \cdot \cos. HCA = CB.$$ [2420a]

The least value of y corresponds to the point L, and is there $= -CL$; the greatest value of y corresponds to the point H,

This law is accurately observed in the open sea; but local circumstances cause the tides to vary a little in our ports, where the tide is rather longer
[2421] in the *ebb* than in the *flow;* and at Brest, the difference is about ten minutes.

The greater the sea is, the more sensible will be the tides. In a fluid mass,
[2421′] the impressions which each particle receives are communicated to the whole mass. It is by this means that the action of the sun, which is insensible in an isolated particle, produces in the ocean such remarkable effects; and this
[2422] is the reason why the ebb and flow are insensible, in lakes and small seas, such as the Black sea, and the Caspian sea.

Local circumstances vary the times and height of the tides very much.

We have seen, in [2294—2296], *the great influence of the depth of the sea upon the height of the tides. The local circumstances of each port may vary*
[2422′] *this height considerably. The undulations of the sea, confined in a strait, may become very great. The reflection of the waters by the opposite shores may also increase the tides. It is from these causes that the tides, which are*
[2423] *generally very small in the Islands of the South Sea, become very considerable in our ports.*

If the ocean covered a spheroid of revolution, and suffered no resistance in its motion, the time of full sea depending on the sun would be the same as that of the passage of the sun over the superior or inferior meridian. But
[2424] this is not the actual state of the ocean; and *local circumstances cause the hour of the tides to vary considerably, even in very neighboring ports.* To form a correct idea of these variations, we shall suppose that there is a wide
[2424′] canal, communicating with the sea, and running far inland. It is evident that the undulations which take place at the *mouth* of the canal, are successively propagated through the whole of its length, so that the figure of the surface is formed by the undulations of waves, which are incessantly renewed, and pass through all their varieties of form in the course of half a

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and is there $= CH$. Hence it is evident, that while the point A moves on the circumference of the circle from L to A, the tide will ascend from its lowest point L to the point B, through the space $LB =$ versed sine of LA; so that if the circumference of the circle be divided into equal parts, representing hours; the time of low water being marked at L, and the time of high water at H; the versed sine of any arc LA, will
[2420b] represent the whole rise of the tide, from the time of low water to the time corresponding to the point A.

day. These waves produce, at each point of the canal, an ebb and flow, which follow the preceding law; but the hours of the tide are retarded, in proportion as the points are at a greater distance from the mouth of the [2425] canal. What we have said of a canal, will apply to a river, whose surface is elevated or depressed by similar waves, notwithstanding the contrary motion of its waters. These waves are perceived near the mouths of all [2425′] rivers; and *in great rivers they extend far into the interior country. In the strait of Pauxis, in the river Amazon, at eighty myriametres [nearly five hundred miles] distance from the sea, they are quite perceptible.*

Tides are perceived in rivers very far from the sea.

18. *We shall now consider the action of the moon, and shall suppose that this luminary moves uniformly in the plane of the equator.* It is evident, [2425″] that it must produce in the ocean an ebb and flow, similar to that which arises from the action of the sun. The two partial tides, produced by the actions of these bodies, combine together, without troubling each other, and their combination produces the compound tide observed in our ports. This being supposed, *if we accent the quantities L, r, ψ, λ, B, which refer to the sun, to obtain those corresponding to the moon*, the height of the tide, arising from the action of the moon, will be expressed by the function, [2426]

Tides produced by the moon.

Lunar tide.

$$\frac{B'.L'}{r'^3} \cdot \cos. (2nt + 2\varpi - 2\psi' - 2\lambda');$$ [2427]

B' and λ' being two other arbitrary constant quantities. The whole height [2427′] of the tide y, arising from the combined action of the sun and moon, will be

General expression of the tide produced by the sun and moon.

$$y = \frac{B.L}{r^3} \cdot \cos. (2nt + 2\varpi - 2\psi - 2\lambda) + \frac{B'.L'}{r'^3} \cdot \cos. (2nt + 2\varpi - 2\psi' - 2\lambda').$$ [2428]

We see, by this formula, that the height of the tides must vary considerably with the phases of the moon. This height is the greatest, when the two [2428′] cosines, in the expression of y, are equal to unity; and then it is the sum of the quantities $\frac{B.L}{r^3}$ and $\frac{B'.L'}{r'^3}$. It is the least, when the cosine affected [2428″] by the greatest coefficient is 1, and the other cosine is -1; then it is equal to the difference of the two preceding quantities. If $\frac{B.L}{r^3}$ exceed [2428‴] $\frac{B'.L'}{r'^3}$, the *maximum* and *minimum* of the *full sea* will take place when the

[2428''''] first cosine is 1, and therefore at the same hour of the day.* But if $\frac{B'.L'}{r'^3}$ exceed $\frac{B.L}{r^3}$, the least tide will take place when the first cosine [2428v] is -1, or at the instant of the solar low tide; the hour of this tide will therefore be at the interval of a quarter of a day from the hour of the greatest tide.† Hence we have a simple method of finding which of the two quantities $\frac{B.L}{r^3}$ and $\frac{B'.L'}{r'^3}$ is the greatest. All the observations [2429] made in our ports concur in showing that the second, or lunar force, exceeds the first.

The different values which may be given to the constant quantities B, B', λ, λ', give rise to several important remarks. If we have $\lambda = \lambda'$, the [2430] greatest tide will happen at the moment of full or new moon, and the least tide at the moment of the quadrature. For at the time of the greatest tide, [2430'] the two angles $2.(nt+\varpi-\psi-\lambda)$ and $2.(nt+\varpi-\psi'-\lambda')$ are equal to zero, or to a multiple of the circumference; their difference also is [2430''] nothing, or a multiple of the circumference. This, in the case of $\lambda = \lambda'$, requires that the moon should be in conjunction or in opposition with the sun.‡ According to the observations made in our ports, the greatest tide [2430'''] follows the new and full moon about a day and a half [2544]; so that

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* (1802) When $\cos.(2nt+2\varpi-2\psi-2\lambda)=1$, we shall have

[2428a] $2nt+2\varpi-2\psi-2\lambda=0^{\circ}$ or 400°; hence $nt+\varpi-\psi=\lambda$ or $200^{\circ}+\lambda$;

but by [2131c] $nt+\varpi-\psi$ represents the distance of the sun from the superior meridian, or the hour of the day; therefore in this case it is equal to the constant quantity λ or $200^{\circ}+\lambda$.

† (1803) When $\cos.(2nt+2\varpi-2\psi-2\lambda)=-1$, we have

$$2nt+2\varpi-2\psi-2\lambda=200^{\circ} \quad \text{or} \quad 600^{\circ};$$

[2428b] hence we find $nt+\varpi-\psi$ equal to $100^{\circ}+\lambda$ or $300^{\circ}+\lambda$, which is 100° greater than the value λ or $200^{\circ}+\lambda$, found in the last note, and 100° corresponds to one quarter of a day.

‡ (1804) The difference of the angles $2.(nt+\varpi-\psi-\lambda)$, $2.(nt+\varpi-\psi'-\lambda')$, is equal to $2.(\psi'-\psi)+2.(\lambda'-\lambda)$, which must be equal to 0° or a multiple of [2131a] 400°, at the time of the greatest tide [2430'']. When $\lambda'=\lambda$, this difference becomes $2.(\psi'-\psi)$, which must be equal to 0° or a multiple of 400°; therefore $\psi'-\psi$

$\psi' - \psi$ is positive, and equal to the synodical motion of the moon during a day and a half; therefore $\lambda' - \lambda$ is negative, consequently λ exceeds λ'. [2431]

We may form a correct idea of this phenomenon, by supposing, as above, that a wide canal, communicating with the sea, extends very far into the interior country, in the direction of the meridian of the mouth of the canal. [2431'] If we suppose that the full sea takes place at its *mouth*, at the moment of the passage of the sun over the meridian, and that it requires twenty-one decimal hours, or $2\frac{1}{10}$ days, to arrive at the *head* of the canal; it is evident that at this last place, the solar tide will follow the passage of the sun over the meridian by one hour; but as two lunar days are equal to 2,070 solar [2431''] days, the lunar tide will follow the passage of the moon over the meridian by $30'$ only. In this case, the angle λ is one hour, turned into degrees, estimating the whole circumference as one day, which gives $\lambda = 40°$. The angle λ' is the interval of $30'$, turned into degrees in the same manner, making $\lambda' = 12°$. If the interior extremity of the canal be to the eastward of [2431'''] its mouth, by a given number of degrees, we must add it to the preceding values of λ and λ', to obtain their true values. In the hypothesis now under consideration, B and B' are equal, and the angle $\lambda - \lambda'$ is equal to the [2431''''] synodical motion of the moon in twenty-one hours; the difference of the values of λ and λ' produces only a retardation of twenty-one hours in the phenomena of the tides which happen at its mouth, where $\lambda = \lambda'$; and it is evident that this result takes place equally for any system of bodies moving uniformly in the plane of the equator. [2432]

We shall now suppose that the canal, of which we have just spoken, has two mouths. If we put* $\frac{d\psi}{dt} = m'$, and $m = n - m'$, the solar tide [2432']

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is $0°$ or $200°$; consequently the bodies must then be in conjunction or in opposition. Again, since $2 . (\psi' - \psi) + 2 . (\lambda' - \lambda) = 0$ [2431a], and $\psi' - \psi$ is by observation positive, and equal to the synodical motion in a day and a half, $\lambda' - \lambda$ must be negative and [2431b] equal to the same quantity; consequently $\lambda > \lambda'$, as in [2431].

* (1804a) In the original work, the equation $m = n - m'$ was omitted, and $\frac{d\psi}{dt}$ was put equal to m. I have corrected this value of m, to make it correspond with the formulas [2431c] [2432''—2435]; it being evident that the tide at the time t, in the head of the canal, corresponds in [2432''] to the tide at the time $t - T$, at the mouth of the canal; so

which takes place at the first mouth by the action of L, will produce, at the head of the canal, a tide whose height will be expressed by

[2432″] $$\frac{B.L}{r^3}.\cos.(2nt+2\varpi-2\psi-2mT);$$

T being the interval of time required to transmit the first tide from the mouth to the head of the canal. Likewise the tide which takes place at the second mouth, will produce at the extremity of the canal, a tide whose height will be represented by

[2432‴] $$\frac{C.L}{r^3}.\cos.(2nt+2\varpi-2\psi-2mT');$$

[2432⁗] T' being the interval of time required to transmit this tide from the mouth to the extremity of the canal, and the coefficient C depending on the height of the tide at this second mouth. The whole height of the tide y, produced by the action of the body L, at the head of the canal, will therefore be

[2433] $$y=\frac{B.L}{r^3}.\cos.(2nt+2\varpi-2\psi-2mT)+\frac{C.L}{r^3}.\cos.(2nt+2\varpi-2\psi-2mT').$$

If we put*

[2434] $$B_{,}=\sqrt{B^2+C^2+2BC.\cos.2m.(T'-T)},$$

[2434′] $$\sin.2\lambda_{,}=\frac{B.\sin.2mT+C\sin.2mT'}{B_{,}},$$

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that t must be changed into $t-T$, and ψ into $\psi-m'T$; consequently $2nt+2\varpi-2\psi$ becomes

[2432a] $$2n.(t-T)+2\varpi-2.(\psi-m'T)=2nt+2\varpi-2\psi-2.(n-m').T$$
$$=2nt+2\varpi-2\psi-2mT, \quad \text{as in [2432″].}$$

* (1805) We shall put for brevity $\mathcal{N}=2nt+2\varpi-2\psi$; also

[2433a] $$B_{,}.\cos.2\lambda_{,}=B.\cos.2mT+C.\cos.2mT', \quad B_{,}.\sin.2\lambda_{,}=B.\sin.2mT+C.\sin.2mT'.$$

The sum of the squares of $B_{,}.\cos.2\lambda_{,}$, $B_{,}.\sin.2\lambda_{,}$, [2433a], gives by reduction

$$B_{,}^2=B^2.(\cos.^2 2mT+\sin.^2 2mT)+C^2.(\cos.^2 2mT'+\sin.^2 2mT')$$
$$+2BC.(\cos.2mT.\cos.2mT'+\sin.2mT.\sin.2mT')$$

[2433b] $$=B^2+C^2+2BC.\cos.2m.(T'-T), \quad \text{as in [2434].}$$

Dividing the value of $B_{,}.\sin.2\lambda_{,}$ [2433a] by $B_{,}$, we get [2434′].

we shall have

$$y = \frac{B_{,} . L}{r^3} . \cos. (2nt + 2\varpi - 2\psi - 2\lambda_{,}). \qquad [2435]$$

Hence we see that $B_{,}$ and $\lambda_{,}$ *depend on the value of* m, *or on the rapidity of the motion of the body in its orbit;* and it is evident that if this canal had three, or a greater number of mouths, the values of $B_{,}$ and $\lambda_{,}$ would be more complicated. Therefore the ratio of the coefficients $\frac{B.L}{r^3}$ and $\frac{B'.L'}{r'^3}$, given by observation of the tides, is not exactly that of the forces $\frac{L}{r^3}$ and $\frac{L'}{r'^3}$; it may be very different in different ports; and it is only by noticing the difference of the values of B and B', that we can determine the ratio of the forces of the sun and moon, by the phenomena of the tides. [2435'] [2435''] [2436] [2436']

If, in the case we have just considered, where the canal has but two mouths, C is equal to $-B$; that is, if it is high water at the first mouth, at the moment it is low water at the second mouth; and moreover,* [2436'']

Then the expression of y [2433], developed and reduced by means of [24] Int., becomes successively, by using the values [2433a],

$$y = \frac{B.L}{r^3} . \cos. (N - 2mT) + \frac{C.L}{r^3} . \cos. (N - 2mT')$$

$$= \frac{L}{r^3} . \{B.(\cos. N . \cos. 2mT + \sin. N . \sin. 2mT) + C.(\cos. N . \cos. 2mT' + \sin. N . \sin. 2mT')\}$$

$$= \frac{L}{r^3} . \{(B . \cos. 2mT + C . \cos. 2mT') . \cos. N + (B . \sin. 2mT + C . \sin. 2mT') . \sin. N\}$$

$$= \frac{B_{,} . L}{r^3} . \{\cos. 2\lambda_{,} . \cos. N + \sin. 2\lambda_{,} . \sin. N\} = \frac{B_{,} . L}{r^3} . \cos. (N - 2\lambda_{,}) \quad [2435]. \qquad [2433c]$$

We may observe, that the preceding calculation would appear to be more conformable to the expression [2419], if we had used in [2432''] $2\lambda + 2mT$ for $2mT$, and in [2432''', &c.] $2\lambda + 2mT'$ for $2mT'$; but the result of the calculation in [2435'], that $B_{,}$, $\lambda_{,}$, depend on m, would not be affected by these changes. [2433d]

* (1806) When $T' = T$, we have $T' - T = 0$, and $\cos. 2m . (T' - T) = 1$; hence we get $B_{,} = \sqrt{(B^2 + C^2 + 2BC)} = B + C$ [2434]; and if $C = -B$, it becomes $B_{,} = 0$, whence $y = 0$ [2435]. If we change the symbols as in [2433d], the same result will be obtained. [2436a]

[2437] $T = T'$, or in other words, if the two tides require the same time to ascend to the *head* of the canal; we shall have $B_{,} = 0$, and there will be no ebb or flow at that place, depending on the oscillations whose period is half a day. This singular case has been observed at Batsha, a port of the kingdom of Tonquin, and in some other places.

The great variety of local circumstances which have an influence on the tides in each port, must produce a considerable effect in these phenomena; and it is
[2437′] *probable that there is no possible case which does not actually take place on the earth.* But since the constant quantities B and λ would be the same for the sun and moon, if the motions of these bodies were equal [2419b, &c.], it
[2438] is natural to suppose that their differences are proportional to the differences of these motions; therefore we shall adopt this hypothesis, and we shall find that it satisfies the observations with remarkable exactness. Hence we shall put

[2439]
$$\lambda = O - m\,T;$$
$$B = P\,.\,(1 - 2\,m\,Q);$$

[2439′] *O, T, P and Q being the same for the sun and moon.* We shall hereafter give the method of determining these constant quantities, in each port, by observations.

19. *We shall now investigate the effects produced by the inequalities in the*
[2439″] *motions, and in the distances, of the sun and moon; always supposing them to move in the plane of the equator.* The partial forces

[2440]
$$\frac{3\,L}{2\,r^3}\,.\,\{\sin.^2\theta - \tfrac{2}{3}\}, \qquad \text{and} \qquad \frac{3\,L}{2\,r^3}\,.\,\sin.\theta\,.\,\cos.\theta,$$

found in [2404′], will no longer be constant, though they will vary with extreme slowness, the period of this variation being one year. If the duration
[2440′] of this period were infinite, the forces would have no other effect than to change the permanent figure of the sea, which would quickly attain its state of equilibrium. But though this duration is finite, we have seen in [2221″—2221^{v}], that in consequence of the resistance which the sea suffers,
[2440″] we may consider it as being at every instant in a state of equilibrium, by the action of these forces; and we may determine, in this hypothesis, the corresponding height of the tide. Moreover we have seen, that whatever

be the depth of the sea, the height of the tide, arising from the action of these forces, is*

$$-\frac{(1+3\,.\cos.2\,\theta)}{8\,g\,.\left(1-\frac{3}{5\,\rho}\right)}\cdot\frac{L}{r^3}. \qquad [2441]$$

If in the parts of the solar forces [2402—2404] which are multiplied by the sine or cosine of the angle $2\,n\,t+2\,\varpi-2\,\psi$, we substitute the values of [2441′]
r and ψ, we may develop each of these parts in a series of sines and cosines of angles, of the form $(2\,n\,t-2\,q\,t+2\,\varepsilon)$, so that we shall have,† [2441″]

Forces.

$$\frac{3\,L}{2\,r^3}\,.\sin.^2\theta\,.\cos.2\,.(n\,t+\varpi-\psi)=\sin.^2\theta\,.\,\Sigma\,k\,.\cos.2\,.(n\,t-q\,t+\varepsilon)\,; \qquad [2442]$$

$$\frac{3\,L}{2\,r^3}\,.\sin.\theta.\cos.\theta.\cos.2.(nt+\varpi-\psi)=\sin.\theta.\cos.\theta.\,\Sigma\,k.\cos.2.(nt-qt+\varepsilon)\,; \qquad [2443]$$

$$-\frac{3\,L}{2\,r^3}\,.\sin.\theta\,.\sin.2\,.(n\,t+\varpi-\psi)=-\sin.\theta\,.\,\Sigma\,k\,.\sin.2\,.(n\,t-q\,t+\varepsilon)\,; \qquad [2444]$$

the sign of finite integrals Σ is used to denote the sum of all the terms, of the form $k\,.\cos.2\,.(n\,t-q\,t+\varepsilon)$, into which the first member of each of these equations can be resolved. [2445]

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* (1808) Putting $\mathrm{v}=0$ in [2221], which represents the part of $\alpha\,y$ independent of $n\,t$, it becomes as in [2441]. [2441a]

† (1809) The sun being supposed to move in the plane of the equator, in an elliptical orbit, with a mean motion represented by $m\,t$, we shall have

$$\psi=m\,t+A\,.\sin.m\,t+B\,.\sin.2\,m\,t+\&c. \qquad [668]\,; \qquad [2442a]$$

A, B, &c., being of the order of the excentricity of the earth's orbit, and its powers. Substituting this in the sine or cosine of $2\,.(n\,t+\varpi-\psi)$, then developing and reducing by means of [22, 24, &c.] Int., we obtain terms depending on the sines and cosines of angles of the form $2\,.(n\,t-q\,t+\varepsilon)$ [2441″]. Similar remarks may be made relative to the value of $\frac{1}{r^3}$, deduced from [659]. Hence the variable parts of the expressions [2402—2404] may be reduced to the forms represented in the second members of the equations [2442—2444], the symbol k being used as a general expression to denote the coefficient of any one of the terms. [2442b]

The most important of these terms is that depending on the angle

[2445'] $$2nt - 2mt + 2\varpi,^{*}$$

which produces the ebb and flow of the tide, in the case we have just
[2445''] examined, where the sun is supposed to move in the plane of the equator, and to be always at the same distance from the earth. The other terms may be considered as the result of the actions of as many other bodies, moving uniformly in the plane of the equator. Combining together the partial ebb and flow, corresponding to each of these bodies, we shall obtain the total ebb and flow arising from the action of the sun.

[2445'''] If we put l for the mass of the fictitious body, whose action produces the term depending on the angle $2nt - 2qt + 2\varepsilon$, and a for its distance from the centre of the earth; we shall have†

[2446] $$\frac{3l}{2a^3} = k, \qquad \text{or} \qquad \frac{l}{a^3} = \tfrac{2}{3}k.$$

We have seen in the preceding article, that the sun being supposed to move
[2446'] uniformly in the plane of the equator, with an angular motion equal to mt, the part of the expression of the height of the sea, depending on the angle $2nt - 2mt + 2\varpi$, is equal to

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* (1810) This is evident, because ψ [2442a] has the term mt independent of the excentricity of the earth's orbit, that is much larger than the other terms. This term would
[2444a] remain, if the excentricity were to become nothing, or the orbit circular; therefore it must include the terms we have computed in [2406v—2439'], depending on the disturbing forces proportional to $\frac{3L}{2r^3} \cdot {\sin. \atop \cos.} 2 . (nt - mt + \varpi)$, as is observed in [2445''].

† (1811) In the same manner as we have supposed, in the last note, that the chief term
[2446a] depending on the disturbing forces $\frac{3L}{2r^3} \cdot {\sin. \atop \cos.} 2 . (nt - mt + \varpi)$ of the body L, with the mean distance r, and mean motion mt, is put under the form $k . {\sin. \atop \cos.} 2 . (nt - mt + \varpi)$; we may suppose that the terms depending on the fictitious body l, at the distance a, must
[2446b] have the disturbing forces $\frac{3l}{2a^3} \cdot {\sin. \atop \cos.} 2 . (nt - qt + \varepsilon)$, comprised under the general form $k . {\sin. \atop \cos.} 2 . (nt - qt + \varepsilon)$ [2442—2444]. This requires that we should have $\frac{3l}{2a^3} = k$, as in [2446].

$$P\,.\,(1-2\,m\,Q)\,.\,\frac{L}{r^3}\,.\,\cos.\,2\,.\,(n\,t-m\,t+\varpi-O+m\,T).*$$ [2447]

The constant quantities P, Q, O, T, are the same for all the heavenly bodies, whatever be their proper motions [2439′]; therefore the sum of the partial tides, arising from the action of all the bodies l, l', l'', &c., will be,† [2447′]

$$\Sigma\,P\,.\,(1-2\,q\,Q)\,.\,\frac{l}{a^3}\,.\,\cos.\,2\,.\,(n\,t-q\,t+\varepsilon-O+q\,T)\,;$$ [2448]

consequently it will be,‡

$$\begin{aligned}&\tfrac{2}{3}\,P\,.\,\Sigma\,k\,.\,\cos.\,2\,.\,(n\,t-q\,t+\varepsilon-O+q\,T)\\ &\tfrac{2}{3}\,P\,Q\,.\,\frac{d}{dt}\,.\,\Sigma\,k\,.\,\sin.\,2\,.\,(n\,t-q\,t+\varepsilon-O+q\,T)\,;\end{aligned}$$ [2449]

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* (1812) Substituting λ, B, [2439], for $\lambda_{\prime}$, $B_{\prime}$, in [2435], we get, for the height of the tide $\alpha\,y$, an expression of the form [2447]; the term of ψ depending on the mean motion $m\,t$ [2442a] being the only one retained. This represents the part of the tide [2447a] computed in [2406$^{\mathrm{v}}$, &c.], upon the supposition that the body moves uniformly in the plane of the equator, and always at the same distance from the earth.

† (1813) The quantities L, r, m, ϖ, corresponding to the body L, become respectively [2448a] l, a, q, ε, for the body l [2446a, b]; making these changes in [2447], it becomes

$$P\,.\,(1-2\,q\,Q)\,.\,\frac{l}{a^3}\,.\,\cos.\,2\,.\,(n\,t-q\,t+\varepsilon-O+q\,T)\,;$$ [2448b]

which represents the tide produced by the body l. Marking l, a, q, ε, with *one*, *two*, &c. accents, we obtain the tides depending on the bodies l', l'', &c. The sum of all these partial tides represents the whole tide [2448].

‡ (1814) Substituting the value of $\frac{l}{a^3}$ [2446] in [2448], and bringing the constant quantities P, Q, from under the sign Σ, it becomes

$$\tfrac{2}{3}\,P\,.\,\Sigma\,k\,.\,\cos.\,2\,.\,(nt-qt+\varepsilon-O+qT)-\tfrac{4}{3}PQ.\Sigma kq.\cos.2.(nt-qt+\varepsilon-O+qT)\,;$$ [2449a]

which is the same as [2449]; observing that the differential of

$$\sin.\,2\,.\,(n\,t-q\,t+\varepsilon-O+q\,T),$$

taken relatively to t, and divided by dt, considering $n\,t$ as constant, is

$$-2\,q\,.\cos.\,2\,.\,(n\,t-q\,t+\varepsilon-O+q\,T).$$

[2449′] *the differential being taken supposing* nt *to be constant.* But by what precedes, we have*

[2450] $$\Sigma k \,.\, \cos.2\,.\,(nt - qt + \varepsilon - O + qT) = \frac{3L}{2r^3} \,.\, \cos.2\,.\,(nt + \varpi - \psi - \lambda)\,;$$

[2451] the time t being decreased by T, in the variable quantities nt, ψ, r, of the second member of this equation, and $\lambda = O - nT$. Therefore the part of the height of the tide depending upon the action of the sun, and also upon the angle $2nt + 2\varpi - 2\psi$, with the preceding conditions, is represented by [2449, 2450, &c.],

[2452] $$P \,.\, \frac{L}{r^3} \,.\, \cos.2.(nt + \varpi - \psi - \lambda) + PQ \,.\, \frac{d}{dt} \,.\, \left\{ \frac{L}{r^3} \,.\, \sin.2\,.\,(nt + \varpi - \psi - \lambda) \right\}.$$

If we transfer to the moon what we have said relative to the sun, we shall find, that the part of the height of the tide depending upon the lunar action, and the rotatory motion of the earth, is†

[2453] $$P \,.\, \frac{L'}{r'^3} \,.\, \cos.2.(nt + \varpi - \psi' - \lambda) + PQ \,.\, \frac{d}{dt} \,.\, \left\{ \frac{L'}{r'^3} \,.\, \sin.2.(nt + \varpi - \psi' - \lambda) \right\}.$$

[2453′] in which expression the time t must also be decreased by T [2451]. The part independent of the rotatory motion of the earth is

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* (1815) Dividing [2442] by $\sin.^2\theta$, we get

[2450a] $$\Sigma k \,.\, \cos.2\,.\,(nt - qt + \varepsilon) = \frac{3L}{2r^3} \,.\, \cos.2\,.\,(nt + \varpi - \psi)\,;$$

which exists for all values of nt; hence if we change nt into $nt - \lambda$, we find

[2450b] $$\Sigma k \,.\, \cos.2\,.\,(nt - qt + \varepsilon - \lambda) = \frac{3L}{2r^3} \,.\, \cos.2\,.\,(nt + \varpi - \psi - \lambda).$$

If in the first member we change t into $t - T$, and λ into $O - nT$, as is directed in [2451], it becomes $\Sigma k \,.\, \cos.2\,.\,\{n\,.\,(t - T) - q\,.\,(t - T) + \varepsilon - (O - nT)\}$; which by reduction changes into the first member of [2450]. The same changes being made in
[2450c] the second member of [2450b], it becomes as in the second member of [2450], modified as in [2451]. By a similar process we may obtain from [2444] an expression like [2450],
[2450d] depending on the sines of the same angles, instead of the cosines.

[2453a] † (1816) The expression [2453] for the moon, is the same as that for the sun [2452], changing L, r, ψ, into L', r', ψ'; the method of computing both formulas being the same.

$$-\frac{(1+3\,.\,\cos.2\,\theta)}{8\,g\,.\left(1-\frac{3}{5\,\rho}\right)}\,.\,\frac{L'}{r'^3}.\text{*}$$ [2454]

Connecting together all the terms arising from the action of the sun and moon, we shall have, for the approximate value of the height of the tide αy,†

Height of the tide, supposing the sun and moon not to move in the plane of the equator.

$$\alpha y=-\frac{(1+3\,.\,\cos.2\,\theta)}{8\,g\,.\left(1-\frac{3}{5\,\rho}\right)}\,.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}$$

$$+P\,.\left\{\frac{L}{r^3}\,.\,\cos.2\,.\,(nt+\varpi-\psi-\lambda)+\frac{L'}{r'^3}\,.\,\cos.2\,.\,(nt+\varpi-\psi'-\lambda)\right\}$$

$$+PQ\,.\,\frac{d}{dt}\,.\left\{\frac{L}{r^3}\,.\sin.2.(nt+\varpi-\psi-\lambda)+\frac{L'}{r'^3}\,.\,\sin.2\,.\,(nt+\varpi-\psi'-\lambda)\right\};$$ [2455]

in which the time t must be decreased by T [2451], *in the terms multiplied by P and Q, and the differential is to be taken supposing nt to be constant* [2449']. [2455']

20. *Lastly, we shall examine the case of nature, in which the sun and moon do not move in the plane of the equator.* We have given in [2401'—2401'''], the method of obtaining the solar and lunar forces, resolved in directions parallel to three right lines, drawn perpendicular to each other, from which it follows, [2455'']

First. That these forces, resolved in directions parallel to the radius of the earth, are‡

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* (1817) This is similar to the expression [2441], produced by the sun, making the same changes of L, r, ψ, into L', r', ψ', as in the preceding note. [2454a]

† (1818) The terms [2441, 2454] added together give the first line of [2455], the second and third lines are the sum of the two formulas [2452, 2453]. The expression [2455] represents the height of the tide, when the sun and moon move in the equator [2439'']. [2455a]

‡ (1819) The part of the quantity $\alpha V'$ [2401], depending on the body L, is given separately in [2192], and then developed in [2193—2195]. Substituting in this part of $\alpha V'$ the expression

$$\sin.^2 v-\tfrac{1}{2}\,.\cos.^2 v=\sin.^2 v-\tfrac{1}{2}\,.\,(1-\sin.^2 v)=-\tfrac{1}{2}\,.\,(1-3\,.\,\sin.^2 v),$$ [2456a]

it becomes,

$$-\frac{\{1+3.\cos.2\theta\}}{4}.\left\{\frac{L}{r^3}.(1-3.\sin.^2 v)+\frac{L'}{r'^3}.(1-3.\sin.^2 v')\right\}$$

Force in the direction of the radius.

[2456]
$$+6.\sin.\theta.\cos.\theta.\left\{\begin{array}{l}\frac{L}{r^3}.\sin.v.\cos.v.\cos.(nt+\varpi-\psi)\\+\frac{L'}{r'^3}.\sin.v'.\cos.v'.\cos.(nt+\varpi-\psi')\end{array}\right\}$$

$$+\tfrac{3}{2}.\sin.^2\theta.\left\{\begin{array}{l}\frac{L}{r^3}.\cos.^2 v.\cos.2.(nt+\varpi-\psi)\\+\frac{L'}{r'^3}.\cos.^2 v'.\cos.2.(nt+\varpi-\psi')\end{array}\right\};$$

Second. That these forces, resolved in a direction perpendicular to the radius of the earth, and in the plane of the meridian, are*

$$\alpha V'=-\frac{\{1+3.\cos.2\theta\}}{8}.\left\{\frac{L}{r^3}.(1-3.\sin.^2 v)\right.$$

[2456b]
$$+3.\sin.\theta.\cos.\theta.\frac{L}{r^3}.\sin.v.\cos.v.\cos.(nt+\varpi-\psi)$$

$$+\tfrac{3}{4}.\sin.^2\theta.\frac{L}{r^3}.\cos.^2 v.\cos.2.(nt+\varpi-\psi);$$

and by changing L, r, v, into L', r', v', we obtain the corresponding part of $\alpha V'$, depending on the body L', so that the whole expression of $\alpha V'$ becomes

$$\alpha V'=-\frac{\{1+3.\cos.2\theta\}}{8}.\left\{\frac{L}{r^3}.(1-3.\sin.^2 v)+\frac{L'}{r'^3}.(1-3.\sin.^2 v')\right\}$$

[2456c]
$$+3.\sin.\theta.\cos.\theta.\left\{\begin{array}{l}\frac{L}{r^3}.\sin.v.\cos.v.\cos.(nt+\varpi-\psi)\\+\frac{L'}{r'^3}.\sin.v'.\cos.v'.\cos.(nt+\varpi-\psi')\end{array}\right\}$$

$$+\tfrac{3}{4}.\sin.^2\theta.\left\{\frac{L}{r^3}.\cos.^2 v.\cos.2.(nt+\varpi-\psi)+\frac{L'}{r'^3}.\cos.^2 v'.\cos.2.(nt+\varpi-\psi')\right\}.$$

The double of this quantity, or $2\alpha V'$, expresses the force in the direction of the radius [2401'], as in [2456].

[2457a] * (1820) The force in the direction of the meridian $\alpha.\left(\frac{dV'}{d\theta}\right)$ [2401''], is equal to the partial differential of [2456c] relative to θ, divided by $d\theta$. This agrees with [2457]; observing that $3.\sin.\theta.\cos.\theta=\frac{3}{2}.\sin.2\theta$ [31] Int., and its differential, divided by $d\theta$, is equal to $3.\cos.2\theta$.

$$\tfrac{3}{4}\,.\,\sin.\,2\theta\,.\left\{\frac{L}{r^3}\,.\,(1-3\,.\,\sin.^2 v)+\frac{L'}{r'^3}\,.\,(1-3\,.\,\sin.^2 v')\right\}$$

$$+3\,.\,\cos.\,2\theta\,.\left\{\begin{array}{l}\frac{L}{r^3}\,.\,\sin.\,v\,.\,\cos.\,v\,.\,\cos.\,(nt+\varpi-\psi)\\ +\frac{L'}{r'^3}\,.\,\sin.\,v'\,.\,\cos.\,v'\,.\,\cos.\,(nt+\varpi-\psi')\end{array}\right\}$$ Force in the direction of the meridian. [2457]

$$+\tfrac{3}{2}\,.\,\sin.\theta\,.\cos.\theta\,.\left\{\begin{array}{l}\frac{L}{r^3}\,.\,\cos.^2 v\,.\,\cos.\,2\,.\,(nt+\varpi-\psi)\\ +\frac{L'}{r'^3}\,.\,\cos.^2 v'\,.\,\cos.\,2\,.\,(nt+\varpi-\psi')\end{array}\right\};$$

Third. That these forces, resolved perpendicularly to the plane of the meridian, are*

$$-3\,.\,\cos.\theta\,.\left\{\begin{array}{l}\frac{L}{r^3}\,.\,\sin.\,v\,.\,\cos.\,v\,.\,\sin.\,(nt+\varpi-\psi)\\ +\frac{L'}{r'^3}\,.\,\sin.\,v'\,.\,\cos.\,v'\,.\,\sin.\,(nt+\varpi-\psi')\end{array}\right\}$$ Force in the direction of the parallel of latitude. [2458]

$$-\tfrac{3}{2}\,.\,\sin.\,\theta\,.\left\{\begin{array}{l}\frac{L}{r^3}\,.\,\cos.^2 v\,.\,\sin.\,2\,.\,(nt+\varpi-\psi)\\ +\frac{L'}{r'^3}\,.\,\cos.^2 v'\,.\,\sin.\,2\,.\,(nt+\varpi-\psi')\end{array}\right\}.$$

The partial forces

$$-\frac{(1+3\,.\,\cos.\,2\theta)}{4}\,.\left\{\frac{L}{r^3}\,.\,(1-3\,.\,\sin.^2 v)+\frac{L'}{r'^3}\,.\,(1-3\,.\,\sin.^2 v')\right\};$$ [2459]

$$\tfrac{3}{4}\,.\,\sin.\,2\theta\,.\left\{\frac{L}{r^3}\,.\,(1-3\,.\,\sin.^2 v)+\frac{L'}{r'^3}\,.\,(1-3\,.\,\sin.^2 v')\right\},$$

vary with extreme slowness; we may therefore suppose, as in the preceding article [2440″], that the sea is at every moment in equilibrium, by the action of these forces. In this case the value of αy,†

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* (1821) The force in the direction of the parallel of latitude is $\frac{\alpha}{\sin.\theta}\,.\left(\frac{dV'}{d\varpi}\right)$ [2401‴]. [2458a] Substituting $\alpha V'$ [2456c], it becomes as in [2458].

† (1822) Substituting [2456a] in [2221], it becomes

$$-\frac{(1+3\,.\,\cos.\,2\theta)}{8g\,.\left(1-\frac{3}{5\rho}\right)}\,.\,\frac{L}{r^3}\,.\,(1-3\,.\,\sin.^2 v),$$

Oscillations of the first kind.

[2460] $$-\frac{(1+3\,.\cos.2\,\theta)}{8\,g\,.\left(1-\frac{3}{5\,\rho}\right)}\,.\left\{\frac{L}{r^3}\,.\,(1-3\,.\sin.^2\mathrm{v})+\frac{L'}{r'^3}\,.\,(1-3\,.\sin.^2\mathrm{v}')\right\},$$

[2221, &c.], represents the height of the tide arising from the action of these forces. The partial forces depending on the angle $2\,n\,t+2\,\varpi+$ &c., may be reduced to other terms, multiplied by sines and cosines of angles of the form $2\,n\,t-2\,q\,t+2\,\varepsilon$. We may prove, as in the preceding article, that there will result, in the expression of the height of the tide, a quantity equal to*

Oscillations of the third kind.

[2461] $$P.\left\{\frac{L.\cos.^2\mathrm{v}}{r^3}.\cos.2.(nt+\varpi-\psi-\lambda)+\frac{L'.\cos.^2\mathrm{v}'}{r'^3}.\cos.2.(nt+\varpi-\psi'-\lambda)\right\}$$
$$+PQ.\frac{d}{dt}.\left\{\frac{L.\cos.^2\mathrm{v}}{r^3}.\sin.2.(nt+\varpi-\psi-\lambda)+\frac{L'.\cos.^2\mathrm{v}'}{r'^3}.\sin.2.(nt+\varpi-\psi'-\lambda)\right\};$$

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representing the tide depending on this part of the solar force. Accenting the letters L, r, v,

[2460a] we get the corresponding part of the effect of the lunar force,

$$-\frac{(1+3.\cos.2\,\theta)}{8g.\left(1-\frac{3}{5\,\rho}\right)}.\frac{L'}{r'^3}.(1-3\,.\sin.^2\mathrm{v}')$$

The sum of these is [2460].

* (1823) The three forces [2442, 2443, 2444] depending on the angle

$$2\,.\,(n\,t+\varpi-\psi),$$

being multiplied by $\cos.^2\mathrm{v}$, produce the parts of the solar forces [2456, 2457, 2458]

[2461a] respectively, depending on the same angle. Now the part of $\alpha\,y$ depending on the three former forces has been computed in [2452], and by multiplying it by $\cos.^2\mathrm{v}$, we must evidently have the value of $\alpha\,y$ corresponding to the last forces, which will therefore be represented by

[2461b] $$P\,.\left\{\frac{L\,.\cos.^2\mathrm{v}}{r^3}\,.\cos.\,2\,.\,(n\,t+\varpi-\psi-\lambda)\right\}$$
$$+P\,Q\,.\frac{d}{d\,t}.\left\{\frac{L\,.\cos.^2\mathrm{v}}{r^3}\,.\sin.\,2\,.\,(n\,t+\varpi-\psi-\lambda)\right\}.$$

The term $\cos.^2\mathrm{v}$, like r, ψ, being introduced between the braces, or under the sign of differentiation, as it evidently ought to be, by reasoning as in [2442a—b, 2449a, &c.]; v being supposed variable. If we now accent the letters L, r, v, ψ, we shall obtain the corresponding part of $\alpha\,y$ depending on the lunar force, and the sum of both parts is equal to the expression [2461].

in which the time t must be diminished by T, and in taking the differential, nt must be considered as constant [2455']. [2461']

It now remains to consider the part of the preceding forces depending on the angle $nt+\varpi+$ &c. This part may be developed in terms multiplied by sines and cosines of angles of the form $nt-qt+\varepsilon$, q being very [2461''] small in comparison with n.* Each of these terms produces, in the interval of about one day, a flow and ebb, analogous to those produced by the terms depending on the angle $2nt-2qt+2\varepsilon$; with this difference only, that [2461'''] the flow depending on the angle $nt-qt+\varepsilon$, takes place but once in a day; on the contrary, that relative to the angle $2nt-2qt+2\varepsilon$, takes place twice in a day.

We shall easily find, by the analysis of the preceding articles, that the height of the tide, depending on forces whose period is nearly a day, may be represented by the formula,

Oscillations of the second kind.

$$A.\left\{\begin{array}{l}\frac{L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}.\cos.(nt+\varpi-\psi-\gamma)\\+\frac{L'}{r'^3}.\sin.\mathrm{v}'.\cos.\mathrm{v}'.\cos.(nt+\varpi-\psi'-\gamma)\end{array}\right\}$$
$$+B.\frac{d}{dt}.\left\{\begin{array}{l}\frac{L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}.\sin.(nt+\varpi-\psi-\gamma)\\+\frac{L'}{r'^3}.\sin.\mathrm{v}'.\cos.\mathrm{v}'.\sin.(nt+\varpi-\psi'-\gamma)\end{array}\right\};\qquad [2462]$$

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* (1824) We have shown in [2442a—b], that $\frac{3L}{r^3}.\cos.2.(nt+\varpi-\psi)$ may be reduced to a series of terms of the form $k.\cos.2.(nt-qt+\varepsilon)$, in which $\frac{q}{n}$ is a [2461c] fraction of the same order as the ratio of the earth's angular motion mt in its orbit, to the angular rotatory motion nt; and by the same method of reasoning, $\frac{L}{r^3}.\sin.(nt+\varpi-\psi)$, or $\frac{L}{r^3}.\cos.(nt+\varpi-\psi)$ may be reduced to a series of terms of the form

$$k.\cos.(nt-qt+\varepsilon),\quad\text{or}\quad k.\sin.(nt-qt+\varepsilon).$$

Again, the longitude of the sun in the ecliptic is represented by a formula similar to [2442a], depending on angles of the order mt; consequently the declination v, as well as sin. v, [2461d] cos. v, must depend upon angles of the same order; and the expressions

[2462'] *A, B, γ, being three arbitrary constant quantities, which can be determined only by observations in each port; the differentials are to be taken supposing nt to be constant; the time t must be decreased by a constant quantity T', which can only be determined by observation.

If we now connect together all these partial heights of the tide, we shall have for the whole height αy,

General expression of the height of the tide, noticing all the circumstances.

[2463]

$$\alpha y = -\frac{(1+3\,.\cos.2\theta)}{8g\,.\left(1-\frac{3}{5\rho}\right)}\,.\left\{\frac{L}{r^3}\,.(1-3\,.\sin.^2\mathrm{v})+\frac{L'}{r'^3}\,.(1-3\,.\sin.^2\mathrm{v}')\right\}$$
$$+A\,.\left\{\begin{array}{l}\frac{L}{r^3}\,.\sin.\mathrm{v}\,.\cos.\mathrm{v}\,.\cos.(nt+\varpi-\psi-\gamma)\\ +\frac{L'}{r'^3}\,.\sin.\mathrm{v}'\,.\cos.\mathrm{v}'\,.\cos.(nt+\varpi-\psi'-\gamma)\end{array}\right\}$$
$$+B\,.\frac{d}{dt}\,.\left\{\begin{array}{l}\frac{L}{r^3}\,.\sin.\mathrm{v}\,.\cos.\mathrm{v}\,.\sin.(nt+\varpi-\psi-\gamma)\\ +\frac{L'}{r'^3}\,.\sin.\mathrm{v}'\,.\cos.\mathrm{v}'\,.\sin.(nt+\varpi-\psi'-\gamma)\end{array}\right\} \qquad (O)$$
$$+P\,.\left\{\begin{array}{l}\frac{L}{r^3}\,.\cos.^2\mathrm{v}\,.\cos.2\,.(nt+\varpi-\psi-\lambda)\\ +\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'\,.\cos.2\,.(nt+\varpi-\psi'-\lambda)\end{array}\right\}$$
$$+PQ\,.\frac{d}{dt}\,.\left\{\begin{array}{l}\frac{L}{r^3}\,.\cos.^2\mathrm{v}\,.\sin.2\,.(nt+\varpi-\psi-\lambda)\\ +\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'\,.\sin.2\,.(nt+\varpi-\psi'-\lambda)\end{array}\right\}.$$

$$\frac{L}{r^3}\,.\sin.\mathrm{v}\,.\cos.\mathrm{v}\,.\sin.(nt+\varpi-\psi),\qquad \frac{L}{r^3}\,.\sin.\mathrm{v}\,.\cos.\mathrm{v}\,.\cos.(nt+\varpi-\psi),$$

[2461e] may be reduced to a series of terms depending on sines and cosines of angles of the form $nt-qt+\varepsilon$, in which q is small in comparison with n.

* (1825) From the variable forces depending on the angle $2nt+2\varpi-2\psi$ [2402—2404], we have found, by the method explained in [2406^{v}—2452], that the height
[2462a] of the tide, depending on this angle, can be reduced to the forms [2452, 2453], and finally to the form [2461]; O, T, P, Q, and $\lambda=O-mT$, being constant quantities to be determined by observation [2439']. The same method may be used with the forces

**In this expression the differentials must be taken supposing* $n t$ *to be constant; and the time* t *must be diminished by a constant quantity* T', *in* [2463']
the terms multiplied by A, B, [2462']; *and by the constant quantity* T, *in* [2463'']
the terms multiplied by P, Q, [2461']; *these constant quantities, as well as* A, B, γ, P, Q, λ, *must be determined, in each port, by observation.*

[2456, 2457, 2458], depending upon the angle $n t + \varpi - \psi$; by which means the expression [2461] will become of the form [2462]; the constant quantities P, PQ, λ, T, being replaced by A, B, γ, T', respectively, which are to be determined by observation. [2462b]

* (1826) The expression [2463] is the sum of those in [2460, 2461, 2462].

CHAPTER IV.

COMPARISON OF THE PRECEDING THEORY WITH OBSERVATIONS.

[2463‴] 21. We *shall now develop the principal phenomena of the tides, which follow from the preceding expression of* y [2463], *and shall compare these results with observations. We shall distinguish the phenomena into two classes; the one relative to the heights of the tides, the other relative to their intervals. We shall consider the tides at their maximum near the syzygies, and at their minimum near the quadratures.*

ON THE HEIGHTS OF THE TIDES NEAR THE SYZYGIES.

[2464] The times of high and low water are determined by the equation* $\frac{dy}{dt}=0$.

[2464′] Now in taking the differential of the value of αy [2463], we may suppose that the preceding expressions of v, v′, r, r', ψ, and ψ', are constant; because these quantities vary so slowly, that the effect of their variations is insensible in the heights of the tide at full sea and low water. For we know, that near these points of *maximum* and *minimum*, a small error in the value of t produces no sensible effect in the value of y.† We may likewise

[2464″] neglect, without sensible error, the term of the expression of αy multiplied

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* (1827) This equation is given by the usual rule for finding the *maximum* or *minimum*

[2464a] of y, by putting $dy=0$, or rather $\frac{dy}{dt}=0$.

† (1827a) Because at the time of high or low water, the altitude is nearly stationary for a short period of time; the velocity of ascent or descent vanishes in consequence of the

[2464b] equation $\frac{dy}{dt}=0$.

by B [2464e]. For the oscillations depending on the angle $nt+\varpi$, the period of which is nearly equal to one day, are very small in our ports, [2630′]; therefore it is highly probable that the coefficient B is insensible.* We shall also see hereafter [2601, 2624], that Q is but of little importance, [2464‴] so that we shall at first neglect the terms which contain Q as a factor. Hence the equation $\frac{dy}{dt}=0$, will give

$$0=\frac{A}{2P}\cdot\left\{\begin{array}{l}\frac{L}{r^3}\cdot\sin.\mathrm{v}\cdot\cos.\mathrm{v}\cdot\sin.(nt+\varpi-\psi-\gamma)\\ +\frac{L'}{r'^3}\cdot\sin.\mathrm{v}'\cdot\cos.\mathrm{v}'\cdot\sin.(nt+\varpi-\psi'-\gamma)\end{array}\right\}$$

[2465]

$$+\frac{L}{r^3}\cdot\cos.^2\mathrm{v}\cdot\sin.2\cdot(nt+\varpi-\psi-\lambda)+\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'\cdot\sin.2\cdot(nt+\varpi-\psi'-\lambda).$$

The fraction $\frac{A}{2P}$ is very small in our ports, and we shall hereafter [2465′] see that at Brest it does not at the most exceed $\frac{1}{40}$;† therefore we [2465″] may neglect it without any sensible error. Then the preceding equation gives

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* (1828) If the sun and moon did not move in their orbits, the right ascensions ψ and [2464c] ψ' would be constant; and then the differentials of the parts of [2463] depending on B, PQ, taken as in [2463′], upon the supposition that nt is constant, would vanish. Moreover, it is shown in [2601, 2624] that Q must be very small; and in [2630′] it is proved from [2464d] observation that the terms depending on the angle whose period is one day, are small; hence A must be small. Lastly, as B [2462] depends on A, in like manner as the small quantity [2464e] PQ depends on the large term P, in [2461], it is highly probable that B must also be small in comparison with A. Hence if we neglect B, Q, and take the differential of [2463] relatively to t, dividing it by $-2nP\cdot dt$, we shall get [2465].

† (1829) Substituting $\frac{L'}{r'^3}=3\cdot\frac{L}{r^3}$ [2706] in [2704], we get

$$8P\cdot\frac{L}{r^3}=6^{\text{met.}},2490,\qquad\text{or}\qquad\frac{L}{r^3}=\frac{0,78}{P}.$$ [2464f]

Using these and the greatest values of v, v′, which are nearly $\mathrm{v}=26^{\circ}$, $\mathrm{v}'=32^{\circ}$; Putting also $\cos.(\lambda-\gamma)=1$, because from observation λ differs but little from γ, we obtain for the function [2630] the following expression, by successive reductions, and using [31] Int.

General equation to determine the time of high or low water.

[2466] $$\text{tang.}\, 2\,.\,(n\,t+\varpi-\psi'-\lambda)=\frac{\frac{L}{r^3}\,.\,\cos.^2 \mathrm{v}\,.\,\sin.\,2\,.\,(\psi-\psi')}{\frac{L'}{r'^3}\,.\,\cos.^2 \mathrm{v}'+\frac{L}{r^3}\,.\,\cos.^2 \mathrm{v}\,.\,\cos.\,2\,.\,(\psi-\psi')}\,.\text{*}$$

$$0^{\text{met.}},183=A\,.\,\frac{L}{r^3}\,.\{2\sin.\mathrm{v}\,.\cos.\mathrm{v}+6\,.\sin.\mathrm{v}'\,.\cos.\mathrm{v}'\}=A\,.\,\frac{0{,}78}{P}\,.\{\sin.2\,\mathrm{v}+3\,.\sin.2\,\mathrm{v}'\}$$

[2465a] $$=A\,.\,\frac{0{,}78}{P}\,.\,3{,}3=2{,}57\,.\,\frac{A}{P};$$

[2465b] Hence the greatest value of $\frac{A}{2P}=\frac{1}{28}$, instead of $\frac{1}{40}$ [2465″]. This quantity is so small in comparison with the others, that it may be neglected, as in [2465″].

[2466a] * (1830) Putting for brevity $C=2\,.\,(n\,t+\varpi-\psi'-\lambda)$, $D=2\,.\,(\psi-\psi')$, we get $C-D=2\,.\,(n\,t+\varpi-\psi-\lambda)$. Substituting these in [2465], and neglecting A [2465″], we obtain the following expression, which is developed by [22] Int.,

$$0=\frac{L}{r^3}\,.\,\cos.^2\mathrm{v}\,.\,\sin.\,(C-D)+\frac{L'}{r'^3}\,.\,\cos.^2\mathrm{v}'\,.\,\sin.C$$

[2466b] $$=\frac{L}{r^3}\,.\,\cos.^2\mathrm{v}\,.\,\{\sin.C\,.\,\cos.D-\cos.C\,.\,\sin.D\}+\frac{L'}{r'^3}\,.\,\cos.^2\mathrm{v}'\,.\,\sin.C.$$

Dividing this by $\cos.C$, we get

$$0=\frac{L}{r^3}\,.\cos.^2\mathrm{v}\,.\,\{\text{tang.}C\,.\,\cos.D-\sin.D\}+\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'\,.\,\text{tang.}C\,;$$

[2466c] hence we obtain, as in [2466], $\text{tang.}\,C=\dfrac{\frac{L}{r^3}\,.\cos.^2\mathrm{v}\,.\,\sin.D}{\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'+\frac{L}{r^3}\,.\cos.^2\mathrm{v}\,.\,\cos.D}$. From this value of $\text{tang.}C$ we may obtain the following values of $\sin.C$, $\cos.C$, $\cos.(C-D)$, required in the next note, putting for brevity N, M, equal to the numerator and denominator of [2466c]. and using formulas [34′, 34‴] Int.; hence we get

[2466d] $$N=\frac{L}{r^3}\,.\cos.^2\mathrm{v}\,.\sin.D,\qquad M=\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'+\frac{L}{r^3}\,.\cos.^2\mathrm{v}\,.\cos.D,$$

[2466e] $$\text{tang.}C=\frac{N}{M},\qquad \sin.C=\frac{N}{\sqrt{(N^2+M^2)}},\qquad \cos.C=\frac{M}{\sqrt{(N^2+M^2)}},$$

[2466f] $$\cos.(C-D)=\cos.C\,.\cos.D+\sin.C\,.\sin.D=\frac{1}{\sqrt{(N^2+M^2)}}\,.\,(M\,.\cos.D+N\,.\sin.D),$$

[2466g] $$N^2+M^2=\left(\frac{L}{r^3}\,.\cos.^2\mathrm{v}\right)^2+\frac{2L}{r^3}\,.\cos.^2\mathrm{v}\,.\,\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'.\cos.2\,.\,(\psi'-\psi)+\left(\frac{L'}{r'^3}.\cos.^2\mathrm{v}'\right)^2.$$

This last equation is easily deduced from N, M, [2466d], by adding their squares, and putting $\sin.^2D+\cos.^2D=1$.

We must now substitute the value of $nt+\varpi-\psi'$ determined by this equation in the expression of αy; and if what the following function then becomes be represented by (A), or

$$(A)=A\,.\left\{\begin{array}{l}\frac{L}{r^3}\,.\,\sin.\,\mathrm{v}\,.\,\cos.\,\mathrm{v}\,.\,\cos.\,(nt+\varpi-\psi-\gamma)\\+\frac{L'}{r'^3}\,.\,\sin.\,\mathrm{v}'\,.\,\cos.\,\mathrm{v}'\,.\,\cos.\,(nt+\varpi-\psi'-\gamma)\end{array}\right\},$$ (A). [2467]

we shall have*

$$\alpha y=-\frac{(1+3\,.\cos.2\,\theta)}{8g\,.\left(1-\frac{3}{5\rho}\right)}\,.\left\{\frac{L}{r^3}\,.\,(1-3\,.\,\sin.^2\mathrm{v})+\frac{L'}{r'^3}\,.\,(1-3\,.\,\sin.^2\mathrm{v}')\right\}+(A)$$

General expression of the elevation of the tide at high water and low water.

$$\pm P.\sqrt{\left(\frac{L}{r^3}.\cos.^2\mathrm{v}\right)^2+\frac{2L}{r^3}.\cos.^2\mathrm{v}.\frac{L'}{r'^3}.\cos.^2\mathrm{v}'.\cos.2.(\psi'-\psi)+\left(\frac{L'}{r'^3}.\cos.^2\mathrm{v}'\right)^2};$$ [2468]

the sign $+$ *corresponds to full sea, and the sign* $-$ *to low water.* [2468']

Supposing this expression to refer to the morning *full sea*, we shall have the height of the evening *full sea*, by increasing the variable quantities by [2468'']

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* (1831) The first line of the expression of αy [2463] gives that multiplied by $(1+3.\cos.2\theta)$ in [2468]; the term multiplied by A gives (A) [2467]; and by neglecting the terms multiplied by B, PQ [2464'', &c.], there will remain only that multiplied [2468a] by P, which may be proved to be equal to the term depending on P in [2468]. For by using the abridged values [2466a], this term becomes as in [2468b], which is reduced to [2468c] by means of [2466f, e]. This, by using [2466d], is finally reduced to the form [2468e].

$$P\,.\left\{\frac{L}{r^3}\,.\cos.^2\mathrm{v}\,.\cos.\,(C-D)+\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'\,.\cos.C\right\}$$ [2468b]

$$=\frac{P}{\sqrt{(N^2+M^2)}}\,.\left\{\frac{L}{r^3}\,.\cos.^2\mathrm{v}\,.\,(M.\cos.D+N.\sin.D)+\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'\,.\,M\right\}$$ [2468c]

$$=\frac{P}{\sqrt{(N^2+M^2)}}\,.\left\{M.\left(\frac{L'}{r'^3}.\cos.^2\mathrm{v}'+\frac{L}{r^3}.\cos.^2\mathrm{v}.\cos.D\right)+N.\left(\frac{L}{r^3}.\cos.^2\mathrm{v}.\sin.D\right)\right\}$$ [2468d]

$$=\frac{P}{\sqrt{(N^2+M^2)}}\,.\,\{M.M+N.N\}=P\,.\sqrt{(N^2+M^2)}.$$ [2468e]

Substituting the expression [2466g], we obtain the term multiplied by P [2468], the sign $\pm$ being prefixed to the radical to distinguish the two values of αy, corresponding to high [2468f] and low water [2468'], which depend chiefly on this term, because it is much larger than the others.

what they have varied during the interval of the two tides. Therefore we must change the sign of (A); because the angle $nt+\varpi-\psi-\gamma$ *increases*
[2468'''] *in that interval about two right angles*, the difference being so small that it may be neglected, taking into consideration the smallness of (A) [2465b].
[2469] *Hence* $2(A)$ *is the difference of the heights of two tides of the same day.*
[2470] Supposing now that y' is the half sum of the heights of the morning and
y'. evening tide; y' *will be the quantity which we shall hereafter call the mean*
[2471] *absolute height of the tide of one day.* Thus we shall have nearly

Mean absolute height of the tide of one day.

$$y'=-\frac{(1+3\cdot\cos.2\theta)}{8g\cdot\left(1-\frac{3}{5\rho}\right)}\cdot\left\{\frac{L}{r^3}\cdot(1-3\cdot\sin.^2\mathrm{v})+\frac{L'}{r'^3}\cdot(1-3\cdot\sin.^2\mathrm{v}')\right\}$$

[2472]

$$+P\cdot\sqrt{\left(\frac{L}{r^3}\cdot\cos.^2\mathrm{v}\right)^2+\frac{2L}{r^3}\cdot\cos.^2\mathrm{v}\cdot\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'\cdot\cos.2\cdot(\psi'-\psi)+\left(\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'\right)^2}\,;$$

all the variable quantities of this expression correspond to the low water falling between the morning and evening tides, and must therefore refer to the time
[2472'] *which precedes the time of low water by the quantity T.** It is very probable that the part of this expression which is not multiplied by P, corresponds to a different time [2463']; but this part is so small in comparison with the other, that we may, without any sensible error, refer them both to the time corresponding to the greatest quantity.

(A'). *If we put* (A') *for what* (A) [2467] *becomes at the time of the low*
[2473] *water falling between the morning and evening tide, the height at this low water will be*

Height of the tide at low water.

$$-\frac{(1+3\cdot\cos.2\theta)}{8g\cdot\left(1-\frac{3}{5\rho}\right)}\cdot\left\{\frac{L}{r^3}\cdot(1-3\cdot\sin.^2\mathrm{v})+\frac{L'}{r'^3}\cdot(1-3\cdot\sin.^2\mathrm{v}')\right\}+(A')$$

[2474]

$$-P\cdot\sqrt{\left(\frac{L}{r^3}\cdot\cos.^2\mathrm{v}\right)^2+\frac{2L}{r^3}\cdot\cos.^2\mathrm{v}\cdot\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'\cdot\cos.2\cdot(\psi'-\psi)+\left(\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'\right)^2}\,.$$

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* (1832) To obtain the value of y' accurately, we must compute the value of αy
[2472a] [2468] for the morning tide, and then for the evening tide; using the values of r, v, r', v', &c., corresponding to those times respectively; the mean of these two values will be the value of y' [2470]. Now as these quantities r, v, &c., vary but little in the course of half a day, it is evident that this mean value will be obtained nearly, by substituting in

Subtracting this expression from the mean absolute height of the tide of one day [2472], we obtain what we shall hereafter call the total tide, being the [2475]
excess of the half sum of the two high tides of the same day, above the intermediate low water. We shall represent this excess by y'', and we [2476]
shall have

Total tide y''.

$$y''=-(A')+2P.\sqrt{\left(\frac{L}{r^3}.\cos.^2 v\right)^2+\frac{2L}{r^3}.\cos.^2 v.\frac{L'}{r'^3}.\cos.^2 v'.\cos.2.(\psi'-\psi)+\left(\frac{L'}{r'^3}.\cos.^2 v'\right)^2}. \quad [2477]$$

Lastly, the difference of two consecutive low tides is $2.(A')$.* [2478]

About the time of the *maximum* of the tides, or near the syzygies, the [2478']
angle $\psi'-\psi$ is quite small, since it is nothing at the *maximum;* therefore
at the time of full sea we shall have nearly $nt+\varpi-\psi'=\lambda$.† Substituting [2479]
this value in the function (A) [2467], we shall have, by supposing that the
time t must be decreased by T in this function, as it is in the function [2479']
multiplied by P,

[2468] the values r, v, &c., corresponding to the time of low tide, falling between the two [2472b]
tides thus computed; because this time is nearly the mean time between those tides, which must be decreased by T according to the directions in [2463''].

* (1833) This is proved in the same manner as $2.(A)$ is proved to be the difference of the heights of two consecutive high tides in [2469]. [2478a]

† (1833a) Putting $\psi=\psi'$, or $\psi=200^\circ+\psi'$, in the term of αy [2463] multiplied [2479a]
by P, on which the tide chiefly depends, it becomes

$$P.\left\{\frac{L}{r^3}.\cos.^2 v+\frac{L'}{r'^3}.\cos.^2 v'\right\}.\cos.2.(nt+\varpi-\psi'-\lambda). \quad [2479b]$$

This is a maximum when $\cos.2.(nt+\varpi-\psi'-\lambda)=1$; or when the angle $2.(nt+\varpi-\psi'-\lambda)=0$ or $=400^\circ$; corresponding to $nt+\varpi-\psi'=\lambda$ or $nt+\varpi-\psi'=200^\circ+\lambda$. This last value is not mentioned by the author in [2479]; [2479c]
and the same neglect is observable in the value of $\psi'-\psi$ [2478'], which is said to be nothing in the syzygies; whereas at the time of the opposition of the luminaries we have
$\psi'-\psi=\pm 200^\circ$. This neglected case does not affect the coefficient of P in [2479b], [2479d]
but might alter the values of (A), (A'), [2467, 2473, &c.]; it is not however necessary to notice this circumstance, because neither of the quantities (A), (A'), occur in the value of
y' [2472, 2482]; moreover the term (A') in the value of y'' [2477] is neglected in [2479e]
[2481', 2483], on account of its smallness.

[2480] $$(A) = -A\,.\left\{\frac{L}{r^3}\,.\,\sin.\,v\,.\,\cos.\,v + \frac{L'}{r'^3}\,.\,\sin.\,v'\,.\,\cos.\,v'\right\}.\,\cos.\,(\lambda - \gamma).^*$$

Values of (A), (A'), near the syzygies.

(A) being very small, the error of the preceding supposition must be insensible. The function (A') becomes very nearly

[2481] $$(A') = A\,.\left\{\frac{L}{r^3}\,.\,\sin.\,v\,.\,\cos.\,v + \frac{L'}{r'^3}\,.\,\sin.\,v'\,.\,\cos.\,v'\right\}.\,\sin.\,(\lambda - \gamma).$$

We may also, on account of the smallness of these functions, suppose in
[2481'] them $\lambda = \gamma$,† which makes (A') vanish. This being premised, if we

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* (1834) Substituting in [2467] the values of the angles ψ, $nt + \varpi - \psi$, ψ', $nt + \varpi - \psi'$, [2479a, c], we get

[2480a] $$(A) = A\,.\left\{\pm\frac{L}{r^3}\,.\,\sin.\,v\,.\,\cos.\,v \pm \frac{L'}{r'^3}\,.\,\sin.\,v'\,.\,\cos.\,v'\right\}.\,\cos.\,(\lambda - \gamma)\,;$$

which differs from that given by the author [2480] in the *double* sign prefixed to the terms, arising from the *two* values of the angles here used. Now from the time of high water to
[2480b] the following low tide, the angle $nt + \varpi - \psi'$ increases about 100° [2268''', 2473], by which means the value of this angle [2479c] become $100^\circ + \lambda$ or $300^\circ + \lambda$. Substituting these in [2467, 2473], we get

[2480c] $$(A') = A\,.\left\{\mp\frac{L}{r^3}\,.\,\sin.\,v\,.\,\cos.\,v \mp \frac{L'}{r'^3}\,.\,\sin.\,v'\,.\,\cos.\,v'\right\}.\,\sin.\,(\lambda - \gamma)\,;$$

which differs likewise from [2481] in the double sign of the terms. These discrepancies are not of much importance, because these terms are finally neglected [2479e].

† (1835) If we examine the calculation [2432'—2439'], by which the value of λ
[2481a] [2439] was obtained in oscillations of the third kind; we shall find that the same method will apply, with a similar result, in oscillations of the second kind, depending on the angle $nt + \varpi$. For the quantity $\lambda_{,}$, deduced from [2434'] in oscillations of the third kind, is a
[2481b] function of B, C, m, T, T'; the terms T, T', [2432'', 2432'''], being the times required for the waves to move from the mouth to the head of the canal. Now it is highly probable that these times are nearly the same for oscillations of the *second* kind, as for those of the *third*. Moreover, from the nature of the factors B, C, [2433], it is probable that the ratio
[2481c] of these quantities is nearly the same in both cases; so that in oscillations of the second kind, we may change B into $B.a$, and C into $C.a$. These changes being made in [2434'],
[2481d] without altering the values of T, T', the numerator and denominator of that expression will be divisible by a, and the values of $\sin.2\lambda_{,}$, or $\lambda_{,}$, will remain unaltered; hence we shall obtain nearly the same value of λ as in [2439]; consequently the value of λ is nearly the
[2481e] same in the second as in the third oscillations, or $\lambda = \gamma$.

neglect the fourth power of $\psi'-\psi$, in the terms of y', y'', multiplied by P, [2481''']
we shall have near the syzygies,*

Mean absolute tides near the syzygies.

$$y'=-\frac{(1+3\,.\cos.2\,\theta)}{8\,g\,.\left(1-\frac{3}{5\,\rho}\right)}\,.\left\{\frac{L}{r^3}\,.\,(1-3\,.\sin.^2\mathrm{v})+\frac{L'}{r'^3}\,.\,(1-3\,.\sin.^2\mathrm{v}')\right\}$$

$$+P\,.\left\{\frac{L}{r^3}\,.\cos.^2\mathrm{v}+\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'\right\}$$

$$-\frac{2\,P\,.\,\frac{L}{r^3}\,.\cos.^2\mathrm{v}\,.\,\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'}{\frac{L}{r^3}\,.\cos.^2\mathrm{v}+\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'}\,.\left\{(\psi'-\psi)^2+\tfrac{1}{4}\,q^2\right\}; \qquad [2482]$$

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* (1836) In order to make $\psi'-\psi$ small at the times of the syzygies of the full moon, we shall change in these syzygies the values of ψ' into $\psi'-200^{\circ}$, in the formulas [2482a] [2468, 2472, 2474, 2477], which will not alter their values. Now neglecting in all our future developments terms of the order $(\psi'-\psi)^4$, we shall get from [44] Int.

$$\cos.2\,.\,(\psi'-\psi)=1-2\,.\,(\psi'-\psi)^2. \qquad [2482b]$$

Substituting this in the radical or factor of P [2472], it becomes as in [2482c]; and by extracting the root we obtain [2482d], always neglecting the variations of v, v', r, r', in comparison with $\psi'-\psi$;

$$\left\{\left(\frac{L}{r^3}\,.\cos.^2\mathrm{v}+\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'\right)^2-\frac{4\,L}{r^3}\,.\cos.^2\mathrm{v}\,.\,\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'\,.\,(\psi'-\psi)^2\right\}^{\frac{1}{2}} \qquad [2482c]$$

$$=\left(\frac{L}{r^3}\,.\cos.^2\mathrm{v}+\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'\right)-\frac{\frac{2\,L}{r^3}\,.\cos.^2\mathrm{v}\,.\,\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'\,.\,(\psi'-\psi)^2}{\frac{L}{r^3}\,.\cos.^2\mathrm{v}+\frac{L'}{r'^3}\,.\cos.^2\mathrm{v}'}\,. \qquad [2482d]$$

If the angle $\psi'-\psi$ vary, by the quantity q, in the interval of two successive tides, and [2482d'] the value of $\psi'-\psi$ be taken to correspond to the intermediate low water, its value at the time of the preceding high water will be $\psi'-\psi-\frac{1}{2}\,q$, and at the following high water [2482e] $\psi'-\psi+\frac{1}{2}\,q$. The former must be substituted in [2468], or rather in [2482d], for the morning tide; the latter, for the evening tide; half their sum will be the corrected value to be used in y' [2472, 2482]. This half sum is

$$\tfrac{1}{2}\,.\left\{(\psi'-\psi-\tfrac{1}{2}\,q)^2+(\psi'-\psi+\tfrac{1}{2}\,q)^2\right\}=(\psi'-\psi)^2+\tfrac{1}{4}\,q^2, \qquad [2482f]$$

as in the formula [2482].

Total tides near the syzygies.

$$y''=2P\cdot\left\{\frac{L}{r^3}\cdot\cos.^2 v+\frac{L'}{r'^3}\cdot\cos.^2 v'\right\}$$

[2483]

$$-\frac{4P\cdot\frac{L}{r^3}\cdot\cos.^2 v\cdot\frac{L'}{r'^3}\cdot\cos.^2 v'}{\frac{L}{r^3}\cdot\cos.^2 v+\frac{L'}{r'^3}\cdot\cos.^2 v'}\cdot\{(\psi'-\psi)^2+\tfrac{1}{8}q^2\};*$$

[2483'] q being the variation of the arc $\psi'-\psi$, in the interval of two consecutive full tides. The addition of the two terms depending on it, is founded upon
[2484] the principle that the true value of $(\psi'-\psi)^2$, in the expression of y', is the half sum of the squares $(\psi'-\psi)^2$ corresponding to the two consecutive high tides; and it is evident that this half sum is equal to the expression $(\psi'-\psi)^2+\frac{1}{4}q^2$ [2482f], in which the arc $\psi'-\psi$ corresponds to the intermediate *low water*. As the variable quantities of the two preceding
[2485] formulas refer to that *low water*, it follows that the square $(\psi'-\psi)^2$ must, for greater accuracy, be increased by $\frac{1}{4}q^2$ in the expression of y', and by $\frac{1}{8}q^2$ in that of y'' [2483c].

22. *We shall now develop the expressions of* y', y'', [2482, 2483], *in the*
[2485'] *equinoxes and solstices, in order to determine the influence of the declinations of these bodies upon the tides.* The term

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* (1837) If we use the development [2482b], the radical, by which P is multiplied in [2474], will become as in [2482d]; but in this case the value of $\psi'-\psi$ will require no
[2483a] correction, because it is assumed to correspond to the time of low water. Putting also $(A')=0$, as in [2481'], the expression [2474] will become

[2483b]

$$-\frac{(1+3\cdot\cos.2\theta)}{8g\cdot\left(1-\frac{3}{5\rho}\right)}\cdot\left\{\frac{L}{r^3}\cdot(1-3\cdot\sin.^2 v)+\frac{L'}{r'^3}\cdot(1-3\cdot\sin.^2 v')\right\}$$

$$-P\cdot\left\{\frac{L}{r^3}\cdot\cos.^2 v+\frac{L'}{r'^3}\cdot\cos.^2 v'\right\}+\frac{2P\cdot\frac{L}{r^3}\cdot\cos.^2 v\cdot\frac{L'}{r'^3}\cdot\cos.^2 v'}{\frac{L}{r^3}\cdot\cos.^2 v+\frac{L'}{r'^3}\cdot\cos.^2 v'}\cdot(\psi'-\psi)^2.$$

Subtracting this from y' [2482], we get the total tide y'', as in [2483], corresponding to
[2483c] [2477]. The factor of q^2 being the same in [2482] as in [2483], because the one contains the coefficient $2P\times\frac{1}{4}q^2$, the other $4P\times\frac{1}{8}q^2$.

$$-\frac{(1+3\,.\cos.2\,\theta)}{8g.\left(1-\frac{3}{5\,\rho}\right)}\,.\left\{\frac{L}{r^3}\,.\,(1-3\,.\sin.^2\mathrm{v})+\frac{L'}{r'^3}\,.\,(1-3\,.\sin.^2\mathrm{v}')\right\}$$ [2486]

of the expression of y' [2482] is very small.* We may therefore suppose, without any sensible error, that the variable quantities r, v, r', v', which it contains, correspond to the instant of the syzygy. When we take the sum [2486'] of the values of y, corresponding to two consecutive syzygies, we may suppose, in the preceding term, that r' is equal to the mean distance of the moon from the earth in the syzygies; for it is evident that if the moon be in [2487] the apogee in any syzygy, it will be nearly in the perigee in the following syzygy.† r is nearly equal to the mean distance of the earth from the sun [2487'] in the syzygies of the equinoxes; and if we consider as many syzygies near the solstices, in the winter, as in the summer, we may also suppose r to be equal to that mean distance. [2488]

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* (1838) We have nearly $\frac{L}{r^3g}=0^{\mathrm{met.}},16$ [2301], $\frac{L'}{r'^3g}=e\,.\,0^{\mathrm{met.}},16=0^{\mathrm{met.}},48$ [2485a] [2302, 2304'], $1-\frac{3}{5\,\rho}=1-\frac{3}{25}=0,88$ [2297a], and at Brest $\cos.2\,\theta$ is nearly equal to $\frac{1}{10}$ [2318']; so that if we put $\mathrm{v}=0$, $\mathrm{v}'=0$, the expression [2486], at its [2485b] maximum, will become $-\frac{(1+0,3)}{8\times 0,88}\,.\{0^{\mathrm{met.}},16+0^{\mathrm{met.}},48\}=-0^{\mathrm{met.}},12$. In the 24 syzygies [2510], the sum of the *total tides* varies from $128^{\mathrm{met.}}$ to $150^{\mathrm{met.}}$, and the mean value [2485c] of one of the tides is nearly 6 metres, being about 50 times as great as the function [2486], which we have just computed. This function being so very small, we may neglect the [2485d] variations of r, v, r', v', during a few days, without any sensible error in the result.

† (1838a) If we notice only the first power e' of the excentricity of the lunar orbit, putting the mean distance from the earth equal to a', and the mean anomaly $\mathcal{A}'$, which in [669] is represented by $nt+\varepsilon-\varpi$, we shall get, from the first of the formulas [669],

$$r'=a'.\{1-e'.\cos.\mathcal{A}'\},\qquad \text{and}\qquad \frac{1}{r'^3}=\frac{1}{a'^3}\,.\,(1+3\,e'.\cos.\mathcal{A}').$$ [2486a]

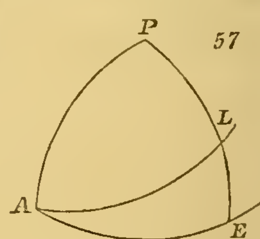
In the following syzygy, $\mathcal{A}'$ is increased by 200° nearly; so that $\cos.\mathcal{A}'$ changes into $-\cos.\mathcal{A}'$, and $\frac{1}{r'^3}=\frac{1}{a'^3}\,.\,(1-3\,e'.\cos.\mathcal{A}')$ nearly. The mean of these two values is $\frac{1}{r'^3}=\frac{1}{a'^3}$; hence we see that the variation in the value of r' is but of little importance, [2486b] when two successive syzygies are used. Similar remarks may be made relative to the distance of the sun from the earth, as in [2488].

[2488'] The part $P \cdot \left\{ \frac{L}{r^3} \cdot \cos.^2 \mathrm{v} + \frac{L'}{r'^3} \cdot \cos.^2 \mathrm{v}' \right\}$ of the expression of y' [2482] varies sensibly in a few days, and as this term is considerable in our ports,
[2488''] we must notice its variations. For this purpose we shall put ε' equal to the
ε', Γ'. inclination of the lunar orbit to the equator, and Γ' equal to the moon's distance from the ascending node of her orbit upon the equator; we shall
[2489] then have* $\sin. \mathrm{v}' = \sin. \varepsilon' . \sin. \Gamma'$, consequently

[2490] $$\cos.^2 \mathrm{v}' = 1 - \tfrac{1}{2} . \sin.^2 \varepsilon' + \tfrac{1}{2} . \sin.^2 \varepsilon' . \cos. 2\Gamma'.\dagger$$

We shall put t for the time elapsed from the *maximum* of the tide, to the time of any observation, t being negative for observations which happen
[2490'] before that *maximum*. Then, by neglecting the powers of t superior to the
[2490''] square, and supposing the motion of the moon in her orbit to be uniform during the time t, which may be admitted without any sensible error, we shall have,‡

[2491] $$\cos.^2 \mathrm{v}' = \cos.^2 \mathrm{v}' - t . \frac{d\Gamma'}{dt} . \sin.^2 \varepsilon' . \sin. 2\Gamma' - t^2 . \left(\frac{d\Gamma'}{dt}\right)^2 . \sin.^2 \varepsilon' . \cos. 2\Gamma'.$$



* (1839) Thus if AL, fig. 57, be the lunar orbit, AE the equator, P its pole, L the place of the moon, PA and PLE
[2489a] arcs drawn perpendicular to the equator AE; then $AL = \Gamma'$,
[2489b] $LE = \mathrm{v}'$, angle $LAE = \varepsilon'$, and we have, from [1345[28]], $\sin. LE = \sin. LAE . \sin. AL$, or $\sin. \mathrm{v}' = \sin. \varepsilon' . \sin. \Gamma'$ [2489].

† (1840) From [2489] and [1] Int., we get successively

[2490a] $$\cos.^2 \mathrm{v}' = 1 - \sin.^2 \mathrm{v}' = 1 - \sin.^2 \varepsilon' . \sin.^2 \Gamma' = 1 - \sin.^2 \varepsilon' . (\tfrac{1}{2} - \tfrac{1}{2} . \cos. 2\Gamma') \quad [2490],$$

from which we find

[2490b] $$\sin.^2 \varepsilon' . \cos. 2\Gamma' = \sin.^2 \varepsilon' - 2 . (1 - \cos.^2 \mathrm{v}') = \sin.^2 \varepsilon' - 2 . \sin.^2 \mathrm{v}',$$

which is used in [2492a].

‡ (1841) Supposing in [607, 608] $\alpha = t$, and considering u as a function of t, we shall have, by retaining only the first and second powers of t,

[2491a] $$u = \mathrm{u} + t . \left(\frac{du}{dt}\right) + \tfrac{1}{2} t^2 . \left(\frac{ddu}{dt^2}\right),$$

u being the value of u when $t = 0$, and t being put equal to nothing after finding the differentials $\left(\frac{du}{dt}\right)$, $\left(\frac{ddu}{dt^2}\right)$. If we put u equal to the value of $\cos.^2 \mathrm{v}'$, given in the

The values of V' *and* Γ', *in the second member of this equation, correspond to the syzygy.* In the equinoxes and solstices, $\sin. 2\Gamma'$ nearly vanishes ; [2491′]

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second member of [2490], we may consider Γ' as the only variable quantity. For the inclination of the orbit ε' varies only by quantities which are of a less order than those of the motion of the moon's node, when compared with the real motion of the moon in her orbit. Hence we have [2491b]

$$u = \cos.^2 V' = 1 - \tfrac{1}{2} . \sin.^2 \varepsilon' + \tfrac{1}{2} . \sin.^2 \varepsilon' . \cos. 2\Gamma' ;$$

$$\left(\frac{du}{dt}\right) = -\sin.^2 \varepsilon' . \sin. 2\Gamma' . \left(\frac{d\Gamma'}{dt}\right) ; \qquad [2491c]$$

$$\tfrac{1}{2} . \left(\frac{ddu}{dt^2}\right) = -\sin.^2 \varepsilon' . \cos. 2\Gamma' . \left(\frac{d\Gamma'}{dt}\right)^2 - \tfrac{1}{2} . \sin.^2 \varepsilon' . \sin. 2\Gamma' . \left(\frac{dd\Gamma'}{dt^2}\right).$$

Substituting these in [2491a], and for the first term u using in the second member of [2491d] the value $\cos.^2 V'$ corresponding to $t = 0$, we shall have, for the general value of $\cos.^2 V'$ corresponding to the time t, as in that second member,

$$\cos.^2 V' = \cos.^2 V' - t . \frac{d\Gamma'}{dt} . \sin.^2 \varepsilon' . \sin. 2\Gamma' - t^2 . \sin.^2 \varepsilon' . \cos. 2\Gamma' . \left(\frac{d\Gamma'}{dt}\right)^2 \qquad [2491d]$$

$$- \tfrac{1}{2} t^2 . \sin.^2 \varepsilon' . \sin. 2\Gamma' . \left(\frac{dd\Gamma'}{dt^2}\right) ;$$

in which we may neglect the term $\left(\frac{dd\Gamma'}{dt^2}\right)$ on account of its smallness, supposing, as in [2490″], the increment $\left(\frac{d\Gamma'}{dt}\right)$ to be constant. To make a rough estimate of the value of this neglected quantity, we may take the three longitudes of the moon, from the Nautical Almanac of 1830, to represent the values of Γ', July $20^{d.}\ 0^{h.}$, $20^{d.}\ 12^{h.}$, $21^{d.}\ 0^{h.}$; and denoting them as in [754v] by β, β', β'', and putting $i=0$, $i'=1$, $i''=2$, we have

$$\beta = 123^d\ 18^m\ 17^s, \qquad \beta' = 129^d\ 56^m\ 16^s, \qquad \beta'' = 136^d\ 29^m\ 19^s. \qquad [2491e]$$

From these we obtain, as in [755],

$$\delta\beta = 6^d\ 37^m\ 59^s, \qquad \delta\beta' = 6^d\ 33^m\ 3^s, \qquad \delta^2\beta = \tfrac{1}{2} . (\delta\beta' - \delta\beta) = -148^s ;$$

and if we change, in [757, &c.], α into Γ', s into t, we shall get from [758] nearly

$$\left(\frac{d\Gamma'}{dt}\right) = \delta\beta = 6^d\ 37^m\ 59^s, \qquad \left(\frac{dd\Gamma'}{dt^2}\right) = 2\,\delta^2\beta = -296^s. \qquad [2491f]$$

These quantities are to each other in the ratio of about 80 to 1 ; therefore $\left(\frac{dd\Gamma'}{dt^2}\right)$ is much smaller than $\left(\frac{d\Gamma'}{dt}\right)$, and may be neglected in comparison with it, particularly as it is multiplied in [2491d] by the very small quantity $\sin. 2\Gamma'$ [2491′] ; hence this formula becomes as in [2491].

therefore, by noticing only the syzygies which happen near these points, the term of the preceding expression of $\cos.^2 v'$ [2491], multiplied by the first
[2491''] power of t, disappears from the sum of the values of y', particularly if we use a sufficient number of observations to make the positive and negative values of $\sin. 2\Gamma'$ mutually destroy each other; therefore we shall have*

[2492] $$\cos.^2 v' = \cos.^2 v' - t^2 . \left(\frac{d\Gamma'}{dt}\right)^2 . \{\sin.^2 \varepsilon' - 2\sin.^2 v'\}.$$

[2492'] *We shall take for the unit of time, the interval between two consecutive tides of the morning and evening, near the syzygies, which interval is about*
[2493] $1^{\mathrm{day}},0271$;† *and shall put* υ *for the mean synodical motion of the moon in*
υ. *that interval.* We shall have in the syzygies, by noticing the argument of *variation*, which near these points always increases the horary motion.

[2494] $$\left(\frac{d\Gamma'}{dt}\right)^2 = 1,165 . \upsilon^2 ;$$

consequently‡

[2495] $$\cos.^2 v' = \cos.^2 v' - 1,165 . t^2 . \upsilon^2 . \{\sin.^2 \varepsilon' - 2 . \sin.^2 v'\}.$$

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* (1842) Neglecting as in [2491''] the term depending on the first power of t [2491], and then subtituting [2490b], we get successively, as in [2492],

[2492a] $$\cos.^2 v' = \cos.^2 v' - t^2 . \left(\frac{d\Gamma'}{dt}\right)^2 . \sin.^2 \varepsilon' . \cos. 2\Gamma' = \cos.^2 v' - t^2 . \left(\frac{d\Gamma'}{dt}\right) . (\sin.^2 \varepsilon' - 2 . \sin.^2 v').$$

The term multiplied by $2 . \sin.^2 v'$ is so small, that we may use in it the value of v' corresponding to the syzygy.

† (1843) We shall find in [2745] that the time of high water upon any day t after the
[2493a] syzygy is $0^{\mathrm{day}},39664 + 0^{\mathrm{day}},027052 . t$; so that on the day of the syzygy, the time of high water is $0^{\mathrm{day}},39664$, and on the following day it is $0^{\mathrm{day}},39664 + 0^{\mathrm{day}},027052$; the
[2493b] interval is $1^{\mathrm{day}},027052$, as in [2493] nearly.

‡ (1844) By Burg's tables, the moon's motion in one day is $13^d\, 10^m\, 35^s$, the sun's
[2493c] $59^m\, 8^s$; the argument of variation increases, in that time, by the difference of these quantities $12^d\, 11^m\, 27^s$, and the corresponding variation is $14^m\, 29^s$. Adding these two
[2493d] last quantities, we obtain $12^d\, 25^m\, 56^s$, or $13^{\circ}\, 81'\, 36''$ nearly, for the diurnal synodical motion of the moon in the syzygies; so that the real motion is to the synodical as
[2493e] $13^d\, 10^m\, 35^s + 14^m\, 29^s$ is to $12^d\, 25^m\, 56^s$, or as 1,0793 to 1. Hence the actual motion

The variation of $\frac{1}{r'^3}$* may be neglected, when we consider at the same time two consecutive syzygies. We may also neglect the variations of $\frac{1}{r^3}$ and $\sin.^2 \text{v}$, because these two quantities alter but very little in the interval of a few days. They also depend on the action of the sun, which we shall hereafter find [2706] to be only one third part of that of the moon. [2495′] [2495″]

It now remains to consider the term

$$-\frac{2P \cdot \frac{L}{r^3} \cdot \cos.^2 \text{v} \cdot \frac{L'}{r'^3} \cdot \cos.^2 \text{v}'}{\frac{L}{r^3} \cdot \cos.^2 \text{v} + \frac{L'}{r'^3} \cdot \cos.^2 \text{v}'} \cdot (\psi' - \psi)^2 \qquad [2496]$$

of the expression of y' [2482]. If we put ε and Γ for the sun, what we have called ε' and Γ' for the moon, we shall have very nearly, by observing that ε' differs but little from ε, and that $\sin.(\Gamma' + \Gamma)$ almost vanishes in the syzygies of the equinoxes and solstices,† [2496′] ε, Γ.

$$(\psi' - \psi) \cdot \cos. \text{v} \cdot \cos. \text{v}' = (\Gamma' - \Gamma) \cdot \cos. \left(\frac{\varepsilon + \varepsilon'}{2}\right); \qquad [2497]$$

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of the moon, in the interval taken for unity, is $1{,}0793 \cdot \upsilon$, provided υ represent the mean synodical motion in the syzygies; in which case we shall have $\frac{d\Gamma}{dt} = 1{,}0793 \cdot \upsilon$, whose square is $\left(\frac{d\Gamma}{dt}\right)^2 = 1{,}165 \cdot \upsilon^2$, as in [2494]; substituting this in [2492], we get [2495]. [2493f] But if υ represent, as in [2493], the mean synodical motion throughout the whole orbit, we shall have $\frac{d\Gamma}{dt} = 1{,}101 \cdot \upsilon$; because the ratio of $13^d\, 10^m\, 35^s + 14^m\, 29^s$ to $12^d\, 11^m\, 27^s$ is as $1{,}101$ to 1; hence $\left(\frac{d\Gamma}{dt}\right)^2 = 1{,}211 \cdot \upsilon^2$; therefore the factor $1{,}211$ must be used instead of $1{,}165$ in [2495—2502], whenever we use the mean value of υ. [2493g]

* (1845) This appears from the demonstration in [2486a—b].

† (1846) If in fig. 57, page 678, we suppose AE to represent the equator, P its pole, AL the ecliptic, PA, PLE, circles of declination, we shall have, in the triangles AEL, PLA, the following equations, deduced from [1345[27], [15]] respectively, [2497a]

$$\cos. LE \cdot \cos. AE = \cos. AL, \qquad \sin. PL \cdot \sin. APE = \sin. PAL \cdot \sin. AL,$$

which changes the preceding term [2496] into

[2498] $$-\frac{2P\cdot\frac{L}{r^3}\cdot\frac{L'}{r'^3}\cdot\left\{\cos.\left(\frac{\varepsilon+\varepsilon'}{2}\right)\right\}^2\cdot t^2\,\nu^2}{\frac{L}{r^3}\cdot\cos.^2\mathrm{v}+\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'}.$$

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or in symbols,

[2497b] $\cos.\mathrm{v}\,.\cos.\psi=\cos.\Gamma,\qquad \cos.\mathrm{v}\,.\sin.\psi=\cos.\varepsilon\,.\sin.\Gamma.$

Accenting these letters, we get two similar equations corresponding to the lunar orbit,

[2497c] $\cos.\mathrm{v}'.\cos.\psi'=\cos.\Gamma',\qquad \cos.\mathrm{v}'.\sin.\psi'=\cos.\varepsilon'.\sin.\Gamma'.$

Multiplying the first of these equations by the fourth, also the second by the third, then taking the difference of these products, we get [2497d]. Reducing this by means of [22, 18, 19] Int., we find [2497e], and by using [27, 28] Int., we finally obtain [2497f].

[2497d] $\cos.\mathrm{v}\,.\cos.\mathrm{v}'.\{\sin.\psi'.\cos.\psi-\cos.\psi'.\sin.\psi\}=\cos.\varepsilon'.\sin.\Gamma'.\cos.\Gamma-\cos.\varepsilon\,.\cos.\Gamma'.\sin.\Gamma\,;$

$\cos.\mathrm{v}\,.\cos.\mathrm{v}'.\sin.(\psi'-\psi)$

$=\tfrac{1}{2}.\cos.\varepsilon'.\{\sin.(\Gamma'+\Gamma)+\sin.(\Gamma'-\Gamma)\}-\tfrac{1}{2}.\cos.\varepsilon\,.\{\sin.(\Gamma'+\Gamma)-\sin.(\Gamma'-\Gamma)$

[2497e] $=\sin.(\Gamma'-\Gamma).\{\tfrac{1}{2}.\cos.\varepsilon'+\tfrac{1}{2}.\cos.\varepsilon\}+\sin.(\Gamma'+\Gamma).\{\tfrac{1}{2}.\cos.\varepsilon'-\tfrac{1}{2}.\cos.\varepsilon\}$

[2497f] $=\sin.(\Gamma'-\Gamma).\cos.\tfrac{1}{2}.(\varepsilon'+\varepsilon).\cos.\tfrac{1}{2}.(\varepsilon'-\varepsilon)-\sin.(\Gamma'+\Gamma).\sin.\tfrac{1}{2}.(\varepsilon'+\varepsilon).\sin.\tfrac{1}{2}.(\varepsilon'-\varepsilon).$

This expression admits of several reductions. For in the syzygies of the equinoxes, Γ, Γ', are nearly equal to 0° or 200°; and in the syzygies of the solstices, they are nearly equal to
[2497g] 100° or 300°. In both these cases $\sin.(\Gamma'+\Gamma)$ becomes nearly equal to $\sin.0^\circ$ or $\sin.200^\circ$, and must therefore be very small, as in [2496']. The inclination of the lunar orbit to the ecliptic is about 6°, so that $\tfrac{1}{2}.(\varepsilon'-\varepsilon)$ must generally be less than half this
[2497h] quantity, or 3°, whose sine is less than $\tfrac{1}{20}$, and cosine nearly equal to 1. Substituting these in [2497f], and neglecting the term depending on $\sin.\tfrac{1}{2}.(\varepsilon'-\varepsilon).\sin.(\Gamma'+\Gamma)$ on account
[2497i] of its smallness, we get $\cos.\mathrm{v}\,.\cos.\mathrm{v}'.\sin.(\psi'-\psi)=\sin.(\Gamma'-\Gamma).\cos.\tfrac{1}{2}.(\varepsilon'+\varepsilon)$. Developing $\sin.(\psi'-\psi)$, $\sin.(\Gamma'-\Gamma)$, by [43] Int., retaining only the first power of the quantities $\psi'-\psi$, $\Gamma'-\Gamma$, we obtain the expression [2497]. The square of this being substituted in [2496], it becomes

[2497k] $$-\frac{2P\cdot\frac{L}{r^3}\cdot\frac{L'}{r'^3}\cdot\cos.^2\tfrac{1}{2}.(\varepsilon'+\varepsilon)}{\frac{L}{r^3}\cdot\cos.^2\mathrm{v}+\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'}\cdot(\Gamma'-\Gamma)^2\,;$$

and by [2493], the synodical motion of the moon, in the time t from the syzygy, is $t\nu$;
[2497l] putting this for $\Gamma'-\Gamma$, in the preceding expression, it becomes as in [2498]. The quantities we have here neglected are very small, since the whole term [2498] rarely exceeds a third of a metre, as we may easily prove by the following rough calculation. For by

This being premised, if we put Y' *for the sum of the values of* y', *corresponding to* $2i$ *syzygies of the equinoxes*, we shall have* [2498']

Expression of of the sum of the mean absolute heights Y' in $2i$ syzygies of the equinoxes or solstices.

$$Y' = -\frac{2i.(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)}.\left\{\frac{L}{r^3}.(1-3.\sin.^2V)+\frac{L'}{r'^3}.(1-3.\sin.^2V')\right\}$$
$$+2iP.\left\{\frac{L}{r^3}.\cos.^2V+\frac{L'}{r'^3}.\cos.^2V'\right\}$$
$$-2iP.\frac{L'}{r'^3}.(t^2+\tfrac{1}{16}).v^2.\left\{1,165.(\sin.^2\varepsilon'-2.\sin.^2V')+\frac{\frac{2L}{r^3}.\left(\cos.\frac{\varepsilon+\varepsilon'}{2}\right)^2}{\frac{L}{r^3}.\cos.^2V+\frac{L'}{r'^3}.\cos.^2V'}\right\}. \quad [2499]$$

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supposing $\cos.^2 v = \cos.^2 v' = \cos.^2 \tfrac{1}{2}.(\varepsilon'+\varepsilon)$, and $\frac{L'}{r'^3} = 3.\frac{L}{r^3}$ [2706], the numerator [2497m] and denominator of [2498] is divisible by $\frac{L}{r^3}.\cos.^2 v$, and then that expression becomes successively, by using [2713], and neglecting the sign,

$$\frac{2P}{4}.\frac{L'}{r'^3}.t^2v^2 = \frac{4^{\text{met.}},6847}{4}.t^2v^2 = 1^{\text{met.}},17.t^2v^2. \quad [2497n]$$

Now tv [2493] represents the synodical motion, and at its maximum, in Table I [2510, &c.], it corresponds to about $2\tfrac{1}{2}$ days, before or after the syzygy; in which time this motion is $\tfrac{5}{2}\times 13^{\circ},8 = 34^{\circ}$ nearly [2493d]; its sine being nearly equal to $\tfrac{1}{2}$, we get [2497o] $tv=\tfrac{1}{2}$. Hence the expression [2497n] becomes $1^{\text{met.}},17\times\tfrac{1}{4} = 0^{\text{met.}},3$ nearly, and the part neglected in [2497h, &c.] is considerably less than $\tfrac{1}{20}$ of this quantity. We may finally remark, that when the moon's node is situated in the first point of Aries, the origin of the quantities Γ', ψ', is the same as that of Γ, ψ. In other situations of the node, the origin will differ a little; but this difference is corrected nearly, by supposing as in [2498] that the origin [2497p] of the angles $\psi'-\psi$, $\Gamma'-\Gamma$, is at the time of the syzygy.

* (1847) The first line of the expression of y' [2482], being taken for $2i$ syzygies, the sum of them will produce the first line of the expression of Y' [2499]; since it is evident [2500a] that the substitution of the mean values of $\sin.^2 v$, $\sin.^2 v'$, represented by $\sin.^2 V$, $\sin.^2 V'$, [2499'], will give nearly a correct result. In the same way, the second line of [2482] becomes, by substituting the value of $\cos.^2 v'$ [2495],

$$P.\left\{\frac{L}{r^3}.\cos.^2 v+\frac{L'}{r'^3}.\cos.^2 v'\right\}-1,165.P.\frac{L'}{r'^3}.t^2v^2.\{\sin.^2\varepsilon'-2.\sin.^2 v'\}; \quad [2500b]$$

and as tv represents the mean synodical motion $\Gamma'-\Gamma$, we have $t^2v^2=(\Gamma'-\Gamma)^2$; and to this we must add $\tfrac{1}{4}q^2$ nearly, for the same reason that $\tfrac{1}{4}q^2$ was added to $(\psi'-\psi)^2$ in [2500c] [2482, 2482f]; hence [2500b] becomes

[2499'] In this expression $\cos.^2 V$, $\cos.^2 V'$, $\sin.^2 V'$ and $\left(\cos.\frac{\varepsilon+\varepsilon'}{2}\right)^2$, are the mean of all the corresponding values of $\cos.^2 \mathrm{v}$, $\cos.^2 \mathrm{v}'$, $\sin.^2 \mathrm{v}'$ and

V, V'. $\left(\cos.\frac{\varepsilon+\varepsilon'}{2}\right)^2$, in the $2i$ syzygies. *The same expression also represents*

[2500] *the sum of the values of y' in $2i$ syzygies of the solstices, when half of them are taken in the winter solstice.*

We shall now consider the expression of y'' [2477, 2483]. The term*

[2501] $$A.\left\{\frac{L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}+\frac{L'}{r'^3}.\sin.\mathrm{v}'.\cos.\mathrm{v}'\right\}$$

[2500d] $$P.\left\{\frac{L}{r^3}.\cos.^2\mathrm{v}+\frac{L'}{r'^3}.\cos.^2\mathrm{v}'\right\}-1,165.P.\frac{L'}{r'^3}.(t^2v^2+\tfrac{1}{4}q^2).\{\sin.^2\varepsilon'-2.\sin.^2\mathrm{v}'\}.$$

[2500d'] But from [2482d', 2492'], we have nearly $q=\frac{1}{2}v$; hence $t^2v^2+\frac{1}{4}q^2=(t^2+\frac{1}{16}).v^2$. Substituting this value in [2500d], and then taking the sum for $2i$ syzygies, we get

[2500e] $$2iP.\left\{\frac{L}{r^3}.\cos.^2V+\frac{L'}{r'^3}.\cos.^2V'\right\}-2iP.\frac{L'}{r'^3}.(t^2+\tfrac{1}{16}).v^2.1,165.\{\sin.^2\varepsilon'-2.\sin.^2V'\},$$

as in the second and third lines of [2499]. The term of [2482] depending on $(\psi'-\psi)^2$ is given in [2496, 2498], and the similar term depending on $\frac{1}{4}q^2$ [2500c], must evidently be to this as $(\psi'-\psi)^2$ to $\frac{1}{4}q^2$, or nearly as t^2v^2 to $\frac{1}{16}v^2$ [2500d']; so that the

[2500f] two terms must be represented by $-\dfrac{2P.\frac{L}{r^3}.\frac{L'}{r'^3}.\left(\cos.\frac{\varepsilon+\varepsilon'}{2}\right)^2.(t^2+\frac{1}{16}).v^2}{\frac{L}{r^3}.\cos.^2\mathrm{v}+\frac{L'}{r'^3}.\cos.^2\mathrm{v}'}$, and for the

[2500g] $2i$ syzygies this becomes $-2iP.\dfrac{L'}{r'^3}.(t^2+\tfrac{1}{16}).v^2.\dfrac{\frac{2L}{r^3}.\left(\cos.\frac{\varepsilon+\varepsilon'}{2}\right)^2}{\frac{L}{r^3}.\cos.^2V+\frac{L'}{r'^3}.\cos.^2V'}$, which

is the last term of [2499], corresponding to $2i$ syzygies of the equinoxes [2498', 2487']. The same expression will also answer for $2i$ syzygies of the solstices, if half of them be

[2500h] in the winter solstice, and half in the summer solstice; and in this case, by [2488], or [2486a, b], the variations of r may be neglected, by taking as many summer as winter syzygies. It may be observed that in all these formulas the values V, V', *are taken to correspond to the instant of the syzygy, although the corresponding tide does not take place*

[2500i] *till* $1\frac{1}{2}$ *days after* [2544], the time T of formulas [2463, &c.] being nearly equal to this quantity.

* (1848) The term (A') [2477] was neglected in [2481', 2483], on account of its

[2501a] smallness in comparison with the other terms of Y''. For $\sin.(\gamma-\lambda)$ [2481] is very

is very small in our ports; it is nothing in the syzygies of the equinoxes; it also disappears from the sum of the values of y'', if we consider two [2501′]
consecutive syzygies, and as many solstices of winter as of summer. Therefore by putting Y'' *for the sum of the values of* y'' *corresponding to* $2i$ Y''.
*syzygies of the equinoxes, we shall have** [2501″]

Expression of Y'' in $2i$ syzygies of the equinoxes or solstices.

$$Y''=4iP.\left\{\frac{L}{r^3}.\cos.^2V+\frac{L'}{r'^3}.\cos.^2V'\right\}$$

$$-4iP.\frac{L'}{r'^3}.(t^2+\tfrac{1}{32}).\upsilon^2.\left\{1,165.(\sin.^2\varepsilon'-2.\sin.^2V')+\frac{\frac{2L}{r^3}.\left(\cos.\frac{\varepsilon+\varepsilon'}{2}\right)^2}{\frac{L}{r^3}.\cos.^2V+\frac{L'}{r'^3}.\cos.^2V'}\right\}; \qquad [2502]$$

this expression also represents the sum of the values of y'' *in* $2i$ *syzygies of the solstices.*

We shall now see what the terms depending on Q, which we have heretofore neglected, add to these expressions of Y', Y''. For this purpose we shall resume the expression of αy [2463]. In the equinoxes and solstices,†

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small [2481e]; $A<\frac{1}{11}P$ [2465b]; and in the equinoxes v, v′, must vanish or be very small; hence the whole expression [2481] may be neglected in the syzygies of the equinoxes. [2501a′]
In the solstices, if we take two consecutive syzygies, v′ changes into $-$v′, by which the sign of the terms of the factor $\frac{L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}+\frac{L'}{r'^3}.\sin.\mathrm{v}'.\cos.\mathrm{v}'$ [2481], depending [2501b]
on v′, is changed, and the mean value of this factor becomes nearly $\frac{L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}$. If we take two more observations at the other solstice, the sign of v will change, and the sum of the two terms will nearly vanish; therefore the factor [2501b], which occurs in [2481], will [2501c]
be so very small, that it may be neglected in the method we have here used.

* (1849) We may derive [2483] from [2482], by putting in [2482] $1+3.\cos.\theta=0$, [2502a]
changing P into $2P$, and q^2 into $\frac{1}{2}q^2$. The same changes being made in [2499], which was derived from [2482], we obtain [2502], depending on [2483]; observing that $\frac{1}{4}q^2$ produced the term $\frac{1}{16}\upsilon^2$ [2500d′]; therefore $\frac{1}{8}q^2$ [2483] will produce $\frac{1}{32}\upsilon^2$ [2499]. [2502b]

† (1850) We have $\frac{d.\cos.^2\mathrm{v}'}{dt}=-2.\sin.\mathrm{v}'.\cos.\mathrm{v}'.\frac{d\mathrm{v}'}{dt}$. This vanishes in the syzygies of the equinoxes, where $\mathrm{v}'=0$ nearly; and in the syzygies of the solstices, where [2502c]
$\frac{d\mathrm{v}'}{dt}=0$ nearly.

[2502′] $\frac{d \,.\, \cos.^2 \mathrm{v}'}{dt}$ vanishes, and we may neglect the differential of $\frac{1}{r'^3}$ divided by dt, when we consider the aggregate of two consecutive syzygies.* We shall also neglect

[2503] $$PQ \,.\, \frac{d}{dt} \,.\, \left\{ \frac{L}{r^3} \,.\, \cos.^2 \mathrm{v} \,.\, \sin.\, 2 \,.\, (nt + \varpi - \psi - \lambda) \right\},$$

on account of the slowness of the variations of v, ψ, r, and because $\frac{L}{r^3}$ is

[2503′] but a third part of $\frac{L'}{r'^3}$.† The term of the formula [2463] depending on Q, will thus increase the expression of αy, by the quantity

[2504] $$-2\, PQ \,.\, \frac{d\psi'}{dt} \,.\, \frac{L'}{r'^3} \,.\, \cos.^2 \mathrm{v}' \,.\, \cos.\, 2 \,.\, (nt + \varpi - \psi' - \lambda).$$

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* (1851) In [2486a] we have $\frac{1}{r'^3} = \frac{1}{a'^3} \,.\, (1 + 3\,e' \,.\, \cos.\, \mathcal{A}')$; $\mathcal{A}'$ being the moon's

[2502d] mean anomaly. Its differential is $\left(\frac{d \,.\, \frac{1}{r'^3}}{dt}\right) = -\frac{3\,e}{a'^3} \,.\, \left(\frac{d\mathcal{A}'}{dt}\right) \,.\, \sin.\, \mathcal{A}'$. In the next syzygy, $\mathcal{A}'$ becomes nearly $\mathcal{A}' + 200^{\circ}$, and this expression changes its sign, so that the sum of the two expressions is nearly equal to nothing. It may also be observed, that $\mathcal{A}'$ is

[2502e] proportional to the *mean* motion of the *moon*; therefore $\left(\frac{d\mathcal{A}'}{dt}\right)$ and $\left(\frac{d \,.\, \frac{1}{r'^3}}{dt}\right)$ must be of the *same order*. The like may be observed relative to the differentials of v′, ψ'.

† (1852) The terms depending on Q in [2463] are

[2503a] $$PQ \,.\, \frac{d}{dt} \,.\, \left\{ \frac{L}{r^3} \,.\, \cos.^2 \mathrm{v} \,.\, \sin.2.(nt + \varpi - \psi - \lambda) + \frac{L'}{r'^3} \,.\, \cos.^2 \mathrm{v}' \,.\, \sin.\, 2 \,.\, (nt + \varpi - \psi' - \lambda) \right\}.$$

Now we have shown, in the last note, that the differential of the quantities depending on r', v′, ψ', must be multiplied by terms of the order of the mean motion of the moon; and in like manner, those depending on r, v, ψ, must be multiplied by terms of the order of the mean motion of the sun in its apparent orbit; so that the last are by this multiplication

[2503b] reduced to about one thirteenth part of the other quantities. Moreover $\frac{L}{r^3} = \frac{1}{3} \,.\, \frac{L'}{r'^3}$ [2497m]; therefore the terms depending on the solar force are of the order of $\frac{1}{3 \times 13}$, or one

Now we have*

$$\frac{d\psi'}{dt}\cdot\cos.^2 v' = \frac{d\Gamma'}{dt}\cdot\cos.\varepsilon' = m'\cdot\cos.\varepsilon'\,; \qquad [2505]$$

$m't$ being the mean motion of the moon; hence the preceding term [2504] becomes [2505′]

$$-2\,m'PQ\cdot\cos.\varepsilon'\cdot\frac{L'}{r'^3}\cdot\cos.2\cdot(nt+\varpi-\psi'-\lambda). \qquad [2506]$$

From this it is evident that in the syzygies of the equinoxes, where $\cos.v'=1$ very nearly, the term depending on Q changes, in the expressions of αy, Y', Y'', [2463, 2499, 2502], the quantity L' into $L'.(1-2m'Q\cdot\cos.\varepsilon')$;† [2507]

Terms of Y', Y'', depending on Q.

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thirty-ninth part of those depending on the moon; and as the lunar forces, depending on Q, are quite small [2601, 2719], we may very safely neglect the term

$$PQ\cdot\frac{d}{dt}\cdot\left\{\frac{L}{r^3}\cdot\cos.^2 v\cdot\sin.2\cdot(nt+\varpi-\psi-\lambda)\right\}, \qquad [2503c]$$

depending on the solar force. Lastly, as we neglect the variations of r', v', [2502′, &c.], we must consider ψ' as the only variable quantity in the term depending on Q in [2503a], by which means it becomes as in [2504]. [2503d]

* (1853) Using fig. 57, page 678, and the same symbols, we have

$$\cos.\varepsilon'\cdot\text{tang.}\,\Gamma' = \text{tang.}\,\psi' \qquad [1345^{29}]\,;$$

and by supposing ε' to be constant, the differential will be $\cos.\varepsilon'\cdot\frac{d\Gamma'}{\cos.^2\Gamma'}=\frac{d\psi'}{\cos.^2\psi'}$ [54] Int. [2505a]
Substituting $\cos.\Gamma'=\cos.\psi'\cdot\cos.v'$ [2497c], it becomes

$$\cos.\varepsilon'\cdot\frac{d\Gamma'}{\cos.^2\psi'\cdot\cos.^2 v'}=\frac{d\psi'}{\cos.^2\psi'}. \qquad [2505b]$$

Multiplying this by $\frac{\cos.^2\psi'\cdot\cos.^2 v'}{dt}$, we get the second expression [2505]; and if $m't$ be the mean motion of the moon, we shall have $\frac{d\Gamma'}{dt}=m'$ nearly, neglecting the perturbations of the moon's motion, which may be done, particularly when the observations are made in all parts of the orbit, as in Tables I, II, &c., [2508, &c.]. Substituting this in the second, we get the third expression [2505], by means of which we obtain [2506] from [2504]. [2505c]

* (1854) In the syzygies of the equinoxes, where $\cos.v'$ is nearly equal to 1, the small term [2506] may be put equal to

$$-2\,m'PQ\cdot\cos.\varepsilon'\cdot\frac{L'}{r'^3}\cdot\cos.^2 v'\cdot\cos.2\cdot(nt+\varpi-\psi'-\lambda). \qquad [2507a]$$

and in the solstices, where we have very nearly $\cos. v' = \cos. \varepsilon'$, L' changes
[2507′] into $L' . \left(1 - \frac{2 m' Q}{\cos. \varepsilon'}\right)$;* so that the difference of the values of L', in these two cases, may serve to determine Q.

23. *We shall now compare the preceding formulas with observations. At*
[2507″] *the beginning of the eighteenth century, at the request of the Academy of Arts and Sciences, a great number of observations of the ebb and flow of the tide were made in our ports. These were continued every day at Brest for six successive years;† and although they are not complete, yet they form, by the*

The larger term, depending on $P . \frac{L'}{r'^3}$ [2463], is

[2507b′] $$P . \frac{L'}{r'^3} . \cos.^2 v' . \cos. 2 . (n t + \varpi - \psi' - \lambda);$$

the sum of both terms is

[2507b] $$P . \frac{L' . (1 - 2 m' Q . \cos. \varepsilon')}{r'^3} . \cos.^2 v' . \cos. 2 . (n t + \varpi - \psi' - \lambda).$$

This may be derived from the second of these terms, by changing, as in [2507], L' into $L' . (1 - 2 m' Q . \cos. \varepsilon')$; and this change must evidently be made in the values of Y', Y'', [2499, 2502].

* (1855) In the solstices, where v' is nearly equal to ε', we may put, in formula [2506], $\cos. \varepsilon' = \frac{\cos.^2 v'}{\cos. \varepsilon'}$, and then this term becomes

[2507c] $$-2 m' P Q . \frac{\cos.^2 v'}{\cos. \varepsilon'} . \frac{L'}{r'^3} . \cos. 2 . (n t + \varpi - \psi' - \lambda).$$

Connecting this with the term of [2463] depending on $P . \frac{L'}{r'^3}$ [2507b′], the sum becomes

[2507d] $$P . \frac{L' . \left(1 - \frac{2 m' Q}{\cos. \varepsilon'}\right)}{r'^3} . \cos.^2 v' . \cos. 2 . (n t + \varpi - \psi' - \lambda),$$

which may be derived from the second of the preceding terms, by changing as in [2507′] L' into $L' . \left(1 - \frac{2 m' Q}{\cos. \varepsilon'}\right)$; and the same changes must evidently be made in the values of Y', Y'', [2499, 2502].

† (1855a) The observations have since been continued at Brest, as will be seen in
[2507e] Book XIII; where the author has discussed the observations made between the years 1807 and 1823.

number of observations, as well as by the magnitude and regularity of the tides, the most complete and useful collection we have of this kind. We shall [2507‴] *compare the preceding formulas with this collection.* The observations being affected by various circumstances foreign from the action of the sun and moon, we must consider a great number of observations, in order that the effects of these accidental causes may mutually destroy each other, and that the aggregate may give the effect of the regular causes only ; we must also [2507⁗] endeavor to combine the observations in the most advantageous manner to ascertain any particular phenomenon that may be required. In this manner, and *for the purpose of investigating the effect of the declination of the moon, we have considered at the same time two consecutive syzygies,* the result of [2507v] which is nearly independent of the variation of the distance of the moon from the earth. To compare the observations made on this point with the theory, *we have taken from the abovementioned collection twenty-four syzygies of the equinoxes, and twenty-four syzygies of the solstices, always taking two* [2507vi] *consecutive syzygies. The following are the days of these syzygies at Brest.*

SYZYGIES OF THE EQUINOXES.

Years.

1711. 28 August, 12 September, 26 September, 12 October.

1712. 1 September, 15 September.

Times of the syzygies of the equinoxes.

1714. 25 August, 8 September, 23 September, 8 October.

1715. 18 February, 5 March, 20 March, 4 April, 28 August, 13 September, 27 September, 12 October. [2508]

1716. 23 February, 8 March, 23 March, 6 April, 1 September, 15 September.

SYZYGIES OF THE SOLSTICES.

Years.

1711. 16 June, 30 June, 25 November, 9 December.

1712. 19 June, 3 July, 28 November, 13 December.

Times of the syzygies of the solstices.

1714. 29 May, 12 June, 27 June, 11 July, 21 November, 7 December, 21 December. [2509]

1715. 5 January, 17 June, 1 July, 26 November, 10 December, 25 December.

1716. 9 January, 5 June, 19 June.

In each of the syzygies, we have taken the mean between the two
[2509′] *absolute* heights of the same day, or in other words, the *mean absolute height of the tide.* We have considered *the day preceding the syzygy, which is denoted by* -1; *the day of the syzygy, which is denoted by* 0; *and the four following days, which are denoted by* 1, 2, 3, 4. It happened several times that only one tide was observed in a day; we have deduced from it
[2509″] the *mean absolute height*, by allowing half its excess above the tide which was not observed, which quantity had been determined nearly, both for the syzygies of the equinoxes and solstices, by a previous discussion of the observations.

Several times also the height of the low water falling between the two
[2509‴] tides of the same day was not observed. To obtain then the total tide, we have supposed, conformably to the theory, that the difference of the total
[2509⁗] tides, in two consecutive days, is nearly double of the difference of the corresponding mean absolute heights.* We have taken a mean result between the total tides deduced from this supposition and from the observations of the two days between which was included that one under consideration.

Sometimes the law of the observed low tides evidently indicated an error in the sign in one of these heights. In this case we have almost always
[2509ᵛ] neglected the observation of the low water, and have computed the total tide by the rule just mentioned. With these precautions we have calculated *the following table, which represents the sums of the mean absolute tides and the total tides, corresponding to each of the days which we have considered in the preceding syzygies.*†

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* (1856) The term multiplied by $(1+3\,.\cos.2\theta)$ [2482] is very small [2485*b*, *c*],
[2509*a*] and varies but little in the course of a day. If we neglect this, the remaining terms of [2482] are very nearly equal to *half* of the total tide [2483]. The only term in which any difference is found, is that multiplied by $\frac{1}{4}q^2$, which is quite small, being by [2500*d*′] of
[2509*b*] the order $\frac{1}{16}v^2$, or $\frac{1}{16}\cdot\frac{1}{25}$; because v [2493] is the mean synodical motion in $1^{\text{day}},027$, or about $\frac{1}{5}$ of the radius.

† (1857) This table is divided into three parts, as in [2510, 2511, 2512]. The sums of the numbers corresponding to each particular day are equal to the numbers in Table II
[2513*a*] respectively. Thus the total tide corresponding to the day of the syzygy in Table II, is $269^{\text{met.}},037 = 142^{\text{met.}},068 + 62^{\text{met.}},002 + 64^{\text{met.}},967$.

TABLE I.

Observations of the tides at Brest;

SYZYGIES OF THE EQUINOXES.

Days.	Mean absolute heights of the tides.	Total tides.
—1	129met.,890	128met.,988
0	136 ,079	142 ,068
1	139 ,851	149 ,342
2	139 ,962	150 ,066
3	137 ,479	143 ,826
4	131 ,653	131 ,770

in 24 syzygies of the equinoxes; [2510]

SYZYGIES OF THE SUMMER SOLSTICE.

Days.	Mean absolute heights of the tides.	Total tides.
—1	62met.,004	58met.,097
0	63 ,914	62 ,002
1	65 ,028	64 ,095
2	65 ,157	64 ,995
3	64 ,147	63 ,425
4	61 ,914	59 ,186

in 12 syzygies of the summer solstice; [2511]

SYZYGIES OF THE WINTER SOLSTICE.

Days.	Mean absolute heights of the tides.	Total tides.
—1	65met.,211	61met.,098
0	66 ,456	64 ,967
1	67 ,121	67 ,202
2	66 ,424	67 ,500
3	65 ,998	66 ,187
4	63 ,574	61 ,425

in 12 syzygies of the winter solstice. [2512]

24. We shall consider at first the whole of these observations, and we shall have, for the forty-eight syzygies, [2512′]

TABLE II.

Days.	Mean absolute heights.	Total tides.
—1	257met.,105	248met.,183
0	266 ,449	269 ,037
1	272 ,000	280 ,639
2	271 ,543	282 ,561
3	267 ,624	273 ,438
4	257 ,141	252 ,381

Combination of all the 48 preceding observations of the tides. [2513]

If we examine the variations of the absolute heights of the total tides of this table, we shall see that the greatest tides do not take place on the day of the syzygy, but between the first and the second days. To determine the distance of the maximum of the tides from the syzygy, in the preceding
[2513'] observations, *we shall take for the unit of time, the interval of two morning tides or of two evening tides in the syzygies; and for epoch, the time of the low water falling between the two tides on the day which precedes the syzygy.* We shall also put, for any day near this phase,

Unit of time and epoch.

[2514]

$$a+bx-cx^2 = \text{the absolute height of the tide ;*}$$

Absolute height of the tide.

x being the number of intervals taken for unity which this tide follows after the epoch. If this formula refer to a morning tide, the expression of the
[2515] tide of the evening of the same day will be $a+b\,.\,(x+\tfrac{1}{2})-c\,.\,(x+\tfrac{1}{2})^2$, considering those inequalities whose period is nearly a day, which are the only ones necessary to be noticed here, because the effects of the other
[2515'] inequalities mutually compensate each other in the observations of Table II, [2513]. If we add the two preceding expressions, the half of their sum will be what we have called [2471] the *mean absolute height of the tide*; therefore the expression of this height is

Mean absolute height of the tide.

[2516]

$$a-\tfrac{1}{16}c+b\,.\,(x+\tfrac{1}{4})-c\,.\,(x+\tfrac{1}{4})^2.$$

The expression of the intermediate low water, according to the theory, is of the form†

[2517]

$$a'-b\,.\,(x+\tfrac{1}{4})+c\,.\,(x+\tfrac{1}{4})^2\,;$$

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* (1858) The expressions of y', y'', [2482, 2483], contain the quantities r, v, r', v',
[2514a] corresponding to the time of the syzygy; and the quantity $(\psi'-\psi)^2$, which is proportional to the square of the time from the syzygy t'; so that either of these expressions might be put under the form $a'-ct'^2$. If we alter the epoch of t', making $t'=b'+x$, we get

[2514b]

$$a'-c\,.\,(b'+x)^2, \qquad \text{or} \qquad (a'-b'^2c)-2b'cx-cx^2\,;$$

which by putting $a'-b'^2c=a$, $-2b'c=b$, becomes $a+bx-cx^2$, as in [2514].

† (1859) The quantity multiplied by $(1+3\,.\cos.2\theta)$ [2472, 2474] is very small in
[2517a] comparison with the other terms [2485b]; and by [2481'] the term (A') is also extremely small; hence the variations of these quantities may be neglected in comparison with those of the remaining term of [2472, 2474], multiplied by P. This term has a *positive* sign in
[2517b] [2472], and a *negative* sign in the height at low water [2474], but in other respects is equal;

$x+\frac{1}{4}$ being the interval between the epoch and that low tide; therefore by putting this time equal to t, we shall have an expression of the following form [2517′]

Total tide.

$$a''+2b.t-2c.t^2 = \text{the height of the total tide.} \quad [2518]$$

The *maximum* of this tide takes place when $t=\frac{b}{2c}$;* this value is also [2519]
that of a, corresponding to the *maximum* of the function $a+bx-cx^2$. [2520]

To determine $\frac{b}{2c}$ by observation, we must use the total tides of Table II [2513]; but the mean absolute heights of this table having been observed more carefully than the total tides, we shall make use of the sums of both of them. [2520′]

Therefore let f, f', f'', f''', f'''', f^{v},† be the six sums obtained by adding the absolute height and the total tide of each day in Table II [2513]. [2520″]
The analytical expression of these sums will be of the form $k+ib.t-ic.t^2$; [2521]
and by supposing successively $t=0$, $t=1$, $t=2$, $t=3$, $t=4$, $t=5$, we shall have the values of f, f', f'', f''', f'''', f^{v}; hence we may deduce‡

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hence its variation during the interval x, or from $x=0$ to $x=x$, must be the same in both expressions, but with different signs. Now by hypothesis [2514], this variation in [2472] is $bx-cx^2$, therefore it must be at the same instant $-bx+cx^2$ in [2474]; [2517c]
and at the time of *low water* x changes into $x+\frac{1}{4}$, consequently this expression is then $-b.(x+\frac{1}{4})+c.(x+\frac{1}{4})^2$; so that if we put a' for the height when $x+\frac{1}{4}=0$, we shall have for the height at low water $a'-b.(x+\frac{1}{4})+c.(x+\frac{1}{4})^2$, as in [2517]. [2517d]
Subtracting this from [2516], we get the *total* tide

$$a-\tfrac{1}{16}c-a'+2b.(x+\tfrac{1}{4})-2c.(x+\tfrac{1}{4})^2;$$

which by putting $x+\frac{1}{4}=t$, $a-\frac{1}{16}c-a'=a''$, becomes as in [2518]. [2517e]

* (1860) This is found by taking the differential of [2518], and putting $it=0$, [2519a]
according to the usual rule for finding the *maximum* [2464a].

† (1861) From Table II [2513] we get $f=257^{\text{met.}},105+248^{\text{met.}},183=505^{\text{met.}},288$; [2520a]
and in like manner we get f', f'', &c., as in the table [2522a].

‡ (1861a) Putting in [2516] $x+\frac{1}{4}=t$ [2517e], the absolute height becomes $a-\frac{1}{16}c+b.t-c.t^2$. Adding this to the total tide [2518], we get

$$a-\tfrac{1}{16}c+a''+3b.t-3c.t^2;$$

[2522] $$12\,ic = f''' + f'' - f^{v} - f;$$

[2522'] $$ib = 5\,ic + \frac{f^{v} + f'''' + f''' - f'' - f' - f}{9};$$

consequently

[2523] $$\frac{b}{2c} = \tfrac{5}{2} + \frac{2\,.\,(f^{v} + f'''' + f''' - f'' - f' - f)}{3\,.\,(f''' + f'' - f^{v} - f)}.$$

Substituting the numerical values of f, f', &c., we shall find*

[2524] $$\frac{b}{2c} = 2{,}58176.$$

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which by putting $a - \frac{1}{16}c + a'' = k$, and $i = 3$, becomes $k + ib\,.\,t - ic\,.\,t^2$, as in formula [2521].
[2522a] Substituting successively $t = 0$, 1, 2, 3, 4, 5, we obtain the six equations in the adjoined table, which may be combined in various manners to find k, b, c. The principle of the least squares [849k] is now most
[2522b] generally used, and we shall hereafter [2524c] compute these equations by this method; at present we shall restrict ourselves to the method of the author. Adding the third and fourth equations, and from the sum subtracting the first and sixth, we get

f	$= 505^{\text{met.}},288$	$= k$
f'	$= 535\ ,486$	$= k + ib - ic$
f''	$= 552\ ,639$	$= k + 2ib - 4ic$
f'''	$= 554\ ,104$	$= k + 3ib - 9ic$
f''''	$= 541\ ,062$	$= k + 4ib - 16ic$
f^{v}	$= 509\ ,522$	$= k + 5ib - 25ic$

$$f''' + f'' - f^{v} - f = 12\,ic \qquad [2522].$$

Again, by adding the three last equations, and subtracting the sum of the three first we obtain $f^{v} + f'''' + f''' - f'' - f' - f = 9\,ib - 45\,ic$; Dividing this by 9, and transposing $5\,ic$, we get [2522']. Again dividing [2522'] by $2\,ic$, we have

[2522c] $$\frac{b}{2c} = \tfrac{5}{2} + \frac{f^{v} + f'''' + f''' - f'' - f' - f}{18\,ic};$$

and by substituting for $18\,ic$ its value deduced from [2522], $\frac{3}{2}\,.\,(f''' + f'' - f' - f)$, it becomes as in [2523].

* (1862) The values of f, f', &c., found in the last note, give

$$f''' + f'' - f^{v} - f = 91^{\text{met.}},933, \quad \text{and} \quad f^{v} + f'''' + f''' - f'' - f' - f = 11^{\text{met.}},275;$$

hence from [2523],

[2524a] $$\frac{b}{2c} = \tfrac{5}{2} + \frac{2 \times 11{,}275}{3 \times 91{,}933} = \tfrac{5}{2} + 0{,}08176 = 2{,}58176;$$

which differs a little from that in [2544], where the value of $\frac{b}{2c}$ is determined more accurately, by a different method.

Therefore in the syzygies of Table II [2513], the interval from the time of low water falling between the two tides on the day of the syzygy, to the time of the maximum of the total tide, is 1,58176. We shall hereafter see, [2525]
[2745 or 2493b], that the interval taken for unity is very nearly $1^{\text{day}},02705$; [2526]
therefore if we multiply it by 1,58176, the product $1^{\text{day}},62455$ will express [2527]
the interval between the *maximum* of the total tide and the time of low tide on the morning of the syzygy. This time is the middle hour between the two high tides of that day, and we find by a mean of the results, that in the preceding observations it was $0^{\text{day}},39657$.* We also find by a mean result, [2528]
that the hour of the syzygy at Brest, in the same observations, was [2747] $0^{\text{day}},45667$, so that it followed the time of low water $0^{\text{day}},06010$; therefore [2529]
it preceded the *maximum* of the total tide by $1^{\text{day}},56445$. But as an error [2530]
of a few metres in these observations might materially affect the result of

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If we combine the equations [2522a] by the method of the least squares [849k], we shall get the three following equations [2524c]. The first is the sum of all these equations depending on the coefficient of k [2522a], which is equal to unity. The second is the sum [2524b]
of the products of these equations by the coefficients of ib, namely 0, 1, 2, 3, 4, 5. The third is the sum of the products of the same equations by the coefficients of $-ic$, namely 0, 1, 4, 9, 16, 25.

$$3198,101=6k+15ib-55ic, \quad 8014,934=15k+55ib-225ic, \quad 29128,020=55k+225ib-979ic. \qquad [2524c]$$

From these equations we easily obtain, by the common methods of elimination, the following values nearly, $k=504,6$, $ib=39,47$, $ic=7,669$. Hence

$$\frac{b}{2c}=\frac{ib}{2ic}=\frac{39,47}{15,338}=2,5733;$$

which is rather more correct than that found in [2524]. If we repeat the operation [2524—2530] with this new value, we find

$$(2,5733-1,0000)\times 1^{\text{day}},02705=1^{\text{day}},61586; \qquad [2524d]$$

subtracting $0^{\text{day}},06010$ [2529], we get $1^{\text{day}},55576$, instead of $1^{\text{day}},56445$ [2530]; the former being rather nearer to the value finally assumed in [2544].

* (1863) In the Table [2738], the time of high water at Brest, on the day of the syzygy of the equinox, is $0^{\text{day}},39708$, and on that of the solstice, $0^{\text{day}},39606$ [2739]; the mean is $0^{\text{day}},39657$, as in [2528]. Subtracting this from $0^{\text{day}},45667$ [2529], we get [2530a]
$^{\text{day}},06010$ [2529], and then from [2527] we obtain

$$1^{\text{day}},62455-0^{\text{day}},06010=1^{\text{day}},56445 \qquad [2530].$$

this computation of the interval, it is proper to determine this important element of the theory of the tides in a more accurate manner.

[2530′] *For this purpose we shall consider the absolute heights of the tides on the second day before and on the fifth day after the syzygy; they are nearly at equal distances on each side of the maximum of the tides, and at this distance they vary in the most sensible manner.* We have added the absolute heights
[2530″] of the tides on the morning and evening of the second day before each syzygy, and when only one height had been observed, have doubled it. The abovementioned collection of observations contains one hundred syzygies, in
[2531] which such observations could be procured. We find $1009^{met.},470$ for the sum of the absolute heights of the tides of the second day preceding the syzygy, and $1010^{met.},886$ for the sum of the absolute heights of the tides on the fifth day following the syzygy. But among the heights which precede the syzygies, 86 correspond to the morning and 114 to the evening; therefore *by taking for the unit of time the interval from the low water on the*
[2532] *second day which precedes the syzygy to the low water which happens about one day later*, the mean hour to which the first sum refers, follows that of the low water on the second day before the syzygy, by $\frac{14}{400}$, or 0,035 of
[2533] this interval.* In the second sum, there are as many morning as evening tides; the hour to which it corresponds is therefore that of the low tide on
[2533′] the fifth day after the syzygy. Thus the middle of the interval comprised between the times to which these sums correspond, is not exactly the middle of the interval comprised between the time of low tide in the morning of the

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[2533a] * (1864) The interval taken for unity [2532] being the time elapsed between the low water two days before the syzygy and the low water one day before the syzygy, the time from low water to the high water immediately preceding is $-\frac{1}{4}$, and to the high water immediately following is $+\frac{1}{4}$. The 86 morning tides will therefore *decrease* the sum of the hours by $-86\times\frac{1}{4}$, and the 114 evening tides will *increase* it by $114\times\frac{1}{4}$; the whole sum will
[2533b] therefore be *increased* by $(114-86)\times\frac{1}{4}=7$; This being divided by the whole number of observations 200, gives 0,035 for the correction of the mean of the times arising from the inequality in the number of observations. This quantity must be subtracted from the mean of the times in the above observations, to obtain the value corresponding to the case
[2533c] where the number of observations in the morning and evening are equal. If we take the mean of this corrected time, and that relative to the fifth day after the syzygy, this correction
[2533d] will be decreased by half the quantity [2533b], or 0,0175, which is to be applied to the mean of all the observations [2534].

second day before the syzygy, and that of the low tide corresponding to the fifth day after the syzygy, but is nearer to this second limit by 0,0175 [2533d]. [2534]

If the two sums $1009^{\text{met.}},470$ and $1010^{\text{met.}},886$ were equal, this middle would be the instant of the *maximum* of the tides; but the second sum exceeds the first by $1^{\text{met.}},416$; therefore the instant of this maximum is rather nearer to the hour of low tide on the fifth day after the syzygy. The absolute height of the tides on that day, and on the second day before the syzygy, varies at Brest by $0^{\text{met.}},14803$,* during one half of the interval taken for unity. Supposing therefore that the instant of the maximum of the tides approaches towards the second limit by a hundredth part of this interval; the sum of the 200 heights relative to that second limit will be increased by $0^{\text{met.}},59212$,† and the sum of the 200 heights relative to the first limit will be diminished by the same quantity; so that the difference of these two sums will be the double of this quantity, or $1^{\text{met.}},18424$; and since observation gives $1^{\text{met.}},416$ [2534″] for this difference, the maximum of the tides must by this means be brought towards the second limit by the quantity 0,01196.‡ Adding this to 0,0175 [2534], we obtain 0,02946 [2534′] [2534″] [2535] [2535′] [2536] [2537] [2538]

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* (1865) This quantity was determined by observations; but we may obtain nearly the same result from formula [2514] $a+bx-cx^2$, by applying it to the absolute heights, given in [2513], for the times -1, 0, 1; from which we get [2535a]

$$a-b-c=257^{\text{met.}},105 \qquad a=266^{\text{met.}},449, \qquad a+b-c=272^{\text{met.}},000.$$

Half the sum of the first and third of these equations, subtracted from the second, gives $c=1^{\text{met.}},896$. Half the difference of the first and third gives $b=7^{\text{met.}},447$. Hence the preceding formula is $266^{\text{met.}},449+7^{\text{met.}},447\,.\,x-1^{\text{met.}},896\,.\,x^2$. When $x=-2$, this becomes $243^{\text{met.}},971$; and when $x=-1\frac{1}{2}$, it becomes $251^{\text{met.}},013$; the difference of these two quantities $7^{\text{met.}},042$, divided by 48, the number of the syzygies [2512′], gives $0^{\text{met.}},147$, corresponding to one syzygy; being nearly the same as in [2535]. [2535b] [2535c]

† (1866) The variation of the height of the tide in the time $\frac{1}{2}$ being $0^{\text{met.}},14803$ [2535], its variation in the time $\frac{1}{100}$ will be $\frac{2}{100}\times 0^{\text{met.}},14803=0^{\text{met.}},0029606$. Multiplying this by the number of tides 200, we get $0^{\text{met.}},59212$ [2536]. [2536a]

‡ (1867) This is obtained by supposing the variations of the heights to be proportional to the variations of the times; whence we get, by using the numbers [2537, 2534″ 2535′], the ratio $1^{\text{met.}},18424$, to $1^{\text{met.}},416$ as 0,01 to 0,01196; therefore a variation of the heights $1^{\text{met.}},416$ corresponds to a variation of the time 0,01196 [2538]. [2537a]

[2538′] for the time by which the *maximum* of the tides approaches nearer to the hour of low tide on the fifth day after the syzygy, than the middle time between this low tide and that of the second day before the syzygy. This
[2539] time estimated in parts of a day is nearly $0^{\text{day}},03022$.*

Now the middle of the interval comprised between these two low tides,
[2539′] is the same as that comprised between the low tide on the morning of the syzygy and the low tide corresponding to the third day which follows it;† the interval of two consecutive morning low tides being then $1^{\text{day}},02705$
[2540] [2536], this second middle time is distant $1^{\text{day}},54058$ [2539*b*] from the low tide on the morning of the day of the syzygy. The hour of this low
[2541] water may be supposed $0^{\text{day}},39657$,‡ as in the observations of Table II

[2538*a*] * (1867*a*) This is obtained by multiplying 0,02946 [2538] by the interval taken for unity, which is nearly $1^{\text{day}},02705$ [2526].

† (1868) The mean of the times of the total tides, in the equinoxes and solstices [2740],
[2539*a*] for the days 0, 1, 2, 3, counted from the syzygy, are $0^{\text{day}},39657$, $1^{\text{day}},42407$, $2^{\text{days}},45051$, $3^{\text{days}},47772$. The mean of the first and last gives the middle interval of these days $1^{\text{day}},93714$; and the mean of the second and third gives in like manner the middle interval of these two days $1^{\text{day}},93729$; differing from the preceding value only 0,00015, which is in conformity with the principles assumed in [2539′], extended to the days -2, $+5$. Now the interval of time from the low tide on the day 0, to the low tide on the day 3, is
[2539*b*] $3 \times 1^{\text{day}},02705 = 3^{\text{days}},08115$ [2526], the half of which, or $1^{\text{day}},54058$, represents, as in [2540], the time in which the middle of this interval follows the low tide on the morning of the day of the syzygy.

‡ (1869) This is the time of the total tide given in [2740] corresponding to the syzygies of Table VI [2738, 2739]; it represents the middle time between the high tides of the same day [2737], and is evidently the time of low water falling between these two tides, so that
[2541*a*] for the syzygies used in [2740] the time of low tide is $0^{\text{day}},39657$; the mean hour of these
[2541*b*] syzygies is $0^{\text{day}},45667$ [2747]. Now the mean of the times of the 100 syzygies above considered is $0^{\text{day}},46013$ [2542], which differs from the preceding only $0^{\text{day}},00346$; and as the daily variation of the time of the tide is $0^{\text{day}},02705$ [2526], this difference will affect the time of the tide by the very small quantity $0^{\text{day}},00346 \times 0,027051 = 0^{\text{day}},00009$, which may be neglected, and the hour of low tide assumed to be $0^{\text{day}},39657$, for these 100 syzygies. Subtracting this time from the mean time of the syzygy $0^{\text{day}},46013$ [2542],
[2541*c*] we get $0^{\text{day}},06356$ for the time which this low tide precedes the syzygy. Now the middle of the interval computed in [2539*b*] followed after this low tide by the time $1^{\text{day}},54058$, consequently this middle interval happened after the syzygy by the difference of these two

[2513], because the mean hour at Brest of the hundred syzygies we have considered is $0^{\text{day}},46013$, nearly as in the observations of that table; therefore the syzygy happened after the morning low water by $0^{\text{day}},06356$ [2541c], consequently it preceded the middle of the interval comprised between these two limits by $1^{\text{day}},47702$. Adding it to $0^{\text{day}},03022$ [2539], *we have* $1^{\text{day}},50724$ *for the interval by which the maximum of the tide at Brest follows the syzygy.* This value will hereafter be used.

Time of high water happens [2542] about a day and a half after the syzygy at Brest. [2543] [2544]

25. *We shall now determine the law of the variations of the mean absolute heights of the tide, and of the total tides of Table II* [2513]. *For this purpose, we shall take for unity the interval of two consecutive tides of the morning and evening, near the syzygies;* and we shall put k for the quantity by which the time of the maximum of the tides follows the middle of the interval between the six days of observation which we have considered. We shall put $a - b \cdot t^2$* for the general expression of the mean absolute heights of the tides of Table II [2513], t being the interval from the time of *maximum*. The mean absolute heights of the tides corresponding to the days $-1,\ 0,\ 1,\ 2,\ 3,\ 4,$ will be†

k. [2545] [2546]

$$a - b\cdot(\tfrac{5}{2}+k)^2, \qquad a - b\cdot(\tfrac{3}{2}+k)^2, \qquad a - b\cdot(\tfrac{1}{2}+k)^2,$$
$$a - b\cdot(\tfrac{1}{2}-k)^2, \qquad a - b\cdot(\tfrac{3}{2}-k)^2, \qquad a - b\cdot(\tfrac{5}{2}-k)^2. \qquad [2547]$$

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quantities, or $1^{\text{day}},54058 - 0^{\text{day}},06356 = 1^{\text{day}},47702$ [2543]. This requires an addition of $0^{\text{day}},03022$ [2539], on account of the unequal number of the observed morning and evening tides, and the inequality of the sums of the absolute heights [2531]. Hence the corrected value $1^{\text{day}},47702 + 0^{\text{day}},03022 = 1^{\text{day}},50724$, which represents the interval from the syzygy to the *maximum* tide at Brest, as in [2544]. [2541d] [2541e]

* (1870) This formula is the same as that in [2514], altering the epoch by the quantity $\frac{b}{2c}$ [2524]. For by putting $x = t + \frac{b}{2c}$ in [2514], we get

$$a + b\cdot\left(t+\frac{b}{2c}\right) - c\cdot\left(t+\frac{b}{2c}\right)^2 = a + \frac{bb}{4c} - c\cdot t^2; \qquad [2546a]$$

and if we change $a + \frac{bb}{4c}$ into a, c into b, it becomes of the form assumed in [2546]; so that the quantities a, b, of this article are different from those in § 24 [2515, &c.]. The values of these coefficients are computed in [2553a—c].

† (1871) The mean of the times $-1,\ 0,\ 1,\ 2,\ 3,\ 4,$ is $\frac{3}{2}$; this added to k will give the time of the maximum of the tides [2545] $\frac{3}{2}+k$. Subtracting this from the [2547a]

If from the sum of the third and fourth, we subtract the sum of the extremes,
[2548] we shall have $12\,b$ for their difference. The results of Table II [2513]
[2549] give $29^{\text{met.}},297$ for this difference; hence we deduce $b = 2^{\text{met.}},4414$.

[2550] If in like manner we represent the total tides of Table II [2513] by $a' - b' \,.\, t^2$, we shall find in the same manner*

[2551] $$b' = 5^{\text{met.}},2197.$$

[2552] According to the theory explained in § 22, $b = \frac{1}{2}\,b'$; consequently $b = 2^{\text{met.}},6098$.† The difference between this value and $2^{\text{met.}},4414$ [2549], given by the absolute mean heights of the tides, is within the limits of the errors of the observations; but the absolute height being observed more
[2552'] carefully than the low water, we shall take for b the third of the sum of the

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times [2547a], we obtain the values of t which correspond to those days respectively, $-(\frac{5}{2}+k)$,
[2547b] $-(\frac{3}{2}+k)$, $-(\frac{1}{2}+k)$, $\frac{1}{2}-k$, $\frac{3}{2}-k$, $\frac{5}{2}-k$. Substituting these in the formula $a - b\,.\,t^2$ [2546], we get the expressions [2547]; putting
[2547c] these equal to the mean absolute heights of the tide in [2513], we obtain the annexed system of equations.

$$\begin{aligned}
a - b\,.\,(\tfrac{5}{2}+k)^2 &= 257^{\text{met.}},105 = f\\
a - b\,.\,(\tfrac{3}{2}+k)^2 &= 266\;\;,449 = f'\\
a - b\,.\,(\tfrac{1}{2}+k)^2 &= 272\;\;,000 = f''\\
a - b\,.\,(\tfrac{1}{2}-k)^2 &= 271\;\;,543 = f'''\\
a - b\,.\,(\tfrac{3}{2}-k)^2 &= 267\;\;,624 = f''''\\
a - b\,.\,(\tfrac{5}{2}-k)^2 &= 257\;\;,141 = f^{\text{v}}
\end{aligned}$$

From these we may eliminate a, k, and obtain a large coefficient for b, by subtracting the sum of the extreme equations from the sum of the third and fourth, which
[2547d] gives $12\,b = 29,297$, whence $b = 2^{\text{met.}},4414$. Moreover, it is evident that this value
[2547e] of $12\,b$ is equal to $f'' + f''' - f - f^{\text{v}}$, as in [2548].

* (1872) We must change a, b, into a', b', in the equations [2547b], and for the numbers of the second members of these equations put the total tides of [2513], namely $248^{\text{met.}},183$,
[2547f] $269^{\text{met.}},037$, $280^{\text{met.}},639$, $282^{\text{met.}},561$, $273^{\text{met.}},438$, $252^{\text{met.}},381$; then subtracting the sum of the first and last of these equations from the sum of the third and fourth, we get, as
[2547g] in [2547d], $12b' = 62^{\text{met.}},636$; hence $b' = 5^{\text{met.}},2197$, as in [2551].

† (1873) If we put a for the terms of [2499] independent of t^2, and b for the coefficient
[2553a] of t^2, it will become $Y' = a - b\,.\,t^2$. Also if we put a' for the terms of [2502]
[2553b] independent of t^2, and b' for the coefficient of t^2, we shall get $Y'' = a' - b'\,.\,t^2$. Comparing the coefficient of t^2 [2499] with that in [2502], we find that the former is half of the latter, or $b = \frac{1}{2}\,b'$, as in [2552]. Substituting in this the value of b' [2551], we
[2553c] obtain $b = 2,60985$ [2552], instead of $b = 2,4414$ [2549].

two values of b and b' given by the observations,* and for b' the double of this quantity; hence we shall have

$$b = 2^{\text{met.}},5537 \; ; \qquad b' = 2\,b = 5^{\text{met.}},1074. \qquad [2553]$$

To determine a, a', we shall observe that the sum of the six preceding expressions of the mean absolute heights of the tides, is

$$6\,a - b \,.\, \{\tfrac{35}{2} + 6\,.\,k^2\}.\dagger \qquad [2554]$$

By the observations of Table II [2513], this formula is equal to $1591^{\text{met.}},862$; therefore we have

$$a = \frac{1591^{\text{met.}},862 + (\frac{35}{2} + 6\,k^2)\,.\,2^{\text{met.}},5537}{6} \; ; \qquad [2555]$$

in the same manner we have

$$a' = \frac{1606^{\text{met.}},239 + (\frac{35}{2} + 6\,k^2)\,.\,5^{\text{met.}},1074}{6}. \qquad [2556]$$

The mean hour of the syzygy in Table II [2513], is $0^{\text{day}},45667$ [2747]; adding to it $1^{\text{day}},50724$ [2544], the distance of the *maximum* of the tide

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* (1874) This is equivalent to the mean of three values of b, $2^{\text{met.}},4414$, $2^{\text{met.}},60985$, $2^{\text{met.}},60985$; one of which was deduced from the absolute mean heights of the tide, and two from the total tides; so that the total tides have the greatest influence, which seems inconsistent with the remarks in [2552′]; but as very great accuracy is not to be expected in the results of observations of the tides, we shall not make any alteration in these numbers. [2553d]

† (1875) Taking the sum of the equations [2547b], we get

$$6\,a - b\,.\,(\tfrac{35}{2} + 6\,k^2) = 1591^{\text{met.}},862, \qquad [2555a]$$

from which we find a [2555], using the value of b [2553]. In like manner, if we change a, b, into a', b', and use the total tides, as in [2547f], we get

$$6\,a' - b'\,.\,(\tfrac{35}{2} + 6\,k^2) = 1606^{\text{met.}},239.$$

Substituting b' [2553], we obtain [2556]. In the preceding equation the coefficient of a' is the same as in the method of the least squares; and if we wish to use this method, we may proceed in the following manner. Instead of k [2559′], we shall put $k = 0{,}026055 + \frac{k'}{b}$; [2555b]
substituting this in the equations [2547b], neglecting k'^2 on account of its smallness, we obtain the same number of equations of the form $a - 6{,}381\,.\,b - 5{,}052\,.\,k' = 257^{\text{met.}},105$, &c. From these last equations we get the values of a, b, k', as in [849k]. [2555c]

[2557] from the syzygy, we get $1^{day},96391$ for the distance of this *maximum* from the midnight which precedes the syzygy at Brest. The middle time between the two tides on the day of the syzygy is $0^{day},39657$ [2541]; adding to
[2558] it $\frac{3}{2}$ of the interval taken for unity, which is $1^{day},02705$ [2526], we have $1^{day},93715$ for the time clapsed from the midnight preceding the syzygy to the middle of the interval between the six days of observation. If we subtract this from $1^{day},96391$, we shall obtain the value of k [2545]
[2559] expressed in days, equal to $0^{day},02676$; dividing it by 1,02705 [2526],
[2559'] we shall find, in parts of the interval taken for unity, $k = 0,026055$; which gives*

[2560] $$a = 272^{met.},760, \qquad a' = 282^{met.},606.$$

Hence the expression of the numbers, relative to the mean absolute heights of the tides of Table II [2513], is

Absolute heights.

[2561] $$272^{met.},760 - 2^{met.},5537 \,.\, t^2;$$

and the expression of the numbers of the same table, relative to the total tides, is

Total tides.

[2562] $$282^{met.},606 - 5^{met.},1074 \,.\, t^2;$$

the values of t relative to all these numbers being respectively

[2563] $$\begin{array}{ccc} -2,526055, & -1,526055, & -0,526055, \\ 0,473945, & 1,473945, & 2,473945. \end{array}$$

Substituting these values in the two preceding formulas [2561, 2562], and
[2563'] comparing the results with the numbers of Table II [2513];† we shall find the differences to be very small, and within the limit of the errors to which the observations are liable.

e,
[2564] *We shall now compare these formulas, given by observation, with those of* § 22, *deduced by the theory of gravity.* Putting e for the height of the *zero*

[2562a] * (1876) This value of k being substituted in [2555, 2556] gives a, a', [2560]; substituting these, and b, b', [2553], in [2546, 2550, &c.], we get [2561, 2562]. The same value of k being substituted in the coefficients of b [2547b], they become as in [2563].

† (1877) Multiplying the squares of the coefficients [2563] by $b = 2,5537$ [2553],
[2563a] we get the quantities $b \,.\, (\frac{5}{2} + k)^2$, $b \,.\, (\frac{3}{2} + k)^2$, &c., [2547]; $16^{met.},295$,

of the scale of observation above the level of the equilibrium which the sea would have independently of the action of the sun and moon; also h for the sum of the squares of the declinations of the sun at the times of the phases in the syzygies of Table II [2513], and h' for the like sum relative to the moon, we shall have, by [2499, 2502],* h, [2564'] h'.

$$a = 48\,\mathrm{e} - \frac{3\,.\,(1+3\,.\,\cos.2\theta)}{8g\,.\left(1-\frac{3}{5\rho}\right)}\,.\left\{\frac{L}{r^3}\,.\,(h-32)+\frac{L'}{r'^3}\,.\,(h'-32)\right\}$$

$$+P\,.\left\{\frac{h\,.L}{r^3}+\frac{h'\,.\,L'}{r'^3}\right\}-\frac{b}{16}=272^{\text{met.}},760\,; \qquad [2565]$$

$$a' = 2P\,.\left\{\frac{h\,.L}{r^3}+\frac{h'\,.\,L'}{r'^3}\right\}-\frac{b}{16}=282^{\text{met.}},606. \qquad [2566]$$

$5^{\text{met.}},947$, $0^{\text{met.}},707$, $0^{\text{met.}},574$, $5^{\text{met.}},548$, $15^{\text{met.}},630$, respectively. Subtracting these from $a=272^{\text{met.}},760$, we get

The corresponding numbers of formula [2561],	256,465, 266,813, 272,053, 272,186, 267,212, 257,130;
The numbers corresponding in Table II are	257,105, 266,449, 272,000, 271,543, 267,624, 257,141; [2563b]
The differences or errors for 48 absolute heights,	—0,640, +0,364, +0,053, +0,643, —0,412, —0,011.

Doubling the numbers 16,295, &c. [2563a], we obtain the corresponding numbers of formula [2562], $32^{\text{met.}},590$, $11^{\text{met.}},894$, $1^{\text{met.}},414$, $1^{\text{met.}},148$, $11^{\text{met.}},096$, $31^{\text{met.}},260$. Subtracting these from $a'=282,606$,

We get - - - - - - - -	250,016, 270,712, 281,192, 281,458, 271,510, 251,346;
The corresponding numbers of Table II are	248,183, 269,037, 280,639, 282,561, 273,438, 252,381; [2563c]
The differences for 48 total tides, - -	+1,833, +1,675, +0,553, —1,103, —1,928, —1,035.

Hence we see that the greatest error in the mean absolute heights for forty-eight syzygies is $0^{\text{met.}},643$, and for the total tides, neglecting its sign, $1^{\text{met.}},928$; the mean errors for one tide are $\frac{0,643}{48}=0^{\text{met.}},013$ and $\frac{1,928}{48}=0^{\text{met.}},04$; so that even the extreme errors are very small. We also see that the greatest errors are in the total tides, as is observed in [2552']. [2563d]

* (1878) We have $1-3\,.\,\sin.^2V=1-3\,.\,(1-\cos.^2V)=3\,.\cos.^2V-2$, and in $2i$ or 48 syzygies, the sum is $48\,.\,(3\,.\,\cos.^2V-2)$; but as V varies in the different syzygies, the actual values must be found at each syzygy, and their sum taken as in [2564'], putting $2i\,.\,\cos.^2V=48\,.\,\cos.^2V=h$; and then the preceding expression becomes $2i\,.\,(1-3\,.\,\sin.^2V)=48\,.\,(3\,.\,\cos.^2V-2)=3\,.\,(h-32)$. In like manner we find [2565a] [2565b]

[2567] We have seen in § 11 [2301], that $\frac{3L}{4r^3g} = 0^{\text{met.}},12316$. The latitude of Brest being $535685''$, we have very nearly $2\theta = 928630''$.* Neglecting in comparison with unity the very small factor $\frac{3}{5\rho}$, because the mean

[2567′] density of the earth is several times greater than that of the sea,† we shall have

[2568]
$$\frac{(1+3\,.\cos.2\theta)}{8g\,.\left(1-\frac{3}{5\rho}\right)}\,.\,\frac{L}{r^3} = 0^{\text{met.}},02745.$$

We shall hereafter see [2706], that in the mean distances of the moon from

[2568′] the earth $\frac{L'}{r'^3} = 3\,.\frac{L}{r^3}$; but the distance of the moon in the syzygies is

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[2565c] $2i\,.\cos.^2V' = h'$, $2i\,.(1-3\,.\sin.^2V') = 3\,.(h'-32)$. These are to be substituted in [2499], and the whole is to be put under this form $Y' = a - b\,.t^2$ [2553a], and then the expression of a will become as in [2565]. For the two first lines of [2499] produce in

[2565d] a the terms multiplied by $(1+3\,.\cos.2\theta)$ and P [2565]; and the term $48c$ must evidently be introduced in [2565], for the height of the zero of the scale. Again, $-b$ [2553a] is put for the coefficient of t^2 in [2499]; and as t^2 makes part of the factor

[2565e] $t^2+\frac{1}{16}$ [2499], the corresponding terms are $-b\,t^2-\frac{1}{16}b$, of which the quantity $-\frac{1}{16}b$ makes part of the value of a in the expression of Y' [2565c]; hence the complete expression of a becomes as in [2565]. Substituting in it the value of a [2560], we obtain the equation [2565]. In precisely the same manner we may put Y'' [2502] under the

[2565f] form $Y'' = a' - b'\,t^2 = a' - 2b\,t^2$ [2553b, 2553], and we shall obtain a' [2566]; then substituting a' [2560], we get the equation [2566].

* (1879) In [323iv] θ is put equal to the angle formed by the axis of the earth, and the

[2567a] radius drawn from the centre of the earth to the particle of the fluid. The complement of this angle is sometimes called the *reduced latitude*, or the *geocentric latitude;* the difference between this and the actual latitude is the *reduction* computed in [1579s], and is about $19'$ for Brest. In calculating the tides, the oblateness of the earth is neglected, and instead

Reduced, or geocentric, latitude.

[2567b] of θ the complement of the geographical latitude [2128xi] may be used. The author seems however to have taken the geocentric latitude; for the geographical latitude is nearly $53^\circ\,74'$ [2318′], the reduced latitude $53^\circ\,55'$ nearly [2567].

† (1880) The smallness of this fraction appears from [2280c]. The value of 2θ gives

[2568a] $\cos.2\theta = 0,1118$, $\frac{1+3\,.\cos.2\theta}{6} = 0,2225$; multiplying this by $\frac{3L}{4r^3g} = 0^{\text{met.}},12316$ [2567], we obtain [2568] nearly.

less by about $\frac{1}{120}$ than its mean distance, by reason of the argument of [2568″]
variation, which always decreases the moon's distance in these parts of the
orbit;* therefore in the syzygies we must put $\frac{L'}{r'^3}=\frac{123}{40}\cdot\frac{L}{r^3}$. In the 48 [2569]
syzygies of Table II [2513] we have

$$h=44{,}13399, \qquad h'=44{,}50884\,; \tag{2570}$$

hence we find

$$\frac{3\,.\,(1+3\,.\,\cos.2\,\theta)}{8g}\cdot\left\{\frac{L}{r^3}\cdot(h-32)+\frac{L'}{r'^3}\cdot(h'-32)\right\}=4^{\text{met.}},1666\,; \tag{2571}$$

consequently†

$$48\,\mathrm{e}-4^{\text{met.}},1666+\tfrac{1}{2}\times 282^{\text{met.}},606-\tfrac{1}{32}\times 2^{\text{met.}},5537=272^{\text{met.}},760\,; \tag{2572}$$

which gives

$$\mathrm{e}=2^{\text{met.}},827. \tag{2573}$$

Then by reducing $\frac{L'}{r'^3}$ to the mean distance of the moon from the earth,

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* (1881) In the expression of the lunar parallax, by Burg's tables, the constant term is 10558″,64; and the term depending on the angular distance of the sun and moon, or the argument of variation, is 80″,25, multiplied by the cosine of the double of the angular distance of the sun and moon, as appears in Book VII, § 26 [5603]. The distance varies [2569a]
nearly in the ratio of these quantities 10558″,64, 80″,25, or as 131 to 1, instead of 120 to 1 [2568″]. This correction *increases* the parallax, and *decreases* the distance; so that for r' [2569b]
we must write $r'.(1-\frac{1}{120})$, and for $\frac{1}{r'^3}$, $(1+\frac{3}{120})\cdot\frac{1}{r'^3}=\frac{123}{120}\cdot\frac{1}{r'^3}$ nearly; and instead of $\frac{L'}{r'^3}=3\,.\,\frac{L}{r^3}$ [2568‴], we must put $\frac{L'}{r'^3}=3\times\frac{123}{120}\cdot\frac{L}{r^3}=\frac{123}{40}\cdot\frac{L}{r^3}$, as in [2569]. [2569c]
Substituting this value and that of $\frac{1+3\,.\,\cos.2\,\theta}{8g}\cdot\frac{L}{r^3}$ [2568, 2567′], also h, h', [2570], in the first member of [2571], we obtain its value, as in the second member of that expression nearly.

† (1882) From [2566] we get $P.\left\{\frac{h.L}{r^3}+\frac{h'.L'}{r'^3}\right\}=\frac{282^{\text{met.}},606}{2}+\frac{b}{32}$. Substituting [2572a]
this and b [2553] in [2565], also the numerical value of the expression [2571], it becomes as in [2572]. From this equation we obtain the value of e [2573]. The observations of the quadratures give, in [2693], $\mathrm{e}=2{,}778$. The cause of this difference is discussed [2572b]
in [2694].

[2574] $$2P\,.\,\tfrac{41}{40}\,.\,h'\,.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}+2P\,.\{h-\tfrac{41}{40}\,.\,h'\}\,.\,\frac{L}{r^3}=282^{\text{met.}},756\;;\text{*}$$

the fraction $2P\,.\{h-\tfrac{41}{40}\,.\,h'\}\,.\,\frac{L}{r^3}$ being very small, we may suppose in it

[2575] $$\frac{L}{r^3}=\tfrac{1}{4}\,.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\};$$

thus we shall have,

[2576] $$2P\,.\{\tfrac{123}{160}\,.\,h'+\tfrac{40}{160}\,.\,h\}\,.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}=282^{\text{met.}},756\;;$$

hence we deduce,

[2577] $$2P\,.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}=6^{\text{met.}},2490\;;$$

[2574a] * (1883) This is done by changing as in [2569c] $\frac{L'}{r'^3}$ into $\frac{123}{120}\,.\,\frac{L'}{r'^3}=\frac{41}{40}\,.\,\frac{L'}{r'^3}$; by which means [2566] becomes $2P\,.\left\{\frac{h\,.\,L}{r^3}+\frac{41}{40}\,.\,\frac{h'\,.\,L'}{r'^3}\right\}-\frac{1}{16}\,b=282^{\text{met.}},606$. Adding the term $\frac{41\,h'\,.\,L}{r^3}$ to the factor of $2P$, and then subtracting the same quantity from this factor, it may be reduced to the form $\frac{41}{40}\,.\,h'\,.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}+(h-\frac{41}{40}\,.\,h')\,.\,\frac{L}{r^3}$; but by [2553], $282^{\text{met.}},606+\frac{1}{16}\,b=282^{\text{met.}},765$; therefore the preceding equation will give

[2574b] $2P\,.\,\frac{41}{40}\,.\,h'\,.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}+2P\,.\,(h-\frac{41}{40}\,.\,h')\,.\,\frac{L}{r^3}=282^{\text{met.}},765$, nearly as in [2574]; the difference of the two last figures of the decimal in the second member being but of little importance. Now the factor $h-\frac{41}{40}\,.\,h'$ is about $\frac{1}{30}$ of $-h$ [2570]; and as this is multiplied by $\frac{L}{r^3}$, which is only one quarter part of the factor $\frac{L}{r^3}+\frac{L'}{r'^3}$ [2706], connected with the other term, it must be so small that no sensible error can arise from the substitution of the value of $\frac{L}{r^3}$ [2575]; then it becomes

[2574c] $$2P\,.\,\tfrac{41}{40}\,.\,h'\,.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}+2P\,.\,(\tfrac{1}{4}\,h-\tfrac{41}{160}\,.\,h')\,.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}=282^{\text{met.}},765.$$

This is easily reduced to the form [2576], and by substituting the values of h, h', [2570], we obtain by division, the expression [2577]. The equation [2575] may be put under the

[2574d] form $\frac{L'}{r'^3}=\frac{3}{4}\,.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}$, which will be used hereafter.

which is the expression of the total tide that would take place at Brest, if the sun and moon moved uniformly in the plane of the equator. For we [2577′] have seen in [2507], that L' in the syzygies of the equinoxes becomes $L'.(1-2m'Q.\cos.\varepsilon')$; and in the syzygies of the solstices it becomes [2577″] $L'.\left(1-\frac{2m'Q}{\cos.\varepsilon'}\right)$ [2507′]; so that in the whole of the syzygies of Table II [2513] L' must be changed into* $L'.\left\{1-m'Q.\left(\cos.\varepsilon'+\frac{1}{\cos.\varepsilon'}\right)\right\}$; now [2577‴] we have

Formula.

$$\cos.\varepsilon'+\frac{1}{\cos.\varepsilon'}=2+4.\frac{(\sin.\tfrac{1}{2}\varepsilon')^4}{\cos.\varepsilon'};\qquad [2578]$$

and this last term may be neglected in comparison with the first, on account of the smallness of $(\sin.\frac{1}{2}\varepsilon')^4$; therefore in the syzygies of Table II, L' must be changed into $L'.(1-2m'Q)$, as if the moon moved in the plane [2579] of the equator.

Variation of the tides near high water.

We shall now determine the variation of the tides, near their maximum, by the theory of gravity. For this purpose we shall resume the expresion of Y'' [2502]; the angle ε having varied sensibly, in the interval of the 48 [2579′] syzygies of Table II [2513], it will be found with sufficient accuracy for our

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* (1884) This is the mean of the values [2577″] for the solstices and equinoxes. If we add $0=2-2$ to the first member of [2578], we get by successive reductions, and using [1] Int.,

$$\cos.\varepsilon'+\frac{1}{\cos.\varepsilon'}=2+\left(\cos.\varepsilon'-2+\frac{1}{\cos.\varepsilon'}\right)=2+\frac{\cos.^2\varepsilon'-2.\cos.\varepsilon'+1}{\cos.\varepsilon'}$$

$$=2+\frac{(1-\cos.\varepsilon')^2}{\cos.\varepsilon'}=2+\frac{(2.\sin.^2\tfrac{1}{2}\varepsilon')^2}{\cos.\varepsilon'},\qquad [2578a]$$

as in [2578]. Now we have nearly $4.(\sin.\frac{1}{2}\varepsilon')^4=4.(\sin.13^\circ)^4=4.(\frac{1}{5})^4=\frac{1}{156}$; [2578b] which may be neglected on account of its smallness; considering also that it is multiplied in [2577‴] by the small quantity $m'Q$, so that we may put $\cos.\varepsilon'+\frac{1}{\cos.\varepsilon'}=2$; and then the expression [2577‴] is simply $L'.(1-2m'Q)$. If the moon move in the equator, the [2578c] angle ε' will be nothing, and the expressions

$$L'.(1-2m'Q.\cos.\varepsilon')\quad\text{and}\quad L'.\left(1-2m'Q.\frac{1}{\cos.\varepsilon'}\right)\quad[2577'']\qquad[2578d]$$

become $L'.(1-2m'Q)$, as in [2579].

purpose, by taking for $\cos.^2\varepsilon'$ the mean of the squares of the cosines of the declinations of the moon, in the twenty-four syzygies of the solstices in
p, p'. [2580] that table. Hence if we put p and p' for the sums of the squares of the cosines of the declinations of the sun and moon, in the twenty-four syzygies
q, q'. [2580'] of the equinoxes; also q and q' for the like sums in the twenty-four syzygies of the solstices; we may suppose nearly*

[2581] $$\sin.^2\varepsilon' = \frac{24-q'}{24}\,; \qquad \left(\cos.\frac{\varepsilon+\varepsilon'}{2}\right)^2 = \frac{q+q'}{2.24}.$$

The term multiplied by t^2 in the expression of Y'', will thus become, for the twenty-four equinoctial syzygies,†

[2582] $$-48\,P.\tfrac{123}{160}.\left(\frac{L}{r^3}+\frac{L'}{r'^3}\right).t^2.\upsilon^2.\left\{\frac{q+q'}{p+\frac{123}{40}.p'}+1{,}165.\left(\frac{2p'-q'-24}{24}\right)\right\};$$

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* (1885) From the expression of q' [2580'] it is evident that $\frac{1}{24}q'$ represents nearly
[2581a] the *mean* value of $\cos.^2\varepsilon'$; hence we shall have nearly $\cos.^2\varepsilon'=\frac{1}{24}q'$, and $\sin.^2\varepsilon'=1-\cos.^2\varepsilon'=\frac{1}{24}.(24-q')$ [2581]. In like manner $\cos.^2\varepsilon=\frac{1}{24}q$ nearly; and as ε' differs but very little from ε, it is evident that the mean of these values $\frac{q+q'}{2.24}$
[2581b] differs but very little from the square of the cosine of the mean angle $\frac{1}{2}.(\varepsilon+\varepsilon')$, [2581].

† (1886) The term multiplied by t^2 in the value of Y'' [2502] is

[2582a] $$-4\,iP.\frac{L'}{r'^3}.t^2.\upsilon^2.\left\{1{,}165.(\sin.^2\varepsilon'-2.\sin.^2V')+\frac{\frac{2L}{r^3}.\{\cos.\frac{1}{2}.(\varepsilon+\varepsilon')\}^2}{\frac{L}{r^3}.\cos.^2V+\frac{L'}{r'^3}.\cos.^2V'}\right\};$$

and as there are 24 syzygies in the equinoxes, and 24 in the solstices, [2507vi], we must
[2582b] put in both $2\,i=24$. We must also write, as in [2574a], $\frac{41}{40}.\frac{L'}{r^3}$ for $\frac{L'}{r'^3}$, to reduce r' to the mean distance. Then putting for the term $\frac{41.L'}{40.r^3}$, which occurs without the braces, its value $\frac{123}{160}.\left(\frac{L}{r^3}+\frac{L'}{r^3}\right)$ [2574d]; and for $\frac{41}{40}.\frac{L'}{r'^3}$, within the braces, its
[2582c] value $\frac{123}{40}.\frac{L}{r^3}$ [2569c], the preceding expression becomes

[2582d] $$-48\,P.\tfrac{123}{160}.\left(\frac{L}{r^3}+\frac{L'}{r^3}\right).t^2.\upsilon^2.\left\{1{,}165.(\sin.^2\varepsilon'-2.\sin.^2V')+\frac{2.\{\cos.\frac{1}{2}.(\varepsilon+\varepsilon')\}^2}{\cos.^2V+\frac{123}{40}.\cos.^2V'}\right\}.$$

and the term relative to the twenty-four syzygies of the solstices will be*

$$-48\,P\,.\tfrac{123}{160}\,.\left(\frac{L}{r^3}+\frac{L'}{r'^3}\right).\,t^2\,.\,v^2\,.\left\{\frac{q+q'}{q+\frac{123}{40}\,.\,q'}-1{,}165\,.\left(\frac{24-q'}{24}\right)\right\};$$ [2583]

in which $\frac{L'}{r'^3}$ *corresponds to the mean distance of the moon from the earth.*

Now we find

$$p=23{,}68196,\qquad p'=23{,}75355,\qquad q=20{,}45203,\qquad q'=20{,}75529.$$ [2584]

v is the mean synodical motion of the moon in the interval $1^{\text{day}},02705$ [2584′]
[2493, 2746]; and this motion is $141866''$,† noticing the argument of [2585]

Now in the syzygies of the equinoxes we have

$$\cos.^2V=\frac{p}{24},\qquad \cos.^2V'=\frac{p'}{24}\ [2580'],\qquad \sin.^2V'=1-\cos.^2V'=\frac{24-p'}{24}.$$ [2582e]

Substituting these and [2581] in [2582d], we get [2582]. [2582f]

* (1887) In the syzygies of the solstices, we have

$$\cos.^2V=\frac{q}{24},\qquad \cos.^2V'=\frac{q'}{24}\ [2580'],\qquad \sin.^2V'=1-\cos.^2V'=\frac{24-q'}{24},$$ [2583a]

and $\sin.^2\varepsilon'$ as in [2581]. Substituting these in [2582d], we get the expression for the solstices [2583].

† (1888) The daily synodical motion of the moon in the syzygies, noticing the variation, is $13^{\circ},8136$ [2493d]. Multiplying this by $1^{\text{day}},02705$ [2584′], we get the value of v in seconds $141872''$, as in [2585] nearly. This is reduced to parts of the radius [2587a]
by dividing by $636619'',8$ [1970h]; hence we have $v=0{,}222852$, whose square is $v^2=0{,}049663$. Multiplying this by $48\,.\tfrac{123}{160}$, we get $48\,.\tfrac{123}{160}\,.\,v^2=1{,}83256$. But [2587b]
by [2577], $P.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}=3^{\text{met.}},1245$. Multiplying this by the preceding equation, we find $48\,P\,.\tfrac{123}{160}\,.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}.\,v^2=5^{\text{met.}},72585$; hence the factor, containing t^2 without [2587c]
the braces in [2582, 2583], is $-5^{\text{met.}},72585\,.\,t^2$; and by substituting the values of p, p', q, q', [2584], these expressions become $-3^{\text{met.}},2040\,.\,t^2$, $-1^{\text{met.}},8977\,.\,t^2$, the former [2587d]
of which represents the value of the coefficient of t^2 in the 24 syzygies of the equinoxes [2510], and the latter in the 24 syzygies of the solstices [2511, 2512]; and their sum $-5^{\text{met.}},1017\,.\,t^2$ represents its value for all the syzygies in Table I [2510—2512]. We [2587e]
deduced from observation in [2562] $-5^{\text{met.}},1074\,.\,t^2$ for the value of this term, which can hardly be said to differ from the theory.

Decrease of tides in the syzygies by the theory.

variation, which always increases this motion in the syzygies. Thus we shall

[2586] have $-3^{\text{met.}},2040 \,.\, t^2$ *for the value of the term multiplied by* t^2 *in the*

[2587] *expression of* Y'' *for the twenty-four equinoctial syzygies;* and $-1^{\text{met.}},8977 \,.\, t^2$ *for the value of the like term relative to the twenty-four solstitial syzygies;* this term relative to the whole of the syzygies will therefore be

[2588] $$-5^{\text{met.}},1017 \,.\, t^2.$$

The observations of the preceding article [2562] have given $-5^{\text{met.}},1074 . t^2$ for this term, which agrees perfectly with the theory.

[2588′] 26. *We shall now examine separately the observations of the syzygies of the equinoxes, and those of the solstices, in Table I* [2510, &c.]. If we determine, by the method of the preceding article, the expressions of the absolute heights and total tides of the syzygies of the equinoxes, given by the observations of Table I [2510], we shall find*

[2589] $$140^{\text{met.}},432 - 1^{\text{met.}},5811 \,.\, t^2$$ [Absolute height in the equinoxes.]

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* (1889) The calculation may be made, as in [2547a—g],

[2589a] by forming equations similar to those in [2547b]. In the present case, the first members of these equations are the same as in [2547b]. The second members are as in the adjoined table; in

[2589b] which column (A) contains the absolute heights [2510]; column (A'') the total tides of [2510]; column (A') the absolute heights, found by adding the numbers of [2511] to those of [2512], for

[2589c] the same day; and column (A''') the similar sums, formed from the total tides of [2511, 2512]. Hence we obtain $12\,b$, $12\,b'$, as in [2547e, g]. Thus for the equinoxes we have, by using the numbers in columns (A), (A''),

ABSOLUTE HEIGHTS.	
(A). Equinoxes.	(A'). Solstices.
129met.,890	127met.,215
136 ,079	130 ,370
139 ,851	132 ,149
139 ,962	131 ,581
137 ,479	130 ,145
131 ,653	125 ,488
TOTAL TIDES.	
(A''). Equinoxes.	(A'''). Solstices.
128met.,988	119met.,195
142 ,068	126 ,969
149 ,342	131 ,297
150 ,066	132 ,495
143 ,826	129 ,612
131 ,770	120 ,611

$$12\,b = 139^{\text{met.}},851 + 139^{\text{met.}},962 - 129^{\text{met.}},890 - 131^{\text{met.}},653$$
$$= 18^{\text{met.}},270, \qquad \text{or} \qquad b = 1^{\text{met.}},5225;$$

[2589d]

$$12\,b' = 149^{\text{met.}},342 + 150^{\text{met.}},066 - 128^{\text{met.}},988 - 131^{\text{met.}},770$$
$$= 38^{\text{met.}},650, \qquad \text{or} \qquad b' = 3^{\text{met.}},2208.$$

Hence we get the corrected values as in [2552′],

[2589e] $$b = \tfrac{1}{3} \,.\, (1^{\text{met.}},5225 + 3^{\text{met.}},2208) = 1^{\text{met.}},5811, \qquad \text{and} \qquad b' = 2b = 3^{\text{met.}},1622;$$

as in [2589, 2590]. In like manner for the solstices, by using columns (A'), (A'''), we get

for the expression of the absolute heights of the tides of the equinoxes; and

$$150^{met.},235 - 3^{met.},1623 \cdot t^2 \qquad \left[\begin{smallmatrix}\text{Total tide in}\\ \text{the equinoxes.}\end{smallmatrix}\right]$$ [2590]

for the expression of the total tides. In like manner we shall find

$$132^{met.},328 - 0^{met.},9725 \cdot t^2 \qquad \left[\begin{smallmatrix}\text{Absolute height}\\ \text{in the solstice.}\end{smallmatrix}\right]$$ [2591]

for the expression of the absolute heights of the tides of the solstices, and

$$132^{met.},371 - 1^{met.},9451 \cdot t^2 \qquad \left[\begin{smallmatrix}\text{Total tide in}\\ \text{the solstice.}\end{smallmatrix}\right]$$ [2592]

for the expression of the corresponding total tides.

In the first place we see that *the total tides, counted from the maximum,* [2593] *decrease more rapidly in the syzygies of the equinoxes than in those of the solstices. This result of observation is wholly conformable to the theory; which, as we have just seen* [2586, 2587], *gives* $-3^{met.},2040 \cdot t^2$ *and* $-1^{met.},8977$ *for the coefficients of* t^2, *differing but very little from the* [2593'] *numbers* $-3^{met.},1623$ *and* $-1^{met.},9451$, *given by observation* [2590, 2592].

Total tides decrease more rapidly in the syzygies of the equinoxes than in those of the solstices.

$$12\,b = \overset{met.}{132,149} + \overset{met.}{131,581} - \overset{met.}{127,215} - \overset{met.}{125,488} = \overset{met.}{11,027}, \quad \text{or} \quad b = \overset{met.}{0,9189};$$

$$12\,b' = 131,297 + 132,495 - 119,195 - 120,611 = 23,986, \quad \text{or} \quad b' = 1,9988;$$ [2589f]

and the corrected values [2552'] become, in this case, as in [2591, 2592],

$$b = \tfrac{1}{3} \cdot (0^{met.},9189 + 1^{met.},9988) = 0^{met.},97256, \qquad b' = 2\,b = 1^{met.},9451.$$ [2589g]

In the equinoxes we have, as in [2555a], for the sum of the tides of column (A), $6\,a - b \cdot (\frac{35}{2} + 6\,k^2) = 814^{met.},914$; and for the sum of the tides of column (A''), [2589h] $6\,a' - b' \cdot (\frac{35}{2} + 6\,k^2) = 846^{met.},060$. Dividing these by 6; substituting the value of $\frac{1}{6} \cdot (\frac{35}{2} + 6\,k^2) = 2,9173$ [2559'], and those of b, b', [2589e], we get successively

$$a = 135^{met.},8190 + 2,9173 \cdot b = 135^{met.},8190 + 4^{met.},6125 = 140^{met.},432 \quad [2589];$$

$$a' = 141\ \ ,0100 + 2,9173 \cdot b' = 141\ \ ,0100 + 9\ \ ,2250 = 150\ \ ,235 \quad [2590].$$ [2589i]

In like manner, by taking one sixth part of the sum of the tides of the solstices, in columns (A'), (A'''), we get the corresponding values of a, a', as in the following equations, which are reduced to numbers by means of the values of b, b' [2589g]:

$$a = \tfrac{1}{6} \cdot 776,948 + 2,9173 \cdot b = 129^{met.},491 + 2^{met.},837 = 132^{met.},328 \quad [2591];$$

$$a' = \tfrac{1}{6} \cdot 760,179 + 2,9173 \cdot b' = 126\ \ ,697 + 5\ \ ,674 = 132\ \ ,371 \quad [2592].$$ [2589k]

[2593″] If we suppose $t=0$ in the preceding expressions, we shall have $8^{met.},104$ for the excess of the mean absolute height of the tides in the equinoxes [2589], over those of the solstices [2591]; and $17^{met.},864$ for
[2593‴] the excess of the corresponding total tides of the equinoxes [2590], over those of the solstices [2592]. This last quantity is rather more than
[2593⁗] double of the former. By § 22, it ought to exceed twice the first expression by the quantity*

[2594] $$\frac{6.(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)}.\left\{\frac{L}{r^3}.(p-q)+\tfrac{41}{40}.\frac{L'}{r'^3}.(p'-q')\right\}.$$

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* (1890) Subtracting twice the value of Y' [2499] from Y'' [2502], we get

[2594a] $$Y''-2Y'=\frac{4i.(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)}.\left\{\frac{L}{r^3}.(1-3.\sin.^2V)+\frac{L'}{r'^3}.(1-3.\sin.^2V')\right\}$$
$$+4iP.\frac{L'}{r'^3}.\tfrac{1}{32}.v^2.\left\{1,165.(\sin.^2\epsilon'-2.\sin.^2V')+\frac{\frac{2L}{r^3}.\left(\cos.\frac{\epsilon'+\epsilon}{2}\right)^2}{\frac{L}{r^3}.\cos.^2V+\frac{L'}{r'^3}.\cos.^2V'}\right\};$$

in which the coefficient of $\frac{1}{32}$ is the same as that of $-t^2$ in [2502]. In the syzygies of the equinoxes of Table I, this coefficient is $3^{met.},1623$ [2590], and in those of the
[2594b] solstices $1^{met.},9451$ [2592]. Hence the greatest value of this term is $3^{met.},1623.\frac{1}{32}.v^2$, or $0^{met.},1.v^2$ nearly; and as $v^2=\frac{1}{25}$ [2509b], it becomes $0^{met.},004$, which is so small that it may be neglected. Then [2594a] becomes, by writing $\frac{41}{40}.\frac{L'}{r'^3}$ for $\frac{L'}{r'^3}$, as in [2574a],

[2594c] $$Y''-2Y'=\frac{4i.(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)}.\left\{\frac{L}{r^3}.(1-3.\sin.^2V)+\tfrac{41}{40}.\frac{L'}{r'^3}.(1-3.\sin.^2V')\right\}.$$

We shall now put Y'_e, Y''_e, for the values of Y', Y'', in the equinoxes; and Y'_s,
[2594d] Y''_s, for their values in the solstices. In the former case we have $\sin.^2V=1-\frac{1}{24}p$ and $\sin.^2V'=1-\frac{1}{24}p'$ [2582e], and in the latter we have $\sin.^2V=1-\frac{1}{24}q$, $\sin.^2V'=1-\frac{1}{24}q'$, [2583a]; hence

[2594e] $$Y''_e-2Y'_e=\frac{4i.(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)}.\left\{\frac{L}{r^3}.\{1-3.(1-\tfrac{1}{24}p)\}+\tfrac{41}{40}.\frac{L'}{r'^3}.\{1-3.(1-\tfrac{1}{24}p')\}\right\};$$
$$Y''_s-2Y'_s=\frac{4i.(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)}.\left\{\frac{L}{r^3}.\{1-3.(1-\tfrac{1}{24}q)\}+\tfrac{41}{40}.\frac{L'}{r'^3}.\{1-3.(1-\tfrac{1}{24}q')\}\right\};$$

This, reduced to numbers, is equal to $2^{\text{met.}},050$; by observation it is $1^{\text{met.}},656$ [2494h]; the difference is within the limits of the errors to [2595]
which such observations are liable.

The excess $17^{\text{met.}},864$ [2593‴] of the total tides of the syzygies of the [2595′]
equinoxes, above those of the solstices, is the effect of the declinations of
the sun and moon, which weaken the action of these bodies upon the sea. [2595″]
This excess is by § 22 equal to*

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and by subtracting the last of these equations from the preceding, and putting $2i=24$ [2507vi], it becomes

$$(Y''_e - Y''_s) - 2.(Y'_e - Y'_s) = \frac{6.(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)} \cdot \left\{ \frac{L}{r^3}.(p-q) + \tfrac{41}{40}.\frac{L'}{r'^3}.(p'-q') \right\}. \qquad [2594f]$$

Now $Y''_e - Y''_s$ expresses the excess of the *total* tides of the equinoxes above those of the solstices; and $Y'_e - Y'_s$ represents the same excess for the *absolute* tides; and the former exceeds the double of the latter, by the quantity in the second member of [2594f], which is the same as that in [2594]. Putting $\frac{L'}{r'^3} = 3.\frac{L}{r^3}$ [2568′], using also [2568, 2584], this expression becomes

$$\frac{6.(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)} \cdot \frac{L}{r^3} \cdot \{p-q+\tfrac{123}{40}.(p'-q')\}$$

$$= 6 \times 0^{\text{met.}},02745 . \{p-q+\tfrac{123}{40}.(p'-q)\} = 2^{\text{met.}},050. \qquad [2594g]$$

This difference by observation [2593″, 2593‴] is $17^{\text{met.}},864 - 2 \times 8^{\text{met.}},104 = 1^{\text{met.}},656$, [2594h]
as in [2595].

* (1891) The chief term of the value of Y'' [2502] which is most materially affected by variations in the declinations V, V', is

$$4iP.\left\{ \frac{L}{r^3}.\cos.^2 V + . \frac{L'}{r'^3}.\cos.^2 V' \right\}. \qquad [2596a]$$

It is upon this term that nearly all the variations of the tides on account of the declinations depend; so that in computing the effect of a change of declination, we may put simply

$$Y'' = 4iP.\left\{ \frac{L}{r^3}.\cos.^2 V + \tfrac{41}{40}.\frac{L'}{r'^3}.\cos.^2 V' \right\}; \qquad [2596b]$$

the coefficient $\frac{41}{40}$ being prefixed to $\frac{L'}{r'^3}$, to reduce it to its mean distance [2574a]. In the equinoxes we must change L' into $L'.(1-2m'Q.\cos.\varepsilon')$ [2507], and in the solstices into $L'.\left(1-\frac{2m'Q}{\cos.\varepsilon'}\right)$ [2507]; hence we have

[2596] $$2P.\left\{\frac{L}{r^3}.(p-q)+\tfrac{41}{40}.\frac{L'}{r'^3}.\left(p'.(1-2m'Q.\cos.\varepsilon')-q'.\left[1-\frac{2m'Q}{\cos.\varepsilon'}\right]\right)\right\};$$

or

[2597] $$2P.\left\{\frac{L}{r^3}.(p-q)+\tfrac{41}{40}.\frac{L'}{r'^3}.(1-2m'Q).(p'-q')\right\}$$
$$+2P.\frac{L'}{r'^3}.\tfrac{41}{40}.2m'Q.(1-\cos.\varepsilon').\left(p'+\frac{q'}{\cos.\varepsilon'}\right);$$

[2596c] $$Y''_e=4iP.\left\{\frac{L}{r^3}.\cos.^2V+\tfrac{41}{40}.\frac{L'}{r'^3}.(1-2m'Q.\cos.\varepsilon').\cos.^2V'\right\};$$

[2596d] $$Y''_s=4iP.\left\{\frac{L}{r^3}.\cos.^2V+\tfrac{41}{40}.\frac{L'}{r'^3}.\left(1-\frac{2m'Q}{\cos.\varepsilon'}\right).\cos.^2V'\right\}.$$

[2496e] In the first of these expressions we must put $\cos.^2V=\frac{1}{24}p$, $\cos.^2V'=\frac{1}{24}p'$ [2580]; and in the second $\cos.^2V=\frac{1}{24}q$, $\cos.^2V=\frac{1}{24}q'$, [2580']; and $2i$ being equal to 24, we shall get from [2596c, d],

[2596f] $$Y''_e=2P.\left\{\frac{L}{r^3}.p+\tfrac{41}{40}.\frac{L'}{r'^3}.(1-2m'Q.\cos.\varepsilon').p'\right\};$$

[2596f'] $$Y''_s=2P.\left\{\frac{L}{r^3}.q+\tfrac{41}{40}.\frac{L'}{r'^3}.\left(1-\frac{2m'Q}{\cos.\varepsilon'}\right).q'\right\};$$

[2596g] $$Y''_e-Y''_s=2P.\left\{\frac{L}{r^3}.(p-q)+\tfrac{41}{40}.\frac{L'}{r'^3}.\left([1-2m'Q.\cos.\varepsilon'].p'-\left[1-\frac{2m'Q}{\cos.\varepsilon'}\right].q'\right)\right\};$$

as in [2596]. The coefficients of p' and q' may be put under the forms [2596h, h'], and then the expression [2596g] becomes as in [2596i, k], or as in [2597].

[2596h] $$1-2m'Q.\cos.\varepsilon'=1-2m'Q+2m'Q.(1-\cos.\varepsilon');$$

[2596h'] $$1-\frac{2m'Q}{\cos.\varepsilon'}=1-2m'Q+\left(2m'Q-\frac{2m'Q}{\cos.\varepsilon'}\right)=1-2m'Q-\frac{2m'Q}{\cos.\varepsilon'}.(1-\cos.\varepsilon');$$

[2596i] $$Y''_e-Y''_s=2P.\left\{\frac{L}{r^3}.(p-q)+\tfrac{41}{40}.\frac{L'}{r'^3}.\left(\begin{array}{c}p'.(1-2m'Q)+2m'Q.(1-\cos.\varepsilon').p'\\-q'.(1-2m'Q)+2m'Q.(1-\cos.\varepsilon').\frac{q'}{\cos.\varepsilon'}\end{array}\right)\right\}$$

[2596k] $$=2P.\left\{\begin{array}{c}\frac{L}{r^3}.(p-q)+\tfrac{41}{40}.\frac{L'}{r'^3}.(1-2m'Q).(p'-q')\\+\frac{L'}{r'^3}.\tfrac{41}{40}.2m'Q.(1-\cos.\varepsilon').\left(p'+\frac{q'}{\cos.\varepsilon'}\right)\end{array}\right\}.$$

now by the preceding article, we have*

$$2P.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}.(1-2m'Q)\right\}=6^{\text{met.}},2490\,; \qquad [2598]$$

moreover we shall hereafter find [2706, 2707], that $\frac{L'}{r'^3}.(1-2m'Q)$ is [2598']

very nearly equal to $\frac{3L}{r^3}$; lastly we may suppose that $\cos.^2\varepsilon'=\frac{q'}{24}$; [2599]

hence the preceding expression becomes,

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* (1892) Making in [2577] the change required in [2579], we get [2598]; but from [2497m, 2579], or from [2707, 2707a], we have $\frac{L}{r^3}=\frac{1}{3}.\frac{L'}{r'^3}.(1-2m'Q)$; and if [2598$a$] we substitute this in [2597], it becomes

$$\begin{aligned}&2P.\frac{L'}{r'^3}.(1-2m'Q).\left\{\frac{p-q}{3}+\tfrac{41}{40}.(p'-q')\right\}\\&+2P.\frac{L'}{r'^3}.\tfrac{41}{40}.2m'Q.(1-\cos.\varepsilon').\left(p'+\frac{q'}{\cos.\varepsilon'}\right).\end{aligned} \qquad [2598b]$$

Substituting the value of $\frac{L}{r^3}$ [2598a] in [2598], we get

$$2P.\frac{L'}{r'^3}.(1-2m'Q)=\tfrac{3}{4}\times 6^{\text{met.}},249=4^{\text{met.}},68675\,; \qquad [2598c]$$

hence [2598b] becomes as in [2598f]. Now from [2584, 2599] we obtain

$$\cos.\varepsilon'=\sqrt{(\tfrac{1}{24}q')}=0,929948, \qquad 1-\cos.\varepsilon'=0,070052, \qquad [2598d]$$

$$\frac{q'}{\cos.\varepsilon'}=22,319, \qquad p'+\frac{q'}{\cos.\varepsilon'}=46,073, \qquad \frac{p-q}{3}+\tfrac{41}{40}.(p'-q')=4,14985. \qquad [2598e]$$

Substituting these in [2598f], we get [2598g],

$$4^{\text{met.}},68675.\left\{\frac{p-q}{3}+\tfrac{41}{40}.(p'-q')+\tfrac{41}{40}.\frac{2m'Q}{1-2m'Q}.(1-\cos.\varepsilon').\left(p'+\frac{q'}{\cos.\varepsilon'}\right)\right\} \qquad [2598f]$$

$$=19^{\text{met.}},449+15^{\text{met.}},505.\frac{2m'Q}{1-2m'Q}. \qquad [2598g]$$

This differs a little from [2600]; and if we put it equal to the observed excess $17^{\text{met.}},864$ [2595'], we get $\frac{2m'Q}{1-2m'Q}=\frac{17,864-19,449}{15,505}=-0,1022$, whence $2m'Q=-0,114$; which differs a little from [2601]. We shall see in [2624, 2719] that other observations give quite different results, and it seems to require a much greater number to determine the value [2598h] of Q with any considerable degree of accuracy.

[2600] $$19^{\text{met.}},494 + \frac{2\,m'Q}{1-2\,m'Q} \times 16^{\text{met.}},953.$$

Putting this equal to the observed excess $17^{\text{met.}},864$ [2593‴], we may thence determine $2\,m'Q$, and we shall find

Value of Q.

[2601] $$2\,m'Q = -\,0,10637.$$

[2601′] *Therefore it appears from the preceding observations, that the rapidity of the motion of the moon in its orbit increases the action of the moon upon the tides at Brest about one tenth part, as it retards the time of the maximum of the tides a day and a half; but this delicate element ought to be determined by a greater number of observations.*

The tide at Brest is increased by the moon's motion.

Effect of the variation of the sun's distance on the tides.

27. *Lastly we shall compare the tides in the syzygies of the winter solstice, with those of the syzygies of the summer solstice, in Table I* [2511, 2512], *to obtain the effect of the variation of the distance of the sun from the earth, upon the heights of the tides.* If we add together the total heights of the days 1, 2,
[2602] in the winter solstices of Table I [2512], we shall have $134^{\text{met.}},702$* for the sum. The like sum of the heights of the syzygial tides of the summer
[2602′] solstices is $129^{\text{met.}},090$ [2511], being less than the former by $5^{\text{met.}},612$; this proves that the greater proximity of the sun in winter than in summer has an influence upon the heights of the tides.

To compare in this respect the theory of gravity with observation, we shall put l for the sum of the squares of the cosines of the declinations of the
l, l'. sun, in the syzygies of the summer solstices of Table I [2511]; and l' for
[2603] the like sum relative to the declinations of the moon. Then the sun being about one sixtieth part nearer in winter than the mean distance, the value of
[2603′] $\frac{L}{r^3}$ is increased about one twentieth part; and for a contrary reason, it is
[2603″] decreased about one twentieth part in summer. This being premised, the formulas of § 22 will give†

[2602a] * (1893) $134^{\text{met.}},702 = 67^{\text{met.}},202 + 67^{\text{met.}},500$ [2512],
$129^{\text{met.}},090 = 64^{\text{met.}},095 + 64^{\text{met.}},995$ [2511].

† (1894) The term of Y'' [2502] multiplied by $t^2 + \frac{1}{32}$, is small in comparison with
[2603a] the first term of that expression, and of nearly the same value in winter as in summer;

$$4P.\left\{\frac{L}{r^3}\cdot(\tfrac{21}{20}q-2l)+\tfrac{41}{40}\cdot\frac{L'}{r'^3}\cdot(q'-2l')\right\}$$ [2604]

for the excess which has been found by observation equal to $5^{\text{met.}},612$ [2602′]; now we have*

$$\frac{L'}{r'^3}=\frac{3L}{r^3}; \qquad 2P.\frac{L}{r^3}=\tfrac{1}{4}\times 6^{\text{met.}},2490.$$ [2605]

We have found $l=10,16776$, $l'=10,34131$; hence the preceding function becomes $4^{\text{met.}},257$;† which differs but $1^{\text{met.}},355$ from the result of observations [2602′]. [2605′] [2606]

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therefore, in calculating the difference of the summer and winter tides, we may notice only that first term, and put

$$Y''=4\,i\,P.\left\{\frac{L}{r^3}.\cos.^2V+\frac{L'}{r'^3}.\cos.^2V'\right\}.$$ [2603b]

Now q [2580′] represents the sum of $\cos.^2V$ for *both* the *winter* and *summer* solstices of Table I; and q' the sum of $\cos.^2V'$ for the same solstices. Moreover, l, l' [2603], represent these sums respectively, for the *summer* solstices; therefore $q-l$ and $q'-l'$ represent the corresponding sums for the *winter* solstices; and as there are twelve observations in each solstice, we have for the *summer* solstice, $\cos.^2V=\frac{1}{12}l$, $\cos.^2V'=\frac{1}{12}l'$; and for the *winter* solstice $\cos.^2V=\frac{q-l}{12}$, $\cos.^2V'=\frac{q'-l'}{12}$. In the *summer* solstice, on account of the variation of the sun's distance, we must write $\frac{19}{20}.\frac{L}{r^3}$ for $\frac{L}{r^3}$; and in the winter solstice, $\frac{21}{20}.\frac{L}{r^3}$ for $\frac{L}{r^3}$, as in [2603′]. We must also write $\frac{41}{40}.\frac{L'}{r'^3}$ for $\frac{L'}{r'^3}$, [2574a]. Lastly, as the number of observations in each solstice is $2i$ or 12, on each day [2602], we shall have $4\,i=48$. Hence the expression [2603b] becomes in the summer solstices as in [2603f], and in the winter solstices as in [2603g]; their difference is [2604]. [2603c] [2603d] [2603e]

$$48P.\left\{\tfrac{19}{20}.\frac{L}{r^3}.\tfrac{1}{12}l+\tfrac{41}{40}.\frac{L'}{r'^3}.\tfrac{1}{12}l'\right\}=4P.\left\{\tfrac{19}{20}.\frac{L}{r^3}.l+\tfrac{41}{40}.\frac{L'}{r'^3}.l'\right\};$$ [2603f]

$$48P.\left\{\tfrac{21}{20}.\frac{L}{r^3}.\frac{q-l}{12}+\tfrac{41}{40}.\frac{L'}{r'^3}.\frac{q'-l'}{12}\right\}=4P.\left\{\tfrac{21}{20}.\frac{L}{r^3}.(q-l)+\tfrac{41}{40}.\frac{L'}{r'^3}.(q'-l')\right\}.$$ [2603g]

* (1895) We have $\frac{L'}{r'^3}=3.\frac{L}{r^3}$ nearly [2568′]; substituting this in [2577], we get $2P.\frac{L}{r^3}=\frac{6^{\text{met.}},249}{4}$ [2605]. [2604a]

† (1896) Substituting $\frac{L'}{r'^3}=3.\frac{L}{r^3}$ [2605] in [2604], it becomes as in the first [2605a]

Effect of the variation of the distance of the moon on the tides.

28. *The effect of the variation of the distances from the earth is much more sensible in the moon than in the sun.* In order to compare on this point the
[2606'] theory with observations, we have added the total tides *of the first and second days after the syzygy,* in twelve syzygies near the perigee, where the semi-diameter of the moon exceeds $30'$; and in the twelve preceding or following syzygies near the apogee, where the moon's semi-diameter is less than $28'$. We have selected these two days, because they include between
[2606''] them the instant of the *maximum* of the tides, to which they are very near. The following table contains these syzygies, and the corresponding total tides.

TABLE III.

[2607]

Observed total tides at Brest, in twelve syzygies near the perigee, and in twelve syzygies near the apogee.

Day of the Syzygy.		Total tide in the perigee.	Total tide in the apogee.
1714.	Jan. 16	$13^{met.},305$	
	Jan. 30		$10^{met.},654$
	April 14	13, 529	
	April 29		10 ,778
	August 10		10 ,453
	August 25	14 ,126	
	Sept. 8		10 ,614
	Sept. 23	14 ,539	
	Oct. 8		10 ,681
	Oct. 23	13 ,470	
1715.	March 5	14 ,300	
	March 20		10 ,985
	April 4	14 ,061	
	April 18		10 ,372
	Oct. 12	14 ,415	
	Oct. 27		10 ,451
	Nov. 11	13 ,711	
	Nov. 26		9 ,986
1716.	May 6		10 ,244
	May 21	13 ,186	
	June 5		9 ,592
	June 19	13 ,479	
	July 4		9 ,750
	July 19	12 ,135	

[2605a'] member of [2605b]; and by using the values $4P.\frac{L}{r^3}=3^{met.},1245$ [2605], q, q', [2584], l, l', [2605'], we finally get the last expression [2605b], being the same as in [2606].

We perceive by this table, that the total tides corresponding to the semi-diameters of the moon which exceed 30′, are always greater than those corresponding to the semi-diameters which are less than 28′. If we add [2607′] together the total tides relative to the greatest semi-diameters, we shall have $164^{\text{met.}},256$ for their sum; that of the total tides relative to the twelve least semi-diameters is $124^{\text{met.}},560$. The difference of these two sums is [2607″] $39^{\text{met.}},6961$. We shall now see what it ought to be by the theory. [2608]

If we neglect, as we have done in the value of y'' [2483], the quantity (A') [2481′], which in the present case is insensible, on account of its smallness, and because the declinations of the moon are alternately north [2608′] and south in the observations of Table III [2607],* it is evident from this expression of y'', that we shall obtain, by the following process, the part of the required difference relative to the terms depending on P. *First*, Finding the mean semi-diameter of the moon in the twenty-four observations of the [2609] table [2607], which is 2917″. *Second*. Multiplying in each observation the square of the cosine of the declination of the moon, by the cube of the [2610] ratio of its semi-diameter to 2917″.† *Third*. Computing the sum of these [2611] products relatively to the twelve observations in which the semi-diameter exceeds 30′, which sum is found to be equal to 13,5846; and then [2612]

$$4\,P\,.\,\frac{L}{r^3}\,.\left\{\tfrac{21}{20}\,.\,q-2\,l+\tfrac{123}{40}\,.\,q'-\tfrac{123}{40}\,.\,2\,l'\right\}$$

$$=3^{\text{met.}},1245\,.\left\{\tfrac{21}{20}\,.\,q-2\,l+\tfrac{123}{40}\,.\,q'-\tfrac{123}{40}\,.\,2\,l'\right\}=4^{\text{met.}},257. \qquad [2605b]$$

* (1897) The function (A') [2481] is very small, as is observed in [2481′]. Its greatest term, which depends on the moon, is $A\,.\,\frac{L'}{r'^3}\,.\,\sin.v'\,.\,\cos.v'\,.\,\sin.(\lambda-\gamma)$. Now in Table [2608a] III [2607], two *consecutive* syzygies are always taken; and if in the one case $\sin.v'$ be *positive*, in the following syzygy it will be of nearly the same value, but *negative;* so that their sum must nearly vanish, and by this means (A') will become very small. [2608b]

† (1898) In the observations of Table III [2607], which include the first and second days after the syzygy [2606′], the time from the *maximum* of the tides is about half a day, so [2611a] that what is called t in [2562] is $\pm\frac{1}{2}$ nearly. Hence the term depending on t in [2562] is to the constant quantity of that formula, as $5,1074\times\frac{1}{4}$ to 282,606, or as 1 to 221; consequently the quantity depending on t [2562], which was deduced from the observations of Table II [2513], must be very small in comparison with the other terms. The [2611b] same result would be obtained from the observations of Table III [2607], Therefore

subtracting from it the sum of the same products, relatively to the twelve
[2613] observations in which the semi-diameter of the moon is below $28'$, which is found to be 9,3628. *Fourth.* Multiplying the difference of these two
[2614] sums, 4,2218, by $4P.\frac{L'}{r'^3}$, r' being in this case the mean distance of
[2615] the moon in the syzygies; this gives $4P.\frac{L'}{r'^3}.4{,}2218$ for the part of the required difference depending on P.

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in finding the effect of the variations of v', r', in the value of y'' [2483], we may neglect the part depending on t^2, or $(\psi'-\psi)^2+\frac{1}{8}q^2$, on account of its smallness,
[2611b'] and put $y''=2P.\left\{\frac{L}{r^3}.\cos.^2 v+\frac{L'}{r'^3}.\cos.^2 v'\right\}$. Now in two consecutive syzygies, the variation of $\frac{L}{r^3}.\cos.^2 v$ must be very small in comparison with that of $\frac{L'}{r'^3}.\cos.^2 v'$; and as these observations are taken at different times of the year, the effect of the changes of $\frac{L}{r^3}.\cos.^2 v$ in finding the *difference* of the heights of the tide, in the perigee and apogee of the moon, must be nearly insensible, because the effect in one season is contrary to what it is in another. Therefore we may neglect this term, and we shall have
[2611c] $y''=2P.\frac{L'}{r'^3}.\cos.^2 v'$, from which the proposed difference of the tides [2608] is to be computed. This expression of y'' is for *one* tide; but on each day of Table III we have taken the *sum* of the *two* days 1, 2, after the syzygy [2606'], therefore it must be doubled,
[2611d] hence $y''=4P.\frac{L'}{r'^3}.\cos.^2 v'$; r' being the distance corresponding to these two tides. If
[2611e] we wish to denote by r' the mean distance of the moon in the syzygies, it will become

$$y''=4P.\frac{L'}{r'^3}.\left(\frac{\text{moon's mean dist.}}{\text{moon's actual dist.}}\right)^3.\cos.^2 v'=4P.\frac{L'}{r'^3}.\left(\frac{\text{moon's actual sem. diam.}}{\text{moon's mean sem. diam.}}\right)^3.\cos.^2 v',$$

[2611f]
$$=4P.\frac{L'}{r'^3}.\left(\frac{\text{moon's sem. diam.}}{2917''}\right)^3.\cos.^2 v'.$$

In the tides of the perigee, $\left(\frac{\text{moon's sem. diam.}}{2917''}\right)^3.\cos.^2 v'$ is equal to 13,5846 [2612]; and in the tides of the apogee, 9,3628 [2613]; hence in the former case we have
[2611g] $y''=4P.\frac{L'}{r'^3}.13{,}5846$, and in the latter $y''=4P.\frac{L'}{r'^3}.9{,}3628$; their difference is $4P.\frac{L'}{r'^3}.4{,}2218$, as in [2615].

To obtain the part depending on Q, we shall observe that by § 20, this part adds to the expression of y'' the term* $\quad -2PQ \cdot \frac{d\psi'}{dt} \cdot \frac{L'}{r'^3} \cdot \cos.^2 \mathrm{v}'$. [2616]

We have by [2505], $\quad \frac{d\psi'}{dt} \cdot \cos.^2 \mathrm{v}' = \frac{d\Gamma'}{dt} \cdot \cos.\varepsilon'$; and if we take the [2617]

mean distance of the moon from the earth for unity, we shall have nearly

$\frac{d\Gamma'}{dt} = \frac{m'}{r'^2}$;† thus the preceding term becomes $\quad -2m'PQ \cdot \frac{L'}{r'^5} \cdot \cos.\varepsilon'$; [2618]

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* (1899) This calculation is made upon the principles explained in [2502′, &c.]. The variations of r, v, may be neglected, for the reasons stated in the last note, so that the term of αy [2463] depending on Q will be as in [2504]; and near the maximum of the tide we have $\cos.2.(nt+\varpi-\psi'-\lambda)=1$ [2479c]; hence [2504] becomes as in [2616]. [2616a]

† (1900) The mean motion of the moon in the time dt being $m'dt$ [2505′], the actual motion, when at the distance r', is $\frac{m'dt}{r'^2}$ [585], the mean distance from the earth being [2618a] unity [2618]. Putting this equal to $d\Gamma'$ [2488″], we get $\frac{d\Gamma'}{dt} = \frac{m'}{r'^2}$ [2618]; if we substitute this in [2617], we get $\frac{d\psi'}{dt} \cdot \cos.^2 \mathrm{v}' = \frac{m'}{r'^2} \cdot \cos.\varepsilon'$; hence the expression [2616] becomes $\quad -2m'PQ \cdot \frac{L'}{r'^5} \cdot \cos.\varepsilon' = -2m'PQ \cdot \frac{L'}{r'^5} \cdot \sqrt{(\tfrac{1}{24}q')}$ [2599, 2619]; [2618b] and the value of αy [2463] is increased by the term

$$-2m'PQ \cdot \frac{L'}{r'^5} \cdot \sqrt{(\tfrac{1}{24}q')} \cdot \cos.2.(nt+\varpi-\psi'-\lambda) \qquad [2616a, 2504].$$

Now $\cos.2.(nt+\varpi-\psi'-\lambda)$ is nearly equal to 1 at high water, and -1 in the following low water, consequently the effect of the total tide [2476] is [2618c]

$$-4m'PQ \cdot \frac{L'}{r'^5} \cdot \sqrt{(\tfrac{1}{24}q')}; \qquad [2618d]$$

and as the tides of two days [2606′] are included in each of the numbers of Table III [2607], this expression must be doubled; by which means it becomes

$$-8m'PQ \cdot \frac{L'}{r'^5} \cdot \sqrt{(\tfrac{1}{24}q')}. \qquad [2618e]$$

If we wish r' to represent the mean distance of the moon from the earth, we must change, as in [2611f, &c.], $\frac{1}{r'^5}$ into $\frac{1}{r'^5} \cdot \left(\frac{\text{moon's sem. diam.}}{2917''}\right)^5$, and then the preceding expression [2618f]

[2619] cos. ε' may be supposed equal to $\sqrt{\frac{q'}{24}}$, q' being the sum of the squares of the cosines of the declinations of the moon, in the twenty-four syzygies of the solstices of Table II [2513], which sum is by the preceding article [2584] equal to 20,75529. Therefore we shall have the part of the
[2619′] required difference, relative to Q, *first*, by computing the sum of the fifth power of the ratio of the semi-diameter of the moon, in each observation in the perigee to 2917″, and subtracting from it the similar sum relative to the observations of the apogee; *second*, by multiplying the remainder by

[2619″] $$-8\,m'\,P\,Q\,.\,\frac{L'}{r'^3}\,.\sqrt{\frac{q'}{24}}.$$

[2620] Thus we find $-8\,m'P\,Q\,.\,\frac{L'}{r'^3}\,.\,6{,}9091$, r' being its value at the mean distance of the moon in the syzygies.

Adding together the two parts depending on P, Q, the required difference will be*

[2618g] becomes $-8\,m'\,P\,Q\,.\,\frac{L'}{r'^5}\,.\sqrt{(\frac{1}{24}q')}\,.\left(\frac{\text{moon's sem. diam.}}{2917''}\right)^5$ [2620]. Finding the sum of the terms of this form in the tides of the perigee, and another sum for those of the apogee, we shall get the difference of these two sums [2608], equal to [2618h], in which Σ is the sign of finite integrals, and r'^5 is changed into r'^3, observing that $r'=1$ [2618].

[2618h] $$-8\,m'P\,Q\,.\,\frac{L'}{r'^3}\,.\sqrt{(\tfrac{1}{24}q')}\,.\,\Sigma\left\{\left(\frac{\text{moon's perig. sem. diam.}}{2917''}\right)^5-\left(\frac{\text{moon's apog. sem. diam.}}{2917''}\right)^5\right\}.$$

According to the author, the quantity $\sqrt{(\frac{1}{24}q')}$, multiplied by the terms under the sign Σ, is 6,9091, hence the expression is as in [2620]; and putting $6{,}9091=4{,}2218+2{,}6873$,
[2618i] it becomes $-4\,P\,.\,\frac{L'}{r'^3}\,.\,2\,m'Q\,.\,4{,}2218-8\,m'PQ\,.\,\frac{L'}{r'^3}\,.\,2{,}6873.$

* (1901) Connecting the two terms [2615, 2618i], which express the differences of the tides arising from P, Q, we get

[2621a] $$4\,P\,.\,\frac{L'}{r'^3}\,.\,4{,}2218\,.\,(1-2\,m'\,Q)-8\,m'\,P\,Q\,.\,\frac{L'}{r'^3}\,.\,2{,}6873.$$

This is easily reduced to the same form as in [2621]; observing that its last term has the factor $1-2\,m'Q$ common to the numerator and denominator.

$$4P.\frac{L'}{r'^3}.(1-2m'Q).4{,}2218-\frac{8m'PQ.\frac{L'}{r'^3}.(1-2m'Q).2{,}6873}{1-2m'Q}.$$ [2621]

We have by § 25*

$$2P.\frac{L'}{r'^3}.(1-2m'Q)=\tfrac{123}{160}\times 6^{\text{met.}}{,}2490\,;$$ [2622]

hence the preceding difference becomes,

$$40^{\text{met.}}{,}562-\frac{2m'Q}{1-2m'Q}.25^{\text{met.}}{,}319.$$ [2623]

Putting this equal to the observed difference [2608] $39^{\text{met.}},6961$, we find $2m'Q=0{,}03425$, which is insensible, and has a contrary sign to the value determined in the preceding article [2601], by the phenomena of the tides, relatively to the declinations. We see by the greatness of the coefficient of $2m'Q$, in the preceding difference, that the phenomena of the tides, depending on the variation of the distance of the moon from the earth, are very proper for determining it; and it follows that $2m'Q$ *is very small, and even insensible at Brest.* [2624] [2625]

The evening tides at Brest exceed the morning tides in the summer solstice, and the contrary in the winter solstice.

*By means of the inequalities of the second kind, the period of which is nearly equal to one day, the evening tides exceed the morning tides at Brest in the summer solstice, and the contrary takes place in the winter solstice.** To [2626]

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* (1902) We have $2P.\frac{L'}{r'^3}.(1-2m'Q)=\frac{3}{4}\times 6^{\text{met.}},249$ [2598c]; r' being the mean distance of the moon. To reduce this to the mean distance in the syzygies, as in the above calculation, we must multiply it by $\frac{41}{40}$ [2574a], and we obtain [2622]. Substituting this in [2621], we get [2622a]

$$2\times 4{,}2218\times(\tfrac{123}{160}\times 6^{\text{met.}}{,}2490)-\frac{2\times 2{,}6873\times(\tfrac{123}{160}\times 6^{\text{met.}}{,}2490).2m'Q}{1-2m'Q};$$ [2622b]

which by reduction becomes as in [2623]. Putting this equal to the value $39^{\text{met.}},696$ found by observation [2608], we obtain $\frac{2m'Q}{1-2m'Q}=\frac{40{,}562-39{,}696}{25{,}819}=0{,}03354$; whence $2m'Q=0{,}0324$, which differs a little from [2624]. [2622c]

* (1903) The chief term of αy [2463] depending on oscillations of the second kind, is that represented by (A) [2467], which nearly vanishes in the equinoxes, on account of the [2626a]

determine the quantity of this phenomenon, we have added, in seventeen syzygies near the summer solstice, the excess of the evening tides over those of the morning, on the first and second days after the syzygy. The
[2626′] *maximum* of the tides falling nearly in the middle of these two days of observation, the daily variation of the height of the tides, arising from the inequalities of the third kind, is nearly insensible in the result, which ought therefore to contain only the *excess* of the evening tides over those of the morning, arising from the inequalities of the second kind. The sum of
[2627] these quantities, in the thirty-four days of observation is $6^{\text{met.}},131$.

We have likewise added the excess of the morning tides over those of the evening, in the eleven syzygies near the winter solstices. The sum of these
[2628] quantities, in the twenty-two days of observation, is $4^{\text{met.}},109$. Taking a mean between these two results, *the excess of an evening tide over that in the morning, in the syzygies of the summer solstices; or of a morning tide over*

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smallness of v and v′. At the time of new moon ψ' is nearly equal to ψ, and $nt+\varpi-\psi$ [2131c] represents the distance of the sun from the meridian. If we put this equal to h, we
[2626b] get $nt+\varpi-\psi-\gamma=nt+\varpi-\psi'-\gamma=h-\gamma$; hence at the time of new moon, the expression [2467] becomes

[2626c] $$(A)=A\,.\left\{\frac{L}{r^3}\,.\,\sin.\mathrm{v}\,.\,\cos.\mathrm{v}+\frac{L'}{r'^3}\,.\,\sin.\mathrm{v}'\,.\,\cos.\mathrm{v}'\right\}.\cos.(h-\gamma).$$

If this represent the evening tide, the preceding morning tide will be found by decreasing h about 200°, which will change the sign of [2626c]; so that if h correspond to the evening tide, the value of (A) in the morning tide will be

[2626d] $$(A)=-A\,.\left\{\frac{L}{r^3}\,.\,\sin.\mathrm{v}\,.\,\cos.\mathrm{v}+\frac{L'}{r'^3}\,.\,\sin.\mathrm{v}'\,.\,\cos.\mathrm{v}'\right\}.\cos.(h-\gamma),$$

nearly. Subtracting this from [2626c], we get the *excess* of the evening tide on the day of
[2626e] the new moon $2A\,.\left\{\frac{L}{r^3}\,.\,\sin.\mathrm{v}\,.\,\cos.\mathrm{v}+\frac{L'}{r'^3}\,.\,\sin.\mathrm{v}'\,.\,\cos.\mathrm{v}'\right\}.\cos.(h-\gamma)$. The same expression also answers nearly for the preceding or following full moon. For at the time of full moon $\psi'=\psi\pm 200^\circ$; also v′ becomes generally of a different sign from that it had in
[2626f] [2626c, &c.]; but the product of the two negative quantities $\sin.\mathrm{v}'.\cos.(nt+\varpi-\psi'-\gamma)$ does not change its sign in [2467]; consequently the resulting expression [2626e] remains unaltered, h being as in [2626b], and v′ is taken with the same sign as at the new moon. Lastly, if the expression [2626e] correspond to the new moon of the summer solstice, it will have a different sign at the time of the new moon of the winter solstice, on account of the
[2626g] change in the signs of sin. v, sin. v′; and from what has been said, it is evident that the same result obtains at the time of the full moon. This agrees with the remarks in [2626].

that in the evening, in the syzygies of the winter solstices, arising from the inequalities of the second kind, is $0^{\text{met.}},183$.* [2629]

This excess is by § 21 equal to the second member of [2630]; hence we have†

$$0^{\text{met.}},183 = 2A \cdot \left\{ \frac{L}{r^3} \cdot \sin. \text{v} \cdot \cos. \text{v} + \frac{L'}{r'^3} \cdot \sin. \text{v}' \cdot \cos. \text{v}' \right\} \cdot \cos. (\lambda - \gamma). \qquad [2630]$$

It is probable that $\cos. (\lambda - \gamma)$ differs but little from unity; a long series of observations *of the low water in the morning and evening* will show exactly its value. [2630′]

ON THE HEIGHTS OF THE TIDES NEAR THE QUADRATURES.

29. *To determine these heights by theory, we shall resume the complete expressions of* y', y'', [2472, 2477], and shall observe *that if we change* ψ' *into* $100° + \psi'$, *or into* $300° + \psi'$, *according as the moon is near its first or last quarter*, we may reduce y' and y'' into series, supposing $\psi' - \psi$ to be small, as is the case near the quadratures. Then neglecting the terms multiplied by Q, we get‡ [2631] [2631′]

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* (1904) This is equal to the sum of $6^{\text{met.}},131$, $4^{\text{met.}},109$, [2627, 2628], divided by the number of observations $34 + 22 = 56$ [2627, 2628]; observing that the observations of two days are used at each syzygy [2626′]. [2627a]

† (1905) The function [2630] is the same as [2626e], observing that at the time of new moon we have nearly, by [2479c], $nt + \varpi - \psi' = nt + \varpi - \psi = \lambda$; consequently λ is the same as h [2626b], and $\cos. (h - \gamma)$ [2626e] changes into $\cos. (\lambda - \gamma)$ [2630]. [2630a]

‡ (1906) To find the time of high water, we must proceed in the same manner as in [2464, &c.], and we shall obtain in the quadratures formulas exactly similar to [2466, 2467, 2468, 2472, 2474, 2477]. Before the development of these new values of y', y'', [2472, 2477], in series, as in [2482, 2483], it is necessary to change, in the terms multiplied by P, ψ' into $100° + \psi'$, when the moon is in the *first quarter*, and ψ' into $300° + \psi'$ in the *last quarter*, by which means the quantity $\psi' - \psi$, will always be small near the quadratures, [2631′]. These changes being made in [2466], it becomes in the quadratures [2632a] [2632b]

$$\text{tang.}\, 2 \cdot (nt + \varpi - \psi' - \lambda) = \frac{-\frac{L}{r^3} \cdot \cos.^2 \text{v} \cdot \sin.\, 2 \cdot (\psi - \psi')}{\frac{L'}{r'^3} \cdot \cos.^2 \text{v}' - \frac{L}{r^3} \cdot \cos.^2 \text{v} \cdot \cos.\, 2 \cdot (\psi - \psi')}; \qquad [2632c]$$

[2632] $$y' = -\frac{(1+3\,.\cos.2\,\theta)}{8g.\left(1-\frac{3}{5\rho}\right)}.\left\{\frac{L}{r^3}.(1-3\,.\sin.^2\mathrm{v})+\frac{L'}{r'^3}.(1-3\,.\sin.^2\mathrm{v}')\right\}$$

Mean absolute height of the tide of one day in the quadratures.

$$+P.\left\{\frac{L'}{r'^3}.\cos.^2\mathrm{v}'-\frac{L}{r^3}.\cos.^2\mathrm{v}\right\}$$

$$+\frac{2\,P.\frac{L}{r^3}.\cos.^2\mathrm{v}.\frac{L'}{r'^3}.\cos.^2\mathrm{v}'}{\frac{L'}{r'^3}.\cos.^2\mathrm{v}'-\frac{L}{r^3}.\cos.^2\mathrm{v}}.\{(\psi'-\psi)^2+\tfrac{1}{4}q^2\};$$

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and the term under the radical sign, by which P is multiplied in [2472, 2477], becomes

[2632d] $$\left\{\left(\frac{L}{r^3}.\cos.^2\mathrm{v}\right)^2-2\,.\frac{L}{r^3}.\cos.^2\mathrm{v}.\frac{L'}{r'^3}.\cos.^2\mathrm{v}'.\cos.2\,.(\psi'-\psi)+\left(\frac{L'}{r'^3}.\cos.^2\mathrm{v}'\right)^2\right\}^{\frac{1}{2}}.$$

These formulas [2632c, d] may be derived from the original expressions [2466, 2472], by
[2632e] merely changing the sign of L, as appears by inspection; and it is evident that this simple
[2632f] principle of derivation may be used in obtaining the factors of P [2632, 2633], from those in [2482, 2483], respectively. We may observe that we have inserted the terms depending on q^2 [2632, 2633], which were accidentally omitted by the author, though he finally introduced them in [2636—2640], by changing t^2 into $t^2+\frac{1}{16}$, or $t^2+\frac{1}{32}$, as in
[2632g] [2500c—d']. The terms multiplied by $(1+3\,.\cos.2\,\theta)$ [2468] appear in the same form in y' [2472, 2632]; but they do not occur in y'' [2477, 2633]. The expression (A), which occurs in [2468] vanishes in the value of y' [2472], on account of the change of signs mentioned in [2468''']; and for the same reason it does not appear in [2632]. The only
[2632h] term which remains to be noticed, in the values of y', y'', [2472, 2477], is that represented by $-(A')$ in y'' [2477]. Now by adding 100° to the angle $nt+\varpi$ [2467, 2480b], we get (A') [2473], and then $-(A')$ [2632k]. Substituting in this the values [2479],
[2632i] $nt+\varpi-\psi'=\lambda$, $nt+\varpi-\psi=\lambda+\psi'-\psi$, we get [2632$l$]. Finally putting, as in [2631'], $\psi'-\psi=100^\circ$ in the first quarter, $\psi'-\psi=300^\circ$ in the last quarter, we get for $-(A)$ the expression [2632m], being the same as the terms multiplied by A in [2633].

[2632k] $$-(A')=A.\left\{\frac{L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}.\sin.(nt+\varpi-\psi-\gamma)+\frac{L'}{r'^3}.\sin.\mathrm{v}'.\cos.\mathrm{v}'.\sin.(nt+\varpi-\psi'-\gamma)\right\}$$

[2632l] $$=A.\left\{\frac{L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}.\sin.(\lambda-\gamma+\psi'-\psi)+\frac{L'}{r'^3}.\sin.\mathrm{v}'.\cos.\mathrm{v}'.\sin.(\lambda-\gamma)\right\}$$

[2632m] $$=A.\left\{\pm\frac{L}{r^3}.\sin.\mathrm{v}.\cos.\mathrm{v}.\cos.(\lambda-\gamma)+\frac{L'}{r'^3}.\sin.\mathrm{v}'.\cos.\mathrm{v}'.\sin.(\lambda-\gamma)\right\}.$$

$$y'' = A \cdot \frac{L'}{r'^3} \cdot \sin. v' \cdot \cos. v' \cdot \sin. (\lambda - \gamma) \pm A \cdot \frac{L}{r^3} \cdot \sin. v \cdot \cos. v \cdot \cos. (\lambda - \gamma)$$

$$+ 2P \cdot \left\{ \frac{L'}{r'^3} \cdot \cos.^2 v' - \frac{L}{r^3} \cdot \cos.^2 v \right\}$$ [2633]

$$+ \frac{4P \cdot \frac{L}{r^3} \cdot \cos.^2 v \cdot \frac{L'}{r'^3} \cdot \cos.^2 v'}{\frac{L'}{r'^3} \cdot \cos.^2 v' - \frac{L}{r^3} \cdot \cos.^2 v} \cdot \{ (\psi' - \psi)^2 + \tfrac{1}{8} q^2 \} ;$$

Total tide in the quadratures.

the sign + is to be used in the first quarter of the moon, and the sign — in the second quarter. [2633′]

*The excess of the evening tide over the morning tide at Brest, in the quadratures of the equinoxes, arising from the inequalities of the second kind, is** [2633″]

$$\pm 2A \cdot \frac{L'}{r'^3} \cdot \sin. v' \cdot \cos. v' \cdot \cos. (\lambda - \gamma).$$ [2634]

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* (1907) It appears from [2469, 2467], that the difference of the two tides of the same day is $2(A)$. If we notice only the part of αy [2463] depending on this quantity, or upon (A) [2467], and put for brevity $A' = A \cdot \frac{L'}{r'^3} \cdot \sin. v' \cdot \cos. v'$, we shall have, in the equinoxes, where $v = 0$, $\alpha y = A' \cdot \cos. (nt + \varpi - \psi' - \gamma)$. We shall represent this function, at the time of high water, in the morning by αy_m, and in the evening by αy_e. Then in the equinoxes, we have in the evening tide of the first quarter, when the moon is above the meridian, $nt + \varpi - \psi' = \lambda$ [2479]; and in the morning tide this angle is decreased by 200°; hence we get [2634a] [2634b] [2634c]

$$\alpha y_e = A' \cdot \cos. (\lambda - \gamma), \quad \alpha y_m = -A' \cdot \cos. (\lambda - \gamma), \quad \alpha y_e - \alpha y_m = 2A' \cdot \cos. (\lambda - \gamma).$$ [2634d]

In the last quarter, we have in the morning tide, when the moon is above the meridian, $nt + \varpi - \psi' = \lambda$ [2479], and in the evening tide this angle is increased by 200°; hence we get from [2634b],

$$\alpha y_m = A' \cdot \cos. (\lambda - \gamma), \quad \alpha y_e = -A' \cdot \cos. (\lambda - \gamma), \quad \alpha y_e - \alpha y_m = -2A' \cdot \cos. (\lambda - \gamma).$$ [2634e]

Now if we resubstitute the value of A', we get generally in the equinoxes, as in [2634],

$$\alpha y_e - \alpha y_m = \pm 2A \cdot \frac{L'}{r'^3} \cdot \sin. v' \cdot \cos. v' \cdot \cos. (\lambda - \gamma),$$ [2634f]

the *upper* sign corresponding to the *first* quarter, the *lower* sign to the *last* quarter; we must also notice the signs of v', considering northern declinations as positive, southern as negative. [2634g]

The evening tide at Brest exceeds the morning tide in the quadratures of the vernal equinox, and the contrary in those of the autumnal equinox.

[2634'] *It follows from the preceding article, that the evening tide at Brest exceeds that of the morning, in the quadratures of the vernal equinox;* the contrary takes place in the quadratures of the autumnal equinox.*

[2634''] If in $2i$ quadratures, near the equinoxes, we notice the absolute heights and the total tides, on the days near the quadrature, putting Y' for the sum

[2635] of the mean absolute heights, we shall find, by the same analysis by which Y' was computed in [2499],†

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* (1908) In the vernal equinox, v' is positive in the first quarter, and negative in the last quarter [2634g]; therefore the second member of [2634f] becomes in both cases

[2634h] $$+2A\cdot\frac{L'}{r'^3}\cdot\sin. v'\cdot\cos. v'\cdot\cos.(\lambda-\gamma),$$

considering $\sin. v'$ as positive; and this expression is positive, as is evident from

[2634i] [2630, 2630']. In the autumnal equinox, v' is negative in the first quarter, and positive in the last quarter [2634g]; and then the second member of [2634f] becomes in both cases of a different sign from that in [2634h], as in [2634'].

† (1909) If we notice no other inequality of the moon's motion, except that depending on

[2636a] the argument of variation, and take the sum of the values of y' [2632] for $2i$ quadratures, we shall obtain [2636]. For the term multiplied by the factor $1+3\cdot\cos. 2\theta$ is the same in both, changing the terms depending on v, v', into their mean values V, V', and multiplying by $2i$. The next term $P\cdot\left\{\frac{L'}{r'^3}\cdot\cos.^2 v'-\frac{L}{r^3}\cdot\cos.^2 v\right\}$ produces

$$2iP\cdot\left\{\frac{L'}{r'^3}\cdot\cos.^2 V'-\frac{L}{r^3}\cdot\cos.^2 V\right\}.$$

The only remaining term is that depending on $(\psi'-\psi)^2+\frac{1}{4}q^2$; and to determine this

[2636b] factor, we shall put, as in [2637], Γ, Γ', for the angular motions of the sun and moon in their orbits, in the time taken for unity, noticing the equation depending on the argument of variation. If we suppose ψ, Γ, to represent the sun's right ascension and longitude

[2636c] respectively, we shall have $\tan g.\psi=\tan g.\Gamma\cdot\cos.\varepsilon$ [670], and when ψ, Γ, are small, we shall obtain nearly $\psi=\Gamma\cdot\cos.\varepsilon$. If ψ, Γ, be nearly equal to 100°, we may put

[2636d] $\psi=100^\circ-\psi_{,}$, $\Gamma=100^\circ-\Gamma_{,}$, and the preceding equation becomes

[2636e] $$\text{cotang.}\,\psi_{,}=\text{cotang.}\,\Gamma_{,}\cdot\cos.\varepsilon, \qquad\text{or}\qquad \tan g.\Gamma_{,}=\tan g.\psi_{,}\cdot\cos.\varepsilon\,;$$

[2636e'] hence we get nearly $\Gamma_{,}=\psi_{,}\cdot\cos.\varepsilon$. In like manner in the lunar orbit, when the moon is in the quadrature, about 100° distant from the equinox, we have, as in the last equation,

[2636f] $\Gamma'=\psi'\cdot\cos.\varepsilon'$; and as ε' differs but little from V', we shall get $\psi'=\frac{\Gamma'}{\cos. V'}$ nearly.

$$Y' = -\frac{2i.(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)}.\left\{\frac{L}{r^3}.(1-3.\sin.^2V)+\frac{L'}{r'^3}.(1-3.\sin.^2V')\right\}$$ [2636]

$$+2iP.\left\{\frac{L'}{r'^3}.\cos.^2V'-\frac{L}{r^3}.\cos.^2V\right\}$$

$$+2iP.\frac{L'}{r'^3}.(t^2+\tfrac{1}{16}).\{\Gamma'-\Gamma.\cos.V'.\cos.\varepsilon\}^2.\left\{1{,}0611.\sin.^2\varepsilon'+\frac{\frac{2L}{r^3}.\cos.^2V}{\frac{L'}{r'^3}.\cos.^2V'-\frac{L}{r^3}.\cos.^2V}\right\};$$

Sum of the mean absolute heights of the tide in $2i$ quadratures of the equinoxes.

Hence $\psi'-\psi$ varies, in the time taken for unity, by the quantity $\frac{\Gamma'}{\cos.V'}-\Gamma.\cos.\varepsilon$; and in the time t this variation is $\psi'-\psi=t.\left\{\frac{\Gamma'}{\cos.V'}-\Gamma.\cos.\varepsilon\right\}$. Squaring this, and [2636g] adding the quantity $\frac{1}{4}q^2$, which has the same effect as in [2500d'], to change the factor t^2 into $t^2+\frac{1}{16}$, we get $(\psi'-\psi)^2+\frac{1}{4}q^2=\frac{(t^2+\frac{1}{16})}{\cos.^2V'}.\{\Gamma'-\Gamma.\cos.V'.\cos.\varepsilon\}^2$. [2636h] Multiplying this by the quantity with which it is connected in [2632], putting V, V', for v, v',

$$\frac{2P.\frac{L}{r^3}.\cos.^2V.\frac{L'}{r'^3}.\cos.^2V'}{\frac{L'}{r'^3}.\cos.^2V'-\frac{L}{r^3}.\cos.^2V},$$ [2636i]

rejecting the factor $\cos.^2V'$, which occurs in the numerator and denominator, and multiplying by $2i$, we get the last term of [2636]; so that there now remains only the part depending on $1{,}0611.\sin.^2\varepsilon'$, which arises from the variation of the moon's declination, [2636k] as we shall now prove. In the quadratures of the equinoxes, we have $v'=\varepsilon'$ nearly; substituting this in [2492], we obtain $\cos.^2v'=\cos.^2v'+t^2.\left(\frac{d\Gamma'}{dt}\right)^2.\sin.^2\varepsilon'$; the terms in [2636l] the second member being valued at the epoch, where $t=0$. Hence we may find the variation of the term $P.\frac{L'}{r'^3}.\cos.^2v'$ [2632], or that of $2iP.\frac{L'}{r'^3}.\cos.^2V'$ [2636]; and by changing the factor t^2 into $t^2+\frac{1}{16}$, for the same reason as in [2636h], this last [2636m] expression becomes $2iP.\frac{L'}{r'^3}.(t^2+\frac{1}{16}).\left(\frac{d\Gamma'}{dt}\right)^2.\sin.^2\varepsilon'$. Now $\left(\frac{d\Gamma'}{dt}\right)$ [2492] represents [2636n] the mean velocity of the moon in the quadratures, and this is measured by the motion Γ' [2637] in the time taken for unity. Substituting this in the preceding expression, then multiplying and dividing it by the factor $(\Gamma'-\Gamma.\cos.V'.\cos.\varepsilon)^2$, putting also for brevity m equal to $\left(\frac{\Gamma'}{\Gamma'-\Gamma.\cos.V'.\cos.\varepsilon}\right)^2$, it becomes [2636o]

$$2iP.\frac{L'}{r'^3}.(t^2+\tfrac{1}{16}).\{\Gamma'-\Gamma.\cos.V'.\cos.\varepsilon\}^2.m.\sin.^2\varepsilon'.$$ [2636p]

[2636'] t being the number of intervals from the *minimum* of the mean absolute height of the tide to the time under consideration; *each interval being equal*
[2637] *to the time from the high water on one day to the corresponding high water on*
Γ, Γ'. *the following day, near the quadratures of the equinoxes.* Γ and Γ' are the angular motions of the sun and moon during this interval, noticing the argument of variation, which always decreases the moon's motion in the
$\varepsilon, \varepsilon'$. [2637'] quadratures. ε, ε', represent the inclinations of the orbits of these bodies to the equator.

[2637''] *The value of* Y' *relative to* $2i$ *quadratures, of which* i *are near the winter solstice, and* i *near the summer solstice, is*

To calculate the numerical value of m, we have as in [2493c, &c.], the moon's mean motion in one day, $13^d\,10^m\,35^s$; the sun's, $59^m\,8^s$; the equation of variation, $-14^m\,37^s$ nearly; subtracting this from $13^d\,10^m\,35^s$, we get the daily motion of the moon, corrected for this

[2636q] variation, $12^d\,55^m\,58^s$; hence $\Gamma = \frac{59^m\,8^s}{12^d\,55^m\,58^s}\,.\,\Gamma' = 0{,}0762\,.\,\Gamma'$. Substituting this in m [2636o], and putting $V' = \varepsilon' = 23^d\,28^m$, we get

$$m = \left(1 - \frac{\Gamma}{\Gamma'}\,.\,\cos.V'\,.\,\cos.\varepsilon'\right)^{-2} = (1 - 0{,}0762\,.\,\cos.^2 23^d\,28^m)^{-2}$$

[2636r]

$$= (1 - 0{,}0641)^{-2} = 1{,}142.$$

Hence the expression [2636p] becomes

$$2iP\,.\,\frac{L'}{r'^3}\,.\,(t^2 + \tfrac{1}{16})\,.\,(\Gamma' - \Gamma\,.\,\cos.V'\,.\,\cos.\varepsilon)\,.\,1{,}142\,.\,\sin.^2\varepsilon'\,;$$

which agrees with the corresponding term of [2636], except that the numerical coefficient is
[2636s] 1,0611, instead of 1,142, making it too small by about a fourteenth part, the author having probably neglected to square the expression of m [2636r]. This difference is however of but very little importance, on account of the smallness of the term. For $\sin.^2\varepsilon'$ is nearly $\frac{1}{6}$;
[2636t] and if we put $V' = V$, $\frac{L'}{r'^3} = 3\,.\,\frac{L}{r^3}$, the last term of [2636] with which this factor is connected is equal to 1, so that the sum of both the terms between the braces is
[2636u] nearly $\frac{1}{6} + 1 = \frac{7}{6}$, and the error $\frac{1}{14} \times \frac{1}{6} = \frac{1}{84}$, or $\frac{1}{98}$ part of $\frac{7}{6}$; so that the error is $\frac{1}{98}$ part of the term connected with the factor t^2, and is less than $\frac{1}{98} \times 10^{\text{met.}},9040$, or $0^{\text{met.}},1$, [2716], observing that the coefficient of t^2 is less in [2636] than in [2639].

We may moreover remark, that the calculation [2636q], for the variation of the moon's motion, is altered a little if the time used differs from *one* day. For example, if we take $1\frac{1}{2}$ or 2 days from the quadrature, corresponding to the extreme observations from the maximum
[2636v] tide [2646, &c.]. The time we have assumed [2636q] corresponds nearly to the mean of the observations in these tables, and it is sufficiently accurate in calculations of this kind. Similar remarks may be made relative to the notes in other parts of this work, as in [2493d].

$$Y' = -\frac{2i.(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)} \cdot \left\{ \frac{L}{r^3}.(1-3.\sin.^2 V) + \frac{L'}{r'^3}.(1-3.\sin.^2 V') \right\} * \qquad [2638]$$

Sum of the mean absolute heights of the tide in $2i$ quadratures of the solstices, half in each solstice.

$$+ 2\,i\,P.\left\{ \frac{L'}{r'^3}.\cos.^2 V' - \frac{L}{r^3}.\cos.^2 V \right\}$$

$$+2iP.\frac{L'}{r'^3}.(t^2+\tfrac{1}{16}).\left\{ \Gamma'.\cos.\varepsilon' - \frac{\Gamma}{\cos.\varepsilon} \right\}^2 . \left\{ \frac{\frac{2L}{r^3}.\cos.^2 V}{\frac{L'}{r'^3}.\cos.^2 V' - \frac{L}{r^3}.\cos.^2 V} - 1{,}0611.\mathrm{tang.}^2\varepsilon' \right\}.$$

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* (1910) The two first lines of the second member of [2638] are deduced from [2632] in the same manner as the similar terms of [2636]. The remaining term, multiplied in [2632] by $(\psi'-\psi)^2+\frac{1}{4}q^2$, is computed in the following manner. Near the solstices, [2638a]
the motion of the sun in right ascension ψ, is equal to its motion in longitude $\Gamma.t$ divided by $\cos.\varepsilon$, or $\psi=\frac{\Gamma}{\cos.\varepsilon}.t$ [2636e′, &c.]. The moon being then in the quadratures, must be near the equinoxes, and we shall have as in [2636c, &c.], $\psi'=\Gamma'.t.\cos.\varepsilon'$; [2638b]
hence $\psi'-\psi=t.\left\{\Gamma'.\cos.\varepsilon'-\frac{\Gamma}{\cos.\varepsilon}\right\}$. Squaring this and adding $\frac{1}{4}q^2$, which is the same as to change t^2 into $t^2+\frac{1}{16}$ [2636h], we get

$$(\psi'-\psi)^2+\tfrac{1}{4}q^2=(t^2+\tfrac{1}{16}).\left\{\Gamma'.\cos.\varepsilon'-\frac{\Gamma}{\cos.\varepsilon}\right\}^2. \qquad [2638c]$$

Substituting this in [2632], and multiplying it by $2i$, we obtain, for $2i$ quadratures of the solstices, the term

$$2\,i\,P.\frac{L'}{r'^3}.(t^2+\tfrac{1}{16}).\left\{\Gamma'.\cos.\varepsilon'-\frac{\Gamma}{\cos.\varepsilon}\right\}^2.\frac{\frac{2L}{r^3}.\cos.^2 V.\cos.^2 V'}{\frac{L'}{r'^3}.\cos.^2 V'-\frac{L}{r^3}.\cos.^2 V}; \qquad [2638d]$$

which is the same as the first of the terms multiplied by $t^2+\frac{1}{16}$ in [2638], observing that in the quadratures of the solstices, $\cos.^2 V'$ is nearly equal to unity. The only remaining term [2638e]
is that depending on $\mathrm{tang.}^2\varepsilon'$, arising from the variation of $\cos.^2 V'$. Now in the quadratures of the solstices, v' is nearly equal to nothing, and the value of $\cos.^2\mathrm{v}'$ [2492] becomes $\cos.^2\mathrm{v}'-t^2.\left(\frac{d\Gamma'}{dt}\right)^2.\sin.^2\varepsilon'=\cos.^2\mathrm{v}'-t^2.\Gamma'^2.\sin.^2\varepsilon'$ [2636n]; consequently the term $P.\frac{L'}{r'^3}.\cos.^2\mathrm{v}'$, produces the correction $-P.\frac{L'}{r'^3}.\Gamma'^2.t^2.\sin.^2\varepsilon'$. If we [2638f]
put for brevity $n=\left(1-\frac{\Gamma}{\Gamma'.\cos.\varepsilon.\cos.\varepsilon'}\right)^{-2}$, we shall have identically [2638f′]

[2638'] By putting Y'' for the total tides corresponding to Y' [2636, 2638], *we shall have, in the* $2i$ *quadratures of the equinoxes,**

[2639] $$Y''=4iP.\left\{\frac{L'}{r'^3}.\cos.^2V'-\frac{L}{r^3}.\cos.^2V\right\}$$

Sum of the total tides in $2i$ quadratures of the equinoxes.

$$+4iP.\frac{L'}{r'^3}.(t^2+\tfrac{1}{32}).\{\Gamma'-\Gamma.\cos.V'.\cos.\varepsilon\}^2.\left\{1{,}0611.\sin.^2\varepsilon'+\frac{\frac{2L}{r^3}.\cos.^2V}{\frac{L'}{r'^3}.\cos.^2V'-\frac{L}{r^3}.\cos.^2V}\right\};$$

$$\Gamma'^2=\left(\Gamma'.\cos.\varepsilon'-\frac{\Gamma}{\cos.\varepsilon}\right)^2.\frac{1}{\cos.^2\varepsilon'}.\left(1-\frac{\Gamma}{\Gamma'.\cos.\varepsilon.\cos.\varepsilon'}\right)^{-2}$$

[2638g] $$=\left(\Gamma'.\cos.\varepsilon'-\frac{\Gamma}{\cos.\varepsilon}\right)^2.\frac{1}{\cos.^2\varepsilon'}.n.$$

Substituting this in the correction [2638f], also putting $\sin.\varepsilon'=\text{tang}.\varepsilon'.\cos.\varepsilon'$, multiplying by $2i$, and changing as above t^2 into $t^2+\frac{1}{16}$, it becomes

[2638h] $$-2iP.\frac{L'}{r'^3}.(t^2+\tfrac{1}{16}).\left\{\Gamma'.\cos.\varepsilon'-\frac{\Gamma}{\cos.\varepsilon}\right\}^2.n.\text{tang.}^2\varepsilon'$$

for the corresponding term of [2638]; in which we must substitute the numerical value of n. If we put $\varepsilon=\varepsilon'=23^d\,28^m$, we shall have, by using [2636q, 2638f],

[2638i] $$n=\left(1-\frac{\Gamma}{\Gamma'.\cos.\varepsilon.\cos.\varepsilon'}\right)^{-2}=(1-0{,}0762.\sec.^2 23^d\,28^m)^{-2}=(1-0{,}09056)^{-2}$$
$$=(0{,}90944)^{-2}=1{,}209;$$

hence [2386h] becomes $-2iP.\frac{L'}{r'^3}.(t^2+\frac{1}{16}).\left(\Gamma'.\cos.\varepsilon'-\frac{\Gamma}{\cos.\varepsilon}\right)^2\times 1{,}209.\text{tang.}^2\varepsilon'$.

[2638k] This is of the same form as the term in [2638], but the coefficient n is put equal to 1,0611, instead of 1,209; therefore the value of this term is about one eighth part more than that
[2638l] given by the author in [2638]. This difference is not however of much importance, as is evident from [2636s—u].

* (1911) In the quadratures of the equinoxes $v=0$, hence the term multiplied by
[2639a] $\sin.v$ vanishes in [2633]. If we take two successive quadratures, $\sin.v'$ must be positive in the one and negative in the other, therefore the term multiplied by $\sin.v'$ must nearly vanish from [2633], especially when we consider the smallness of A [2630], and of $\lambda-\gamma$
[2639b] [2481']; consequently A may be neglected in finding Y'' [2639]. If we change P into $2P$, and q^2 into $\frac{1}{2}q^2$, in [2632], it produces the terms depending on P in [2633]; and by making the same changes in [2636], which was derived from [2632], we obtain the
[2639c] corresponding terms of Y'' [2639]; observing that the change of q^2 into $\frac{1}{2}q^2$, makes the factor $t^2+\frac{1}{16}$ [2636h] become $t^2+\frac{1}{32}$ as in [2639]. We may also observe that the numerical coefficient 1,0611 [2639], ought to be changed to 1,142, as in [2636s].

and in the $2i$ *quadratures of the solstices,** [2639′]

$$Y''=4iP\cdot\left\{\frac{L'}{r'^3}\cdot\cos.^2V'-\frac{L}{r^3}\cdot\cos.^2V\right\}$$ [2640]

$$+4iP\cdot\frac{L'}{r'^3}\cdot(i^2+\tfrac{1}{32})\cdot\left\{\Gamma'\cdot\cos.\varepsilon'-\frac{\Gamma}{\cos.\varepsilon}\right\}^2\cdot\left\{\frac{\frac{2L}{r^3}\cdot\cos.^2V}{\frac{L'}{r'^3}\cdot\cos.^2V'-\frac{L}{r^3}\cdot\cos.^2V}-1{,}0611\cdot\text{tang.}^2\varepsilon'\right\}.$$

Sum of the total tides in $2i$ quadratures of the solstices, half in each solstice.

Lastly we find, as in § 22, that in order to notice the terms depending on Q, it is sufficient to change L' into $L'\cdot\left(1-\frac{2m'Q}{\cos.V'}\right)$ in the expressions [2641] relative to the equinoxes ;† and L' into $L'\cdot(1-2m'Q\cdot\cos.\varepsilon')$, in the [2641′] expressions relative to the solstices.

30. *To compare these results of the theory with observations, we have selected from the beforementioned collection* [2507″], *the observations in twenty-four* [2641″] *syzygies near the equinoxes, and in twenty-four syzygies near the solstices; taking always two consecutive quadratures.* The days of these quadratures at Brest are as follows.

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* (1912) In the quadratures of the solstices $v'=0$ nearly; hence we may neglect the term multiplied by $A\cdot\sin.v'$, in computing Y'' from [2633]. If we take as many winter as summer solstices, the term $A\cdot\sin.v$ will in the one case be positive, and in the [2640a] other negative; so that their sum will nearly vanish in finding Y''. This term may also be neglected on account of the smallness of A, even when the number of observations in each solstice is not exactly the same. The terms depending on P [2640] are deduced from those [2640b] in [2638], by the principle of derivation used in [2639*b*, *c*]; observing that the numerical coefficient 1,0611 [2641], ought to be changed into 1,209 [2638*k*].

† (1913) The terms relative to Q are noticed in [2507, 2507′], by changing L' into $L'\cdot(1-2m'Q\cdot\cos.\varepsilon')$ when the *moon* is in the *equinoxes*, and into $L'\cdot\left(1-\frac{2m'Q}{\cos.\varepsilon'}\right)$ [2641a] when the *moon* is in the *solstices*. The former case corresponds to the *quadratures* of the *solstices*, the latter to the *quadratures* of the *equinoxes*, as in [2641, &c.], changing ε' into [2641b] V', to which it is nearly equal.

QUADRATURES OF THE EQUINOXES.

Years.

1711. September 5, September 19, October 4, October 18.

[2642] 1712. March 15, March 29, August 24, September 8, September 22, October 7.

Times of the quadratures of the equinoxes at Brest.

1714. August 18, September 1, September 17, September 30.

1715. March 12, March 28, August 22, September 6.

1716. March 1, March 15, March 30, April 14, September 8, September 23.

QUADRATURES OF THE SOLSTICES.

1711. June 23, July 7.

[2643] 1712. May 27, June 12, June 25, July 11.

Times of the quadratures of the solstices at Brest.

1714. May 21, June 5, June 20, July 4, December 14, December 28.

1715. May 26, June 8, June 24, July 8, November 18, December 3, December 17.

1716. January 2, May 28, June 12, June 27, July 11.

We should have taken as many quadratures in the winter, as in the summer solstice, but could not for want of observations.

Mean absolute height of the tide.

In each of these quadratures we have taken a mean between the absolute heights of two consecutive tides, to obtain what we have called the *mean*
[2644] *absolute height of the tide* [2471]. We have in the first place taken the two tides on the day of the quadrature, then the two following tides, afterwards the
[2644′] two tides which follow them, and finally the two tides which follow these last; so that it has often occurred that the two tides which we have combined
[2645] together did not happen on the same day. The *total tide* is the excess of the mean absolute height above the intermediate low tide. The numbers of these tides are denoted by 0, 1, 2, 3, commencing with that on the day of the quadrature. Several times the height at low water was not observed;
[2645′] sometimes only one of the two daily tides was observed. To supply these defects, we have used the same methods as in the tides of the syzygies [2509″, &c.]. Hence we have obtained the following results:

Total tide.

TABLE IV.

QUADRATURES OF THE EQUINOXES.

Observations of the tides at Brest; in 24 quadratures of the equinoxes;

Numbers of the tides.	Mean absolute heights.	Total tides.	
0	99met.,511	69met.,835	[2646]
1	94, 282	58 ,638	
2	96 ,059	62 ,383	
3	105 ,639	81 ,342	

QUADRATURES OF THE SOLSTICES.

in 24 quadratures of the solstices.

Numbers of the tides.	Mean absolute heights.	Total tides.	
0	106met.,117	82met.,244	[2647]
1	102 ,997	76 ,289	
2	103 ,220	76 ,654	
3	106 ,760	84 ,498	

31. *We shall in the first place consider the whole of these observations;* and we shall have, for the forty-eight quadratures, the following results:* [2647′]

TABLE V.

Combination of all the 48 preceding observations of the tides.

Numbers of the tides.	Mean absolute heights.	Total tides.	
0	205met.,628	152met.,079	[2648]
1	197 ,279	134 ,927	
2	199 ,279	139 ,037	
3	212 ,399	165 ,840	

We shall take for the unit of time the interval between two morning tides, or two evening tides, in the quadratures; and for epoch the mean time between the two high tides on the day of the quadrature, at Brest. We shall suppose, for any day near this phase,† [2648′] Unit of time and epoch. [2648″]

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* (1914) The numbers of Table V are the sums of the numbers in [2646, 2647] corresponding to the same day and tide; thus on the day 0, the absolute tide 205met.,628 is equal to 99met.,511 + 106met.,117. [2648a]

† (1915) The computation in this article is exactly like those in § 24. The formula [2649] corresponds to [2514]; observing that b, c, have different signs and values from those in [2514, &c.]. Formula [2650] corresponds to [2515], [2651] to [2516], [2652] to [2517], [2653] to [2518], [2654] to [2519], and [2655] to [2520]. [2649a]

[2649] $$a - bx + cx^2 = \text{the mean absolute height of the tide ;}$$

[2649′] x representing the number of intervals taken for unity, between the epoch and this tide, supposing the tide to follow after the epoch. If the formula [2649] correspond to a morning tide, the expression of the evening tide

[2650] of the same day will be $a - b \,.\, (x + \tfrac{1}{2}) + c \,.\, (x + \tfrac{1}{2})^2$, noticing only those inequalities in which the period is nearly half a day. If we take the half sum of the two expressions [2649, 2650], we shall get the mean absolute height of the tide; hence we have

Absolute height of the tide.

[2651] $$a + \tfrac{1}{16} c - b \,.\, (x + \tfrac{1}{4}) + c \,.\, (x + \tfrac{1}{4})^2 = \text{the mean absolute height of the tide.}$$

The expression of the intermediate low water is, according to the theory, of the form

[2652] $$a' + b \,.\, (x + \tfrac{1}{4}) - c \,.\, (x + \tfrac{1}{4})^2 = \text{the height at low water ;}$$

[2652′] therefore by putting $x + \tfrac{1}{4} = t$, the expression of the total tide will be of the form,

Total tide.

[2653] $$m - 2bt + 2ct^2 = \text{the height of the total tide.}$$

[2654] The *minimum* of this tide takes place when $t = \dfrac{b}{2c}$; this value of t is

[2655] likewise the value of x corresponding to the *minimum* of the formula $a - bx + cx^2$.

To determine $\dfrac{b}{2c}$, we may use the total tides of Table V [2648];

[2655′] but the absolute heights of the same table having been more carefully observed than the total tides, we shall use their sum; and shall put f, f', f'', f''', for the four sums obtained by adding the mean absolute heights to the corresponding total tides in the table. The analytical expression of

[2655″] these sums will be of the form $k - ib \,.\, t + ic \,.\, t^2$.* Supposing successively

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* (1916) This expression is similar to that in [2521].

[2656a] If we put successively $t=0$, $t=1$, $t=2$, $t=3$, and call the resulting values f, f', f'', f'''; then substitute for these quantities the sums of the numbers in [2648], corresponding to each of these days; we shall obtain the annexed equations, which are similar to those in [2522a].

$$\begin{aligned}
f &= 357^{\text{met.}},707 = k \\
f' &= 332\phantom{^{\text{met.}}},206 = k - ib + ic \\
f'' &= 338\phantom{^{\text{met.}}},316 = k - 2ib + 4ic \\
f''' &= 378\phantom{^{\text{met.}}},239 = k - 3ib + 9ic
\end{aligned}$$

Adding together the first and last equations, then

[2656b] subtracting the sum of the second and third, we obtain $4ic = f - f' - f'' + f''' = 65^{\text{met.}},424$

$t=0$, $t=1$, $t=2$, $t=3$, we shall have the values of f, f', f'', f'''; from which we may obtain

$$4ic = f - f' - f'' + f''' ;$$ [2656]

$$ib = 3ic + \frac{f + f' - f'' - f'''}{4} ;$$ [2657]

consequently

$$\frac{b}{2c} = \tfrac{3}{2} + \frac{f + f' - f'' - f'''}{2\cdot(f - f' - f'' + f''')} = 1,2964.$$ [2658]

We shall hereafter find that the interval taken for unity is $1^{\text{day}},0521$ [2809]; [2659]
hence the interval from the epoch to the time of the *minimum* of the tide, [2660]
estimated in days, is $1^{\text{day}},3639$. The hour of the epoch at Brest in these [2661]
observations is $0^{\text{day}},6121$,* and the mean hour of the quadrature $0^{\text{day}},4683$ [2662]

[2656]; which gives $ic = 16^{\text{met.}},356$. Again, adding the first and second equations, and subtracting the sum of the third and fourth, we get

$$f + f' - f'' - f''' = -26^{\text{met.}},642 = 4ib - 12ic, \qquad \text{whence}$$

$$ib = 3ic + \tfrac{1}{4}\cdot(f + f' - f'' - f''') = 3ic - 6^{\text{met.}},6605,$$ [2656c]

as in [2657]. Substituting in [2657] the value of ic deduced from [2656], we get

$$ib = \tfrac{3}{4}(f - f' - f'' + f''') + \tfrac{1}{4}\cdot(f + f' - f'' - f''') ;$$ [2656d]

dividing this by half of [2656], we obtain, as in [2658],

$$\frac{b}{2c} = \tfrac{3}{2} + \frac{f + f' - f'' - f'''}{2\cdot(f - f' - f'' + f''')} = \tfrac{3}{2} - \frac{26,642}{2\times 65,424} = 1,2964.$$ [2656d']

The interval taken for unity is 1,052067 [2809], or as it is called above, 1,0521; multiplying this by 1,2964, we get the interval in days $1^{\text{day}},3639$, as in [2660]. [2656e]

We may combine the equations [2656a] by the method of the least squares, as in [2524b—d], and we shall obtain the three following equations, similar to [2524c], [2656f]

$$1406,468 = 4k - 6ib + 14ic ; \quad 2143,555 = 6k - 14ib + 36ic ; \quad 5089,621 = 14k - 36ib + 98ic.$$ [2656g]

hence we get $k = 357,817$, $ib = 42,297$, $ic = 16,356$, $\dfrac{b}{2c} = \dfrac{ib}{2ic} = 1,2930$; [2656h]
and the interval betwe n the quadrature and the minimum tide $1^{\text{day}},5042$, found as in [2661—2663]. These numbers differ but little from those obtained by the author [2658, &c.].

* (1917) This time $0^{\text{day}},61215$ is the mean of the times $0^{\text{day}},60566$, $0^{\text{day}},61863$, [2661a]
corresponding to the number 0 in Table VII [2806, 2807]; which by [2648''] is the hour of the epoch.

Time of the minimum high water happens about a day and a half after the quadrature at Brest

[2811′]; so that the quadrature preceded the epoch by $0^{day},1438$. Adding this quantity to $1^{day},3639$, we have $1^{day},5077$, for the interval by which
[2663] the *minimum* of the tides follows after the quadrature. This differs but very little from $1^{day},50724$, found in [2544] to be the interval by which the *maximum* of the tides follows after the syzygy; therefore these two intervals are equal, as they ought to be by the theory [2729—2730, 2797, 2797′]. We shall
[2663′] suppose them both to be $1^{day},50724$.

We shall now determine the law of the variations of the mean absolute heights, and the total tides, in the forty-eight preceding quadratures. For this purpose we shall take *for the unit of time, the interval of two consecutive*
k. *morning tides, or evening tides, near the quadratures; and shall put k for*
[2663″] *the interval by which the time of minimum* of the tides precedes the middle of the interval between the four days of observation. We shall put $a+b\,.\,t^2$,
[2664] *for the general expression of the mean absolute heights* of Table V [2648]; t being the time from the *minimum* of these heights. The mean absolute heights, corresponding to the numbers 0, 1, 2, 3, will be*

[2665] $$a+b\,.\,(\tfrac{3}{2}-k)^2;\quad a+b\,.\,(\tfrac{1}{2}-k)^2;\quad a+b\,.\,(\tfrac{1}{2}+k)^2;\quad a+b\,.\,(\tfrac{3}{2}+k)^2.$$

If from the sum of the two extremes we subtract the sum of the two middle
[2666] terms, we shall have $4\,b$ for the difference,† which by Table V [2648] is
[2667] equal to $21^{met.},469$; hence we deduce $b=5^{met.},3672$.

In like manner, if we represent the heights of the total tides of Table V
[2668] [2648] by $a'+b'\,.\,t^2$, we shall find, by a similar computation,

[2669] $$b'=10^{met.},9887.‡$$

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* (1918) This is similar to the expressions [2547], neglecting the terms corresponding to
[2665a] the days -0 and $+4$, changing the sign of b, as in [2649a]. The sign of k is also changed; because in [2663″] it is supposed to *precede* the middle time, but in [2545] it *follows* that time.

† (1919) The sum of the two extremes is $2\,a+b\,.\,(\tfrac{9}{2}+2\,k^2)$, the sum of the two
[2666a] middle terms is $2\,a+b\,.\,(\tfrac{1}{2}+2\,k^2)$; the difference of these sums is $4\,b$, and by using the absolute heights [2648], we get

[2666b] $$4\,b=205^{met.},628+212^{met.},399-197^{met.},279-199^{met.},279=21^{met.},469;$$

hence $b=5^{met.},3672$, as in [2667].

[2668a] ‡ (1920) The formula [2668] is similar to [2550], changing the sign of b', as we have done with those of b, e, [2649a]. Then we find $4\,b'$, as in the last note, by means of the

According to the theory, $b=\frac{1}{2}b'=5^{\text{met.}},4943$ [2668d]; the difference between this value of b and the preceding, is within the limits of the errors of the observations. [2669′]

If we take for b, one third of the sum of the two values of b and b', and put b' equal to the double of b,* we shall have, [2669″]

Values of b, b'.

$$b=5^{\text{met.}},4520\,; \qquad b'=10^{\text{met.}},9040=2\,b. \qquad [2670]$$

To determine a and a', we shall observe that the sum of the four preceding expressions of the absolute heights of the tides [2665], is $4a+b.(5+4k^2)$. This sum is by Table V [2648] equal to $814^{\text{met.}},585$;† therefore we have, [2671]

$$a=\frac{814^{\text{met.}},585-(5+4\,k^2)\,.\,5^{\text{met.}},4520}{4}. \qquad [2672]$$

In like manner we shall find

$$a'=\frac{591^{\text{met.}},883-(5+4\,k^2)\,.\,10^{\text{met.}},9040}{4}. \qquad [2673]$$

To determine k, we shall observe that the mean hour of the quadrature at Brest, in the forty-eight quadratures of Table V [2648], is $0^{\text{day}},46829$ [2674] [2811′]. Adding to it $1^{\text{day}},50724$ [2663′], the distance from the quadrature to the *minimum* of the tides, we have $1^{\text{day}},97553$, for the time [2675] elapsed from the midnight preceding the quadrature to the *minimum* of the tide. The middle time at Brest, between the two tides on the day of the

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total tides [2648], from which we get

$$4\,b'=152^{\text{met.}},079+165^{\text{met.}},840-134^{\text{met.}},927-139^{\text{met.}},037=43^{\text{met.}},955, \qquad [2668b]$$

or $b'=10^{\text{met.}},9887$. If we compare [2636, 2639], also [2638, 2640], we shall find that the coefficient of t^2 in Y', is half of that in Y'', at the same season; that is, the coefficient in [2636] is half of that in [2639], and in [2638] is half of that in [2640]. Therefore the [2668c] coefficient of t^2, in the formula found from adding the expressions [2636, 2638], is half of that in the sum of [2639, 2640]; so that the value b, in all the observations of Table V [2668d] [2648], is half of b'.

* (1921) This method of finding b, b', is liable to the objections made in [2553d]. [2670a]

† (1922) This quantity is the sum of the absolute heights of [2648]. Putting it equal to $4\,a+b\,.\,(5+4\,k^2)$, and using b [2670], we get a [2672]. The similar equation in a', b', using the sum of the total tides $591^{\text{met.}},883$ [2648], and b' [2670], is

$$4\,a'+b'.(5+4\,k^2)=591^{\text{met.}},883, \qquad \text{whence we get } a' \text{ [2673].} \qquad [2671a]$$

quadrature, is $0^{day},61215$ [2661a], in the observations of Table V [2648].
[2676] Adding to it $\frac{3}{2}$ of the interval taken for unity, which interval is $1^{day},05207$
[2677] [2809]; we have $2^{days},19025$, for the time from the midnight preceding the quadrature, to the middle time between the extreme observations of the table. If we subtract from it $1^{day},97553$ [2675], the difference $0^{day},21472$
[2678] is the value of k, expressed in days. Dividing it by $1^{day},05207$ [2809],
[2679] we obtain $k = 0,204093$, expressed in parts of the interval taken for unity; hence we deduce*

Values of a, a'.

[2680]
$$a = 196^{met.},604, \qquad a' = 133^{met.},886.$$

Hence the expression of the numbers of Table V [2648], relative to the absolute heights of the tides, is

Absolute heights.

[2681]
$$196^{met.},604 + 5^{met.},4520 \, . \, t^2 \, ;$$

and the expression of the numbers of the same table, relative to the total tides, is

Total tides.

[2682]
$$133^{met.},886 + 10^{met.},9040 \, . \, t^2 \, ;$$

[2683] *We shall now compare these formulas, deduced from observation, with those in* § 29, *given by the theory of gravity.* Let e be the height of the *zero* of
e.
the scale of observation, above the level of equilibrium which the sea would
[2683′] take, neglecting the action of the sun and moon; also h the sum of the squares of the cosines of the declinations of the sun, at the times of the
[2683″] phases, in the quadratures of Table V [2648]; and h' the similar sum relative to the moon; we shall obtain, from § 29,†

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* (1923) Substituting $k = 0,204093$ [2679], in a, a', [2672, 2673], we get [2680].
[2680a] Using these values of a, a', also b, b', [2670], we obtain from [2664, 2668] the formulas [2681, 2682] respectively.

† (1924) In the $4i = 48$ quadratures of Table V [2648], we have $4i \, . \cos.^2 V = h$, $4i \, . \cos.^2 V' = h'$, [2683′, &c.]; hence we get, as in [2565b],

[2683a]
$$4i \, . \, (1 - 3 \, . \sin.^2 V) = 3 \, . \, (h - 32), \qquad 4i \, . \, (1 - 3 \, . \sin.^2 V') = 3 \, . \, (h' - 32).$$

[2683b] Substituting these in the sum of the formulas [2636, 2638], we obtain the value of Y', corresponding to 48 quadratures; supposing 24 to be in the equinoxes [2634″], 12 in the summer solstice, and 12 in the winter solstice [2637″]. To each of these tides we must add the constant height e, making 48 e. If we neglect the part multiplied by $t^2 + \frac{1}{16}$, the rest of the terms will be the same as in the first member of [2684], excluding b. Now by

$$48\,e - \frac{3.(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)}.\left\{\frac{L}{r^3}.(h-32)+\frac{L'}{r'^3}.(h'-32)\right\}$$
[2684]
$$+P.\left\{\frac{h'.L'}{r'^3}-\frac{h.L}{r^3}\right\}+\frac{b}{16}=196^{\text{met.}},604\,;$$

we have for Brest [2568],

$$\frac{(1+3.\cos.2\theta)}{8g.\left(1-\frac{3}{5\rho}\right)}.\frac{L}{r^3}=0^{\text{met.}},02745.$$ [2685]

But in the forty-eight quadratures under consideration, the value of $\frac{L}{r^3}$ is not exactly equal to its mean value. Table V [2648] includes twenty-four [2685′] quadratures of the equinoxes [2642], eighteen quadratures of the summer, and six of the winter solstices [2643]. Now we have seen in [2603′, &c.], that in the quadratures of the summer solstices, $\frac{L}{r^3}$ is decreased one [2686] twentieth part, and in the quadratures of the winter solstices it is increased one twentieth part; we must therefore multiply the mean value of $\frac{L}{r^3}$ by [2686′] $\frac{79}{80}$,* to obtain the mean value in these forty-eight quadratures. Moreover, [2686″] $\frac{L'}{r'^3}$ is less by one fortieth part in the quadratures than in the mean distances, [2686‴]

[2664], the coefficient of t^2 is b; therefore the coefficient of $t^2+\frac{1}{16}$ is also equal to b, producing the terms $b.t^2+\frac{1}{16}b$, of which the part $\frac{1}{16}b$ is to be added to the first [2683c] member of [2684] to obtain a [2664], and then by substituting a [2680], we obtain the equation [2684].

* (1925) In 18 summer solstices it is decreased by $18\times\frac{1}{20}$, and in 6 winter solstices is increased by $6\times\frac{1}{20}$; so tha in the whole 24 solstices it is decreased by $12\times\frac{1}{20}$, [2686a] and the mean decrease for one solstice is $\frac{12}{24}\times\frac{1}{20}=\frac{1}{40}$. Now there are also 24 tides of the equinoxes, upon which no allowance of this kind is to be made; therefore the [2686b] correction, for the whole 48 observations, is only one half of the preceding value or $\frac{1}{80}$; so that we must change $\frac{L}{r^3}$ into $\frac{79}{80}.\frac{L}{r^3}$. [2686c]

on account of the argument of variation;* and as it is nearly equal to $\frac{3L}{r^3}$ in the mean distances, it must be supposed in the quadratures equal to

[2687] $\frac{117}{40} \cdot \frac{L}{r^3}$. Lastly, we have found in the preceding quadratures,

[2688] $$h = 44{,}16767, \qquad h' = 44{,}45074.$$

This being premised, we have†

[2689] $$\frac{3\,.\,(1+3\,.\cos.2\theta)}{8g\,.\left(1-\frac{3}{5\rho}\right)} \cdot \left\{ \frac{L}{r^3} \cdot (h-32) + \frac{L'}{r'^3} \cdot (h'-32) \right\} = 3^{\text{met.}},989.$$

The expression of the total tides of Table V [2648], compared with the formulas of § 29, gives‡

[2690] $$2P\,.\left\{ \frac{h'\,.\,L'}{r'^3} - \frac{h\,.\,L}{r^3} \right\} + \frac{b}{16} = 133^{\text{met.}},886.$$

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* (1926) The variation *increases* the distance in the quadratures, as much as it *decreases* it in the syzygies, as is evident from its chief term being as the cosine of the double of the elongation of the sun and moon [2569a]; and as $\frac{L'}{r'^3}$ is increased a fortieth part in the syzygies [2569c], it must be decreased in the quadratures by the same quantity. Now if

[2687a] we denote by $\frac{L'}{r'^3}$ the mean value of that quantity, we must use in the quadratures

[2687b] $\frac{39}{40} \cdot \frac{L'}{r'^3}$, instead of $\frac{L'}{r'^3}$; and as we have supposed in the mean distances $3 \cdot \frac{L}{r^3} = \frac{L'}{r'^3}$, [2568′], we must change $\frac{L'}{r'^3}$ into $3 \times \frac{39}{40} \cdot \frac{L}{r^3} = \frac{117}{40} \cdot \frac{L}{r^3}$, as in [2687].

† (1927) Changing $\frac{L'}{r'^3}$ into $\frac{117}{40} \cdot \frac{L}{r^3}$ [2687], also $\frac{L}{r^3}$ into $\frac{79}{80} \cdot \frac{L}{r^3}$ [2686c];

[2688a] we obtain for the part of [2684] included in the first member of [2689], the following expression, neglecting the sign, and using the values [2685, 2688],

$$\left\{ \frac{3\,.\,(1+3\,.\cos.2\theta)}{8g\,.\left(1-\frac{3}{5\rho}\right)} \cdot \frac{L}{r^3} \right\} \cdot \left\{ \tfrac{79}{80} \cdot (h-32) + \tfrac{117}{40} \cdot (h'-32) \right\}$$

[2688b] $$= 3 \times 0^{\text{met.}},02745\,.\left\{ \tfrac{79}{80}\,.\,(h-32) + \tfrac{117}{40}\,.\,(h'-32) \right\} = 3 \times 0^{\text{met.}},02745 \times 48{,}434 = 3^{\text{met.}},989.$$

‡ (1928) Adding the formulas [2639, 2640], we obtain the value of Y'' for $2i$ quadratures of the equinoxes, and $2i$ quadratures of the solstices. Then putting, as in

This equation requires a small correction, arising from having in Table V [2648], eighteen quadratures in the summer, and only six in the winter. For the low water in these quadratures corresponds to the solar high water in the evening; which, in summer, exceeds at Brest the solar high water in the morning by $0^{\text{met.}},0457$.* Therefore we must increase $133^{\text{met.}},886$ by [2691]
six times $0^{\text{met.}},0457$, to render it independent of the inequalities of which the period is nearly one day; then we have

$$2P.\left\{\frac{h'.L'}{r'^3}-\frac{h.L}{r^3}\right\}=133^{\text{met.}},819; \qquad [2692]$$

[2668], b', or $2b$ [2670], for the coefficient of t^2, or $t^2+\frac{1}{32}$; also $4i=48$, and as in [2683a], $4i.\cos.^2V=h$, $4i.\cos.^2V'=h'$; we get for this sum,

$$Y''=2P.\left\{\frac{h'.L'}{r'^3}-\frac{h.L}{r^3}\right\}+b'.(t^2+\tfrac{1}{32})=2P.\left\{\frac{h'.L'}{r'^3}-\frac{h.L}{r^3}\right\}+\tfrac{1}{16}b+b'.t^2. \qquad [2690a]$$

Comparing this with [2668], we get $a'=2P.\left\{\frac{h'.L'}{r'^3}-\frac{h.L}{r^3}\right\}+\frac{1}{16}b$; and if we substitute a' [2680], it becomes as in [2690].

* (1929) The excess of the *evening* tide in summer is $0^{\text{met.}},183$ [2629, 2630]; and [2691a]
as $\frac{L}{r^3}$ is nearly one fourth part of $\frac{L'}{r'^3}+\frac{L}{r^3}$ [2575], the part of this excess, depending on the *solar* force $\frac{L}{r^3}$, is nearly one quarter of $0^{\text{met.}},183$, or $0^{\text{met.}},0457$; therefore the evening solar tide exceeds the morning solar tide, in summer, by $0^{\text{met.}},0457$, and the contrary takes place in winter; or in other words, the *evening solar tide in summer* exceeds [2691b]
the *mean value* of the solar tide, by the half of this quantity, or $\frac{1}{2}\times 0^{\text{met.}},0457$. Therefore if we wish to obtain the total tide, by using the *mean* solar tide of that day, we must *add* $\frac{1}{2}\times 0^{\text{met.}},0457$ to the total tide obtained in the *summer* quadratures, and the contrary in the *winter* quadratures. Now in Table V [2648], there are 18 summer, and 6 winter [2691c]
quadratures; so that the quantity ought to be *increased* by $18\times\frac{1}{2}\times 0^{\text{met.}},0457$, and *decreased* by $6\times\frac{1}{2}\times 0^{\text{met.}},0457$; making on the whole an increase of

$$12\times\tfrac{1}{2}\times 0^{\text{met.}},0457=0^{\text{met.}},2742. \qquad [2691d]$$

Adding this to $133^{\text{met.}},886$, we get $134^{\text{met.}},1602$ for the corrected value of the second member of [2690]; and by using b [2670], we obtain

$$2P.\left\{\frac{h'.L'}{r'^3}-\frac{h.L}{r^3}\right\}+0,3407=134^{\text{met.}},1602, \qquad [2691e]$$

which is easily reduced to the form [2692].

hence we deduce*

e.

[2693] $$e = 2^{met.},778.$$

[2694] The observations of the syzygies have given in [2573], $e = 2^{met.},827$. It is a question whether the small difference of these two values arises from the errors of the observations; or from the circumstance that the great tides at Brest do not run out entirely, on the *ebb*, to the point determined by the theory. It is presumed that it depends on the last cause, but this can be
[2694'] determined only by means of a much greater number of observations.

We shall resume the equation [2692],

[2695] $$2P \cdot \left\{ \frac{h' \cdot L'}{r'^3} - \frac{h \cdot L}{r^3} \right\} = 133^{met.},819.$$

To reduce the values of $\frac{L}{r^3}$, $\frac{L'}{r'^3}$, to the mean distances of the sun and moon, we must multiply the sum of the squares of the cosines of the declinations of the sun, in the quadratures of the solstices of Table V [2648],
[2696] by $\frac{39}{40}$,† in order to correct for the effect of the greater number of summer than of winter solstices. Then adding to the product, the amount of the squares of the cosines of the declinations of the sun, in the quadratures of

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* (1930) If we substitute in [2684] the numerical value of the expressions [2689,
[2693a] 2692, 2670], it becomes $48\,e - 3^{met.},989 + \frac{1}{2} \times 133^{met.},819 + \frac{1}{16} \times 5,4520 = 196^{met.},604$, whence $48\,e = 133^{met.},343$, and $e = 2^{met.},778$, as in [2693].

† (1931) It appears from [2603', &c.], that the force $\frac{L}{r^3}$ is *diminished* $\frac{1}{20}$ in summer,
[2696a] *increased* $\frac{1}{20}$ in winter, and is at its mean value in the equinoxes. In Table V [2648], there are 18 observations in summer, and 6 in winter; consequently the whole variation, for
[2696b] the 24 observations, is $-18 \times \frac{1}{20} + 6 \times \frac{1}{20} = -12 \times \frac{1}{20}$. Dividing this by 24, the number of observations, we get the mean diminution equal to $\frac{1}{40}$; so that we ought to write, in formula [2695], $\frac{39\,h \cdot L}{40\,r^3}$ for $\frac{h \cdot L}{r^3}$, or $\frac{39}{40}\,h$ for h, in these observations of the solstices. Now the whole value of h [2688] is divided, in [2708, &c., 2714], into two
[2696c] parts, $p = 23,68841$, $q = 20,47926$; the former being the part for the quadratures of the equinoxes, the latter for those of the solstices. Therefore for h, in [2695], we must use
[2696d] the following expression, $p + \frac{39}{40}\,q = 23,68841 + \frac{39}{40} \times 20,47926 = 43,6557$, as in [2697].

the equinoxes; the sum will be the value of h which must be used. Hence we find $h = 43{,}6557$ [2696d]. [2697]

In the quadratures, the value of $\frac{L'}{r'^3}$ must be decreased by one fortieth part, on account of the argument of variation [2687b]. This is equivalent to a decrease of h' [2688, 2683″] in the like ratio; by which means it becomes $43{,}3395$; therefore we have,* [2697′] [2698]

$$2P \,.\, \left\{ 43{,}3395 \,.\, \frac{L'}{r'^3} - 43{,}6557 \,.\, \frac{L}{r^3} \right\} = 133^{\text{met.}}{,}819\,; \qquad [2699]$$

r and r' being the mean distances of the sun and the moon from the earth. We may put this equation under the following form,

$$2P \,.\, 43{,}3395 \,.\, \left\{ \frac{L'}{r'^3} - \frac{L}{r^3} \right\} - 2P \,.\, 0{,}3162 \,.\, \frac{L}{r^3} = 133^{\text{met.}}{,}819. \qquad [2700]$$

In the small term $2P \,.\, 0{,}3162 \,.\, \frac{L}{r^3}$, we may suppose†

$$\frac{L}{r^3} = \tfrac{1}{2} \,.\, \left\{ \frac{L'}{r'^3} - \frac{L}{r^3} \right\}; \qquad [2701]$$

therefore we shall have,

$$2P \,.\, 43{,}1814 \,.\, \left\{ \frac{L'}{r'^3} - \frac{L}{r^3} \right\} = 133^{\text{met.}}{,}819\,; \qquad [2702]$$

hence we deduce,

$$2P \,.\, \left\{ \frac{L'}{r'^3} - \frac{L}{r^3} \right\} = 3^{\text{met.}}{,}0990. \qquad [2703]$$

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* (1932) This is deduced from [2695], by substituting h [2697] and h' [2698].

† (1933) The equation [2701] follows from the value $\frac{L'}{r^3} = 3 \,.\, \frac{L}{r^3}$ [2568′], which has been so often used. The error of this assumption must be inconsiderable, on account of the smallness of the coefficient $0{,}3162$, in comparison with $43{,}3395$, by which the other term of [2700] is multiplied. Substituting therefore for $-2P \,.\, 0{,}3162 \,.\, \frac{L}{r^3}$ the value $-2P \,.\, 0{,}3162 \times \tfrac{1}{2} \,.\, \left\{ \frac{L'}{r'^3} - \frac{L}{r^3} \right\}$, the formula [2700] becomes as in [2702]. Dividing this by $43{,}1814$, we get [2703]. [2701a] [2701b]

We have found in [2577],

[2704] $$2P.\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\right\}=6^{\text{met.}}2490;$$

hence we get,*

[2705] $$\frac{L'}{r'^3}=2{,}9677.\frac{L}{r^3}.$$

The lunar force $\frac{L'}{r'^3}$, is nearly three times the solar force $\frac{L}{r^3}$.

Therefore we may suppose very nearly,

[2706] $$\frac{L'}{r'^3}=3.\frac{L}{r^3}.$$

But we must observe *that this ratio is not exactly that of the masses of the sun and moon, divided respectively by the cubes of their mean distances from the earth.* For it appears from § 25, that L' and L being these masses, and

[2706′] $m\,t$, $m'\,t$, the mean motions of these bodies about the earth, *the ratio just*

[2707] *found is that of* $\frac{L'.(1-2m'Q)}{r'^3}$ † *to* $\frac{L.(1-2mQ)}{r^3}$; *it cannot therefore be taken for the ratio of* $\frac{L'}{r'^3}$ *to* $\frac{L}{r^3}$, *except when* Q *is nothing or*

[2707′] *insensible;* and we have seen above [2601, 2624], that this is nearly the case at Brest.

We shall now determine the variation of the tides near their minimum, as it appears from the theory. For this purpose, we shall resume the values of

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* (1934) Taking the sum and difference of the equations [2703, 2704], we get

[2705a] $$4P.\frac{L'}{r'^3}=9^{\text{met.}},3480; \qquad 4P.\frac{L}{r^3}=3^{\text{met.}},1500.$$

[2705b] Dividing the former by the latter, we obtain the ratio of $\frac{L'}{r'^3}$ to $\frac{L}{r^3}$ [2705]. From a discussion of the later observations at Brest [2507c], it was found necessary to decrease this ratio from 2,9677 to 2,3533, as will be seen in Book XIII, [11905].

† (1935) The method used, in [2577′—2579], to prove, in the syzygies, that L' ought to be changed into $L'.(1-2m'Q)$, may be applied in the present case, in the

[2707a] quadratures; by merely reading *quadratures when the moon is in the solstices or equinoxes,* instead of *syzygies of the solstices or equinoxes respectively.* It is also evident, from the same principles, that L ought to be changed into $L.(1-2mQ)$, but m and Q being both very small, we may neglect $2mQ$.

p, p'.

Y'' [2639, 2640]. Let p and p' be the sum of the squares of the cosines of the declinations of the sun and moon, in the quadratures of the equinoxes of Table V [2648]; q and q' the like sums in the quadratures of the solstices of the same table; we may suppose, in these expressions, [2708]

$q\ q'$. [2708']

$$\cos.^2 \varepsilon = \frac{q}{24}; \qquad \cos.^2 \varepsilon' = \frac{p'}{24}. \qquad [2709]$$

The term multiplied by t^2, in the expression of Y'' [2639], relative to the twenty-four equinoctial quadratures [2646], becomes*

Coefficient of t^2 at the time of the equinoxes.

$$\tfrac{39}{40}.48P.\frac{L'}{r'^3}.t^2.\left\{\Gamma' - \frac{\Gamma}{24}.\sqrt{qp'}\right\}^2.\left\{1{,}0611.\left(\frac{24-p'}{24}\right) + \frac{2p.\frac{L}{r^3}}{\frac{39}{40}p'.\frac{L'}{r'^3} - p.\frac{L}{r^3}}\right\}; \qquad [2710]$$

Γ' and Γ *being the mean motions of the moon and sun, in the quadratures, in the interval taken for unity*, and which is, in the equinoctial quadratures, equal to $1^{\text{day}}{,}057493$† [2833]. We must observe, that in the quadratures Γ' is always decreased, on account of the equation of the variation [2637']. Γ', Γ, [2711]

The term multiplied by t^2, in the expression of Y'' [2640], relative to the twenty-four quadratures of the solstices of Table V [2647], becomes, by decreasing $\frac{L}{r^3}$ one fortieth part, because there are eighteen summer solstices, and only six winter solstices, [2696], [2711']

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* (1936) This expression in [2639] is

$$4iP.\frac{L'}{r'^3}.t^2.\{\Gamma' - \Gamma.\cos.V'.\cos.\varepsilon\}^2.\left\{1{,}0611.\sin.^2\varepsilon' + \frac{\frac{2L}{r^3}.\cos.^2 V}{\frac{L'}{r'^3}.\cos.^2 V' - \frac{L}{r^3}.\cos.^2 V}\right\}. \qquad [2710a]$$

The term $\frac{L'}{r'^3}$ must be multiplied by $\frac{39}{40}$, because $\frac{L'}{r'^3}$ is less by one fortieth part in the quadratures than its mean value [2686''']. Then putting, as in [2708, &c.], $2i = 24$, [2710b]
$\cos.V' = \sqrt{(\frac{1}{24}p')}$, $\cos.\varepsilon = \sqrt{(\frac{1}{24}q)}$, $\sin.^2\varepsilon' = 1 - \cos.^2\varepsilon' = \frac{24-p'}{24}$, $\cos.V = \sqrt{(\frac{1}{24}p)}$, [2710c]
it becomes as in [2710].

† (1937) This is easily deduced from formula [2833], by which the daily difference of the tides is $0^{\text{day}}{,}057493$ in the equinoxes, which agrees exactly with the above. In the solstices, the difference is $1^{\text{day}}{,}046643$ [2834], as in [2712']. [2711a]

Coefficient of t^2 at the time of the solstices.

[2712] $$\tfrac{39}{40}\,.\,48\,P\,.\,\frac{L'}{r'^3}\,.\,t^2\,.\left\{\Gamma'.\left(\frac{p'}{24}\right)^{\frac{1}{2}}-\frac{\Gamma}{\left(\frac{q}{24}\right)^{\frac{1}{2}}}\right\}^2.\left\{\frac{2\,q\,.\,\frac{L}{r^3}}{\frac{q'.L'}{r'^3}-\frac{q.L}{r^3}}-1{,}0611\,.\left(\frac{24-p'}{p'}\right)\right\},\ *$$

Γ, Γ'. Γ and Γ' being the motions of the sun and moon in these quadratures, during

[2712′] the interval taken for unity; which is equal to $1^{\text{day}}{,}046643$ [2834], in the quadratures of the solstices.

In these expressions, $\frac{L}{r^3}$ and $\frac{L'}{r'^3}$ are reduced to the mean distances of the sun and moon from the earth, in which case we have†

[2713] $$\frac{L'}{r'^3}=3\,.\,\frac{L}{r^3}\,; \qquad 2\,P\,.\,\frac{L'}{r'^3}=\tfrac{3}{4}\times 6^{\text{met.}}{,}2490=4^{\text{met.}}{,}6867.$$

Values of p, p', q, q'.

We have found

[2714] $$p=23{,}68841\,;\quad p'=20{,}69652\,;\quad q=20{,}47926\,;\quad q'=23{,}75422.$$

This being premised, we shall have $7^{\text{met.}}{,}819$‡ for the term multiplied by t^2 in the expression of Y'', relative to the twenty-four equinoctial

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* (1938) The coefficient of t^2, in the value of Y'' in the quadratures of the solstices [2640], must be multiplied by $\frac{39}{40}$ [2696], and $2\,i$ being 24, it becomes

[2712a] $$\tfrac{39}{40}\times 48\,P.\frac{L'}{r'^3}\,.\,t^2\,.\left\{\Gamma'.\cos.\varepsilon'-\frac{\Gamma}{\cos.\varepsilon}\right\}^2.\left\{\frac{\frac{2\,L}{r^3}\,.\cos.^2 V}{\frac{L'}{r'^3}.\cos.^2V'-\frac{L}{r^3}.\cos.^2V}-1{,}0611\,.\,\text{tang.}^2\,\varepsilon'\right\}.$$

Now putting as in [2710c, 2708′] $\cos.\varepsilon'=\sqrt{(\frac{1}{24}\,p')}$, $\cos.\varepsilon=\sqrt{(\frac{1}{24}\,q)}$, $\cos.^2V=\frac{1}{24}\,q$,

[2712b] $\cos.^2V'=\frac{1}{24}\,q'$; and for $\text{tang.}^2\,\varepsilon'$ writing $\frac{\sin.^2\varepsilon'}{\cos.^2\varepsilon'}=\frac{1-\frac{1}{24}\,p'}{\frac{1}{24}\,p'}=\frac{24-p'}{p'}$, it becomes as in [2712].

† (1939) Substituting $\frac{L}{r^3}=\frac{1}{3}\,.\,\frac{L'}{r'^3}$ [2706], in [2704], it becomes

[2713a] $$2\,P\,.\left\{\frac{L'}{r'^3}+\frac{L'}{3r'^3}\right\}=6^{\text{met.}}{,}2490\,; \qquad \text{hence} \qquad 2\,P\,.\,\frac{L'}{r'^3}=\tfrac{3}{4}\times 6^{\text{met.}}2490.$$

‡ (1940) If we substitute the values [2713], in the expressions [2710, 2712], they become respectively as in [2714a, b],

quadratures, and $2^{met.},894$ for the same term relative to the twenty-four [2715]
solstitial quadratures. The sum of these two quantities is $10^{met.},713$, which
differs but very little from the result $10^{met.},9040$, given in [2670], by the [2716]
observations of Table V [2648].

32. *We shall now consider separately the tides in the quadratures of the equinoxes, and those in the quadratures of the solstices.* We shall find, by the method in the preceding article,* (Heights of the tides at Brest in the quadratures of the equinoxes.)

$$94^{met.},033 + 3^{met.},747 \,.\, t^2, \qquad \left[\text{Absolute heights.}\right] \qquad [2717]$$

$$58^{met.},370 + 7^{met.},495 \,.\, t^2. \qquad \left[\text{Total tides.}\right] \qquad [2717']$$

$$\tfrac{39}{40} \times 18 \times 6^{met.},2490 \,.\, t^2 \,.\, \{\Gamma' - \tfrac{1}{24}\Gamma \,.\, \sqrt{(q\,p')}\}^2 \,.\, \left\{ 1,0611 \,.\, \left(\frac{24-p'}{24}\right) + \frac{2p}{\frac{117}{40} \,.\, p' - p} \right\}; \qquad [2714a]$$

$$\tfrac{39}{40} \times 18 \times 6^{met.},2490 \,.\, t^2 \,.\, \left\{ \Gamma' \,.\, \sqrt{(\tfrac{1}{24}p')} - \frac{\Gamma}{\sqrt{(\frac{1}{24}q)}} \right\}^2 \,.\, \left\{ \frac{2q}{3q'-q} - 1,0611 \,.\, \left(\frac{24-p'}{p'}\right) \right\}; \qquad [2714b]$$

and if we use the values of p, p', q, q', [2714], they change into

$$157^{met.},021 \,.\, t^2 \,.\, \{\Gamma' - 0,857817 \,.\, \Gamma\}^2, \qquad 62^{met.},465 \,.\, t^2 \,.\, \{\Gamma' - 1,165750 \,.\, \Gamma\}^2. \qquad [2714c]$$

In the former, Γ, Γ', must be taken for the time $1^{day},057493$ [2711]; and in the latter, for the time $1^{day},046643$ [2712']; so that if we multiply the preceding expressions by [2714d]
$1,057493^2$, and $1,046643^2$, respectively, they will become,

$$175^{met.},595 \,.\, t^2 \,.\, \{\Gamma' - 0,857817 \,.\, \Gamma\}^2, \quad \text{and} \quad 68^{met.},428 \,.\, t^2 \,.\, \{\Gamma' - 1,165750 \,.\, \Gamma\}^2; \qquad [2714e]$$

and Γ', Γ, will represent the motions in one day. Now the sun's motion in one day is $59^m\, 8^s,3$ [2636q], or in parts of the radius, $\Gamma = 0,017203$; the mean daily motion of [2714f]
the moon is $13^d\, 10^m\, 35^s$, the correction for the variation is nearly $-14^m\, 37^s$, making $12^d\, 55^m\, 58^s$, which in parts of the radius is $\Gamma' = 0,22572$; hence

$$\Gamma' - 0,857817 \,.\, \Gamma = 0,21096, \qquad \Gamma' - 1,165750 \,.\, \Gamma = 0,20567. \qquad [2714g]$$

Substituting these in [2714e], they become $7^{met.},815$ and $2^{met.},894$ nearly as above. The sum of these two terms is $10^{met.},709$, which differs but very little from $10^{met.},9040$, found by observation in [2670, 2682]. The numbers [2715] are erroneous in the original, being 2,794, 10,613.

* (1941) Applying the method [2663″—2669] to the equinoctial tides of Table IV [2717a]
[2646], we find $b = \tfrac{1}{4} \,.\, (99^{met.},511 + 105^{met.},639 - 94^{met.},282 - 96^{met.},059) = 3^{met.},702$,
$b' = \tfrac{1}{4} \,.\, (69^{met.},835 + 81^{met.},342 - 58^{met.},638 - 62^{met.},383) = 7^{met.},539$; from which we [2717b]

Heights of the tides at Brest, in the quadratures of the solstices.

for the expressions of the absolute heights, and of the total tides, in the equinoxes of Table IV [2646, 2647]; the expressions of the same quantities, relative to the tides in the solstices of this table, are*

[2718] $$102^{met.},571 + 1^{met.},705 \,.\, t^2, \qquad \left[\begin{smallmatrix}\text{Absolute}\\\text{heights.}\end{smallmatrix}\right]$$

[2718'] $$75^{met.},517 + 3^{met.},410 \,.\, t^2. \qquad \left[\begin{smallmatrix}\text{Total}\\\text{tides}\end{smallmatrix}\right]$$

The tides increase more rapidly in the equinoxes than in the solstices.

In the first place, we see *that the tides increase more rapidly in the equinoxes than in the solstices*, which is conformable to the theory. According to observations [2717'], the coefficient of t^2, relative to the total tides, is

[2718''] $7^{met.},495$ in the equinoxes, and $3^{met.},410$ [2718'] in the solstices; and we

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obtain, as in [2669''], the corrected values $b = \frac{1}{3}\,.\,(3^{met.},702 + 7^{met.},539) = 3^{met.},747$, $b' = 2b = 7^{met.},494$, [2717, &c.]. The sum of the equinoctial *absolute* tides of Table IV

[2717c] [2646] is $99^{met.},511 + 94^{met.},282 + 96^{met.},059 + 105^{met.},639 = 395^{met.},491$, and the sum of the *total* tides of the equinoxes,

[2717d] $$69^{met.},835 + 58^{met.},638 + 62^{met.},383 + 81^{met.},342 = 272^{met.},198\,;$$

from these we may obtain a, a', by the same method as in [2672, 2673], using the preceding values of b, b'. Hence we have

[2717e] $$a = \tfrac{1}{4}.\{395^{met.},491 - (5+4k^2).3^{met.},747\}, \quad a' = \tfrac{1}{4}.\{272^{met.},198 - (5+4k^2).7^{met.},494\};$$

substituting $(5 + 4k^2) \times 3^{met.},747 = 19^{met.},359$ [2679], we get

[2717f] $$a = 94^{met.},033, \qquad a' = 58^{met.},370, \qquad [2717, 2717'].$$

* (1942) In the same manner as b, b', [2717b, &c.], were computed from the *equinoctial* tides [2646], we may find, from the *solstitial* tides [2647],

[2718a]
$$b = \tfrac{1}{4}.\{106^{met.},117 + 106^{met.}760 - 102^{met.},997 - 103^{met.},220\} = \tfrac{1}{4} \times 6^{met.},660 = 1^{met.},665,$$
$$b' = \tfrac{1}{4}.\{82^{met.},241 + 84^{met.},498 - 76^{met.},289 - 76^{met.},654\} = \tfrac{1}{4} \times 13^{met.},799 = 3^{met.},450.$$

Hence the corrected values [2669'] are

[2718b] $$b = \tfrac{1}{3}\,.\,(1^{met.},665 + 3^{met.},450) = 1^{met.},705, \qquad b' = 2b = 3^{met.},410.$$

The sum of the *absolute* tides in the solstices [2647] is $419^{met.},094$, and the sum of the *total* tides in the solstices of the same table is $319^{met.},685$; hence we get, as in the last note,

[2718c] $$a = \tfrac{1}{4}.\{419^{met.},094 - (5+4k^2).1^{met.}705\}, \quad a' = \tfrac{1}{4}.\{319^{met.},685 - (5+4k^2).3^{met.},410\};$$

substituting $(5 + 4k^2)\,.\,1^{met.},705 = 8^{met.},809$ [2679], we obtain

$$a = 102^{met.},571, \qquad a' = 75^{met.},517, \qquad [2718, 2718'].$$

have seen, in the preceding article [2714—2715], that the theory gives $7^{\text{met.}},819$ and $2^{\text{met.}},894$, for these coefficients. The differences are within the limits of the errors of the observations, and of the elements employed in the calculation.

If we subtract the first term of the expression of the total tides in the equinoxes [2717'], from the first term of the like expression in the solstices, [2718'''] [2718'], the difference $17^{\text{met.}},147$ is the effect of the declinations of the sun and moon. To render it independent of the oscillations whose period is nearly a day, we must add to it, as we have seen in the preceding article [2691], six times $0^{\text{met.}},0457$, and then it becomes $17^{\text{met.}},421$. [2718'''']

According to the formulas of § 29, this effect is equal to*

$$\tfrac{39}{40}\cdot 2P\cdot\left\{\frac{L}{r^3}\cdot(p-q)+\frac{L'}{r'^3}\cdot\left[(1-2m'Q.\cos.\varepsilon').q'-\left(1-\frac{2m'Q}{\cos.\varepsilon'}\right)\cdot p'\right]\right\}, \quad [2718^{\text{v}}]$$

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* (1943) Putting $B=7^{\text{met.}},495$ and $B'=3^{\text{met.}},410$ for the coefficients of t^2, in [2718c'] the expressions of the total tides [2717', 2718'], corresponding to [2639, 2640], we get, for the terms independent of t^2, in the *equinoxes*,

$$4iP\cdot\left\{\frac{L'}{r'^3}\cdot\cos.^2V'-\frac{L}{r^3}\cdot\cos.^2V\right\}+\tfrac{1}{32}B, \qquad [2639]; \quad [2718d]$$

and in the *solstices* $4iP\cdot\left\{\frac{L'}{r'^3}\cdot\cos.^2V'-\frac{L}{r^3}\cdot\cos.^2V\right\}+\tfrac{1}{32}B'$. The moon being [2718e] in the quadratures, we must write $\tfrac{39}{40}\cdot\frac{L'}{r'^3}$ for $\frac{L'}{r'^3}$ [2697']. We must also change L' into $L'.\left(1-\frac{2m'Q}{\cos.V'}\right)$, in the equinoxes [2641], and L' into $L'.(1-2m'Q.\cos.\varepsilon')$, [2718f] in the solstices [2641']. Moreover, in the solstices we must change $\frac{L}{r^3}$ into $\tfrac{39}{40}\cdot\frac{L}{r^3}$, for the reasons stated in [2696]. Hence these expressions become respectively,

$$4iP\cdot\left\{\tfrac{39}{40}\cdot\frac{L'}{r'^3}\cdot\left(1-\frac{2m'Q}{\cos.V'}\right).\cos.^2V'-\frac{L}{r^3}\cdot\cos.^2V\right\}+\tfrac{1}{32}B, \qquad \text{and}$$

$$4iP\cdot\left\{\tfrac{39}{40}\cdot\frac{L'}{r'^3}\cdot(1-2m'Q\cdot\cos.\varepsilon')\cdot\cos.^2V'-\tfrac{39}{40}\cdot\frac{L}{r^3}\cdot\cos.^2V\right\}+\tfrac{1}{32}B'; \quad [2718g]$$

each of which corresponds to $2i$ observations. In the first of these expressions we have $2i.\cos.^2V=p$, $2i.\cos.^2V'=p'$, [2708, 2714]; and in the last, $2i.\cos.^2V=q$, $2i.\cos.^2V'=q'$. Hence in the *equinoxes* we have,

or*

$$\tfrac{39}{40} \cdot 2P \cdot \Big\{ \frac{L}{r^3} \cdot (p-q) + \frac{L'}{r'^3} \cdot (1-2m'Q) \cdot (q'-p') \Big\}$$

[2718vi]

$$+ \tfrac{39}{40} \cdot 2P \cdot \Big\{ \frac{L'}{r'^3} \cdot (1-2m'Q) \cdot (1-\cos.\varepsilon') \cdot \Big(q' + \frac{p'}{\cos.\varepsilon'}\Big) \cdot \frac{2m'Q}{1-2m'Q} \Big\};$$

[2718h]

$$2P \cdot \Big\{ \tfrac{39}{40} \cdot \frac{L'}{r'^3} \cdot \Big(1 - \frac{2m'Q}{\cos.V'}\Big) \cdot p' - \frac{L}{r^3} \cdot p \Big\} + \tfrac{1}{32} B,$$

in which we may put $V' = \varepsilon'$; and in the *solstices*,

[2718h']

$$2P \cdot \Big\{ \tfrac{39}{40} \cdot \frac{L'}{r'^3} \cdot (1-2m'Q \cdot \cos.\varepsilon') \cdot q' - \tfrac{39}{40} \cdot \frac{L}{r^3} \cdot q \Big\} + \tfrac{1}{32} B';$$

subtracting the first of these functions from the second, we obtain the excess of the value of a' in the solstices, over that in the equinoxes,

[2718i]

$$2P \cdot \Big\{ \frac{L}{r^3} \cdot (p - \tfrac{39}{40} q) + \tfrac{39}{40} \cdot \frac{L'}{r'^3} \cdot \Big[(1-2m'Q \cdot \cos.\varepsilon') \cdot q' - \Big(1 - \frac{2m'Q}{\cos.\varepsilon'}\Big) \cdot p'\Big] \Big\} + \frac{B'-B}{32}; \text{ or}$$

[2718k]

$$\tfrac{39}{40} \cdot 2P \cdot \Big\{ \frac{L}{r^3} \cdot (p-q) + \frac{L'}{r'^3} \cdot \Big[(1-2m'Q \cdot \cos.\varepsilon') \cdot q' - \Big(1 - \frac{2m'Q}{\cos.\varepsilon'}\Big) \cdot p'\Big] \Big\}$$

[2718k']

$$+ \tfrac{1}{40} \cdot 2P \cdot \frac{L}{r^3} \cdot p + \frac{B'-B}{32};$$

which differs from [2718v] in the terms [2718k'], omitted by the author.

* (1944) The factor $1 - 2m'Q \cdot \cos.\varepsilon'$, connected with q' in [2718v], may be put under the following form, being the same as in [2596h], multiplying the last term by $1 - 2m'Q$, and then dividing by the same quantity:

[2718l]

$$1 - 2m'Q + (1-2m'Q) \cdot (1-\cos.\varepsilon') \cdot \frac{2m'Q}{1-2m'Q}.$$

Substituting this in [2718k], it produces the terms depending on q' in [2718vi]. In like manner, the factor $-\Big(1 - \frac{2m'Q}{\cos.\varepsilon'}\Big)$, connected with p', may be put under the following form, which is deduced by the same method from [2596h']:

[2718m]

$$-(1-2m'Q) + (1-2m'Q) \cdot (1-\cos.\varepsilon') \cdot \frac{1}{\cos.\varepsilon'} \cdot \frac{2m'Q}{1-2m'Q}.$$

Substituting this in [2718k], it produces the terms depending on p' in [2718vi]. The terms [2718k'], omitted by the author, ought to be added to [2718vi].

in which expression we may suppose $\cos.\varepsilon' = \sqrt{\frac{p'}{24}}$ [2709], and then by putting*

$$\frac{L'}{r'^3}\cdot(1-2m'Q) = \frac{3L}{r^3}; \qquad 2P\cdot\left\{\frac{L}{r^3}+\frac{L'}{r'^3}\cdot(1-2m'Q)\right\} = 6^{\text{met.}},2490.$$ [2718vii]

it becomes,

$$18^{\text{met.}},861 + \frac{2m'Q}{1-2m'Q}\cdot 15^{\text{met.}},015.$$ [2718viii]

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* (1945) The first of the equations [2718vii] is deduced from [2706, 2707], the second from [2577, 2579]. Substituting the first in the second, we get

$$2P\cdot\frac{L}{r^3} = \tfrac{1}{4}\times 6^{\text{met.}},2490; \qquad 2P\cdot\frac{L'}{r'^3}\cdot(1-2m'Q) = \tfrac{3}{4}\times 6^{\text{met.}},2490;$$ [2719a]

hence [2718vi] becomes

$$\tfrac{39}{40}\times\tfrac{1}{4}\times 6^{\text{met.}},2490\,.\,\{(p-q)+3\,.\,(q'-p')\}$$
$$+\tfrac{39}{40}\times\tfrac{3}{4}\times 6^{\text{met.}},2490\,.\,(1-\cos.\varepsilon')\,.\left(q'+\frac{p'}{\cos.\varepsilon'}\right)\cdot\frac{2m'Q}{1-2m'Q}.$$

If we put $\cos.\varepsilon' = \sqrt{(\frac{1}{24}p')}$ [2709], and use the values [2714], it becomes

$$18^{\text{met.}},861 + \frac{2m'Q}{1-2m'Q}\cdot 15^{\text{met.}},015,$$ [2719b]

as in [2718viii]. The neglected term depending on P [2718k'], is

$$\tfrac{1}{40}\times 2P\cdot\frac{L}{r^3}\cdot p = \tfrac{1}{40}\times\tfrac{1}{4}\times 6^{\text{met.}},2490\times 23,68841 = 0^{\text{met.}},925;$$

and that depending on B, B', [2718k', c'], is $\frac{1}{32}\cdot(B'-B) = -0^{\text{met.}},128$; their sum [2719c] $0^{\text{met.}},925 - 0^{\text{met.}},128 = 0^{\text{met.}},797$, must be added to the term $18^{\text{met.}},861$ [2718viii]; making its corrected value equal to $19^{\text{met.}},658$. If we put the expression [2718viii] equal to the value by observation [2718''''], $17^{\text{met.}},421$, we get

$$\frac{2m'Q}{1-2m'Q} = \frac{17,421-18,861}{15,015} = -0^{\text{met.}},0959, \quad\text{and}\quad 2m'Q = -0^{\text{met.}},1061, \quad\text{as in [2719].}$$ [2719d]

Using the corrected value $19^{\text{met.}},658$, instead of $18^{\text{met.}},861$, it becomes

$$\frac{2m'Q}{1-2m'Q} = \frac{17,421-19,658}{15,015} = -0^{\text{met.}},1489, \qquad\text{whence}\qquad 2m'Q = -0^{\text{met.}},175.$$

We have found in [2601], $2m'Q = -0^{\text{met.}},10637$; and in [2624], $2m'Q = 0^{\text{met.}},03425$. These numbers differ very much from each other, and the true value can be obtained only [2719e] by increasing greatly the number of observations.

Putting this equal to the result obtained from observation $17^{met.},421$ [2718'''''], we obtain,

[2719] $$2m'Q = -0,1061\,;$$

Remarks on the value of $2m'Q$.

which agrees with the result deduced from the observations in the syzygies [2601]; but it is of a contrary sign to that found in [2624], by comparing the observations in the perigee and apogee. Hence it follows that *we may*
[2720] *neglect the terms depending on Q, until its exact value shall be determined by a very great number of observations.*

33. We have seen in [2634'], that the evening tides at Brest exceed
[2720'] those of the morning, in the quadratures of the vernal equinox; and that the contrary takes place in the quadratures of the autumnal equinox. To verify this phenomenon, we have added together, in eleven quadratures near the vernal equinox, the excess of the evening over the morning tide, on the
[2720''] first and second days after the quadrature. This sum is $3^{met.},143$. We have likewise found $3^{met.},385$, for the sum of the excess of the morning
[2721] tides over those of the evening, in thirteen quadratures near the autumnal
[2722] equinox. The mean between these observations, gives $0^{met.},138$* *for the excess of an evening tide over that of the morning, in the quadratures of the vernal equinox; or of a morning tide over that of the evening, in the quadratures of the autumnal equinox.*

Difference of the morning and evening tides.

We have found in [2629] $0^{met.},183$, *for the excess of an evening tide over that of the morning, in the syzygies of the summer solstices.* This excess is
[2723] to the preceding, according to the theory, as† $\frac{L}{r^3}+\frac{L'}{r'^3}$ to $\frac{L'}{r'^3}$, or as

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* (1946) The eleven quadratures of the vernal equinox contain one observation on the first, and one on the second day after the quadrature [2720''], so that the mean is

$$\tfrac{1}{22}\times 3^{met.},143 = 0^{met.},143\,;$$

those of the autumnal equinox give $\frac{1}{26}\times 3^{met.},385 = 0^{met.},130$; the mean of both is
[2722a] $0^{met.},137$ nearly as in [2722].

† (1947) In the syzygies of the solstices, this excess is

[2722b] $$2A\,.\left\{\frac{L}{r^3}\,.\,\sin.\,v\,.\,\cos.\,v+\frac{L'}{r'^3}\,.\,\sin.\,v'\,.\,\cos.\,v'\right\}.\,\cos.\,(\lambda-\gamma)\qquad[2630]\,;$$

4 to 3, which is very nearly the same as the ratio of the numbers $0^{\text{met.}},183$ and $0^{\text{met.}},138$. [2723']

Lastly we have found that the influence of the variation of the moon's distance, is perceived as sensibly in the observations of the tides of the quadratures, as in those of the syzygies [2608, &c.]. [2724]

ON THE HOURS OF HIGH WATER, AND THE INTERVALS BETWEEN THE TIDES, NEAR THE SYZYGIES.

Equation to determine the time of high water.

34. We shall resume the equation [2466],

$$\text{tang.}\, 2\,.\,(nt+\varpi-\psi'-\lambda)=\frac{\frac{L}{r^3}\,.\cos.^2 \mathrm{v}\,.\,\sin.\,2\,.\,(\psi-\psi')}{\frac{L'}{r'^3}\,.\cos.^2 \mathrm{v}'+\frac{L}{r^3}\,.\cos.^2 \mathrm{v}\,.\,\cos.\,2\,.\,(\psi-\psi')}\,; \qquad [2725]$$

and shall put it under the following form,*

$$\text{tang.}\, 2\,.\,(nt+\varpi-\psi-\lambda)=\frac{\frac{L'}{r'^3}\,.\cos.^2 \mathrm{v}'\,.\,\sin.\,2\,.\,(\psi'-\psi)}{\frac{L}{r^3}\,.\cos.^2 \mathrm{v}+\frac{L'}{r'^3}\,.\cos.^2 \mathrm{v}'\,.\,\cos.\,2\,.\,(\psi'-\psi)}\,. \qquad [2726]$$

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and in the quadratures of the equinoxes, where v is nearly equal to nothing, the excess is

$$2A\,.\,\frac{L'}{r'^3}\,.\,\sin.\,\mathrm{v}'\,.\,\cos.\,\mathrm{v}'\,.\,\cos.\,(\lambda-\gamma) \qquad [2634]. \qquad [2723a]$$

If we put $\mathrm{v}=\mathrm{v}'$ in the first of these expressions, it will be to the last as $\frac{L}{r^3}+\frac{L'}{r'^3}$ to $\frac{L'}{r'^3}$, which by the first equation [2713] is as 4 to 3. Now by [2629] the excess in the syzygies is $0^{\text{met.}},183$, and $4:3::0^{\text{met.}},183:0^{\text{met.}},137$, which agrees with the value found in [2722a]. [2723b]

* (1948) The equation [2466] or [2725] is derived from [2465]. Now in this last equation, we may change L, r, v, ψ, into L', r', v', ψ', respectively, and the contrary, without altering its value, as is evident by inspection; therefore we may make the same changes in [2725], by which means it becomes as in [2726]. We may also obtain [2726] directly from [2725], by means of the formula [29] Int., putting [2726a]

$$a=2\,.\,(nt+\varpi-\psi'-\lambda), \qquad b=2\,.\,(\psi'-\psi), \qquad [2726b]$$

and reducing; but it is unnecessary to go through this calculation, as the former method is the most simple.

[2726′] The angle $\psi'-\psi$ being small near the syzygies, we may neglect its third power, and we shall have,*

[2727] $$nt+\varpi-\psi-\lambda=\frac{(\psi'-\psi)\cdot\frac{L'}{r'^3}\cdot\cos.^2 v'}{\frac{L'}{r'^3}\cdot\cos.^2 v'+\frac{L}{r^3}\cdot\cos.^2 v}.$$

Hour of the total tide.

We shall now consider *the middle time between the two hours of high water of*
[2728] *the same day. This time we shall call the hour of the total tide.* The
[2728′] preceding equation will also be satisfied at this hour,† provided the variable quantities nt, ψ, and ψ', correspond to this hour. Now $nt+\varpi-\psi$ is

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* (1949) About the time of new moon, $\psi'-\psi$ is small; but at the full moon, it is nearly
[2727a] equal to 200° [2479d], and then we must change ψ into $\psi\pm 200°$ in [2726]. This change does not alter the form of the equation, but it renders $\psi'-\psi$ so small near the syzygies, that we may neglect the third power of $\psi'-\psi$, as in [2726′], and consider the angle $nt+\varpi-\psi-\lambda$ as of the same order as $\psi'-\psi$. Hence by formulas [45, 43, 44] Int., we may put

[2727b] $$\text{tang.}\,2\cdot(nt+\varpi-\psi-\lambda)=2\cdot(nt+\varpi-\psi-\lambda),$$
$$\sin.\,2\cdot(\psi'-\psi)=2\cdot(\psi'-\psi),\qquad \cos.\,2\cdot(\psi'-\psi)=1.$$

Substituting these in [2726], and then dividing by 2, we get [2727].

† (1950) The quantities in [2727] are supposed to correspond to *the hour of the maximum*
[2728a] *of the total tide* [2728], taken as an epoch. If we put M for the coefficient of $\psi'-\psi$, in the second member of [2727]; also $\delta.(nt)$, $\delta\psi$, $\delta\psi'$, δM, for the variations of nt, ψ, ψ', M, respectively, during the interval from high water, to the hour of the following total tide [2728], or the epoch; the equations [2727] for this high water, will become as in [2728b]. In the high water following this epoch, the equation [2727] will become as in [2728c] nearly, neglecting the second differences of $\delta.(nt)$, $\delta\psi$, &c.

[2728b] $$nt-\delta.(nt)+\varpi-(\psi-\delta\psi)-\lambda=(M-\delta M)\cdot(\psi'-\delta\psi'-\psi+\delta\psi);$$

[2728c] $$nt+\delta.(nt)+\varpi-(\psi+\delta\psi)-\lambda=(M+\delta M)\cdot(\psi'+\delta\psi'-\psi-\delta\psi).$$

Half the sum of these two equations is

[2728d] $$nt+\varpi-\psi-\lambda=M\cdot(\psi'-\psi)+\delta M\cdot(\delta\psi'-\delta\psi);$$

and by neglecting $\delta M\cdot(\delta\psi'-\delta\psi)$, of the second order of differences, we get,

[2728e] $$nt+\varpi-\psi-\lambda=M\cdot(\psi'-\psi);$$

in which nt, ψ, M, correspond to the epoch, or to the time of the total tide, as in [2727—2728′]; observing that all these times are to be decreased by $1^{\text{day}},50724$ [2544], in finding ψ, ψ', &c.

the horary angle of the sun [2131c], and $\psi' - \psi$ is nothing at the *maximum* of the total tide [2478']; therefore if we put [2729]

$T =$ the apparent hour of the *maximum* of the total tide, [2729'] *T.*

we shall have, for the apparent hour of the total tide, on any day,*

General expression of the apparent hour of the total tide in the syzygies; First form,

$$T + \frac{(\psi' - \psi) \cdot \frac{L'}{r'^3} \cdot \cos.^2 v'}{\frac{L'}{r'^3} \cdot \cos.^2 v' + \frac{L}{r^3} \cdot \cos.^2 v} = \text{hour of the total tide}; \qquad [2730]$$

the second term of this expression being reduced to time, estimating the whole circumference 400° *as one day. We shall put* υ *for the synodical motion of the moon in the syzygies, during the interval between two consecutive tides of the morning, or evening, near the syzygies. This interval we shall take for the unit of time;* and we shall put t for the number of these intervals, counted from the time of the maximum to the time of high water corresponding to t. In the syzygies of the equinoxes $\psi' - \psi$ is nearly equal to $t\upsilon . \cos. \varepsilon'$ [2733e], and we may suppose $\cos.^2 v' = \cos.^2 v$; hence the apparent hour of the total tide is† [2730'] [2731] υ. [2732] [2733]

Second form, in the syzygies of the equinoxes;

$$T + \frac{\frac{L'}{r'^3} \cdot t\upsilon . \cos. \varepsilon'}{\frac{L'}{r'^3} + \frac{L}{r^3}} = \text{hour of the total tide in the equinoxes.} \qquad [2734]$$

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* (1951) Using M as in the last note, the equation [2728e] becomes

$$n t + \varpi - \psi = \lambda + M . (\psi' - \psi). \qquad [2729a]$$

Now at the maximum of the total tide, $\psi' - \psi = 0$ [2478'], and the horary angle $n t + \varpi - \psi$ [2131c] is then equal to T [2729']; hence the preceding equation becomes simply $T = \lambda$. Substituting this in the general expression of the horary angle, or the hour of the total tide [2729a], it becomes $n t + \varpi - \psi = T + M . (\psi' - \psi)$, as in [2730]. [2729b]

† (1952) From [2505] we get $d\psi' = dt . \frac{m' . \cos. \varepsilon'}{\cos.^2 v'}$; in like manner for the sun, $d\psi = dt . \frac{m . \cos. \varepsilon}{\cos.^2 v}$; hence $d\psi' - d\psi = dt . \left\{ \frac{m' . \cos. \varepsilon'}{\cos.^2 v'} - \frac{m . \cos. \varepsilon}{\cos.^2 v} \right\}$; but m is much smaller than m', and ε' is nearly equal to ε [2497h]; therefore we may, without any sensible error, change $\cos. \varepsilon$ into $\cos. \varepsilon'$, and the preceding expression becomes [2733a]

$$d\psi' - d\psi = dt . \cos. \varepsilon' . \left\{ \frac{m'}{\cos.^2 v'} - \frac{m}{\cos.^2 v} \right\}. \qquad [2733b]$$

[2735] In the solstices $(\psi'-\psi).\cos.v'$ [2733f] is nearly equal to $t\upsilon$, and we may also suppose $\cos.^2 v'=\cos.^2 v$; hence the apparent hour of the total tide is,

Third form, in the syzygies of the solstices.

[2736]
$$T+\frac{\frac{L'}{r'^3}\cdot t\upsilon}{\left(\frac{L}{r^3}+\frac{L'}{r'^3}\right)\cdot\cos.v'}=\text{hour of the total tide in the solstices.}$$

In these calculations we must suppose t to be negative in the tides preceding the *maximum*. We shall now compare these formulas with observations.

35. For this purpose we have selected the hours of the total tides of Table I [2510, &c.], on the days 0, 1, 2, 3, taking the middle between the

[2737] hours of high water on the same day; these hours being counted from the apparent midnight preceding. We have obtained the following results.

Observed hours of the total tide at Brest;

TABLE VI.

SYZYGIES OF THE EQUINOXES.

in 24 syzygies of the equinoxes,

[2738]

Days counted from the syzygy.	Apparent hours of the total tide at Brest.
0	0^{day},39708
1	0 ,42222
2	0 ,44733
3	0 ,47359

SYZYGIES OF THE SOLSTICES.

and in 24 syzygies of the solstices.

[2739]

0	0^{day},39606
1	0 ,42592
2	0 ,45369
3	0 ,48186

In the syzygies, we have nearly $\cos.^2 v=\cos.^2 v'$; substituting this in [2733b], we get $d\psi'-d\psi=(m'dt-mdt)\cdot\frac{\cos.\varepsilon'}{\cos.^2 v'}$; and by neglecting the variations of ε', v', its integral

[2733c] is $\psi'-\psi=(m't-mt)\cdot\frac{\cos.\varepsilon'}{\cos.^2 v'}$. Now $m't$ represents the mean motion of the moon in the time t [2505'], mt the mean motion of the sun; therefore $m't-mt$ is the mean synodical motion, which is represented by $t\upsilon$ [2731, 2732]; hence we get,

[2733d]
$$\psi'-\psi=t\upsilon\cdot\frac{\cos.\varepsilon'}{\cos.^2 v'}.$$

We shall in the first place consider all these observations collectively. If we take the mean between the hours of the total tides in this table, corresponding to the same day, in the syzygies of the equinoxes, and in the syzygies of the solstices, we shall have,

$$0^{\text{day}},39657, \qquad 0^{\text{day}},42407, \qquad 0^{\text{day}},45051, \qquad 0^{\text{day}},477725, \qquad [2740]$$

for the apparent hours of the total tides corresponding to the days 0, 1, 2, 3. We shall assume the following general expression for the hour of the total tide,* Hours of the total tides.

$$a + b \,.\, t' = \text{the hour of the total tide}\,; \qquad [2741]$$

t'.

t' being the number of intervals taken for unity [2732], *counted from the time of the total tide on the day of the syzygy.* Subtracting the first of these hours from the fourth, we obtain the value of $3\,b$, from which we get $b = 0^{\text{day}},027052$. If from the sum of the four preceding hours [2740] we [2742] [2743]

In the syzygies of the equinoxes v' is very small, so that $\cos.^2 v'$ is nearly equal to 1; hence the expression becomes $\psi' - \psi = t\,v\,.\cos.\,\varepsilon'$, as in [2733]; substituting this in [2730], and putting $\cos.^2 v' = \cos.^2 v$, we get [2734]. In the syzygies of the solstices v' is nearly equal to ε', therefore we may put $\cos.\,\varepsilon' = \cos.\,v'$ in [2733*d*], and we shall get $\psi' - \psi = t\,v\,.\,\frac{1}{\cos.\,v'}$, as in [2735]. Substituting this and $\cos.^2 v = \cos.^2 v'$ in [2730], we obtain [2736]. [2733*e*] [2733*f*]

* (1953) This assumed form corresponds with those in formulas [2734, 2736]; and if we put successively t equal to 0, 1, 2, 3, we get the annexed system of equations. Then, according to the method of the author, we must subtract the first from the fourth, to

$$\begin{array}{|ll|}\hline a & = 0^{\text{day}},39657 \\ a + b & = 0\ \ ,42407 \\ a + 2\,b & = 0\ \ ,45051 \\ a + 3\,b & = 0\ \ ,477725 \\ \hline\end{array} \qquad [2743a]$$

obtain $3\,b = 0^{\text{day}},477725 - 0^{\text{day}},39657 = 0^{\text{day}},081155$; hence $b = 0^{\text{day}},027052$ [2743]. [2743*b*]
The sum of the four equations [2743*a*] is $4\,a + 6\,b = 1^{\text{day}},748875$; if we subtract $6\,b = 0^{\text{day}},162310$, we get $4\,a = 1^{\text{day}},586565$, or $a = 0^{\text{day}},39664$ [2744]. [2743*c*]
Substituting these values of a, b, in [2741], we find [2745].

The equations [2743*a*] may also be combined by the method of the least squares [849*k*]. [2743*d*]
In this case the two fundamental equations are found, by taking in the first place the sum of all the equations [2743*a*], and then the sum of their products, by the coefficients 0, 1, 2, 3, respectively. These equations are $4\,a + 6\,b = 1^{\text{day}},748875$, $6\,a + 14\,b = 2^{\text{days}},758265$; [2743*e*]
whence $a = 0^{\text{day}},39673$, $b = 0^{\text{day}},026991$; consequently the hour of the total tide [2741] becomes $0^{\text{day}},39673 + 0^{\text{day}},026991\,.\,t'$, which differs but very little from [2745]. [2743*f*]

[2744] subtract $6b$, the difference is $4a$; therefore $a = 0^{\mathrm{day}},39664$; and the expression of the hour of the total tide [2741] becomes,

Hour of the total tide in the syzygies.

[2745] $$0^{\mathrm{day}},39664 + 0^{\mathrm{day}},027052 \,.\, t'.$$

To obtain the constant quantity T of the preceding article [2729'], we shall
[2746] observe, that when the tide follows after the syzygy by $1^{\mathrm{day}},027052$, the hour of the total tide is increased by $0^{\mathrm{day}},027052$ [2745]; and in the syzygies of the preceding table [2738, 2739], the hour of the syzygy at
[2747] Brest, by the mean of the observations, is $0^{\mathrm{day}},45667$. Therefore if we suppose x to be the value of t' [2742], corresponding to the hour of the maximum of the total tide T [2729'], we shall have for this hour,*

[2748] $$0^{\mathrm{day}},39664 + 0,027052 \,.\, x,$$

[2748'] and this time will follow after the syzygy by $1,027052 \,.\, x - 0^{\mathrm{day}},06003$ [2748c]. Now T [2729'] is the hour of the total tide corresponding to the *maximum*, and this tide follows the syzygy by $1^{\mathrm{day}},50724$ [2544]. Putting
[2749] this quantity equal to the preceding expression $1,027052 \,.\, x - 0^{\mathrm{day}},06003$ [2748'], we obtain x, and find,

Value of T at Brest, or the hour of the maximum total tide in the syzygies.

[2750] $$T = 0^{\mathrm{day}},39664 + 0,027052 \,.\, x = 0^{\mathrm{day}},43793.$$

This is the value of T at Brest; and it is, in this port, the hour of the total
[2750'] solar tide, supposing the sun only to act upon the sea, and that it moves uniformly in the plane of the equator. If we subtract from it a quarter of a

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[2748a] * (1954) In the definition of x [2748] an alteration was made in the original work, to render it less obscure. The substitution of x for t' in [2745] gives the hour T of the total tide, at its maximum, $0^{\mathrm{day}},39664 + 0,027052 \,.\, x$ [2748], counted from the preceding
[2748b] midnight; or $0^{\mathrm{day}},39664 + 1,027052 \,.\, x$, counted from the midnight preceding the syzygy as an epoch. If we subtract the hour of the syzygy $0^{\mathrm{day}},45667$ [2747], which is counted from the same epoch, we obtain the interval, from the syzygy to the time of the maximum
[2748c] total tide, $1,027052 \,.\, x - 0^{\mathrm{day}},06003$ [2748']; and as this is equal to $1^{\mathrm{day}},50724$
[2748d] [2544], we get $x = \frac{1,50724 + 0,06003}{1,027052} = 1,526$. Substituting this value of x in [2748], we obtain the hour of the maximum total tide

[2748e] $$T = 0^{\mathrm{day}},39664 + 0^{\mathrm{day}},04128 = 0^{\mathrm{day}},43792 \qquad [2750].$$

day,* *the difference* $0^{\text{day}},18793$ *is, in this hypothesis, the hour of the solar high water at Brest, counted from the apparent midnight, or noon.* [2751]

We shall now determine the value of the coefficient of t', *which results from the theory of gravity.* We have seen in the preceding article [2734], that this coefficient, in the syzygies of the equinoxes, is equal to

Coefficient of t' in the syzygies of the equinoxes.

$$\frac{\frac{L'}{r'^3}\cdot\upsilon\cdot\cos.\varepsilon'}{\frac{L}{r^3}+\frac{L'}{r'^3}}.$$ [2752]

The angle υ [2731] is equal to $14°,1866$ [2585]; dividing it by $400°$, to reduce it to parts of a day, it becomes $\upsilon=0^{\text{day}},0354665$. We may suppose $\cos.\varepsilon'=\sqrt{\frac{q'}{24}}$ [2581*a*], q' being equal to $20,75529$ [2584]. [2753]

Moreover we have, in the syzygies of the equinoxes, $\frac{L'}{r'^3}=\frac{1\,2\,3}{4\,0}\cdot\frac{L}{r^3}$ [2569]; [2753']

but we must decrease this value one thirtieth part, because in the equation $\frac{dy}{dt}=0$ [2464*a*, 2465], the expressions of $\frac{L}{r^3}$ and $\frac{L'}{r'^3}$ are multiplied respectively by $n-m$ and $n'-m'$,† mt and $m't$ being the motions [2754]

* (1954*a*) The interval between the two solar tides of the same day is $0^{\text{day}},5$ [2419—2420]; therefore the hour of the total tide [2728], which falls midway between them, must be distant $0^{\text{day}},25$ from each of them, in the hypothesis assumed [2750']. [2750*a*]

† (1955) In finding, from the value of $\frac{dy}{dt}=0$, the equation [2465], we have neglected the variations of ψ, ψ', [2464'], and have put

$$d\,.\cos.2\,.(nt+\varpi-\psi'-\lambda)=-2\,n\,dt\,.\sin.2\,.(nt+\varpi-\psi'-\lambda);$$ [2752*a*]

whereas it is nearly equal to $-2.(n-m').dt.\sin.2.(nt+\varpi-\psi'-\lambda)$, because $d\psi'$ is nearly equal to $m'dt$ [2505']. To rectify this, we must change, in [2465], $\frac{L'}{r'^3}$ into $\frac{L'}{r'^3}\cdot\frac{n-m'}{n}$; and for a similar reason, $\frac{L}{r^3}$ into $\frac{L}{r^3}\cdot\frac{n-m}{n}$ [2754]. These changes must also be made in [2466], which was deduced from [2465], and in [2725—2736, 2752, 2756]. The same effect may be obtained in [2752, 2756], by changing L' into $L'.(n-m')$, and L into $L.(n-m)$, as in [2754]. Now the sideral revolutions of the sun and moon, also that [2752*b*] [2752*c*]

[2754′] of the sun and moon [2706′]. $m'-m$ is nearly *a thirtieth part* of $n-m$
[2755] [2752e], hence we have $0^{\text{day}},024679$ [2752f], for the coefficient of t' in the syzygies of the equinoxes, resulting from the theory.

Coefficient of t' in the syzygies of the solstices.

In the syzygies of the solstices, this coefficient is, by the preceding article [2736], equal to

[2756]
$$\frac{\frac{L'}{r'^3}\cdot v}{\left(\frac{L}{r^3}+\frac{L'}{r'^3}\right)\cdot \cos. \text{v}'}.$$

[2756′] We may suppose $\cos. \text{v}'=\sqrt{\frac{q'}{24}}$; hence we get $0^{\text{day}},028603$ [2752f] for this coefficient. Adding it to the preceding [2755], and taking half the
[2757] sum, we obtain $0^{\text{day}},026641$, for the coefficient of t', deduced from the theory, relative to the whole of the syzygies of Table VI [2738, 2739],
[2757′] taken collectively. This differs but little from $0^{\text{day}},027502$, deduced from the observations [2745].

To make the results of theory and observation agree, we must increase a
[2757″] little the ratio of $\frac{L'}{r'^3}$ to $\frac{L}{r^3}$; which furnishes another method of

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[2752d] of the earth about its axis, are nearly in sideral days, 366^{days}, $27^{\text{days}},4$ 1^{day}; hence if we put $n=1$, we shall have $m'=\frac{1}{27,4}$, $m=\frac{1}{366}$; consequently

[2752e]
$$\frac{m'-m}{n-m}=\tfrac{1}{30}, \qquad \frac{n-m'}{n-m}=1-\left(\frac{m'-m}{n-m}\right)=\tfrac{29}{30},$$

nearly [2754′]; and the value of $\frac{L'}{r'^3}$ [2753′] must be decreased in the same ratio, by which means it will become $\frac{L'}{r'^3}=\frac{123}{40}\times\frac{29}{30}\cdot\frac{L}{r^3}=\frac{1189}{400}\cdot\frac{L}{r^3}$. Substituting this in the two formulas [2752, 2756], they become respectively $\frac{1189}{1589}\cdot v\cdot\cos. \varepsilon'$, $\frac{1189}{1589}\cdot\frac{v}{\cos. \text{v}'}$; and by using the values of v, ε', [2753], and that of v' [2756′], they finally become $0^{\text{day}},02468$,
[2752f] $0^{\text{day}},02854$; being nearly the same as in [2755, 2756′]. The mean of these two values is $0^{\text{day}},02661$, which differs but very little from [2757]. We may incidentally remark, that
[2752g] although the change of n into $n-m$, or $n-m'$, affects a little the hour of the total tide, it is not necessary to notice this circumstance, in finding the height of the tide, as is evident from [2464a—b].

ascertaining this ratio. But we may determine this important element with greater accuracy, by using the differences of the observed hours of the high water, at nearly three and a half days on each side of the maximum of [2757‴] the tides. For this purpose, we have considered, in the abovementioned collection of observations [2507″, &c.], ninety-eight syzygies, and we have [2758] added the hours of high water in the morning and evening, on the second day before the syzygy; these hours being counted from the apparent time [2758′] of midnight in the morning tides, and from the time of noon in the evening tides. When the hour of high water has been observed but once in a day, we have doubled it; by this means we have procured 196 observations. In [2758″] like manner, we have added the hours of high water in the morning and evening, on the fifth day after the syzygy. The sum of these hours is [2758‴] $16^{\text{days}},997222$, on the second day before the syzygy, and $55^{\text{days}},386111$, on [2759] the fifth day after the syzygy. Their difference, divided by 196, is equal to $0^{\text{day}},195862$; which is the retardation of the tides in the interval of these [2759′] observations.

We shall resume the equation of the preceding article [2726],

$$\text{tang.}\,2\,.\,(nt+\varpi-\psi-\lambda)=\frac{\frac{L'}{r'^3}\,.\,\cos.^2 v'\,.\,\sin.\,2\,.\,(\psi'-\psi)}{\frac{L}{r^3}\,.\,\cos.^2 v+\frac{L'}{r'^3}\,.\,\cos.^2 v'\,.\,\cos.\,2\,.\,(\psi'-\psi)}\,. \qquad [2760]$$

The ninety-eight syzygies which we have considered, having been taken indiscriminately, near the equinoxes, and near the solstices, we may suppose [2760′] $\cos.^2 v'=\cos.^2 v$, and $\psi'-\psi$ equal to the synodical motion of the moon [2760″] from the time of the greatest tide.* Now this time has fallen very near to

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* (1956) In the equinoxes, we have $\psi'-\psi=tv.\cos.\epsilon'$ [2733e]; and in the solstices, where $\cos.v'=\cos.\epsilon'$ nearly, we have $\psi'-\psi=\frac{tv}{\cos.\epsilon'}$ [2733f]. The mean of [2760a] these two values, using [2578a, &c.], is

$$\psi'-\psi=\tfrac{1}{2}\,.\,tv\,.\left\{\cos.\epsilon'+\frac{1}{\cos.\epsilon'}\right\}=tv=\text{the mean synodical motion.} \qquad [2760b]$$

Therefore if we take as many tides in the equinoxes as in the solstices, we may suppose the mean synodical motion, after the maximum, to be $\psi'-\psi$, and before it, $\psi'-\psi$; so that $2\,.\,(\psi'-\psi)$ is nearly equal to the synodical motion, in the whole interval [2761]. [2760c]

the middle of the interval comprised between these observations; therefore
[2761] we may suppose $2.(\psi'-\psi)$ to be equal to the synodical motion of the moon during this interval [2760c]. Moreover, the hour of the greatest tide
[2762] is determined by the equation* $nt+\varpi-\psi-\lambda=0$ [2479c]; therefore
[2762'] $2.(nt+\varpi-\psi-\lambda)$ is the retardation of the tide, in the interval comprised between the observations. This retardation, estimated in degrees, supposing the whole circumference $400°$ to correspond to one day, being put equal to
[2763] μ, we shall have,†

Ratio of the lunar and solar forces.

[2764]
$$\frac{\frac{L'}{r'^3}}{\frac{L}{r^3}}=\frac{\text{tang.}\,\mu}{\sin.2.(\psi'-\psi)-\text{tang.}\,\mu\,.\cos.2.(\psi'-\psi)}.$$

[2765] The preceding observations give $\mu=783448''$;‡ but among the observations
[2765'] which precede the syzygy, 112 correspond to the evening high water; and among those which follow the syzygy, only 100 refer to the evening high
[2765''] water. Hence it is evident, that the mean interval of the observations is
[2766] $7^{\text{days}}165249$.§ If we suppose this interval to be divided exactly in two equal

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* (1957) The maximum of the tide corresponds to $\psi'=\psi$, and $nt+\varpi-\psi-\lambda=0$
[2762a] [2479c]; in which $nt+\varpi-\psi$ [2131c] represents the horary angle, or hour of the day. Hence it appears, that the angle $nt+\varpi-\psi-\lambda$, computed from [2760], for the fifth day after the syzygy, represents also the increment of this angle from the time of the maximum of the tide to that fifth day; and its double, or $2.(nt+\varpi-\psi-\lambda)$, represents the whole retardation, from the second day before, to the fifth day after the syzygy
[2762b] nearly; hence $\mu=2.(nt+\varpi-\psi-\lambda)$ [2762', 2763].

† (1958) Substituting μ [2762b], and $\cos.^2 v=\cos.^2 v'$ [2760''], in [2760], we

[2764a] get $\text{tang.}\,\mu=\dfrac{\frac{L'}{r'^3}\,.\sin.2.(\psi'-\psi)}{\frac{L}{r^3}+\frac{L'}{r'^3}\,.\cos.2.(\psi'-\psi)}$; hence we easily obtain [2764].

[2765a] ‡ (1959) This is deduced from $0^{\text{day}},195862$ [2759'], turned into degrees, estimating one day as equal to $400°$.

§ (1960) If there were as many evening as morning tides, on the second day before the
[2766a] syzygy, and on the fifth day after the syzygy, the whole interval would be 7 days, increased by the retardation of the tides [2759'], making the interval $7^{\text{days}},195862$. This is to be corrected, on account of the greater number of evening than morning observations. There

parts, at the instant of the *maximum* of the tide, and notice the argument of variation, we shall find $983284''$,* for the synodical motion of the moon [2767]

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are 196 observations on the second day before the syzygy [2758″]; 112 in the evening [2765′], and 84 in the morning. In like manner, on the fifth day after the syzygy, there are 196 tides, of which 100 are evening tides [2765″], and 96 morning tides. We shall put, on the second day before the syzygy, T for the hour of the morning tide, counted from the preceding midnight; and $T+t$ for the hour of the evening tide, counted from the preceding noon. Also T', $T'+t$, for the hours of the morning and evening tides respectively, on the fifth day after the syzygy; counted in like manner from the midnight, or noon, immediately preceding. Then we shall have, on the second day before the syzygy, $84\,T+112\,.\,(T+t)=16^{\text{days}},997222$ [2759, 2766c]; and on the fifth day after the syzygy, $96\,T'+100\,.\,(T'+t)=55^{\text{days}},386111$ [2759, 2766d]. Subtracting the first of these equations from the second, and dividing by 196, we get [2766b] [2766c] [2766d] [2766e] [2766f]

$$T'-T-\tfrac{12}{196}\,.\,t=0^{\text{day}},195862 \qquad [2759]. \qquad [2766f']$$

If we now count all these hours, T, $T+t$, T', $T'+t$, from the midnight preceding the morning tide, on the second day before the syzygy, they will become respectively

$$T, \qquad 0^{\text{day}},5+T+t, \qquad 7^{\text{days}}+T', \qquad 7^{\text{days}},5+T'+t. \qquad [2766g]$$

Making these changes in the first members of the equations [2766e, f], and dividing by 196, we get for the mean of all the observations, on the second day before the syzygy, $T+\tfrac{112}{196}\,.\,(0^{\text{day}},5+t)$; and on the fifth day after the syzygy, $7^{\text{days}}+T'+\tfrac{100}{196}\,.\,(0^{\text{day}},5+t)$. [2766h] The difference of these two expressions represents the mean interval between the observations, which is reduced by means of [2766f′],

$$7^{\text{days}}-\tfrac{12}{196}\times 0^{\text{day}},5+(T'-T-\tfrac{12}{196}\,.\,t)=7^{\text{days}}-0^{\text{day}},030613+0^{\text{day}},195862$$
$$=7^{\text{days}},165249, \qquad \text{as in [2766]}. \qquad [2766i]$$

* (1961) The maximum tide happens about a day and a half after the syzygy [2544], being nearly at the middle of the interval between the observations made on the second day before, and on the fifth day after the syzygy, as in [2766]. Now the whole interval being $7^{\text{days}},165249$ [2766], and the moon's mean synodical daily motion $12^d\,11^m\,27^s$ [2493c]; the mean synodical motion, in half that interval, is $43^d\,40^m\,30^s,6$. The equation of variation, corresponding to this angle, in Burg's tables, is nearly $34^m\,22^s,9$. The sum is the corrected motion, in this half interval, $44^d\,14^m\,53^s,5=49^\circ,1646$; multiplying this by 2, we get the motion in the whole interval, $98^\circ,3292$; being nearly as in [2769]. In finding the equation of variation, the angle $43^d\,40^m\,30^s,4$ is counted from the conjunction; a small difference will be found, if we count it from the opposition. [2767a] [2767b] [2767c]

during this interval, or for the value of $2\,.\,(\psi'-\psi)$. This being premised, we shall have,*

[2768] $$\frac{L'}{r'^3}=3{,}053\,.\,\frac{L}{r^3}\,.$$

This value corresponds to the mean distance of the moon from the earth, because the equation of the variation is very nearly evanescent, at the limits
[2768′] of the interval comprised between these observations; but we must increase it by a thirtieth part, as we have seen [2753′]; hence we get,

[2769] $$\frac{L'}{r'^3}=3{,}155\,.\,\frac{L}{r^3}\,;$$

The intervals of the tides prove the lunar force at Brest to be nearly
[2770]
3 times as great as the solar.

which is very nearly equal to 3. *The observations of the heights and of the intervals of the tides concur therefore at Brest, in showing that the action of the moon on the tides is nearly three times that of the sun.*

36. *We shall now consider separately the syzygies of the equinoxes, and those of the solstices,* of Table VI [2738, &c.]. If we use the method of the preceding article, we shall find,†

[2768a] * (1962) Substituting μ [2765], $2.(\psi'-\psi)=983284''$ [2767] in [2764], we get [2768]. The effect of the variation in the lunar parallax, $80'',25\,.\,\cos.2\,.\,(\psi'-\psi)$ [2569a], is less than $2''$, or $\frac{1}{5000}$ part of the whole parallax $10558'',64$; therefore the parallax, or the
[2768b] distance r', may be considered as at its mean value. We must however change $\frac{L'}{r'^3}$ into $\frac{29}{30}\,.\,\frac{L'}{r'^3}$, for the reasons stated in [2753′, &c.]; and then [2768] becomes

$$\tfrac{29}{30}\,.\,\frac{L'}{r'^3}=3{,}053\,.\,\frac{L}{r^3}\,;$$

whence we easily get [2769] nearly.

† (1963) The computation of the quantities in [2771, 2772] is made as in [2743a—c]. The tides of the equinoxes [2738] give
[2771a] the system of equations [2771a], similar to [2743a]. The sum of these equations is $4\,a+6\,b=1^{\mathrm{day}},74022$; the difference of the extreme equations is

[2771b] $$3\,b=0^{\mathrm{day}},17359-0^{\mathrm{day}},39708=0^{\mathrm{day}},07651\,;$$

hence $b=0^{\mathrm{day}},025503$, and

$$a=\tfrac{1}{4}\,.\,(1{,}74022-6\,b)=\tfrac{1}{4}\,.\,(1{,}74022-0{,}153018)=0^{\mathrm{day}},39680,$$

EQUINOXES.	
a	$=0^{\mathrm{day}},39708$
$a+b=0$	$,42222$
$a+2b=0$	$,44733$
$a+3b=0$	$,47359$
SOLSTICES.	
a	$=0^{\mathrm{day}},39606$
$a+b=0$	$,4259$[illegible]
$a+2b=0$	$,45369$
$a+3b=0$	$,48186$

$$0^{\text{day}},39680 + 0^{\text{day}},025503 \,.\, t', \qquad \left[{}^{\text{In the}}_{\text{equinoxes.}}\right] \qquad [2771]$$

for the expression of the hour of the total tide in the syzygies of the equinoxes, and

Hour of the total tide in the syzygies, at Brest.

$$0^{\text{day}},39648 + 0^{\text{day}},028600 \,.\, t', \qquad \left[{}^{\text{In the}}_{\text{solstices.}}\right] \qquad [2772]$$

for that expression in the syzygies of the solstices.

The mean hour of the syzygy at Brest is $0^{\text{day}},51612$ in the first syzygies, [2773]
and $0^{\text{day}},39722$ in the last. Hence we deduce $T = 0^{\text{day}},43725$,* in the [2774]

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as in [2771]. In like manner, the tides of the solstices [2739] give the system [2771b], whose sum is $4\,a + 6\,b = 1^{\text{day}},75753$. The difference of the extreme observations is $3\,b = 0,48186 - 0,39606 = 0^{\text{day}},08580$; hence, as in [2772],

$$b = 0^{\text{day}},02860, \qquad a = \tfrac{1}{4}\,.\,(1,75753 - 6\,b) = 0^{\text{day}},39648. \qquad [2771b']$$

If we combine these equations by the method of the least squares, as in [2347d—f], we shall obtain, from the system [2771a], the two fundamental equations,

$$4\,a + 6\,b = 1^{\text{day}},74022, \qquad 6\,a + 14\,b = 2^{\text{days}},73765\,; \qquad [2771c]$$

hence $a = 0^{\text{day}},39686$, $b = 0^{\text{day}},025464$; and the hour of the tide in the equinoxes becomes $0^{\text{day}},39686 + 0^{\text{day}},025464 \,.\, t'$, which differs but little from [2771]. In like manner, the [2771d]
system [2771b] gives $4\,a + 6\,b = 1^{\text{day}},75753$. $6\,a + 14\,b = 2^{\text{days}},77888$; hence [2771$e$]
$a = 0^{\text{day}},39661$, $b = 0^{\text{day}},028517$; and the hour of the tide in the solstices becomes $0^{\text{day}},39661 + 0^{\text{day}},028517 \,.\, t'$, being nearly as in [2772]. [2771f]

* (1964) These values of T are computed as in [2748a—d], using the expressions [2771, 2772], instead of [2745]. The equation corresponding to [2748b] is in the equinoxes $0^{\text{day}},39680 + 1^{\text{day}},025503 \,.\, x$; and this represents the hour T of the total tide [2774a]
at its maximum, counted from the midnight preceding the syzygy as an epoch [2748b]. Subtracting the hour of the syzygy $0^{\text{day}},51612$ [2773], we obtain the interval from the [2774b]
syzygy to the time of the maximum total tide,

$$1^{\text{day}},025503 \,.\, x + 0^{\text{day}},39680 - 0^{\text{day}},51612 = 1^{\text{day}},50724 \qquad [2544]\,; \qquad [2774c]$$

hence $x = 1,5861$. Substituting this in the expression of T, counted from the midnight preceding the total tide, as in [2748e], we get from [2774a],

$$T = 0^{\text{day}},39680 + 0^{\text{day}},025503 \,.\, x = 0^{\text{day}},43725, \qquad [2774d]$$

as in [2774]. Proceeding in the same manner in the solstices, using [2772] instead of [2771], we get $1^{\text{day}},028600 \,.\, x + 0^{\text{day}},39648 - 0^{\text{day}},39722 = 1^{\text{day}},50724$ [2772, 2774]; hence $x = 1,466$, and $T = 0^{\text{day}},39648 + 0^{\text{day}},028600 \,.\, x = 0^{\text{day}},43841$, as in [2775]. [2774$e$]

[2775] observations of the syzygies of the equinoxes; and $T = 0^{day},43841$, in the observations of the syzygies of the solstices. The difference between
[2775'] these values and $0^{day},43793$ [2750], given by the whole of the observations of the syzygies, taken collectively in the preceding article, is within the limits of the errors of observation.

The daily retardation of the tides at Brest is less in the equinoxes than in the solstices.

It follows from the preceding expressions, that the coefficient of t', or, in
[2776] *other words, the daily retardation of the tide, near the syzygies, is less in the equinoxes than in the solstices.* This result of observation agrees with the theory, which in the preceding article gives $0^{day},024679$ [2755], and $0^{day},028603$ [2756'], for these coefficients, which differ but little from
[2777] $0^{day},025503$, and $0^{day},028600$, deduced from the observations in [2771, 2772].

The daily retardation of the tide varies sensibly with the distance of the moon;

37. *The daily retardation of the tides varies very sensibly with the distance*
[2777'] *of the moon from the earth.* To compare, upon this point, the theory with observation, we have added, in the tides of the perigee, of Table III [2607],
[2777''] the hours of high water in the morning, and in the evening, on the day of the syzygy; these hours being counted from the apparent midnight in the morning, or from the apparent noon in the evening; their sum is
[2778] $3^{days},476389$. We have added, in like manner, the hours of high water in the morning, and in the evening, on the third day after the syzygy; and have found $5^{days},719444$ for the sum. The difference of these two sums,
[2779] $2^{days},243055$, divided by 72, gives $0^{day},031154$ for the daily retardation of the tide.

In the tides of the apogee of the same table, the sum of the hours of high
[2780] water on the day of the syzygy is $3^{days},642361$, and the sum of the hours of high water on the third day after the syzygy is $5^{days},229514$. The
[2781] difference of these two sums $1^{day},587153$, divided by 72, gives $0^{day},022044$ for the daily retardation of the tides. Hence we see that *this retardation is*
[2782] *less in the apogee, than in the perigee, of the moon.* Comparing the preceding results with the semi-diameters of the moon, in the observations of Table III [2607], we find that one minute variation in this semi-diameter corresponds
[2783] to 258''* of variation in the daily retardation of the time of high water.

and is less in the apogee than in the perigee of the moon.

[2783a] * (1965) The difference of the semi-diameters must have been $3',53$ to produce the above result. For the difference of the daily retardations is

We shall now see what the theory gives, relatively to this point. The [2783′]
observations of Table III [2607] having been taken indiscriminately, some near the equinoxes, and others near the solstices; we may suppose $\psi'-\psi$ [2784]
to be equal to the mean synodical motion of the moon in the syzygies [2760b], and $\cos.^2 v' = \cos.^2 v$ [2760″]. In this case, the daily retardation [2784′]
of the tides, near the syzygies, is by § 34 equal to*

Daily retardation near the syzygies.

$$\frac{\frac{L'}{r'^3}\cdot\upsilon}{\frac{L'}{r'^3}+\frac{L}{r^3}}. \qquad [2785]$$

But υ is greater in the perigee of the moon than in the apogee; we shall have very nearly, at these two points of the orbit, $r'^2\upsilon = r_{\prime}'^2\upsilon_{\prime}$ [585, &c.]; [2786]
$r_{\prime}'$ and $\upsilon_{\prime}$ being relative to the mean distance of the moon in the syzygy. [2787]
Hence the preceding expression becomes,

$$\frac{\frac{L'}{r_{\prime}'^3}\cdot\left(\frac{r_{\prime}'}{r'}\right)^5\cdot\upsilon_{\prime}}{\frac{L'}{r_{\prime}'^3}\cdot\left(\frac{r_{\prime}'}{r'}\right)^3+\frac{L}{r^3}}. \qquad [2788]$$

We have seen in [2753′], that $\frac{L'}{r_{\prime}'^3} = \frac{123}{40}\cdot\frac{L}{r^3}$; but this value must be [2788′]

$$0^{\text{day}},031154 - 0^{\text{day}},022044 = 0^{\text{day}},00911 \qquad [2779, 2781], \qquad [2783b]$$

or 911″. Dividing this by 3′,53, we get 258″ [2783].

* (1966) The expression [2785] may be easily deduced from [2730], using formulas [2784, 2784′]; but we may obtain it more simply, by taking the mean of the two functions [2752, 2756]; putting $\cos. v' = \cos. \varepsilon'$, by which means we shall have very nearly [2785a]

$$\cos.\varepsilon' + \frac{1}{\cos. v'} = \cos.\varepsilon' + \frac{1}{\cos.\varepsilon'} = 2 \qquad [2578a]. \qquad [2785b]$$

The value of the synodical motion υ is greater in the perigee than in the apogee. If we put $r_{\prime}'$, $\upsilon_{\prime}$, for the mean values of r', υ, we shall have very nearly $r'^2\upsilon = r_{\prime}'^2\upsilon_{\prime}$ [585, &c.]; [2785c]
observing that the synodical motion differs but about a thirteenth part from the actual motion of the moon in its orbit. The value of $\frac{L'}{r'^3}$ may be put under the form $\frac{L'}{r_{\prime}'^3}\cdot\left(\frac{r_{\prime}'}{r'}\right)^3$; [2785d]
substituting this and $\upsilon = \left(\frac{r_{\prime}'}{r'}\right)^2\cdot\upsilon_{\prime}$ [2785c], in [2785], it becomes as in [2788].

[2789] decreased a thirtieth part [2753′]; hence $\frac{L'}{r_{,}'^{3}} = 2{,}9725 \cdot \frac{L}{r^3}$.* We have also found, in the syzygies of the perigee in Table III [2607], $\frac{r_{,}'}{r'} = 1{,}06057$;

[2790] and in those of the apogee, $\frac{r_{,}'}{r'} = 0{,}93943$. Lastly we have, by reducing $\upsilon_{,}$ into time, in proportion of the whole circumference 400° for one day,

[2791] $\upsilon_{,} = 0^{\text{day}}{,}0354665$. This being premised, the preceding formula [2788] gives $0^{\text{day}}{,}031125$, for the daily retardation of the syzygial tides of the perigee

[2792] in Table III [2607]; and $0^{\text{day}}{,}022272$, for the daily retardation of those of the apogee. These differ but very little from the observed retardations

[2793] $0^{\text{day}}{,}031154$ [2779], and $0^{\text{day}}{,}022044$ [2781].

ON THE HOURS AND INTERVALS OF THE TIDES, NEAR THE QUADRATURES.

38. If in the equation [2726],

Equation to determine the time of high water.

[2794]
$$\text{tang.}\,2\,.\,(nt+\varpi-\psi-\lambda) = \frac{\frac{L'}{r'^3}\,.\cos.^2 v'\,.\sin.2\,.(\psi'-\psi)}{\frac{L}{r^3}\,.\cos.^2 v + \frac{L'}{r'^3}\,.\cos.^2 v'\,.\cos.2\,.(\psi'-\psi)},$$

[2794′] we change ψ' into $100^\circ+\psi'$, or into $300^\circ+\psi'$;† according as the

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* (1967) Substituting υ [2753] and $\frac{L'}{r_{,}'^{3}}$ [2789], in [2788]; then rejecting the factor $\frac{L}{r^3}$, which occurs in the numerator and denominator, we get, for the daily retardation,

[2788a]
$$\frac{2{,}9725 \times 0^{\text{day}}{,}0354665\,.\left(\frac{r_{,}'}{r'}\right)^5}{2{,}9725\,.\left(\frac{r_{,}'}{r'}\right)^3 + 1}.$$

[2788b] In the perigee, $\frac{r_{,}'}{r'} = 1{,}06057$; in the apogee, $\frac{r_{,}'}{r'} = 0{,}93943$ [2790]; and the corresponding values of the retardation [2788a] are $0^{\text{day}}{,}031117$, $0^{\text{day}}{,}022265$, respectively, being nearly as in [2792].

[2794a] † (1968) If we change ψ' into $\psi'+100^\circ$, or $\psi'+300^\circ$, as in [2794′], the formula [2794] will become of the following form; in which $\psi'-\psi$ will be small near the quadratures, as in [2632b, c]:

moon is near its first, or its last quarter; and suppose $\downarrow'-\downarrow$ to be so small near these points, that we may neglect its third power, we [2794''] shall have,

$$nt+\varpi-\downarrow-\lambda=\frac{(\downarrow'-\downarrow)\cdot\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'}{\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'-\frac{L}{r^3}\cdot\cos.^2\mathrm{v}}. \qquad [2795]$$

Putting therefore T for the apparent hour of the minimum of the total tide, T. [2796] the apparent hour of any tide near the quadrature will be,*

Hour of total tide near the quadratures.

$$T+\frac{(\downarrow'-\downarrow)\cdot\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'}{\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'-\frac{L}{r^3}\cdot\cos.^2\mathrm{v}}; \qquad [2797]$$

the angles $\downarrow'$ and $\downarrow$ being counted from the quadrature. [2797']

$\downarrow'\cdot\cos.\mathrm{v}'$ is the motion of the moon in its orbit, near the quadratures of the equinoxes [2636f]. We shall put Γ' for this motion, during the interval Γ'. [2798] of two consecutive tides of the morning or evening, near the quadratures, [2799] which interval we shall here take for unity; t the number of these intervals, t. from the *minimum* of the total tide, unto the tide under consideration; then [2800] we shall have $\downarrow'\cdot\cos.\mathrm{v}'=\Gamma' t$. Now putting Γ for the sun's motion, in [2801] the time taken for unity, we shall have $\downarrow=\Gamma t\cdot\cos.\varepsilon$ [2636c, &c.]; Γ. therefore the apparent time of the total tide, near the quadratures of the [2802] equinoxes, will be,

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$$\text{tang.}\,2\cdot(nt+\varpi-\downarrow-\lambda)=\frac{-\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'\cdot\sin.2\cdot(\downarrow'-\downarrow)}{\frac{L}{r^3}\cdot\cos.^2\mathrm{v}-\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'\cdot\cos.2\cdot(\downarrow'-\downarrow)}. \qquad [2794b]$$

Neglecting the third power of $\downarrow'-\downarrow$, we may use the values [2727b]; by which means the preceding expression, after dividing it by 2, may be reduced to the form [2795], by [2794c] changing the signs of the numerator and denominator.

* (1969) The equation [2795] may be derived from [2727], by changing the sign of L. Making the same change in [2730], which was derived from [2727], we get [2797]; [2797a] T being, in this case, the time of the minimum of the total tide.

Theoretical computation of the hour of the total tide, in the quadratures of the equinoxes,

[2803]
$$T+\frac{\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'\cdot\left\{\frac{\Gamma'}{\cos.^2\mathrm{v}'}-\Gamma\cdot\cos.\varepsilon\right\}\cdot t}{\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'-\frac{L}{r^3}\cdot\cos.^2\mathrm{v}}.\;*$$

[2804] In the quadratures of the solstices, we have† $\psi'=\Gamma' t\cdot\cos.\varepsilon'$ [2638b]; $\psi=\frac{\Gamma t}{\cos.\mathrm{v}}$ [2638a—b]; therefore the hour of the total tide then becomes,

and of the solstices.

[2805]
$$T+\frac{\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'\cdot\left\{\Gamma'\cdot\cos.\varepsilon'-\frac{\Gamma}{\cos.\mathrm{v}}\right\}\cdot t}{\frac{L'}{r'^3}\cdot\cos.^2\mathrm{v}'-\frac{L}{r^3}\cdot\cos.^2\mathrm{v}}.$$

We shall now compare these results with observations.

39. For this purpose, we have determined the hours of the total tides of Table IV [2646, 2647], corresponding to the numbers 0, 1, 2, 3; by taking
[2805′] the mean of the hours of high water in the two tides, corresponding to the same number; these hours being counted from the apparent midnight preceding. We have found the following results:

Observed hours of the total tide at Brest, in the quadratures,

TABLE VII.

QUADRATURES OF THE EQUINOXES.

of the equinoxes,

[2806]

Number of the total tide.	Apparent hours of the total tide at Brest.
0	0^{day},60566
1	0 ,66125
2	0 ,72411
3	0 ,77815

QUADRATURES OF THE SOLSTICES.

and of the solstices.

[2807]

0	0^{day},61863
1	0 ,66311
2	0 ,70933
3	0 ,75856

* (1970) The calculation in this paragraph is similar to [2636c—g]. The equations [2636f, c], for the time t, correspond respectively to [2801, 2802]; and [2636g] changes
[2803a] into $\psi'-\psi=t\cdot\left\{\frac{\Gamma'}{\cos.\mathrm{v}'}-\Gamma\cdot\cos.\varepsilon\right\}$; substituting this in [2797], we get [2803].

† (1971) In the quadratures of the solstices we have nearly $\varepsilon=\mathrm{v}$; hence the values of

We shall first consider the whole of these observations collectively. Taking the mean of the hours of the total tides [2806, 2807], corresponding to the same number in the quadratures of the equinoxes, and in those of the solstices; we obtain

$$0^{\mathrm{day}},61215, \qquad 0^{\mathrm{day}},66218, \qquad 0^{\mathrm{day}},71672, \qquad 0^{\mathrm{day}},76835, \tag{2808}$$

for the apparent hours of the total tides, corresponding to the numbers 0, 1, 2, 3. Using the method in [2740—2745], we find for the expression of these hours,*

Hours of the total tide in the quadratures.

$$0^{\mathrm{day}},61175 + 0^{\mathrm{day}},052067 \,.\, t' ; \tag{2809}$$

t' being the number of intervals taken for unity, counted from the time of the total tide on the day of the quadrature. To deduce the constant quantity T from the formulas of the preceding article, we shall observe that when the tide is $1^{\mathrm{day}},052067$ distant from the quadrature, its hour increases by $0^{\mathrm{day}},052067$ [2809]. In the quadratures of Table VII [2806, 2807], the hour of the quadrature at Brest, taking the mean value, is $0^{\mathrm{day}},46828$. Therefore if we suppose x to be the value of t' [2809′] corresponding to the hour of the minimum of the total tide, we shall have for this hour, [2809′] [2810] [2811] [2811′] [2812]

$$0^{\mathrm{day}},61175 + 0^{\mathrm{day}},052067 \,.\, x ; \tag{2813}$$

ψ', ψ, [2638a—b], become as in [2804], and $\psi' - \psi = \left\{ \mathrm{r}' \,.\, \cos.\varepsilon' - \dfrac{\mathrm{r}}{\cos.\mathrm{v}} \right\} . t$. If we substitute this in [2797], we get [2805]. [2803b]

* (1972) The computation is made as in [2743a—c], using the numbers [2808] instead of [2740]. Hence we get the annexed equations, from which we find,

$$\begin{array}{|l|}\hline a \qquad\quad = 0^{\mathrm{day}},61215 \\ a + b = 0 \quad\ ,66218 \\ a + 2b = 0 \quad\ ,71672 \\ a + 3b = 0 \quad\ ,76835 \\ \hline\end{array} \tag{2808a}$$

$$b = \tfrac{1}{3} \,.\, (0^{\mathrm{day}},76835 - 0^{\mathrm{day}},61215) = 0^{\mathrm{day}},052067,$$

and $\quad a = \tfrac{1}{4} \,.\, (0^{\mathrm{day}},61215 + 0^{\mathrm{day}},66218 + 0^{\mathrm{day}},71672 + 0^{\mathrm{day}},76835) - \tfrac{3}{2}\, b = 0^{\mathrm{day}},61175$; [2808$b$]

substituting these in [2741], we get [2809].

We may also find the values of a, b, by means of the method of the least squares, as in [2743d—f]. For the sum of the equations [2808a] is $4\,a + 6\,b = 2^{\mathrm{days}},75940$; and the sum of the products found by multiplying each of them by 0, 1, 2, 3, respectively, is $6\,a + 14\,b = 4^{\mathrm{days}},40067$. From these we get $a = 0^{\mathrm{day}},61138$, $b = 0^{\mathrm{day}},052314$; and [2741] becomes $0^{\mathrm{day}},61138 + 0^{\mathrm{day}},052314 \,.\, t'$; which differs but little from [2809]. [2808c] [2808d]

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and this last hour will follow after the quadrature by

[2814] $$1^{\text{day}},052067\,.\,x + 0^{\text{day}},14347.$$

Now T is the hour of the total tide corresponding to the *minimum*, and this hour follows after the quadrature by $1^{\text{day}},50724$ [2544]; putting this
[2815] quantity equal to the function $1^{\text{day}},052067\,.\,x + 0^{\text{day}},14347$, we obtain x;*
Value of T, and then we find,

[2816] $$T = 0^{\text{day}},61175 + 0^{\text{day}},052067\,.\,x = 0^{\text{day}},67924.$$

or of the hour of the minimum of the total tide at Brest, in the quadratures.

This is the time of the *minimum* of the total tide at Brest, in the quadratures. This hour must therefore exceed by a quarter of a day the hour of the *maximum* of the total tide, which we have found in [2750] to be
[2817] equal to $0^{\text{day}},43793$. But the difference of these two hours is only
[2818] $0^{\text{day}},24131$, which is less by about $8'\frac{1}{2}$ than a quarter of a day. This seems to indicate an anticipation in the hour of high water at Brest, in proportion as the height of the tide at full sea decreases. We have already noticed a similar effect, in the height of the *zero* of the scale of observation, above the level of the sea, determined by the tides of the syzygies [2573], and by those of the quadratures [2693, &c.]. These are probably the slight
[2818'] variations, arising from the hypothesis we have used, that the two partial solar and lunar tides are placed the one above the other, as if they were separately arranged upon the level surface of the sea; which is not the case except the undulations are infinitely small.

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* (1973) The calculation of this part is made as in [2748*a*, &c.], or rather as in
[2816*a*] [2774*a*—*e*]; using the quantities $0^{\text{day}},61175$, $0^{\text{day}},052067$ [2809], $0^{\text{day}},46828$ [2811'], instead of $0^{\text{day}},39680$, $0^{\text{day}},025503$, $0^{\text{day}},51612$ [2774*a*, *b*], respectively; by which means [2774*e*] becomes as in [2814],

[2816*b*] $$1^{\text{day}},052067\,.\,x + 0^{\text{day}},61175 - 0^{\text{day}},46828 = 1^{\text{day}},50724\,;$$

hence $x = 1,296$. The same changes being made, in the value of T [2774*d*], it becomes
[2816*c*] $T = 0^{\text{day}},61175 + 0^{\text{day}},052067\,.\,x = 0^{\text{day}},67923$, nearly as in [2816]. Subtracting the hour
[2816*d*] of *maximum* $0^{\text{day}},43793$ [2750], the difference is $0^{\text{day}},24131$; this varies $0^{\text{day}},00869$, or $8',69$, from a quarter of a day, which is the difference by the theory. For the *maximum*
[2816*e*] tide corresponds to $\psi' - \psi = 0$, in [2729, 2730], and the *minimum* to $\psi' - \psi = 0$ in [2797]; but in the last case ψ' differs 100° or 300° [2794'] from its value in [2730]; therefore the times of *maximum* and *minimum* must differ 100°, or one quarter of a day;
[2816*f*] observing that 300° is equivalent to -100°, the whole circumference being 400°.

We shall now determine the value of the coefficient of t' which results from the theory of gravity. We have seen in the preceding article [2803], that this coefficient in the quadratures of the equinoxes, is equal to [2818'']

Daily retardation of the hour of the total tide in the quadratures of the equinoxes.

$$\frac{\frac{L'}{r'^3}\cdot\cos.^2 v'\cdot\left\{\frac{\Gamma'}{\cos. v'}-\Gamma\cdot\cos.\varepsilon\right\}}{\frac{L'}{r'^3}\cdot\cos.^2 v'-\frac{L}{r^3}\cdot\cos.^2 v}. \qquad [2819]$$

We may suppose $\cos.^2 v'=\frac{1}{24}p'=\frac{1}{24}\times 20{,}69652$ [2708, 2714]; $\cos. v=\frac{1}{24}p=\frac{1}{24}\times 23{,}68841$ [2708', 2714]. Moreover, in the quadratures, [2820]

$\frac{L'}{r'^3}=\frac{117}{40}\cdot\frac{L}{r^3}$ [2687]; and this value must be decreased one thirtieth part [2821]

[2753'].* Γ' is the mean motion of the moon, near the quadratures, in the interval of two tides, from one day to another, near the quadratures of the equinoxes. This interval is equal to† $1^{day},05750$; and the motion must be decreased on account of the equation of *variation.* Γ is the corresponding mean motion of the sun.‡ Lastly we may suppose [2821'] [2822] [2823]

$$\cos.^2\varepsilon=\frac{1}{24}q=\frac{1}{24}\times 20{,}47926 \qquad [2709, 2714]. \qquad [2824]$$

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* (1974) By this means we shall get $\frac{L'}{r'^3}=\frac{117}{40}\times\frac{29}{30}\cdot\frac{L}{r^3}=2{,}8275\cdot\frac{L}{r^3}$. [2821a]

† (1975) The daily retardation of the hour of the total tide, at this time, is $0^{day},057493$ [2833]; hence the interval of the two daily tides [2821'] is $1^{day},057493$, being nearly as in [2822]. [2822a]

‡ (1976) The mean daily motion of the sun [2493c], is $59^m 8^s=1^\circ,0951$. Multiplying this by the interval $1^{day},05750$, we get the value of Γ in degrees; which is to be reduced to time as in [2731], by putting $400^\circ=1^{day}$; hence we get, [2823a]

$$\Gamma=1^\circ,0951\times 1{,}05750=1^\circ,1580=0^{day},002895. \qquad [2823b]$$

In like manner, from the moon's mean daily motion in the quadratures, corrected for the variation, $12^d\,55^m\,58^s=14^\circ,36975$ [2636q], we obtain, [2823c]

$$\Gamma'=14^\circ,36975\times 1{,}05750=15^\circ,1960=0^{day},037990. \qquad [2823d]$$

These values of Γ, Γ', correspond to the formula [2819], whose numerical value is found in [2825]; the interval being $1^{day},05750$ [2822]. They must be decreased in the ratio $\frac{1{,}04664}{1{,}05750}$, in the formula [2826], where the interval is $1^{day},04644$ [2828]; and they will then become $\Gamma=0^{day},002865$, $\Gamma'=0^{day},037600$. [2823e] [2823f]

[2825] This being premised, we shall find $0^{day},06425$* for the coefficient of t', given by the theory in the quadratures of the equinoxes.

Daily retardation in the quadratures of the solstices.

In the quadratures of the solstices, the coefficient of t [2805] is equal to

[2826]
$$\frac{\frac{L'}{r'^3}\cdot\cos.^2 v'\cdot\left\{\Gamma'.\cos.\varepsilon'-\frac{\Gamma}{\cos.v}\right\}}{\frac{L'}{r'^3}\cdot\cos.^2 v'-\frac{L}{r^3}\cdot\cos.^2 v}.$$

[2827] Here $\cos.^2 v'=\frac{1}{24}q'=\frac{1}{24}\times 23,75422$, $\cos.^2 v=\frac{1}{24}q=\frac{1}{24}\times 20,47926$, [2708', 2714]. Γ and Γ' are the motions of the sun and moon, in the interval of two tides, on successive days, near the quadratures of the
[2828] solstices, which interval is nearly $1^{day},04664$ [2834]. The moon's motion must be decreased, on account of the argument of variation. Moreover,
[2829] $\frac{L'}{r'^3}=\frac{117}{40}\cdot\frac{L}{r^3}$ [2687]; but as there are eighteen summer, and six winter quadratures, in the observations of Table VII [2806, 2807], the value of
[2829'] $\frac{L}{r^3}$ ought to be decreased one fortieth part.† Lastly, we must decrease
[2830] $\frac{L'}{r'^3}$ by a thirtieth part [2753']. This being premised, we find $0^{day},04528$,

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[2825a] * (1977) Substituting in [2819] the values of $\frac{L'}{r'^3}$ [2821a], $\cos.^2 v'=\frac{1}{24}p'$, $\cos.^2 v=\frac{1}{24}p$ [2820]; dividing the numerator and denominator by $\frac{1}{24}\times 2,8275\,.\frac{L}{r^3}$, and then using the values of p, p', [2714], it becomes

$$\frac{p'}{p'-\frac{p}{2,8275}}\cdot\left\{\frac{\Gamma'}{\cos.v'}-\Gamma.\cos.\varepsilon\right\}=\frac{20,69652}{20,69652-8,37787}\cdot\left\{\frac{\Gamma'}{\cos.v'}-\Gamma.\cos.\varepsilon\right\}$$

[2825b]
$$=1,680097\,.\left\{\frac{\Gamma'}{\cos.v'}-\Gamma.\cos.\varepsilon\right\}.$$

[2825c] Now we have Γ, Γ', [2823b, d], $\cos.v'$ [2820], $\cos.\varepsilon$ [2824]; hence

[2825d]
$$1,680097\,.\frac{\Gamma'}{\cos.v'}=0^{day},06874,\qquad 1,680097\,.\,\Gamma\,.\cos.\varepsilon=0^{day},00449,$$

and $0^{day},06874-0^{day},00449=0^{day},06425$, as in [2825].

† (1978) It appears from [2696] that in consequence of the greater number of summer than
[2828a] of winter solstices [2696a, &c.], we must change $\frac{L}{r^3}$ into $\frac{39}{40}\cdot\frac{L}{r^3}$, in the formula [2826].

for the coefficient of t* given by the theory in the quadratures of the solstices. If we add the two coefficients [2825, 2830], relative to the equinoxes and solstices, and take the half of the sum $0^{day},05476$, it will be the coefficient of t in the whole collection of observations of Table VII. This coefficient is by observation $0^{day},052067$ [2809]. The difference is within the limits of the errors of the observations and of the elements used in the calculation. [2831] [2832]

We shall now consider separately the observations of the quadratures of the equinoxes, and those of the quadratures of the solstices, in Table VII [2806, 2807]. If we use the preceding method, we shall find, for the hour of the total tide near the quadratures of the equinoxes,†

Hours of the total tide at Brest, in the quadratures.

$$0^{day},60605 + 0^{day},057493 \,.\, t'; \qquad \left[\begin{smallmatrix}\text{In the}\\ \text{equinoxes.}\end{smallmatrix}\right]$$ [2833]

and for the hour of the total tides near the quadratures of the solstices,

$$0^{day},61744 + 0^{day},046643 \,.\, t'. \qquad \left[\begin{smallmatrix}\text{In the}\\ \text{solstices.}\end{smallmatrix}\right]$$ [2834]

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* (1979) Substituting in [2826], the values of $\frac{L'}{r'^3}$ [2821a], $\frac{L}{r^3}$ [2828a], $\cos.^2 v'$, $\cos.^2 v$ [2827]; dividing the numerator and denominator by $\frac{1}{24} \times 2,8275 \,.\, \frac{L}{r^3}$; using the values of q, q', [2714], we get, [2829a]

$$\frac{q'}{q' - \frac{39}{40} \,.\, \frac{q}{2,8275}} \,.\, \left\{ \Gamma' \,.\, \cos.\, \varepsilon' - \frac{\Gamma}{\cos.\, v} \right\} = \frac{23,75422}{23,75422 - 7,06181} \,.\, \left\{ \Gamma' \,.\, \cos.\, \varepsilon' - \frac{\Gamma}{\cos.\, v} \right\}$$

$$= 1,42306 \,.\, \left\{ \Gamma' \,.\, \cos.\, \varepsilon' - \frac{\Gamma}{\cos.\, v} \right\}. \qquad [2829b]$$

We have Γ, Γ', [2823f], $\cos.\, \varepsilon'$ [2709], $\cos.\, v$ [2827]; hence [2829c]

$$1,42306 \,.\, \Gamma' \,.\, \cos.\, \varepsilon' = 0^{day},04969, \qquad 1,42306 \,.\, \frac{\Gamma}{\cos.\, v} = 0^{day},00441,$$

and $0^{day},04969 - 0^{day},00441 = 0^{day},04528$, as in [2830]. [2829$d$]

† (1980) This calculation is made as in [2771a, b], using the numbers [2806, 2807] instead of [2738, 2739], by which means we get the annexed systems of equations [2833a, b], corresponding to the equinoxes and solstices respectively, similar to [2771a, b]. Then by the method of the author, we have in the equinoxes,

$b = \frac{1}{3} \,.\, (0^{day},77815 - 0^{day},60566) = 0^{day},057493$,

$a = \frac{1}{4} \,.\, (0^{day},60566 + 0^{day},66125 + 0^{day},72411 + 0^{day},77815) - \frac{3}{2}\, b$
$= 0^{day},60605$.

EQUINOXES.	
a	$= 0^{day},60566$
$a + b$	$= 0\;\;,66125$
$a + 2b$	$= 0\;\;,72411$
$a + 3b$	$= 0\;\;,77815$
SOLSTICES.	
a	$= 0^{day},61863$
$a + b$	$= 0\;\;,66311$
$a + 2b$	$= 0\;\;,70933$
$a + 3b$	$= 0\;\;,75856$

[2833a] [2833b]

[2835] The mean hour of the quadrature at Brest, is $0^{\text{day}},44418$ in the first quadratures, and $0^{\text{day}},49239$ in the second. Hence we find T equal to
[2836] $0^{\text{day}},67919$,* by the observations of the quadratures of the equinoxes; and
[2837] T equal to $0^{\text{day}},67904$, by the observations of the quadratures of the
[2838] solstices. The differences between these values and $0^{\text{day}},67924$, found in [2816], by the combination of all the observations in the equinoxes and in the solstices, are within the limits of the errors of the observations.

It follows from the preceding expressions, that the coefficient of t', or in
[2838'] other words, *the daily retardation of the tides, near the quadratures, is greater*

Substituting these in [2741], we get [2833]. In like manner, from the system [2833b], we get in the solstices,

[2833b']
$$b = \tfrac{1}{3}\,.\,(0^{\text{day}},75856 - 0^{\text{day}},61863) = 0^{\text{day}},046643, \qquad \text{and}$$
$$a = \tfrac{1}{4}\,.\,(0^{\text{day}},61863 + 0^{\text{day}},66311 + 0^{\text{day}},70933 + 0^{\text{day}},75856) - \tfrac{3}{2}\,b = 0^{\text{day}},61744\,;$$

hence [2741] becomes as in [2834].

We may also find the values of a, b, by the method of the least squares, as in [2771c—f]. The system of equations [2833a] will by this method give, in the equinoxes,

[2833c]
$$0^{\text{day}},60524 + 0^{\text{day}},058034\,.\,t',$$

instead of [2833]; and the system [2833b] will give, in the solstices,

[2833d]
$$0^{\text{day}},61751 + 0^{\text{day}},046601\,.\,t',$$

instead of [2834]. These differ but little from the results of the methods of the author.

* (1981) Making the calculation as in [2816a—f], we must change, in the quadratures
[2836a] of the equinoxes, $0^{\text{day}},61175$, $0^{\text{day}},052067$ [2809], $0^{\text{day}},46828$ [2811']; into $0^{\text{day}},60605$, $0^{\text{day}},057493$ [2833], $0^{\text{day}},44418$ [2835], respectively; and in the quadratures of the
[2836b] solstices, into $0^{\text{day}},61744$, $0^{\text{day}},046643$ [2834], $0^{\text{day}},49239$ [2836], respectively. In the former case, the equations [2816b, c] become, in the equinoxes,

[2836c]
$$1^{\text{day}},057493\,.\,x + 0^{\text{day}},60605 - 0^{\text{day}},44418 = 1^{\text{day}},50724,$$
$$T = 0^{\text{day}},60605 + 0^{\text{day}},057493\,.\,x\,;$$

[2836d] from the first of these equations we get $x = 1,2722$, and the second gives $T = 0^{\text{day}},67919$, as in [2836]. In the second case, the equations [2816b, c] become, in the solstices,

[2836e]
$$1^{\text{day}},046643\,.\,x + 0^{\text{day}},61744 - 0^{\text{day}},49239 = 1^{\text{day}},50724,$$
$$T = 0^{\text{day}},61744 + 0^{\text{day}},046643\,.\,x\,;$$

[2836f] from which we get, in the solstices, $x = 1,3206$, $T = 0^{\text{day}},67904$, as in [2837].

The daily retardation of the tide in the quadratures, is greater in the equinoxes than in the solstices.

in the equinoxes than in the solstices. This result of observation is conformable to the theory, which has given $0^{\text{day}},06425$ [2825], and $0^{\text{day}},04528$ [2830]. [2839]
These differ but little from the coefficients $0^{\text{day}},057493$, $0^{\text{day}},046643$, deduced [2839′]
from observation [2833, 2834]. *The difference will be still less, if we notice the third powers of* $\psi'-\psi$, *neglected in* [2726′, 2794″], *and which become* [2839″]
*sensible, especially near the quadratures of the equinoxes.**

The daily retardation in the quadratures increases in the perigee, and decreases in the apogee.

40. *The daily retardation of the tides, near the quadratures, increases in the tides of the perigee, and decreases in those of the apogee;* and this [2840]
phenomenon, arising from the variation of the moon's distance, is less in the [2840′]
tides of the quadratures, than in those of the syzygies. To compare the theory, relatively to this part, with observation, we have added, in eleven quadratures, in which the semi-diameter of the moon is less than twenty-eight [2840″]
minutes, the retardation of the tides, both in the morning and in the evening,

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* (1982) If we do not neglect the third power of $\psi'-\psi$, in [2727, &c.], its expression [2734] will contain t'^3; and instead of the expression of the hour of the tide $a+bt'$ [2839a]
[2741], we shall have $a+bt'+ct'^3$. Now if we put successively $t=0$, $t=1$, $t=2$, $t=3$, the numbers in the quadratures of the equinoxes [2806] will give,

$$0^{\text{day}},60566=a, \qquad 0^{\text{day}},66125=a+b+c,$$
$$0^{\text{day}},72411=a+2b+8c, \qquad 0^{\text{day}},77815=a+3b+27c. \qquad [2839b]$$

Combining these equations by the method of the least squares, we obtain the three fundamental equations,

$$4a+6b+36c=2,76917, \quad 6a+14b+98c=4,44392, \quad 36a+98b+794b=27,46418; \qquad [2839c]$$

whence we obtain $a=0^{\text{day}},60464$, $b=0^{\text{day}},05928$, $c=-0^{\text{day}},000142$. This value [2839d]
of $b=0,05928$, comes nearer to the value by the theory $0,06425$ [2825], than the value $0,057493$, computed in [2833]. We may proceed in the same manner with the numbers [2839e]
in the solstices [2807], from which we get,

$$0^{\text{day}},61863=a, \qquad 0^{\text{day}},66311=a+b+c,$$
$$0^{\text{day}},70933=a+2b+8c, \qquad 0^{\text{day}},75856=a+3b+27c. \qquad [2839f]$$

The three fundamental equations deduced from these, by the principle of the least squares, are

$$4a+6b+36c=2,74963, \quad 6a+14b+98c=4,35745, \quad 36a+98b+794c=26,81887; \qquad [2839g]$$

whence we get $a=0^{\text{day}},61861$, $b=0^{\text{day}},044292$, $c=0^{\text{day}},000262$. This value of b [2839h]
differs rather less from that by the theory $0^{\text{day}},04528$ [2830], than the value $0^{\text{day}},046643$, [2839i]
found in [2834], from the observations.

[2840'''] on the day of the quadrature, and in the three following corresponding tides, [2840''''] and have found the sum to be $3^{\text{days}},26667$. We have likewise added, in eleven corresponding quadratures, in which the moon's semi-diameter exceeds twenty-nine minutes and a half, the retardation of the tides, both in the morning and evening, from the day of the quadrature to the third following [2840v] tide corresponding; and have found $3^{\text{days}},39306$ for the sum of these retardations. The sum of the moon's semi-diameters is $30222''$ in the [2840vi] eleven first quadratures, and $32728''$ in the last quadratures; therefore an augmentation of $2506''$ in the sum of these semi-diameters, has produced [2840vii] an increase of $0^{\text{day}},12639$, in the sum of the retardations. Hence it follows, that an increase of one minute in the semi-diameter of the moon, [2841] produces $84''$ in the daily retardation of the tides, near the quadratures.* *This increase is about one third of that corresponding to the same variation* [2841'] *of the moon's semi-diameter in the syzygies, which was found, in* [2783], *to be* $258''$.

Daily retardation of the tide in the syzygies.

The daily retardation of the tides near the syzygies, found in [2788], is

[2842]
$$\frac{\frac{L'}{r_{\prime}'^{3}}\cdot\left(\frac{r_{\prime}'}{r'}\right)^{5}\cdot\upsilon_{\prime}}{\frac{L'}{r_{\prime}'^{3}}\cdot\left(\frac{r_{\prime}'}{r'}\right)^{3}+\frac{L}{r^{3}}}.$$

[2842'] If we suppose $r'=r_{\prime}'-\delta r'$, the increment of the retardation of the tides, corresponding to the diminution $-\delta r'$, will be†

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* (1983) The variation for $2506''$, or $25',06$, in the semi-diameter [2840vii], is

[2841a]
$$3^{\text{days}},39306-3^{\text{days}},26667=0^{\text{day}},12639=12639'' \qquad [2840'''', 2840^{\text{v}}].$$

Hence the variation for one minute is $\frac{12639}{25,06}=504''$; and as there are three days, and [2841b] two tides in each day [2840'''], this must be divided by 6, to obtain the daily variation $84''$, corresponding to an increase of one minute in the moon's semi-diameter.

† (1984) If we put for brevity $\frac{L}{r^3}=\lambda$, $\frac{L'}{r'^3}=\lambda'$, $\frac{\delta r'}{r_{\prime}'}=\rho$, we shall have

[2843a]
$$r'=r_{\prime}'-\delta r'=r_{\prime}'.(1-\rho); \qquad \frac{r_{\prime}'}{r'}=\frac{1}{1-\rho}=1+\rho,$$

neglecting ρ^2, and the higher powers of ρ. Substituting these in [2842], and then developing [2843b] according to the powers of ρ, it becomes, by putting R for its value when $\rho=0$ [2843'],

Increment of the retardation in the syzygies,

$$\frac{\delta r'}{r'_{,}} \cdot \frac{R \cdot \left\{ \frac{2 L'_{,}}{r'^{3}_{,}} + \frac{5L}{r^3} \right\}}{\frac{L'}{r'^{3}_{,}} + \frac{L}{r^3}}; \qquad [2843]$$

R being the mean daily retardation of the tides, near the syzygies. In like manner we shall find, from [2797],* [2843']

and in the quadratures.

$$\frac{\delta r'}{r'_{,}} \cdot \frac{R' \cdot \left\{ \frac{2 L'}{r'^{3}_{,}} - \frac{5L}{r^3} \right\}}{\frac{L'}{r'^{3}_{,}} - \frac{L}{r^3}}, \qquad [2844]$$

for the increment of the retardation of the tides, corresponding to $-\delta r'$ in the quadratures, R' being the mean daily retardation of the tides at that time. In these expressions we may suppose, without any sensible error, that $\frac{L'}{r'^{3}_{,}} = \frac{3L}{r^3}$ [2706]; then they become† $\frac{11 R}{4} \cdot \frac{\delta r'}{r'_{,}}$ and $\frac{R'}{2} \cdot \frac{\delta r'}{r'_{,}}$. [2844'] [2845]

$$\frac{\lambda'.(1+\rho)^5.v_{,}}{\lambda'.(1+\rho)^3+\lambda} = \frac{\lambda'.(1+5\rho).v_{,}}{\lambda'+\lambda+3\rho.\lambda'} = \frac{\lambda'.(1+5\rho).v_{,}}{\lambda'+\lambda} \cdot \left(1 - \frac{3\rho.\lambda'}{\lambda'+\lambda}\right) = \frac{\lambda'.v_{,}}{\lambda'+\lambda} \cdot \left\{1 + 5\rho - \frac{3\rho.\lambda'}{\lambda'+\lambda}\right\}$$

$$= \frac{\lambda'.v_{,}}{\lambda'+\lambda} \cdot \left\{1 + \rho \cdot \left(\frac{2\lambda' + 5\lambda}{\lambda'+\lambda}\right) = R \cdot \left\{1 + \rho \cdot \left(\frac{2\lambda'+5\lambda}{\lambda'+\lambda}\right)\right\}. \qquad [2843c]$$

The term of this expression depending on ρ is the same as in [2843].

* (1985) The retardation of the tide in the syzygies is obtained from [2730], in the quadratures from [2797]; and the last may be derived from the first, by merely changing the sign of L [2797a]. Making the same change in the increment [2843], corresponding to the syzygies, we obtain the increment in the quadratures [2844]; R being supposed to become R' in the quadratures [2844']. [2844a] [2844b]

† (1986) Substituting $\frac{L'}{r'^3} = 3 \cdot \frac{L}{r^3}$ [2706], in [2843, 2844], they become respectively $\frac{11}{4} R \cdot \frac{\delta r'}{r'_{,}}$, $\frac{1}{2} R' \cdot \frac{\delta r'}{r'_{,}}$, as in [2845]. Now we have $R = 0^{\text{day}},027052 = 2705''$ nearly [2745]; $R' = 5207''$ [2809]. Hence the two preceding expressions are [2845a]

$$\tfrac{11}{4} \times 2705'' \cdot \frac{\delta r'}{r'_{,}} \qquad \text{and} \qquad \tfrac{1}{2} \times 5207'' \cdot \frac{\delta r'}{r'_{,}};$$

which are to each other in the ratio of $258''$ to $90''$ nearly as in [2847]. [2845b]

This increment is nearly the same by observation as by the theory.

[2846] But by what precedes, $R = 2705''$ [2745], $R' = 5207''$ [2809]; and *if the first of these retardations be supposed equal to* $258''$ [2783], *the second*
[2847] *will be* $90''$; *the observations give* $84''$ [2841]; *therefore the theory and observations agree upon this point.*

[2847′] 41. *We can now reduce to numbers the expression* [2463], *which represents the height of the tide* αy *above the surface of equilibrium at any time.* We have seen, in [2464*d*, *e*], that the terms of this expression multiplied by B
[2847″] and Q are insensible at Brest. We may also, by reason of the smallness of
[2848] A [2465*b*], suppose $\gamma = \lambda$ [2481*e*], in the term multiplied by A. The constant quantity λ is the interval by which the solar tide at Brest follows
[2848′] after the time of the passage of the sun over the meridian [2419′, 2131*c*]; this interval being reduced to degrees, allowing 400° for a day. Now
[2849] from the combination of all the observations of the syzygial tides, this
[2850] interval is $0^{\text{day}},18793$ [2751]; and by those of the quadratures, the interval
[2851] is $0^{\text{day}},17924$.* The mean of these two results is $0^{\text{day}},18358$; and by reducing it to degrees, we get the value we shall assume for λ, namely,
[2852] $\lambda = 73^\circ,432$. This being premised, the expression of αy for Brest is†

General expression of the height of the tide at Brest.

[2853]
$$\alpha y = -0^{\text{met.}},02745 \,.\, \{i^3 \,.\, (1 - 3 \,.\, \sin.^2 \text{v}) + 3\,i'^3 \,.\, (1 - 3 \,.\, \sin.^2 \text{v}')\}$$
$$+\, 0^{\text{met.}},07179 \,.\, \left\{ \begin{array}{c} i^3 \,.\, \sin. \text{v} \,.\, \cos. \text{v} \,.\, \cos. (\upsilon - 73^\circ,432) \\ +\, 3\,i'^3 \,.\, \sin. \text{v}' \,.\, \cos. \text{v}' \,.\, \cos. (\upsilon + \psi - \psi' - 73^\circ,432) \end{array} \right\}$$
$$+\, 0^{\text{met.}},78112 \,.\, \left\{ \begin{array}{c} i^3 \,.\, \cos.^2 \text{v} \,.\, \cos. 2 \,.\, (\upsilon - 73^\circ,432) \\ +\, 3\,i'^3 \,.\, \cos.^2 \text{v}' \,.\, \cos. 2 \,.\, (\upsilon + \psi - \psi' - 73^\circ,432) \end{array} \right\}.$$

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* (1987) The minimum of the total tide at Brest is $0^{\text{day}},67924$ [2816], in the quadratures, which represents the middle time between the two times of high water [2728].
[2850*a*] This must evidently differ $0^{\text{day}},5$ nearly from the time of the solar high water following after the sun's passage over the meridian, which must therefore be at $0^{\text{day}},17924$.

[2853*a*] † (1988) Substituting [2568, 2706] in the term of [2463] depending on $(1 + 3 . \cos. 2\theta)$, it produces the first line of [2853], having the factor $-0^{\text{met.}},02745$. In the term depending on A [2463], we must put $\gamma = \lambda = 73^\circ,432$ [2848, 2852], $nt + \varpi - \psi = \upsilon$
[2853*b*] [2131*c*, 2853′], $nt + \varpi - \psi' = \upsilon + \psi - \psi'$. If we suppose $\frac{L}{r^3}$, $\frac{L'}{r'^3}$, to correspond
[2853*c*] to the mean distances of the sun and moon, we must multiply these quantities by i^3, i'^3, respectively [2855]. Lastly, substituting the value [2706], we get for this term of [2463] the following expression,

The symbols in this formula are as follows. First, υ *is the sun's horary angle, or the angle which the sun describes by its diurnal motion, from the time of its passage over the meridian of Brest to the time for which the computation is made* [2853b]. Second, v *and* v′ *are the declinations of the sun and moon respectively; the northern declinations being supposed positive, and the southern negative.* Third, ψ *and* ψ' *are the right ascensions of the sun and moon respectively* [2130′′′′, 2426]. Fourth, *i is the ratio of the mean distance of the sun to its actual distance; and* i' *is the actual parallax of the moon, divided by the constant term of this parallax* [2853c]. Fifth, *the quantities* v, v′, ψ, ψ', *i and* i', *correspond to the time which precedes the time under consideration by* $1^{\mathrm{day}},50724$ [2544, 2463′′, 2472′, &c.].

Symbols. υ. [2853′] v, v′. [2854] ψ, ψ'. i, i'. [2855] [2856]

The different causes which modify the oscillations of the sea, on our coasts, and probably also the error of the hypothesis of infinitely small oscillations, which we have used, make the preceding formula vary a little from observations. Thus the time of low water determined by this formula, differs some minutes from the observed time [2818]; because *it takes the tide less time to ebb than to flow at Brest.* We perceive also that, by the operation of the same causes, *the level of the sea* [2573, 2693] *is rather higher in the syzygies than in the quadratures.* These causes seem also to [2856′] [2856′′]

$$A.\frac{L}{r^3}.\{i^3.\sin.\mathrm{v}.\cos.\mathrm{v}.\cos.(\upsilon-73^\circ,432)+3\,i'^3.\sin.\mathrm{v}'.\cos.\mathrm{v}'.\cos.(\upsilon+\psi-\psi'-73^\circ,432)\}. \qquad [2853d]$$

The value of $A.\frac{L}{r^3}$ might be obtained from [2630], by substituting the values of v, v′, also $\gamma=\lambda$ [2853b]. The author has not given the values of v, v′; but if we put $\mathrm{v}=\mathrm{v}'=22^\circ$, we shall obtain the factor $0^{\mathrm{met.}},07179$, used in [2853]. For in this case we shall have $2.\sin.\mathrm{v}.\cos.\mathrm{v}=2.\sin.\mathrm{v}'.\cos.\mathrm{v}'=\sin.2\,\mathrm{v}=\sin.44^\circ=0,63742.$ If we substitute this and [2706] in [2630], we get [2853e]

$$0^{\mathrm{met.}},183=4\,A.\frac{L}{r^3}.(2.\sin.\mathrm{v}.\cos.\mathrm{v})=4\,A.\frac{L}{r^3}.0,63742. \qquad [2853f]$$

Hence $A.\frac{L}{r^3}=\frac{0,183}{4\times0,63742}=0^{\mathrm{met.}},07178,$ and the function [2853d] becomes as in the second term of [2853], depending on the factor $0^{\mathrm{met.}},07179$. Lastly, substituting [2706] in [2704], we get $P.\frac{L}{r^3}=0^{\mathrm{met.}},78112$; hence the term multiplied by P [2463] becomes as in the last term of [2853]; the factors i^3, i'^3, and the values of λ [2853c, b], being used as above. [2853g]

[2856'''] retard the tides in proportion to their magnitudes. Notwithstanding these slight variations, we may use the preceding formula in calculating the heights of the tides, because they may be altered by the winds in a much more sensible manner.

This formula furnishes a simple method of determining the greatest tides
[2856''''] *which happen in each syzygy.* The knowledge of these very high tides is interesting in the labors and operations in the sea ports, and is also useful in preventing the accidents which might happen from the inundations produced by these tides. It is therefore important to ascertain the heights of these tides by previous calculations. This may be done in the following manner. The
[2856^{v}] greatest tide, as we have seen, follows after the time of new moon, or full moon, about a day and a half [2544], and at the time of this high tide, the angles $\upsilon - 73°,432$ and $\upsilon + \psi - \psi' - 73°,432$ are nothing, or equal to two right angles [2479c]; therefore we shall then have,

Height of the greatest tide near the syzygies. First form;

[2857]
$$\alpha y = -0^{\text{met.}},02745 . \{ i^3 . (1 - 3 . \sin.^2 \mathrm{v}) + 3 i'^3 . (1 - 3 . \sin.^2 \mathrm{v}') \}$$
$$+ 0^{\text{met.}},07179 . \{ \pm i^3 . \sin. \mathrm{v} . \cos. \mathrm{v} \pm 3 i'^3 . \sin. \mathrm{v}' . \cos. \mathrm{v}' \}$$
$$+ 0^{\text{met.}},78112 . \{ i^3 . \cos.^2 \mathrm{v} + 3 i'^3 . \cos.^2 \mathrm{v}' \}.$$

In this expression we may neglect the two first terms, which are very small
[2857'] in comparison with the last; besides, they have no sensible effect, except near the solstices, where the tides are considerably decreased by the declinations of the bodies. Then we shall have,

Second form;

[2857'']
$$\alpha y = 0^{\text{met.}},78112 . \{ i^3 . \cos.^2 \mathrm{v} + 3 i'^3 . \cos.^2 \mathrm{v}' \}.$$

In the syzygies of the equinoxes $i = 1$ very nearly, v and v' are nothing,
[2857'''] and the mean value of i'^3 is $\frac{41}{40}$ [2569b—c, 2855]; taking therefore *unity for the mean value of* αy *near the syzygies of the equinoxes, its value for any syzygy whatever will be,**

Third and best form.

[2858]
$$\alpha y = \tfrac{40}{163} . \{ i^3 . \cos.^2 \mathrm{v} + 3 i'^3 . \cos.^2 \mathrm{v}' \}.$$

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* (1989) Putting $i = 1$, $i'^3 = \frac{41}{40}$, $\mathrm{v} = 0$, $\mathrm{v}' = 0$, in [2857''], it becomes
[2854a] $\alpha y = 0^{\text{met.}},78112 \times \frac{163}{40}$, equal to the mean value of the tide in the syzygy, represented by unity in [2857''']. Hence $0^{\text{met.}},78112 = \frac{40}{163}$; substituting this in [2857''], we get [2858].

Thus we shall have, by this very simple formula, the height of the greatest tide, which follows in one or two days after the time of new or full moon. The quantities i, i′, v, v′, correspond to the time of the syzygy. This formula will also determine the greatest ebb of the tide, below the surface of equilibrium. For it follows from the general expression of αy [2853], that the sea falls as much below the surface of equilibrium, at low water, as it is elevated above it in the corresponding high water.* *The height of the tide taken for unity, may be determined, by means of numerous observations of the differences of the elevations of the tide, between high water and low water, upon the first and second days after the syzygies, near the equinoxes. For the half of the mean value of these differences will express very nearly the height of the tide taken for unity.* [2858′] [2858″] Tide taken for unity. [2858‴]

42. *To complete this theory, we must now determine the time of high water, by a simple formula, which may be easily reduced to tables.* For this purpose, we shall resume the equation [2466],

To find the time of high water. First form.

$$\text{tang.}\,2\,.\,(nt+\varpi-\psi'-\lambda)=\frac{\frac{L}{r^3}\,.\,\cos.^2 v\,.\,\sin.\,2\,.\,(\psi-\psi')}{\frac{L'}{r'^3}\,.\cos.^2 v'+\frac{L}{r^3}\,.\,\cos.^2 v\,.\,\cos.\,2\,.\,(\psi-\psi')}\,. \qquad [2859]$$

This equation contains seven variable quantities r, r', v, v′, nt, ψ, ψ'; so that it would be difficult, in its present form, to reduce it to tables; but we may simplify it, by the consideration that the semi-diameters of the sun and moon differ but very little from each other. *We shall put H, H′, for the apparent semi-diameters of the sun and moon respectively, at their mean distances from the earth*, in which situations we have found $\frac{L'}{r_{\prime}'^3}=3\,.\,\frac{L}{r^3}$ [2706]; and *h, h′, for their actual semi-diameters.* We shall then have, by observing that in the preceding formula $\frac{L'}{r'^3}$ must be decreased about one thirtieth part, or more accurately, in the ratio of 2,89811 to 3 [2755], [2859′] [2859″] H, H′. [2859‴] h, h′. [2860]

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* (1990) Neglecting, as in [2857′], the effect produced by the two first terms of [2853], on account of their smallness, and retaining only the last term of that formula, or that which has the factor $0^{\text{met.}},78112$. [2858a]

Second form.

[2861] $$\text{tang.}\,2.(nt+\varpi-\psi'-\lambda)=\frac{\left(\frac{h}{H}\right)^3.\cos.^2\text{v}.\sin.2.(\psi-\psi')}{2{,}89811.\left(\frac{h'}{H'}\right)^3.\cos.^2\text{v}'+\left(\frac{h}{H}\right)^3.\cos.^2\text{v}.\cos.2.(\psi-\psi')}.\text{*}$$

Correction of the semi-diameters for the declinations.

To make use of this equation, we must first construct a table, containing the values of the function,

[2862] $$\left\{\frac{2{,}89811\,.\,H'+H}{3{,}89811}\right\}.\left\{1-\sqrt[3]{\cos.^2\text{v}}\right\},$$

corresponding to all the degrees, from $\text{v}=0$ to $\text{v}=32^\circ$. We must

[2862'] afterwards correct the semi-diameters of the sun and moon h, h',† found in

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* (1991) The ratio of $n-m'$ to $n-m$ is found, in [2752e], to be as 29 to 30;

[2860a] or more accurately as $1-\frac{1}{27{,}322}$ to $1-\frac{1}{366\frac{1}{4}}$, which is the same as 2,89811 to 3; so that instead of [2706], we must put $\frac{L'}{r_{,}'^3}=2{,}89811\,.\,\frac{L}{r_{,}^3}$; $r_{,}'$, $r_{,}$, being the mean values of r', r, respectively. Now we evidently have,

[2860b] $$\frac{L}{r^3}=\frac{L}{r_{,}^3}.\left(\frac{r_{,}}{r}\right)^3=\frac{L}{r_{,}^3}.\left(\frac{h}{H}\right)^3,\qquad \frac{L'}{r'^3}=\frac{L'}{r_{,}'^3}.\left(\frac{r_{,}'}{r'}\right)^3=\frac{L'}{r_{,}'^3}.\left(\frac{h'}{H'}\right)^3=2{,}89811\,.\,\frac{L}{r_{,}^3}.\left(\frac{h'}{H'}\right)^3;$$

substituting these in [2859], and rejecting the factor $\frac{L}{r_{,}^3}$, which occurs in the numerator and denominator, we get [2861]. We may observe that the author, in this section, has

[2860c] erroneously used the factor 2,89841, instead of 2,89811.

† (1992) If we apply a small correction to the semi-diameters h, h', so as to include in them the effect of the factors $\cos.^2\text{v}$, $\cos.^2\text{v}'$ [2861], and put these new values equal to $h_{,}$ and $h_{,}'$ respectively, we shall have,

[2862a] $$\left(\frac{h_{,}}{H}\right)^3=\left(\frac{h}{H}\right)^3.\cos.^2\text{v};\qquad \left(\frac{h_{,}'}{H'}\right)^3=\left(\frac{h'}{H'}\right)^3.\cos.^2\text{v}'.$$

The cube roots of these give $h_{,}=h.(\cos.^2\text{v})^{\frac{1}{3}}$, $h_{,}'=h'.(\cos.^2\text{v}')^{\frac{1}{3}}$, or

[2862b] $$h-h_{,}=h.\{1-(\cos.^2\text{v})^{\frac{1}{3}}\},\qquad h'-h_{,}'=h'.\{1-(\cos.^2\text{v}')^{\frac{1}{3}}\}.$$

These expressions of $h-h_{,}$, $h'-h_{,}'$, are to be subtracted from h, h', respectively, to obtain their corrected values $h_{,}$, $h_{,}'$. Now the greatest value of v is 26°, and that of v', 32°;

[2862c] so that the greatest values of $1-(\cos.^2\text{v})^{\frac{1}{3}}$, $1-(\cos.^2\text{v}')^{\frac{1}{3}}$, are $\frac{1}{18}$ and $\frac{1}{12}$ respectively; and as h is nearly equal to h', we may, in computing $h-h_{,}$, $h'-h_{,}'$, from the formulas [2862b], suppose, for h, h', some *intermediate mean value*. The error of this supposition

the ephemeris; by subtracting from each of them the quantity which corresponds, in this table, to the declination of the body. Then we shall have very nearly,*

Third formula for the time of high water.

$$nt+\varpi-\psi'-\lambda=\tfrac{1}{2}\,\text{ang. tang.}\left\{\frac{\left(\frac{h}{H}\right)^3.\sin.2.(\psi-\psi')}{2{,}89811.\left(\frac{h'}{H'}\right)^3+\left(\frac{h}{H}\right)^3.\cos.2.(\psi-\psi')}\right\};$$ [2863]

in which h, h', represent the semi-diameters of the sun and moon corrected in the abovementioned manner. By this means the declinations of the sun and moon vanish from the expression of $nt+\varpi-\psi'-\lambda$. In strictness, we ought to subtract from the sun's semi-diameter the quantity

$$h.(1-\sqrt[3]{\cos.^2 v}) \qquad [2862b];$$ [2864]

but this being very small, and the value of h differing but little from $\frac{2{,}89811.H'+H}{3{,}89811}$, we may substitute this last quantity for h. The same remark applies to the correction of the semi-diameter of the moon; and as [2865]

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must be very small; because h varies only about $\frac{1}{60}$ part from its mean value, and h' about $\frac{1}{16}$ part, by reason of the change of distance, arising from the excentricities of the orbits. [2862d] Therefore the error arising from the use of H and H', instead of h, h', in the second members of [2862b], does not exceed $\frac{1}{18}\times\frac{1}{60}$ for the sun, or $\frac{1}{12}\times\frac{1}{16}$ for the moon; and in general is much less. Moreover, as H differs but little from H'; since the former is about $15^m\,43^s$, the latter $16^m\,01^s$; we may, in [2862b], use either H or H', [2862e] instead of h, h'. But instead of using H or H', it will be more accurate to use, for h or h', $\frac{2{,}89811.H'+H}{3{,}89811}$. This form is assumed, because the mean lunar force is to the solar, as 2,89811 to 1 [2860]. Substituting this for h or h', in the second members of [2862b], we obtain for $h-h_{\prime}$, $h'-h_{\prime}'$, expressions of the form of [2862]. [2862f]

* (1993) Substituting in [2861] the values [2862a], we get,

$$\text{tang.}\,2.(nt+\varpi-\psi'-\lambda)=\frac{\left(\frac{h_{\prime}}{H}\right)^3.\sin.2.(\psi-\psi')}{2{,}89811.\left(\frac{h_{\prime}'}{H'}\right)^3+\left(\frac{h_{\prime}}{H}\right)^3.\cos.2.(\psi-\psi')}.$$ [2864a]

From this tangent we get half the corresponding arc $nt+\varpi-\psi'-\lambda$, as in [2863], in which h, h', represent the corrected values $h_{\prime}$, $h_{\prime}'$, respectively, [2862a].

the influence of this body upon the hour of high water is to that of the sun in the ratio of 2,89811 to 1, the diameters H' and H are made to enter

[2866] in this proportion in the function $\frac{2,89811 \,.\, H'+H}{3,89811}$. If we now consider that

[2867] the difference between $\frac{h'.H}{h.H'}$ and $\frac{H+h'-h}{H'}$ is $\frac{(H-h).(h'-h)}{h.H'}$, and that it may be neglected on account of the smallness of the factors $H-h$ and $h'-h$, we shall have,*

Fourth and best formula for the time of high water.

[2868]
$$nt+\varpi-\psi'-\lambda=\tfrac{1}{2}\,\text{ang. tang.}\left\{\frac{\sin.2\,.(\psi-\psi')}{2,89811\,.\left(\frac{H+h'-h}{H'}\right)^3+\cos.2\,.(\psi-\psi')}\right\}.$$

We may easily reduce, to a tabular form, this expression of $nt+\varpi-\psi'-\lambda$; and by turning it into time, considering the whole circumference 400° as equal to one day, we shall obtain the law of the retardation of the tides,

[2868'] upon the time of the moon's passage over the superior or inferior meridian; which time is determined by the condition†

[2868a] * (1994) The first member of [2863] is the same as that of [2868]; and if we divide the numerator and denominator of the second member of [2863] by $\left(\frac{h}{H}\right)^3$, we shall find that the numerator is the same as in [2868], but the denominator becomes

[2868b]
$$2,89811\,.\left(\frac{h'.H}{H'.h}\right)^3+\cos.2\,.(\psi-\psi'),$$

which may be reduced to the same form as in [2868], by the following considerations. We

[2868c] have identically $\frac{h'.H}{h.H'}=\frac{H+h'-h}{H'}+\frac{(H-h).(h'-h)}{h.H'}$, as is easily proved by reducing the first term of the second member to the denominator $h.H'$, and neglecting the terms which mutually destroy each other. Now from Burg's tables, we find nearly that

[2868d]
$$\frac{H-h}{h}<\tfrac{1}{60},\quad \frac{h'-h}{H'}<\tfrac{1}{10};\qquad \text{consequently}\qquad \frac{H-h}{h}\,.\,\frac{h'-h}{H'}<\tfrac{1}{60}\times\tfrac{1}{10}<\tfrac{1}{600};$$

therefore this term may be neglected in comparison with the other term of [2868c]; and we may put $\frac{h'.H}{h.H'}=\frac{H+h'-h}{H'}$, without any sensible error; by which means the denominator

[2868e] [2868b] becomes as in [2868]. This last formula [2868] is well adapted to the computation of the hour of high water, by a table of double entry, having $\psi-\psi'$ at the top, $h'-h$ at the side, and $nt+\varpi-\psi'-\lambda$ for the corresponding number.

[2869a] † (1995) This is evident from [2131a,b, 2426].

Equation of the moon's passing the meridian.

$$n\,t + \varpi - \psi' = 0, \qquad \text{or} \qquad n\,t + \varpi - \psi' = 200^{\circ}.$$ [2869]

But to use this table, we must ascertain, in each port, the time by which the *maximum* of the tide follows after the syzygy. We have found at Brest, that this time is $1^{\text{day}},50724$ [2544], and according to observations it is nearly the same in all the ports of France bordering on the ocean; so that the values of $n\,t + \varpi - \psi' - \lambda$ correspond to the values of $\psi - \psi'$, which precede, by $1^{\text{day}},50724$, the time for which the calculation is made. [2869′] Moreover, we must determine the value of the constant quantity λ. This quantity, reduced to time, is the hour of high water which follows after the syzygy by $1^{\text{day}},50724$; and we may determine it by a great number of observations of the hour of high water on the second day after the syzygy.

Recapitulation of the phenomena of the tides, showing the agreement of the theory and observation.

43. *We shall now recapitulate in a few words the principal phenomena of the tides, and their relation with the laws of universal gravitation.* We have [2869″] generally considered these phenomena near their *maxima* and *minima*, and we have divided them into two classes; the one relative to the heights of the tides, the other relative to the hours of the tides, and their intervals. We shall now examine separately these two classes of phenomena.

The heights of the tides in each port, at their *maximum* near the syzygies, and at their *minimum* near the quadratures, are the *data* of the observations, [2869‴] which best show the ratio of the actions of the sun and moon upon the tides; and by means of this ratio, the various phenomena of the tides, which result from the theory of universal gravitation. One of these phenomena, which is very proper for the verification of the theory, is the law of the diminution of the tides from the time of *maximum*, or the law of their increase from [2869⁗] the *minimum*. We have seen in [2593′, 2716], that the theory of gravity accords perfectly with the observations in this respect.

Variation of the heights of the tides near the maximum or the minimum.

These laws of the decrease and increase of the tides vary with the declinations of the sun and moon: we have seen in [2590, 2592], that *their decrease near the syzygies of the equinoxes, is to their corresponding decrease near the syzygies of the solstices, in the ratio of* 13 *to* 8;* and that this result is conformable to the theory of gravity. We have seen likewise, [2870]

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* (1996) These decrements are as $3^{\text{met.}},2040$ to $1^{\text{met.}},8977$, by the theory [2593′]; and as $3^{\text{met.}},1623$ to $1^{\text{met.}},9451$, by observation; being nearly in the ratio of 13 to 8. [2870a]

Variation of the heights, depending on the declinations. [2717′, 2718′], that *the increment of the tides, counted from the minimum near the quadratures of the equinoxes, is to the corresponding increment near*
[2871] *the quadratures of the solstices, as 2 to 1*;* and that the theory of gravity gives nearly the same ratio.

According to this theory, *the height of the total tide, at its maximum near the syzygies of the equinoxes, is to the corresponding height near the syzygies*
[2872] *of the solstices, nearly as the square of the radius is to the square of the cosines of the declinations of these bodies near the solstices*; and we have seen in [2590, 2592], that this differs but little from the result of observations.† By the same theory, *the excess of the heights of the total tides, in their minimum, near the quadratures of the solstices, above their corresponding*
[2873] *heights, near the quadratures of the equinoxes, is the same as the excess of the heights of the total tides in their maximum, near the syzygies of the equinoxes, above their corresponding heights near the syzygies of the solstices*;‡ and we

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[2870b] * (1997) These increments, by observation, are as $7^{met.},495$ to $3^{met.},410$ [2718″]; and by the theory, as $7^{met.},819$ to $2^{met.},894$ [2718″, &c.], being nearly as in [2871].

† (1998) The chief term of the expression of the total tides [2502] is

[2872a] $$4\,i\,P\,.\left\{\frac{L}{r^3}\,.\cos.^2 V+\frac{L'}{r'^3}\,.\cos.^2 V'\right\},$$

which in the syzygies, where $\cos.^2 V=\cos.^2 V'$ nearly, is proportional to the square of the cosine of the declination; so that the height, near the syzygies of the equinoxes, is to that near
[2872b] the syzygies of the solstices, by *the theory*, as the radius is to the square of the cosine of the declination. By the observations of Table I [2510, &c.], the constant term of the heights of the total tides in the equinoxes is $150^{met.},235$ [2590], and that in the solstices $132^{met.},371$ [2592]; which are to each other as 23,71775 to 20,90. The first term 23,71775 is the
[2872c] mean of the values of p, p', [2584], and the last is nearly the mean of the values of q, q', [2584], which represent respectively the squares of the cosines of the declinations of the sun and moon, in the equinoxes, and in the solstices [2580]; so that the theory and observation agree very nearly with each other.

‡ (1998a) The chief term of the total tides [2502], in the syzygies, is

[2873a] $$4\,i\,P\,.\left\{\frac{L'}{r'^3}\,.\cos.^2 V'+\frac{L}{r^3}\,.\cos.^2 V\right\};$$

and in the quadratures [2639, 2640], is

[2873b] $$4\,i\,P\,.\left\{\frac{L'}{r'^3}\,.\cos.^2 V'-\frac{L}{r^3}\,.\cos.^2 V\right\};$$

have seen [2590, 2592, 2717′, 2718′] that this is exactly conformable to the theory.

Variation of the heights of the tide depending on the distances of the sun and moon.

The influence of the moon upon the tides *increases, by the principle of gravity, as the cube of its parallax;* and by [2608, 2623, &c.], this is so exactly conformable to observation, that we might have deduced from observations the law of this influence. [2874]

Retardation of the tides near the maximum or minimum.

The phenomena of the intervals of the tides accord equally well with the theory, as those of their heights. According to the theory, *the daily retardation of the tides, at their maximum, near the syzygies, is only about half what it is at their minimum, near the quadratures.* In the first case it is nearly* 27′ [2757], and in the last case 55′ [2831]. We have seen in [2745, 2809], that the observations differ but little from this result of the theory. [2875]

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the former being the *maximum*, the latter the *minimum* value. Suppose now V_e, V'_e, to be the values of V, V', when the *bodies* are respectively in the *equinoxes*; and V_s, V'_s, their corresponding values in the *solstices*. The *minimum* value [2873b] in the quadratures of the *equinoxes*, is $4\,i\,P\,.\left\{\frac{L'}{r'^3}\,.\cos.^2 V'_s - \frac{L}{r^3}\,.\cos.^2 V_e\right\}$, and in the quadratures of the *solstices*, $4\,i\,P\,.\left\{\frac{L'}{r'^3}\,.\cos.^2 V'_e - \frac{L}{r^3}\,.\cos.^2 V_s\right\}$. The excess of the latter is [2873c]

$$4\,i\,P\,.\left\{\frac{L'}{r'^3}\,.(\cos.^2 V'_e - \cos.^2 V'_s) + \frac{L}{r^3}\,.(\cos.^2 V_e - \cos.^2 V_s)\right\}. \qquad [2873d]$$

The *maximum* value [2873a] near the syzygies of the *equinoxes*, is

$$4\,i\,P\,.\left\{\frac{L'}{r'^3}\,.\cos.^2 V'_e + \frac{L}{r^3}\,.\cos.^2 V_e\right\},$$

and in the *solstices*, $4\,i\,P\,.\left\{\frac{L'}{r'^3}\,.\cos.^2 V'_s + \frac{L}{r^3}\,.\cos.^2 V_s\right\}$; and the *excess* of the former is $4\,i\,P\,.\left\{\frac{L'}{r'^3}\,.(\cos.^2 V'_e - \cos.^2 V'_s) + \frac{L}{r^3}\,.(\cos.^2 V_e - \cos.^2 V_s)\right\}$, which is equal to the preceding expression [2873d], as in [2873]. By observation [2717′, 2718′], the excess in the quadratures is $75^{\rm met.},517 - 58^{\rm met.}{\cdot}370 = 17^{\rm met.},147$; and in the syzygies [2590, 2592] the excess is $150^{\rm met.},235 - 132^{\rm met.},371 = 17^{\rm met.},864$. These two quantities are nearly equal to each other, as they ought to be by the theory [2873d, e]. [2873e]

* (1999) In the syzygies the retardation, by the theory is 26′,641 [2757], by observation 27′,052 [2745]. In the quadratures, the retardation by theory is 54′,76 [2831], and by observation 52′,067 [2809]. [2875a]

The retardation of the tides varies with the declinations of the bodies.
[2876] *According to the theory, it is greater in the syzygies of the solstices, than in those of the equinoxes, in the ratio of 8 to 7.* In the quadratures of the equinoxes, it is greater than in those of the solstices, in the ratio of 13*
[2876′] *to 9.*† We have seen in [2777, 2839′], that the observations give nearly the same ratios.

Effect of the declinations on the retardation of the tides.

The distance of the moon from the earth has an influence on the retardation of the tides. According to the theory, an increase of one minute in the
[2877] *semi-diameter of the moon, produces an increase of* 258″ [2783] *in this retardation, in the syzygies, and only* 90″ [2847] *in the quadratures;* and we have seen in [2847], that this agrees with the observations, and conforms in every respect, relatively to the tides, to the law of universal gravitation.

Effect of the distances of the sun and moon on the retardation.

We have treated fully on the ebb and flow of the sea; because it is one
[2877′] of the results of the attraction of the heavenly bodies most obvious to us, and the law which regulates it can be examined at every moment. It is hoped that the theory of the tides here given will induce observers to attend to the subject, in ports which, like Brest, are well situated for such observations. Accurate observations, continued during a period of the revolution of the moon's nodes, might fix with precision the elements of the
[2877″] theory of the ebb and flow of the tide, and perhaps make sensible the small oscillations depending on the inverse ratio of the fourth power of the distance of the moon from the earth, which have heretofore been confounded with the errors of the observations.

[2876a] * (2000) By the theory [2777], this ratio is 28′,603 to 24′,679; and by observation, 28′,600 to 25′,503, as in [2876].

[2876b] † (2001) This ratio, by the theory [2839′], is 64′,25 to 45′,28; and by observation, 57′,493 to 46′,643 nearly as in [2876′].

CHAPTER V.

ON THE OSCILLATIONS OF THE ATMOSPHERE.

44. Since it is impossible to submit to analysis the motions of the atmosphere, depending upon the variations of the sun's heat, and upon [2877'''] several other circumstances which modify these motions; *we shall restrict ourselves to the consideration of the oscillations arising from the attractions of the sun and moon; supposing the atmosphere to have a uniform temperature, and a variable density, proportional at each point to the compressing force.* [2877''''] In this hypothesis we have found, in the first book [363], the two following equations, using the same symbols as in that article :

Oscillations of the air depending on the attractions of the sun and moon.

Equations of the oscillations of the level strata of the atmosphere. First form.

$$r^2.\delta\theta.\left\{\left(\frac{d\,d\,u'}{d\,t^2}\right)-2\,n\,.\sin.\theta\,.\cos.\theta\,.\left(\frac{d\,v'}{d\,t}\right)\right\}$$

$$+r^2.\delta\varpi.\left\{\sin.^2\theta\,.\left(\frac{d\,d\,v'}{d\,t^2}\right)+2n.\sin.\theta.\cos.\theta.\left(\frac{d\,u'}{d\,t}\right)\right\}=\delta V'-g.\delta y'-g.\delta y; \qquad [2878]$$

$$y'=-\,l\,.\left\{\left(\frac{d\,u'}{d\,\theta}\right)+\left(\frac{d\,v'}{d\,\varpi}\right)+\frac{u'.\cos.\theta}{\sin.\theta}\right\}. \qquad [2879]$$

If we suppose the depth of the sea to be a constant quantity, represented by l', and neglect its density, as in [2280''', &c.]; we shall have, as in [338, 347] of the first book,

Equations of the motion of the surface of the sea.

$$r^2.\delta\theta.\left\{\left(\frac{d\,d\,u}{d\,t^2}\right)-2\,n\,.\sin.\theta\,.\cos.\theta\,.\left(\frac{d\,v}{d\,t}\right)\right\}$$

$$+r^2.\delta\varpi.\left\{\sin.^2\theta\,.\left(\frac{d\,d\,v}{d\,t^2}\right)+2\,n\,.\sin.\theta\,.\cos.\theta\,.\left(\frac{d\,u}{d\,t}\right)\right\}=\delta V'-g.\delta y; \qquad [2880]$$

$$y=-\,l'\,.\left\{\left(\frac{d\,u}{d\,\theta}\right)+\left(\frac{d\,v}{d\,\varpi}\right)+\frac{u\,.\cos.\theta}{\sin.\theta}\right\}; \qquad [2881]$$

the value of V' being the same as in the preceding equations [359″, &c]. Therefore by putting

[2882] $$(l-l').u'+l'u=lu'';$$

[2882′] $$(l-l').v'+l'v=lv'';$$

[2882″] $$(l-l').y'+ly=ly';$$

Equations of the oscillations of the level strata of the atmosphere. Second form.

we shall obtain, from the four preceding equations,*

$$r^2.\delta\theta.\left\{\left(\frac{ddu''}{dt^2}\right)-2n.\sin.\theta.\cos.\theta.\left(\frac{dv''}{dt}\right)\right\}$$

[2883] $$+r^2.\delta\varpi.\left\{\sin.^2\theta.\left(\frac{ddv''}{dt^2}\right)+2n.\sin.\theta.\cos.\theta.\left(\frac{du''}{dt}\right)\right\}=\delta V'-g.\delta y'';$$

[2884] $$y''=-l.\left\{\left(\frac{du''}{d\theta}\right)+\left(\frac{dv''}{d\varpi}\right)+\frac{u''.\cos.\theta}{\sin.\theta}\right\}.$$

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* (2002) Multiplying the equations [2878, 2880] by $l-l'$, l', respectively; adding these products; then substituting the following expressions, which are easily deduced from the differentials of [2282, 2882′],

[2882a]
$$(l-l').\left(\frac{ddu'}{dt^2}\right)+l'.\left(\frac{ddu}{dt^2}\right)=l.\left(\frac{ddu''}{dt^2}\right);\qquad (l-l').\left(\frac{ddv'}{dt^2}\right)+l'.\left(\frac{ddv}{dt^2}\right)=l.\left(\frac{ddv''}{dt^2}\right);$$
$$(l-l').\left(\frac{du'}{dt}\right)+l'.\left(\frac{du}{dt}\right)=l.\left(\frac{du''}{dt}\right);\qquad (l-l').\left(\frac{dv'}{dt}\right)+l'.\left(\frac{dv}{dt}\right)=l.\left(\frac{dv''}{dt}\right);$$

we get,

[2882b]
$$lr^2.\delta\theta.\left\{\left(\frac{ddu''}{dt^2}\right)-2n.\sin.\theta.\cos.\theta.\left(\frac{dv''}{dt}\right)\right\}$$
$$+lr^2.\delta\varpi.\left\{\sin.^2\theta.\left(\frac{ddv''}{dt^2}\right)+2n.\sin.\theta.\cos.\theta.\left(\frac{du''}{dt}\right)\right\}=l.\delta V'-g.(l-l').\delta y'-gl.\delta y.$$

The differential of [2882″] relative to δ, being multiplied by $-g$ gives

[2882c] $$-g.(l-l').\delta y'-gl.\delta y=-gl.\delta y''.$$

Substituting this in the second member of [2882b], then dividing by l, we obtain [2883].

Multiplying [2879] by $\frac{l-l'}{l}$, and adding the product to [2881], we get,

[2882d]
$$\frac{(l-l')}{l}.y'+y=-(l-l').\left\{\left(\frac{du'}{d\theta}\right)+\left(\frac{dv'}{d\varpi}\right)+\frac{u'.\cos.\theta}{\sin.\theta}\right\}$$
$$-l'.\left\{\left(\frac{du}{d\theta}\right)+\left(\frac{dv}{d\varpi}\right)+\frac{u.\cos.\theta}{\sin.\theta}\right\}.$$

These two equations correspond evidently to the oscillations of the sea, supposing its depth to be l. In this case we may determine the value of y'', [2884']
as well as that of y, by the first chapter of this book [2294, &c.]. We can therefore obtain the value of y', by means of the analysis explained in that chapter.

We have observed in the first book [363'''], that k being the height of the barometer, in the state of equilibrium, its oscillations are represented by the formula [363$^{\text{iv}}$]; consequently also, by the following expression,* [2885]

Oscillation of the barometer.

$$\frac{\alpha k.(y+y')}{l}=\frac{\alpha k.(ly''-l'y)}{l.(l-l')}=\text{the oscillation of the barometer.} \qquad [2886]$$

It is evident from [349'—349''], that l is the ratio of the height of the atmosphere to the radius of the earth,† supposing the density of the air, and [2886']
its temperature, to be everywhere the same. Now we have found by experiment, that at the temperature of melting ice, the density of mercury is to that of air, in the ratio of 10320 to 1 nearly; and as the mean height [2887]
of the barometer is about $0^{\text{met.}},76$, it follows that $l=\frac{1}{812}$. At a greater [2888]

The first member of this equation is equal to y'' [2882'']; and the second member may be reduced, by means of [2882—2882'', 2882a], to the form of the second member of [2884]; [2882e]
hence we obtain [2884]. It appears evident by inspection, that the equations [2883, 2884] are the same as those for the oscillations of the sea [2880, 2881], supposing the depth of the sea to be l' instead of l, as in [2884']. [2882f]

* (2003) From [2882''] we find $y'=\frac{l.(y''-y)}{l-l'}$; adding y, we obtain

$$y+y'=\frac{ly''-l'y}{l-l'}; \qquad [2886a]$$

multiplying by $\frac{\alpha k}{l}$, we get [2886]. Substituting $l'=2l$ [2890] in the last equation, [2886b]
we have $y+y'=2y-y''$; which is used in [2892].

† (2004) The radius of the earth being taken for unity, as in [2293']. [2886c]

‡ (2005) The height of the atmosphere of uniform density represented by

$$10320\times 0^{\text{met.}},76=7843^{\text{met.}} \qquad [2887];$$

which is about $\frac{1}{812}$ part of the mean radius of the earth [2891], being rather less than the [2888a]
estimate in note 238, which was made for a different temperature.

temperature, the value of l increases. To obtain some idea of the oscillations of the barometer, we shall suppose the temperature to be such
[2889] that $l=\frac{1}{722,5}$. This is one of the depths of the sea, for which we have determined, in [2306], the value of αy; and in the present case it will
[2890] represent the value of $\alpha y''$. Moreover we shall suppose* $l'=2l$, which is also one of the depths of sea considered in [2307]; and then the value of αy will be that corresponding to this depth. Substituting these values of
[2890'] l, l', and for αy, $\alpha y''$, the quantities found in [2307, 2306]; also
[2891] $k=0^{\text{met.}},76$, and the radius of the earth 6366200 metres nearly [2035b]; we shall have, to determine the oscillations of the barometer,†

Oscillations of the barometer.

$$\frac{\alpha k\,.\,(l y''-l' y)}{l\,.\,(l-l')}=\frac{\alpha k}{l}\,.\,(2y-y'')$$

[2892]

$$=0^{\text{met.}},000010623\,.\left\{\frac{1+3\,.\,\cos.2\theta}{3}\right\}.\{\sin.^2\text{v}-\tfrac{1}{2}.\cos.^2\text{v}+e\,.\sin.^2\text{v}'-\tfrac{1}{2}e\,.\cos.^2\text{v}'\}$$

$$+\,0^{\text{met.}},000010623.\begin{pmatrix}1,0000\\-4,6952\,.\,\sin.^2\theta\\-2,9342\,.\,\sin.^4\theta\\-0,6922\,.\,\sin.^6\theta\\-0,0899\,.\,\sin.^8\theta\\-0,0076\,.\,\sin.^{10}\theta\end{pmatrix}.\,\sin.^2\theta\,.\left\{\begin{array}{c}\cos.^2\text{v}\,.\,\cos.2\,.\,(nt+\varpi-\psi)\\+\,e\,.\cos.^2\text{v}'.\cos.2\,.\,(nt+\varpi-\psi')\end{array}\right\}.$$

[2889a] * (2006) The supposition that $l'=2l$, or that the uniform depth of the sea l' is equal to twice the height of a homogeneous atmosphere l, is used, in order to enable us to simplify the calculation, by employing the numerical coefficients, already computed with considerable labor, in [2306, 2307].

[2891a] † (2007) Substituting in [2306] the value [2285], $x^2=1-\mu^2=1-\cos.^2\theta=\sin.^2\theta$ [2128xii], we get the quantity $\alpha y''$ [2890'], in the following form,

$$\alpha y''=0^{\text{met.}},12316\,.\left\{\frac{1+3.\cos.2\theta}{3}\right\}.\{\sin.^2\text{v}-\tfrac{1}{2}\,.\cos.^2\text{v}+e\,.\sin.^2\text{v}'-\tfrac{1}{2}e\,.\cos.^2\text{v}'\}$$

[2891b]

$$+\,0^{\text{met.}},12316\,.\left\{\begin{array}{l}1,0000+6,1960\,.\sin.^2\theta+3,2474\,.\sin.^4\theta+0,7238\,.\sin.^6\theta\\ \qquad+0,0919\,.\sin.^8\theta+0,0076\,.\sin.^{10}\theta+0,0004\,.\sin.^{12}\theta\end{array}\right\}$$

$$\times\sin.^2\theta\,.\{\cos.^2\text{v}\,.\cos.2\,.\,(nt+\varpi-\psi)+e\,.\cos.^2\text{v}'.\cos.2\,.\,(nt+\varpi-\psi')\}.$$

Substituting $x=\sin.\theta$ [2891a] in [2307], we get αy [2307]; multiplying this by 2, and subtracting $\alpha y''$ [2891b], we get,

If we suppose the sun and moon to be in conjunction, or in opposition, in the plane of the equator; and at their mean distances; in which case $e=3$ very nearly [2301′, 2706]; *we shall have at the equator* $0^{\text{met.}},0006305$,* *for the difference between the greatest elevation and the lowest depression of the*

Oscillations of the barometer [2893]
[2894]
near the equator.

$$\alpha.(2y-y'')=0^{\text{met.}},12316.\left\{\frac{1+3.\cos.2\theta}{3}\right\}.\{\sin.^2 v-\tfrac{1}{2}.\cos.^2 v+e.\sin.^2 v'-\tfrac{1}{2}e.\cos.^2 v'\}$$

$$+0^{\text{met.}},12316.\left\{\begin{array}{l}1-4,6952.\sin.^2\theta-2,9342.\sin.^4\theta-0,6923.\sin.^6\theta\\ -0,0901.\sin.^8\theta-0,0076.\sin.^{10}\theta-\&c.\end{array}\right\} \quad [2891c]$$

$$\times\sin.^2\theta.\{\cos.^2 v\cos.2.(nt+\varpi-\psi)+e.\cos.^2 v'.\cos.2.(nt+\varpi-\psi')\}.$$

Now $l=\frac{\text{Radius}}{722,5}=\frac{6366200}{722,5}$, and $k=0^{\text{met.}},76$; hence

$$\frac{k}{l}\times 0^{\text{met.}},12316=0^{\text{met.}},12316\times\frac{0,76\times 722,5}{6366200}=0^{\text{met.}},000010623. \quad [2891d]$$

Multiplying [2891c] by $\frac{k}{l}$, and substituting the preceding numerical value, we shall get the expression [2892] nearly. We may observe that the second value of [2892] is easily deduced from the first, by putting $l'=2l$ [2890], and rejecting the factor l, from the numerator or denominator. [2891e]

* (2008) Comparing [2301′] with [2706], we get $e=3$. If we suppose the moon to be in conjunction, or in opposition, in the plane of the equator, we have $v=v'=0$, and the factor $\sin.^2 v-\frac{1}{2}.\cos.^2 v+e.\sin.^2 v'-\frac{1}{2}e.\cos.^2 v'$ [2892] becomes $-\frac{1}{2}-\frac{1}{2}e=-2$. At the equator, $\sin.\theta=1$, $\cos.\theta=0$, $\cos.2\theta=-1$, $\frac{1+3.\cos.2\theta}{3}=-\frac{2}{3}$, and the factor $1,0000-4,6952.\sin.^2\theta-2,9342.\sin.^4\theta-\&c.=-7,4191$. Hence the expression [2892] becomes, [2894a] [2894b]

$$0^{\text{met.}},000010623\times\{\tfrac{2}{3}\times 2\}$$
$$-0^{\text{met.}},000010623\times 7,4191\times\{\cos.2.(nt+\varpi-\psi)+3.\cos.2(nt+\varpi-\psi')\}. \quad [2894c]$$

The greatest value of this function is found, by putting

$$\cos.2.(nt+\varpi-\psi)=\cos.2.(nt+\varpi-\psi')=-1;$$

and its least value by putting $\cos.2.(nt+\varpi-\psi)=\cos.2.(nt+\varpi-\psi')=1$. The difference of these two expressions is $0^{\text{met.}},000010623\times 7,4191\times 8=0^{\text{met.}},0006305$, as above. [2894d]

barometer, as it is affected by the attraction of the sun and moon. This quantity, notwithstanding it is so very small, may be determined by a long
[2894'] series of barometrical observations, made between the tropics, where the barometer varies but very little. This phenomenon deserves the attention of observers.

The attraction of the sun and moon produces a wind corresponding to the ebb and flow of the sea.

The action of the sun and moon, produces a wind corresponding to the ebb and flow of the sea. We shall determine the force of this wind at the equator, making the same suppositions as before. For this purpose, we
[2894''] shall resume the first equation of this article [2878], and shall put $\cos.\theta=0$ [2128xii] ; hence we obtain,*

[2895]
$$\frac{d\,d\,v'}{d\,t^2}=-g.\left(\frac{dy'}{d\varpi}\right)-g.\left(\frac{d\,y}{d\varpi}\right)+\left(\frac{dV'}{d\,\varpi}\right).$$

[2896] Now we have $y'+y=2y-y'$ [2886b] ; moreover, from formulas [2193—2195, 2301],†

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* (2008a) Putting $\cos.\theta=0$, $\sin.\theta=1$, in [2878], it becomes

$$r^2.\delta\theta.\frac{d\,d\,u'}{d\,t^2}+r^2.\delta\varpi.\frac{d\,d\,v'}{d\,t^2}=\delta\,V'-g.\delta y'-g.\delta y.$$

This is equivalent to two equations, depending on the variations $\delta\theta$, $\delta\varpi$. Now the part of
[2895a] the second member, depending on $\delta\varpi$, is $\left(\frac{d\,V'}{d\,\varpi}\right).\delta\varpi-g.\left(\frac{dy'}{d\varpi}\right).\delta\varpi-g.\left(\frac{dy}{d\varpi}\right).\delta\varpi$, and by putting it equal to the corresponding part of the first member, we get, by dividing
[2895b] by $\delta\varpi$, $r^2.\frac{d\,d\,v'}{d\,t^2}=\left(\frac{d\,V'}{d\,\varpi}\right)-g.\left(\frac{dy'}{d\varpi}\right)-g.\left(\frac{dy}{d\varpi}\right)$. Substituting in this $r=1$ nearly [2886c], we get [2895].

† (2009) Substituting $\cos.\theta=0$ [2894''] in [2193—2195], and retaining only the terms depending on ϖ, because the other terms independent of ϖ vanish when we take the
[2897a] differential relatively to ϖ, we get $\alpha\,V'=\frac{3\,L}{4\,r^3}.\cos.^2\mathrm{v}.\cos.2.(n\,t+\varpi-\psi)$. This is the part depending on the sun ; the similar term depending on the moon, is

$$\alpha\,V'=\frac{3\,L'}{4\,r'^3}.\cos.^2\mathrm{v}'.\cos.2.(nt+\varpi-\psi')=\frac{3\,L}{4\,r^3}.e.\cos.^2\mathrm{v}'.\cos.2.(n\,t+\varpi-\psi'),$$

[2897b] [2301']. Taking the differential of the sum of these two terms relatively to ϖ, dividing by $d\varpi$, and substituting [2301], we get [2897].

$$\alpha.\left(\frac{dV'}{d\varpi}\right)=-2g.0^{\text{met.}},12316.\{\cos.^2\text{v}.\sin.2.(nt+\varpi-\psi)+e.\cos.^2\text{v}'.\sin.2.(nt+\varpi-\psi')\}.$$ [2897]

Hence by substituting the values of y and y'', we get,*

$$\alpha.\frac{ddv'}{dt^2}=-2g.1^{\text{met.}},0369.\{\cos.^2\text{v}.\sin.2.(nt+\varpi-\psi)+e.\cos.^2\text{v}'.\sin.2.(nt+\varpi-\psi')\};$$ [2898]

which, by integration, gives nearly,

$$\alpha.dv'=H.dt+\frac{g}{n^2}.ndt.1^{\text{met.}},0369.\{\cos.^2\text{v}.\cos.2.(nt+\varpi-\psi)+e.\cos.^2\text{v}'.\cos.2.(nt+\varpi-\psi')\}.$$ [2899]

If we suppose that dt represents one second, ndt will be nearly the [2899′]
one hundred thousandth part of the circumference ;† moreover, $\frac{n^2}{g}$ is $\frac{1}{289}$ [2900]

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* (2009a) The two terms $-g.\left(\frac{dy'}{d\varpi}\right)-g.\left(\frac{dy}{d\varpi}\right)$ [2895] are equivalent to

$$-g.\left(\frac{d.(y'+y)}{d\varpi}\right),\quad\text{or by [2896]},\quad -g.\left(\frac{d.(2y-y'')}{d\varpi}\right).$$ [2898a]

The value of this expression is easily deduced from [2891c], multiplied by $-g$, putting $\sin.\theta=1$, $\cos.\theta=0$, and neglecting the terms which do not contain ϖ, because they vanish in the differential relatively to ϖ. We shall have, after taking this differential, and dividing by $d\varpi$, using [2894b],

$$-\alpha g.\left(\frac{dy'}{d\varpi}\right)-\alpha g.\left(\frac{dy}{d\varpi}\right)$$

$$=-2g\times0^{\text{met.}},12316\times7^{\text{met.}},4191.\{\cos.^2\text{v}.\sin.2.(nt+\varpi-\psi)+e.\cos.^2\text{v}'.\sin.2.(nt+\varpi-\psi')\}$$ [2898b]

$$=-2g\times0^{\text{met.}},91375.\{\cos.^2\text{v}.\sin.2.(nt+\varpi-\psi)+e.\cos.^2\text{v}'.\sin.2.(nt+\varpi-\psi')\}.$$ [2898c]

Adding this to [2897], we get for the sum, a quantity equal to the second member of [2895], multiplied by α, which is equal to $\alpha.\frac{ddv'}{dt^2}$, the first member of that equation multiplied by α; and this sum becomes as in [2898]. The integral of this last expression, relative to t, [2898d] is easily reduced to the form [2899].

† (2010) The angle described by the earth's diurnal motion in the time t, is nt; which may be measured by the arc described by a point at the distance 1 from the axis of revolution. [2900a] Now the day being divided into 100000 centesimal seconds, corresponding to the whole circumference 6,2832, the arc described in one second will be $0{,}000062832=ndt$, dt [2900b] being one second [2899′]. In [1594a, &c.], we have $\frac{n^2}{g}=\frac{1}{289}$; hence

$$\frac{g}{n^2}.ndt\times1^{\text{met.}},0369=289\times0{,}000062832\times1^{\text{met.}},0369=0^{\text{met.}},01883.$$ [2900c]

of the radius of the earth [1594a], which we shall denote by r; hence we shall have,

[2901] $$\alpha . r d v' = rH . dt + 0^{\text{met.}},01883 . \{\cos.^2 v . \cos. 2 . (nt + \varpi - \psi) + e . \cos.^2 v' . \cos. 2 . (nt + \varpi - \psi')\}.$$

If the constant quantity H be finite, there will result, at the equator, a constant wind; and we might thus account for the *trade* winds. But the
[2902] value of this constant quantity depends on the initial motion of the atmosphere, and we have already observed, in [2208, &c.], that whatever depends on this motion at the origin, must have been destroyed a long while since, by the resistances of every kind, which the particles of air have suffered in their oscillations. *Hence we may in general conclude, that the trade winds do not arise from the attractions of the sun and moon upon the atmosphere.*

The trade winds do not depend on the attractions of the sun and moon.

If we suppose these two bodies to be in conjunction or in opposition, in
[2903] the equator, and $e = 3$ [2893], we shall have $0^{\text{met.}},07532$,* *for the greatest space that a particle of air would describe in a second, by means of both their forces.* Now it appears to be impossible to ascertain by observation
[2903'] the existence of so small a wind, in an atmosphere which is otherwise very much agitated. But this is not the case with barometrical observations; particularly if we take into view the extreme accuracy with which the observations of the barometer may be made. These variations, as we have observed in the heights of the tides, may be considerably increased by local circumstances; therefore they deserve the particular attention of observers.

[2904] We do not know to what extent the small oscillations of the atmosphere, depending upon the attractions of the sun and moon, can alter the motions

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Substituting this in the last term of [2899], it becomes as in the last term of [2901]; the
[2900d] first and second terms of [2899] being multiplied by the value of the radius of the earth r, which was taken for unity [2886c], produce the first and second terms of [2901] respectively. Then all the terms of [2901] may be rendered homogeneous, by putting for r its value in metres [2891].

* (2011) If we substitute, in [2901], $H = 0$, $e = 3$, $v = v' = 0$, $\psi' = \psi$ or $\psi' = \psi + 200^\circ$, we shall get,

[2903a] $$\alpha . r d v' = 4 \times 0^{\text{met.}},01883 . \cos. 2 . (nt + \varpi - \psi) = 0^{\text{met.}},07532 . \cos. 2 . (nt + \varpi - \psi).$$

The maximum of this expression is $0^{\text{met.}},07532$. This represents the greatest space $\alpha . r d v'$,
[2903b] described in the time $dt = 1$ second, by a particle of air, in virtue of the forces of attraction of the sun and moon.

produced by the various causes, which operate upon a fluid endowed with such great mobility. For in consequence of this extreme mobility, a great change can be produced by a very slight cause. It is by observation alone that we can get information on this point. We shall merely observe, that [2904′]
if the atmosphere rest immediately upon the solid nucleus of the earth, the differential equations of its motion will be, by what precedes, the same as those of the sea, supposing its depth to be everywhere the same.* Now we have seen in [2255, 2256], that then the oscillations of the second kind, which [2904″]
are the only ones depending on the difference between the northern and southern declinations of the sun and moon,† vanish. These oscillations also vanish, or at least are nearly insensible, when the atmosphere covers a sea in which the oscillations are nothing,‡ or very small, which is the case in our ports. [2904‴]
The sign of the declinations of the two bodies has not therefore any perceptible influence in the modifications of the atmosphere.

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* (2012) If the atmosphere rest on the solid nucleus of the earth, instead of resting upon the sea, the term $g \,.\, \delta y$, depending on the oscillations of the sea [356‴, 359^{iv}], will [2904a]
vanish from [2878]. Then the equations representing the oscillations of the atmosphere [2878, 2879], will become like those for the oscillations of the sea [2880, 2881], the accents on the letters being changed, and the depth of the atmosphere being everywhere equal to the [2904b]
constant quantity l'.

† (2013) The expressions [2193, 2195], on which the oscillations of the first and third kinds depend, are the same in north as in south declinations; which is not the case with [2904c]
[2194], on which those of the second kind depend.

‡ (2013a) If the oscillations of the sea are nothing, the motions of the air will be the same as if it rested on a solid nucleus, and then its undulations will be small, as is observed [2904d]
in [2904″].

FIFTH BOOK.

ON THE MOTIONS OF THE HEAVENLY BODIES ABOUT THEIR OWN CENTRES OF GRAVITY.

The motions of the heavenly bodies about their own centres of gravity, have such a connexion with their figures, and with the oscillations of the fluids which cover them, that we have thought it best to present the analysis
[2904''''] of these motions, immediately after the theories explained in the preceding books. *Among the bodies of the solar system, we shall consider the Earth, the Moon, and the rings of Saturn, which are the only ones in which we can compare the theory with observation*; but the following analysis can be applied generally to all the heavenly bodies.

CHAPTER I.

ON THE MOTIONS OF THE EARTH ABOUT ITS CENTRE OF GRAVITY.

1. We *shall resume the general equations of the motion of a solid body, of any figure*, demonstrated in Chapter vii, Book I. If we retain all the denominations of that chapter, and substitute in the equations [234] the
[2904v] values $p' = Cp$, $q' = Aq$, $r' = Br$, [233], we shall get,*

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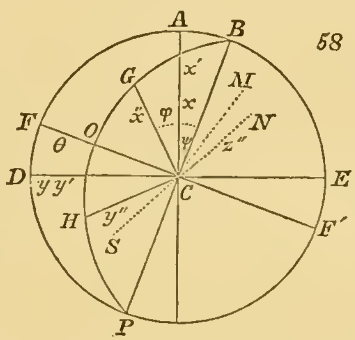
* (2014) In order to recall to mind the notation used in Book I, we have inserted the annexed figure 58; which is similar to that in Vol. I, page 112. The plane of the figure
[2907a] $BFPE$, represents the plane of the ecliptic at a given epoch, or in other words, the fixed ecliptic. $BGOP$ is the plane of the equator, intersecting the ecliptic in the line BCP,
[2907b] and passing through the centre of the earth C. CA is the axis of x, or x' [212''', &c.]; CD, perpendicular to CA, is the axis of y, or y'; and the axis of z, or z', is
[2907c] perpendicular to the plane of the ecliptic. CG is the *first principal axis*, or axis of

General equations of the motions of a solid body about its three axes.

$$dp+\frac{(B-A)}{C}\,.\,qr\,.\,dt=\frac{dN}{C}\,.\cos.\theta-\frac{dN'}{C}\,.\sin.\theta\,; \qquad [2905]$$

$$dq+\frac{(C-B)}{A}\,.\,rp\,.\,dt=-\frac{\{dN.\sin.\theta+dN'.\cos.\theta\}}{A}\,.\sin.\varphi+\frac{dN''}{A}\,.\cos.\varphi\,; \qquad (D') \qquad [2906]$$

$$dr+\frac{(A-C)}{B}\,.\,pq\,.\,dt=-\frac{\{dN.\sin.\theta+dN'.\cos.\theta\}}{B}\,.\cos.\varphi-\frac{dN''}{B}\,.\sin.\varphi. \qquad [2907]$$



58

x'' [226^{iv}]; CH the *second principal axis*, or axis of y''; [2907d]
both these axes being in the plane of the equator; CN is
the *third principal axis*, or axis of z'', drawn perpendicularly [2907e]
to the plane of the equator. Then, as in Vol. I, page 112,
note 81, we shall have the plane angle $BCG=\varphi$; the [2907f]
plane angle $BCH=\varphi+100^{\circ}$, the spherical angle
$FBO=\theta$. We may, as in [212^{iv}], consider x', y', z', [2907g]
as the co-ordinates of a particle dm, placed at M, and [2907h]
referred to the axes CA, CD, z. The co-ordinates
of the same particle, referred to the three principal axes
CG, CH, CN, are x'', y'', z'', [226^{iv}], respectively; and the distance of this [2907i]
particle from the centre C, is $CM=R$ [2908]. If we represent the distance of this [2907k]
particle from the axes of x'', y'', z'', by R_x, R_y, R_z, respectively; we shall have, [2907l]
as in the value of ρ^2, in the second line of note 125*a*, page 150, Vol. I, the first of the values
of R_z^2 [2907*n*]. In like manner we get R_y^2, R_x^2. The second forms of the values
[2907*n*] are easily deduced from the first, by observing that $R^2=x''^2+y''^2+z''^2$ [19*e*], [2907m]

$$R_x^2=y''^2+z''^2=R^2-x''^2, \quad R_y^2=x''^2+z''^2=R^2-y''^2, \quad R_z^2=x''^2+y''^2=R^2-z''^2. \qquad [2907n]$$

Substituting these in the momenta of inertia [229], we get

$$A=S\,.\,R_x^2\,.\,dm, \qquad B=S\,.\,R_y^2\,.\,dm, \qquad C=S\,.\,R_z^2\,.\,dm. \qquad [2907o]$$

The values of x'', y'', z'', in terms of R, μ, ϖ, are easily deduced, from the principle
of orthographic projection, applied to the annexed figure; from which we obtain the following [2907p]
expressions, being the same as in [1480*b*], changing x, y, z, into z'', x'', y'', in order to
conform to the present notation. Hence we obtain,

$$x''=R\,.\sqrt{(1-\mu^2)}\,.\cos.\varpi, \qquad y''=R\,.\sqrt{(1-\mu^2)}\,.\sin.\varpi, \qquad z''=R\,.\,\mu. \qquad [2907q]$$

Substituting these in the second forms [2907*n*], and extracting the square root, we get
R_x, R_y, R_z, [2911—2913]; which are to be substituted in [2907*o*], to obtain the values [2907r]
of A, B, C, [2915—2917]. S is the sign of integration relatively to the particle dm, [2907s]
and it must be extended to the whole mass of the body [214′]; in the third book [1705, &c.],
the author uses the symbol f, which is changed into S in [2952]. [2907t]

[2907′] We must now determine the momenta of inertia A, B, C, and the values of dN, dN', dN''.

Symbols. 2. *We shall first consider the momenta of inertia;* putting

[2908] $R =$ the radius drawn from the centre of gravity of the earth to a particle dm of its mass;

[2909] $\mu =$ the cosine of the angle which R makes with the third axis, or the axis of the equator;

[2910] $\varpi =$ the angle formed by the intersection of the plane which passes through the axis of the equator and the radius R, and the plane passing through the same axis and the first principal axis;

then we shall have [2907r],

[2911] $R\,.\sqrt{1-(1-\mu^2)\,.\cos.^2\varpi} =$ the distance of this particle from the *first principal axis;*

[2912] $R\,.\sqrt{1-(1-\mu^2)\,.\sin.^2\varpi} =$ the distance of this particle from the *second principal axis;*

[2913] $R\,.\sqrt{1-\mu^2} =$ the distance of this particle from the *third principal axis, or the axis of the equator.*

Now the momentum of inertia of a body, relative to any one of its axes, is the sum of the products of each particle of the body, by the square of its distance from that axis [245‴, 229]; and A, B, C, *being, by* [227c, 245, 245″],

[2914] *the momenta of inertia of the earth, corresponding to the first, second and third of the principal axes;* we shall have [2907s],

Momenta of inertia.

[2915] $$A = S\,.R^2\,.dm\,.\{1-(1-\mu^2)\,.\cos.^2\varpi\};$$

[2916] $$B = S\,.R^2\,.dm\,.\{1-(1-\mu^2)\,.\sin.^2\varpi\};$$

[2917] $$C = S\,.R^2\,.dm\,.\{1-\mu^2\}.$$

First form;

The integrals of these expressions must be taken so as to include the whole mass of the earth.

Now we have [1480″],

[2918] $$dm = R^2\,.dR\,.d\mu\,.d\varpi;$$

and by observing that the integral must be taken from $R=0$ to its value at the surface, represented by $R=R'$, we shall have,* [2919]

Second form.

$$A=\tfrac{1}{5}\,.\,S\,.\,R'^5\,.\,d\mu\,.\,d\varpi\,.\,\{1-(1-\mu^2)\,.\,\cos.^2\varpi\}\,; \quad [2920]$$

$$B=\tfrac{1}{5}\,.\,S\,.\,R'^5\,.\,d\mu\,.\,d\varpi\,.\,\{1-(1-\mu^2)\,.\,\sin.^2\varpi\}\,; \quad [2921]$$

$$C=\tfrac{1}{5}\,.\,S\,.\,R'^5\,.\,d\mu\,.\,d\varpi\,.\,\{1-\mu^2\}. \quad [2922]$$

We shall suppose R'^5 to be developed in a series of the following form [1530*l*],

$$R'^5=U^{(0)}+U^{(1)}+U^{(2)}+U^{(3)}+\&c.\,; \quad [2923]$$

$U^{(i)}$ being a rational and integral function of μ, $\sqrt{1-\mu^2}.\cos.\varpi$, $\sqrt{1-\mu^2}.\sin.\varpi$, satisfying the equation of partial differentials [1437],

$$0=\left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{dU^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddU^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i\,.\,(i+1)\,.\,U^{(i)}. \quad [2924]$$

The function $1-(1-\mu^2).\cos.^2\varpi$ is equal to $\tfrac{2}{3}.+\{\tfrac{1}{3}-(1-\mu^2).\cos.^2\varpi\}$. The constant quantity $\tfrac{2}{3}$ is included in the form $U^{(0)}$; and the function [2925] $\tfrac{1}{3}-(1-\mu^2)\,.\cos.^2\varpi$ is of the form $U^{(2)}$,† since it will satisfy the preceding

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* (2015) Substituting [2918] in [2915—2917], and integrating relatively to R, we get [2920—2922]; and by using the development of R'^5 [2923], we finally obtain A, B, C, [2921*a*] under very simple forms in [2936—2938], as will be seen in the subsequent calculations.

† (2016) Putting in [1528*a*, *c*], $B_0^{(0)}=\tfrac{2}{3}$, $B_2^{(0)}=\tfrac{1}{2}$, $B_2^{(2)}=-\tfrac{1}{2}$, and neglecting the other terms, we shall have by means of [6] Int.,

$$Y^{(0)}=\tfrac{2}{3},\quad Y^{(2)}=\tfrac{1}{2}.(\mu^2-\tfrac{1}{3})-\tfrac{1}{2}.(1-\mu^2).\cos.2\varpi=\tfrac{1}{2}.(\mu^2-\tfrac{1}{3})-\tfrac{1}{2}.(1-\mu^2).(2.\cos.^2\varpi-1) \quad [2925a]$$
$$=\tfrac{1}{3}-(1-\mu^2)\,.\cos.^2\varpi,$$

as in [2925]. In like manner, if we neglect all the terms of [1528*a*, *c*], excepting $B^{(0)}=\tfrac{2}{3}$, $B_2^{(0)}=\tfrac{1}{2}$, $B_2^{(2)}=\tfrac{1}{2}$, and reduce by means of [1] Int., we shall have,

$$Y^{(0)}=\tfrac{2}{3},\quad Y^{(2)}=\tfrac{1}{2}.(\mu^2-\tfrac{1}{3})+\tfrac{1}{2}.(1-\mu^2).\cos.2\varpi=\tfrac{1}{2}.(\mu^2-\tfrac{1}{3})+\tfrac{1}{2}.(1-\mu^2).(1-2.\sin.^2\varpi) \quad [2925b]$$
$$=\tfrac{1}{3}-(1-\mu^2)\,.\sin.^2\varpi,$$

as in [2926]. Lastly, putting $B_2^{(0)}=-1$ in [1528*c*], and neglecting the other terms, we shall get $Y^{(2)}=(\tfrac{1}{3}-\mu^2)$, as in [2927]. We may remark that these different values [2925*c*] of $Y^{2)}$ may be put under one general form $Y^{(2)}=D'+E'.\cos.2\varpi$, which will be of [2925*d*]

equation of partial differentials [2924]. In like manner, $1-(1-\mu^2).\sin.^2\varpi$
[2926] is equal to $\frac{2}{3}+\{\frac{1}{3}-(1-\mu^2).\sin.^2\varpi\}$; and the second term of this
[2927] expression is of the form $U^{(2)}$. Lastly, the function $(1-\mu^2)$ is equal to $\frac{2}{3}+(\frac{1}{3}-\mu^2)$, and the part $(\frac{1}{3}-\mu^2)$ is of the form $U^{(2)}$. Hence we have,
Momenta of inertia. by means of the theorem [1476],*

[2928]
$$A=\tfrac{1}{5}.S.d\mu.d\varpi.\{\tfrac{2}{3}.U^{(0)}+[\tfrac{1}{3}-(1-\mu^2).\cos.^2\varpi].U^{(2)}\};$$

[2929]
$$B=\tfrac{1}{5}.S.d\mu.d\varpi.\{\tfrac{2}{3}.U^{(0)}+[\tfrac{1}{3}-(1-\mu^2).\sin.^2\varpi].U^{(2)}\};$$

[2930]
$$C=\tfrac{1}{5}.S.d\mu.d\varpi.\{\tfrac{2}{3}.U^{(0)}+(\tfrac{1}{3}-\mu^2).U^{(2)}\}.$$

Third form.

[2930′] The integrals must be taken from $\mu=-1$ to $\mu=1$, and from $\varpi=0$ to $\varpi=2\pi$ [1470′]; hence we obtain,

Momenta of inertia.

[2931]
$$A=\tfrac{8}{15}.\pi.U^{(0)}+\tfrac{1}{5}.S.U^{(2)}.d\mu.d\varpi.\{\tfrac{1}{3}-(1-\mu^2).\cos.^2\varpi\};$$

[2932]
$$B=\tfrac{8}{15}.\pi.U^{(0)}+\tfrac{1}{5}.S.U^{(2)}.d\mu.d\varpi.\{\tfrac{1}{3}-(1-\mu^2).\sin.^2\varpi\};$$

[2933]
$$C=\tfrac{8}{15}.\pi.U^{(0)}+\tfrac{1}{5}.S.U^{(2)}.d\mu.d\varpi.\{\tfrac{1}{3}-\mu^2\}.$$

Fourth form.

The function $U^{(2)}$ is of this form [1528c],†

[2934]
$$U^{(2)}=H.(\tfrac{1}{3}-\mu^2)+H'.\mu.\sqrt{1-\mu^2}.\sin.\varpi+H''.\mu.\sqrt{1-\mu^2}.\cos.\varpi$$
$$+H'''.(1-\mu^2).\sin.2\varpi+H''''.(1-\mu^2).\cos.2\varpi.$$

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use hereafter; D', E', being functions of μ and constant quantities. For if we compare this form with [2925a, b, c], we shall find that, in the expression of A [2920, 2925a], we
[2925e] must put $D'=\frac{1}{2}.(\mu^2-\frac{1}{3})$, $E'=-\frac{1}{2}.(1-\mu^2)$. In the expression of B [2921, 2925b],
[2925f] we must put $D'=\frac{1}{2}.(\mu^2-\frac{1}{3})$, $E'=\frac{1}{2}.(1-\mu^2)$. In the expression of C [2922, 2925c],
[2925g] we must put $D'=-(\mu^2-\frac{1}{3})$, $E'=0$.

* (2017) When we substitute R'^5 [2923] in [2920—2922], we may neglect all the
[2928a] terms of the form $S.Y^{(i)}.U^{(i')}.d\mu.d\varpi$ [1476], in which i' differs from i; and retain only the terms in which $i'=i$; so that we may neglect all the terms except $U^{(0)}$, $U^{(2)}$. By this means [2920—2922] become as in [2928—2930]. The part depending on $U^{(0)}$,
[2928b] in all three of these last formulas, is $\frac{2}{15}.U^{(0)}.S.d\mu.d\varpi=\frac{2}{15}.U^{(0)}.4\pi=\frac{8}{15}.\pi.U^{(0)}$ [1468a]. Substituting this in [2928—2930], we get [2931—2933].

† (2018) The function [2934] is of the same form as the general expression of $Y^{(2)}$ [1528c], and is also like that in [1753], changing the sign of H. Then the consideration that the axes are principal axes gives the equations [1754] as in [2935]; by means of
[2933a] which the expression [2934] becomes $U^{(2)}=H.(\frac{1}{3}-\mu^2)+H''''.(1-\mu^2).\cos.2\varpi$; and by putting for brevity $D=-H.(\mu^2-\frac{1}{3})$, $E=H''''.(1-\mu^2)$, it becomes
[2933b] $U^{(2)}=D+E.\cos.2\varpi$. The expressions of $Y^{(2)}$ [2925a, b, c], which occur as factors

The consideration that the axes are principal axes, gives as in [1754, &c.], Conditions required to have all three axes principal axes.

$$H' = 0, \qquad H'' = 0, \qquad H''' = 0.$$ [2935]

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of $U^{(2)}$ in [2931, 2933], are of the form $Y^{(2)} = D' + E' . \cos. 2\varpi$ [2925*d*]; D', E', being functions of μ and constant quantities [2925*e*—*g*]. Hence the integrals [2931—2933], or the parts depending on $U^{(2)}$, become of the form, [2933*c*]

$$\tfrac{1}{5} . S . U^{(2)} . Y^{(2)} . d\mu . d\varpi = \tfrac{1}{5} . S . (D + E . \cos. 2\varpi) . (D' + E' . \cos. 2\varpi) . d\mu . d\varpi$$
$$= \tfrac{1}{5} . S . (DD' + DE' . \cos. 2\varpi + D'E . \cos. 2\varpi + EE' . \cos.^2 2\varpi) . d\mu . d\varpi.$$ [2933*d*]

Substituting $\cos.^2 2\varpi = \tfrac{1}{2} + \tfrac{1}{2} . \cos. 4\varpi$, [6] Int., and observing that $S_0^{2\pi} d\varpi . \cos. n\varpi = 0$ [1483*b*], $S_0^{2\pi} d\varpi = 2\pi$ [1467*b*], the expression becomes [2933*e*]

$$\tfrac{1}{5} . S . U^{(2)} . Y^{(2)} . d\mu . d\varpi = \tfrac{1}{5} . S . (DD' + \tfrac{1}{2} EE') . d\mu . d\varpi$$
$$= \frac{2\pi}{5} . S . (DD' + \tfrac{1}{2} EE') . d\mu;$$ [2933*f*]

in which we must substitute the values D, E, [2933*a*], and those of D', E', corresponding to $Y^{(2)}$ [2925*e*, *f*, *g*]; taking the integrals from $\mu = -1$ to $\mu = 1$ [2930']. Now we have generally $S . \mu^{2n} d\mu = \frac{1}{2n+1} . (\mu^{2n+1} + 1)$, which vanishes when $\mu = -1$, n being an integral number or zero; and when $\mu = 1$, it becomes $S_{-1}^{1} \mu^{2n} d\mu = \frac{2}{2n+1}$. Putting now successively $n = 0$, $n = 1$, $n = 2$, &c., we get [2933*g*] [2933*h*]

$$S_{-1}^{1} d\mu = \tfrac{2}{1}; \qquad S_{-1}^{1} \mu^2 d\mu = \tfrac{2}{3}; \qquad S_{-1}^{1} \mu^4 d\mu = \tfrac{2}{5}; \qquad S_{-1}^{1} \mu^6 d\mu = \tfrac{2}{7}; \qquad \&c.$$ [2933*i*]

By means of these formulas, we may obtain the integrals [2933*f*], corresponding to each of the expressions [2931—2933]. For by substituting $D = -H . (\mu^2 - \tfrac{1}{3})$ [2933*a*] and $D' = \tfrac{1}{2} . (\mu^2 - \tfrac{1}{3})$ [2925*e*] in [2933*f*], we get the corresponding term of that expression, which is reduced by the aid of the integrals [2933*i*]; and we finally obtain, for the part of the value of A [2931] depending on H, [2933*k*]

$$\frac{2\pi}{5} . S . DD' . d\mu = -\frac{\pi H}{5} . S . (\mu^2 - \tfrac{1}{3})^2 . d\mu = \frac{\pi H}{5} . S . (-\mu^4 + \tfrac{2}{3}\mu^2 - \tfrac{1}{9}) . d\mu$$
$$= \frac{\pi H}{5} . (-\tfrac{2}{5} + \tfrac{2}{3} \times \tfrac{2}{3} - \tfrac{1}{9} \times \tfrac{2}{1}) = -\frac{8\pi}{9 \times 5^2} . H,$$ [2933*l*]

as in [2936]. The value of D' [2925*f*] corresponding to B, being the same as in the preceding integral, it will produce, in B [2937], the same term $-\frac{8\pi}{9 \times 5^2} . H$. The value of D' [2925*g*] corresponding to C, is equal to that depending on A, multiplied by -2; therefore we must multiply the expression [2933*l*] by -2, to obtain $\frac{16\pi}{9 \times 5^2} . H$ [2933*m*] [2933*n*]

Momenta of inertia about the three principal axes. Fifth form.

These three equations contain all the conditions necessary to make the axes here used the three principal axes. Hence we have

[2936] $$A = \frac{8\pi}{15} \cdot U^{(0)} - \frac{8\pi}{9 \cdot 5^2} \cdot H - \frac{8\pi}{3 \cdot 5^2} \cdot H'''' ;$$

[2937] $$B = \frac{8\pi}{15} \cdot U^{(0)} - \frac{8\pi}{9 \cdot 5^2} \cdot H + \frac{8\pi}{3 \cdot 5^2} \cdot H'''' ;$$

[2938] $$C = \frac{8\pi}{15} \cdot U^{(0)} + \frac{16\pi}{9 \cdot 5^2} \cdot H.$$

General equation of a solid in which all the momenta of inertia are equal.

If we wish to have the three momenta of inertia A, B, C, equal, we must put* $H=0$, $H''''=0$; consequently [2934, 2935],

[2939] $$U^{(2)} = 0.$$

If A, B, C, be equal, the momenta of inertia will be equal relatively to all the axes.

Therefore this last equation satisfies the requisite conditions, to make the three axes become principal axes, and at the same time the momenta of inertia about these axes equal to each other. Now we have shown, in [253″—254′], that *in this case the momenta of inertia are equal relatively to*
[2940] *all the axes; consequently the sphere is not the only body which possesses this property.* The preceding analysis gives the general equation [2923, 2939] of

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[2938]. In like manner, the part of A [2933f] depending on $E' = -\frac{1}{2} \cdot (1-\mu^2)$ [2925e] and E [2933a] is

$$\frac{\pi}{5} \cdot S \cdot E E' \cdot d\mu = -\frac{\pi}{2 \times 5} \cdot H'''' \cdot S \cdot (1-\mu^2)^2 \cdot d\mu = -\frac{\pi}{2 \times 5} \cdot H'''' S (1-2\mu^2+\mu^4) \cdot d\mu$$

[2933o] $$= -\frac{\pi}{2 \times 5} \cdot H'''' \cdot (\tfrac{2}{1} - \tfrac{4}{3} + \tfrac{2}{5}) \qquad = -\frac{8\pi}{3 \times 5^2} \cdot H'''',$$

[2933p] as in [2936]; and as E' [2925f] corresponding to B, is the same as the preceding value of E', with its sign changed, we shall have the part of B depending on H, equal to the preceding value [2933o], with its sign changed; making it $\frac{8\pi}{3 \times 5^2} \cdot H''''$, as in [2937].

[2933q] Lastly, the term $E' = 0$ [2925g] corresponds to the value of C, therefore EE' or H'''' must vanish from C [2938].

* (2019) Making $A = B$ [2936, 2937], we get $H''''=0$; and then putting
[2939a] $B = C$ [2937, 2938], we obtain $H=0$; hence $A = B = C = \frac{8\pi}{15} \cdot U^{(0)}$. In this case the value of $U^{(2)}$ [2933a] vanishes, as in [2939]

all the solids to which it appertains. We have previously spoken of this equation [254′]. We may observe, that *these results are independent of the supposition that the origin of* R' *passes through the centre of gravity of the spheroid; therefore they hold good, whatever point within the body may be* [2941]
selected for the origin of this radius.

Supposing the earth to be composed of an infinite number of strata, of forms which vary from the centre to the surface; the radius R of one of these strata can always be expressed in the following manner [1497, &c.],

Radius R.

$$R = a + \alpha a \,.\, \{ Y^{(1)} + Y^{(2)} + Y^{(3)} + Y^{(4)} + \&c. \}; \qquad [2942]$$

α being a very small constant coefficient, and $Y^{(1)}$, $Y^{(2)}$, &c., functions of the same nature as $U^{(1)}$, $U^{(2)}$, &c.; or in other words, they satisfy the [2942′]
equation of partial differentials [2924]; but these functions may contain a in any manner whatever. If we neglect quantities of the order α^2, we shall have [2942],

$$R^5 = a^5 + 5\,\alpha\, a^5 \,.\, \{ Y^{(1)} + Y^{(2)} + Y^{(3)} + \&c. \}. \qquad [2943]$$

If we now suppose a solid homogeneous body to have a density equal to unity, and a radius equal to that of the stratum just mentioned, we shall [2943′]
obtain, relatively to this solid,*

Momenta of inertia of a solid homogeneous body, about its three axes. Sixth form.

$$A = \frac{8\pi \,.\, a^5}{15} + \alpha \,.\, S \,.\, a^5 \,.\, Y^{(2)} \,.\, d\mu \,.\, d\varpi \,.\, \{ \tfrac{1}{3} - (1-\mu^2) \,.\, \cos.^2 \varpi \}; \qquad [2944]$$

$$B = \frac{8\pi \,.\, a^5}{15} + \alpha \,.\, S \,.\, a^5 \,.\, Y^{(2)} \,.\, d\mu \,.\, d\varpi \,.\, \{ \tfrac{1}{3} - (1-\mu^2) \,.\, \sin.^2 \varpi \}; \qquad [2945]$$

$$C = \frac{8\pi \,.\, a^5}{15} + \alpha \,.\, S \,.\, a^5 \,.\, Y^{(2)} \,.\, d\mu \,.\, d\varpi \,.\, \{ \tfrac{1}{3} - \mu^2 \}. \qquad [2946]$$

Density ρ.

Taking the differentials of these values relatively to a, then multiplying them by the density ρ of the stratum whose radius is a, ρ being any function [2947]
of a, we shall get the momentum of inertia of this stratum. To obtain the momenta of inertia for the whole earth, it will only be necessary to integrate

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* (2020) Comparing [2923, 2943], we get $U^{(0)} = a^5$, $U^{(2)} = 5\,\alpha \,.\, a^5 \,.\, Y^{(2)}$; [2944a]
substituting these in [2931—2933], we get [2944—2946] respectively.

the momenta of the strata relatively to a, from $a=0$ to the *value of* a,
[2947′] *corresponding to the surface of the earth, which we shall denote by* $a=1$.
Hence we shall have,*

Momenta of inertia of a body composed of strata varying in density from the centre to the surface. Seventh form.

[2948] $$A=\frac{8\pi}{15}.S.\rho.d.a^5+\alpha.S.\rho.d.(a^5.Y^{(2)}).d\mu.d\varpi.\{\tfrac{1}{3}-(1-\mu^2).\cos.^2\varpi\};$$

[2949] $$B=\frac{8\pi}{15}.S.\rho.d.a^5+\alpha.S.\rho.d.(a^5.Y^{(2)}).d\mu.d\varpi.\{\tfrac{1}{3}-(1-\mu^2).\sin.^2\varpi\};$$

[2950] $$C=\frac{8\pi}{15}.S.\rho.d.a^5+\alpha.S.\rho.d.(a^5.Y^{(2)}).d\mu.d\varpi.\{\tfrac{1}{3}-\mu^2\};$$

the differential $d.(a^5.Y^{(2)})$ *being taken relatively to the variable quantity* a *only.*

[2951] It follows from the equation [1705], that if we put $\alpha\varphi$ *for the ratio of the centrifugal force to the gravity at the equator* [1726′], we shall have, by the condition of the equilibrium of fluids, covering the surface of the earth,†

[2952] $$S_0^1.\rho.d.(a^5.Y^{(2)})=\tfrac{5}{3}.\{Y^{(2)}+\tfrac{1}{2}\varphi.(\mu^2-\tfrac{1}{3})\}.S_0^1.\rho.d.a^3;$$

the value of $Y^{(2)}$ in the second member of this equation corresponds to the surface of the earth [2947′], and the integrals are to be taken from $a=0$ to $a=1$; therefore we shall have,

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* (2021) The formulas [2948—2950] are easily deduced from [2944—2946]; by
[2949a] taking the differentials of the terms a^5, $a^5.Y^{(2)}$, relatively to a, then multiplying by the density ρ, and finally integrating relatively to a.

† (2022) At the surface of the earth $a=1$ [1702″], and the first integral of the
[2951a] equation [1705] corresponding to that surface vanishes, its limits being $a=1$, $a=1$; consequently that equation becomes, when $i=2$,

[2951b] $$0=-\tfrac{4}{3}\pi.Y^{(2)}.S_0^1.\rho.d.a^3+\tfrac{4}{5}\pi.S_0^1.\rho.d.(a^5.Y^{(2)})+Z^{(2)};$$

observing that $a=1$ in the factors which are free from the sign of integration. Now if we notice only the permanent figure of the earth, we may neglect the attraction of the bodies S, S', &c.; by which means the second of the equations [1632] becomes

[2951c] $$\alpha.Z^{(2)}=-\tfrac{1}{2}g.(\mu^2-\tfrac{1}{3})=-\tfrac{2}{3}\pi.\alpha\varphi.(\mu^2-\tfrac{1}{3}).S_0^1.\rho.d.a^3 \qquad [1727].$$

Substituting this in [2951b], and multiplying by $\frac{5}{4\pi}$, we get, by reduction, the formula [2952].

Momenta of inertia of a solid body.

$$A = \frac{8\pi}{15} \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^5 + \frac{4\,\alpha\pi}{27} \,.\, \varphi \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^3$$
$$+ \frac{5\,\alpha}{3} \,.\, S \,.\, Y^{(2)} \,.\, d\mu \,.\, d\varpi \,.\, \{\tfrac{1}{3} - (1-\mu^2) \,.\, \cos.^2\varpi\} \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^3 \,; \qquad [2953]$$

$$B = \frac{8\pi}{15} \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^5 + \frac{4\,\alpha\pi}{27} \,.\, \varphi \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^3$$
$$+ \frac{5\,\alpha}{3} \,.\, S \,.\, Y^{(2)} \,.\, d\mu \,.\, d\varpi \,.\, \{\tfrac{1}{3} - (1-\mu^2) \,.\, \sin.^2\varpi\} \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^3 \,; \qquad [2954]$$

$$C = \frac{8\pi}{15} \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^5 - \frac{8\,\alpha\pi}{27} \,.\, \varphi \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^3$$
$$+ \frac{5\,\alpha}{3} \,.\, S \,.\, Y^{(2)} \,.\, d\mu \,.\, d\varpi \,.\, \{\tfrac{1}{3} - \mu^2\} \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^3.^* \qquad [2955]$$

Eighth form.

The function $Y^{(2)}$ is of the form† [1528c],

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* (2023) Substituting [2952] in [2948—2950], we obtain [2953—2955] respectively; no reduction being necessary, except in the term multiplied by φ; and this term is easily deduced from the coefficients of H, H'''', [2936—2938], in the following manner. The part of $\alpha \,.\, S \,.\, \rho \,.\, d \,.\, (a^5 \,.\, Y^{(2)})$ [2952] depending on φ is $\frac{5}{6}\,\alpha\varphi \,.\, (\mu^2 - \frac{1}{3}) \,.\, S \,.\, \rho \,.\, d \,.\, a^3$, and this is multiplied in [2948, 2949, 2950], by the same factors as those of $\frac{1}{5}\,U^{(2)}$ in [2931, 2932, 2933]; so that if for a moment we put these two expressions equal to each other, we shall get $U^{(2)} = \frac{25}{6}\,\alpha\varphi \,.\, (\mu^2 - \frac{1}{3}) \,.\, S \,.\, \rho \,.\, d \,.\, a^3$. Comparing this value of $U^{(2)}$ with [2933a], we obtain $H = -\frac{25}{6}\,\alpha\varphi \,.\, S \,.\, \rho \,.\, d \,.\, a^3$, $H'''' = 0$; and if we substitute these values of H, H'''', in A, B, C, [2936—2938], which were deduced from formulas [2931—2933], we shall evidently get the terms of A, B, C, depending on φ, in [2953, 2954, 2955]. Thus the term $-\frac{8\pi}{9\times 5^2} \,.\, H$ [2936] becomes [2953a] [2953b] [2953c]

$$\frac{8\pi}{9\times 5^2} \times \tfrac{25}{6}\,\alpha\varphi \,.\, S \,.\, \rho \,.\, d \,.\, a^3 = \frac{4\,\alpha\pi}{27} \,.\, \varphi \,.\, S \,.\, \rho \,.\, d \,.\, a^3, \qquad [2953d]$$

as in [2953]. Moreover, the term of B depending on H [2937] is the same as that of A [2936], and the term of C [2938] is double of that of A, and of a different sign; and we find the same proportions between the coefficients of φ, in the values of A, B, C, [2953, 2954, 2955]. [2953e]

† (2024) This value corresponds to the general form [1528c], and is similar to those in [1753, 2934]. The consideration of the principal axes has shown, in [1754], that H', H'', H''', which are the coefficients of $\sin.\varpi$, $\cos.\varpi$, $\sin.2\varpi$, [1753], vanish. In like manner, when we use the symbols [2956], we shall have $h'=0$, $h''=0$, $h'''=0$, as in [2957]. Substituting these in [2956], we get [2958]. [2956a] [2956b]

[2956]
$$Y^{(2)} = h \,.\, (\tfrac{1}{3} - \mu^2) + h' \,.\, \mu \,.\, \sqrt{1-\mu^2} \,.\, \sin.\, \varpi + h'' \,.\, \mu \,.\, \sqrt{1-\mu^2} \,.\, \cos.\, \varpi$$
$$+ h''' \,.\, (1-\mu^2) \,.\, \sin.\, 2\varpi + h'''' \,.\, (1-\mu^2) \,.\, \cos.\, 2\varpi.$$

The consideration that the axes are principal axes, gives as in [1754, &c.],

[2957] $$h' = 0, \qquad h'' = 0, \qquad h''' = 0 \,;$$

consequently,

[2958] $$Y^{(2)} = h \,.\, (\tfrac{1}{3} - \mu^2) + h'''' \,.\, (1-\mu^2) \,.\, \cos.\, 2\varpi.$$

We have seen in [2056', &c.], that the variation of gravity is very nearly proportional to the square of the sine of the latitude, so that the value of
[2959] h'''' must be very small [2056'']. In fact the term h'''' is nothing if the earth be a solid of revolution ;* but for greater generality, we shall retain it in the following researches. Then we shall have,†

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* (2025) If the earth be a spheroid of revolution, the angle ϖ evidently vanishes from
[2959a] the expression of the radius, whose origin is in the axis of revolution ; and then we have $h''''=0$ [2958], the radius being the same for the same value of μ, whatever be the value of ϖ, as in [1730b', &c.].

† (2026) If we compare the values of A, B, C, [2953—2955] with those in
[2960a] [2960—2962] respectively, we shall find that the terms multiplied by $\frac{8}{15}\pi$ are exactly the same in both systems ; and by a very slight reduction, it appears that the terms multiplied by φ, in the corresponding expressions of both systems, are also equal. The only remaining term in [2953—2955] is that depending on $Y^{(2)}$, and we shall now show that this produces the terms of [2960—2962], depending on h, h'''', in the following manner. If we put
[2960b] for a moment $U^{(2)} = \frac{25}{3}\alpha \,.\, Y^{(2)} \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^3$ in [2931, 2932, 2933], it will produce the terms depending on $Y^{(2)}$ in [2953, 2954, 2955] respectively, as is evident by inspection ; therefore if we make the same substitution in [2936, 2937, 2938], which were deduced from [2931, 2932, 2933], by performing the integrations relatively to μ, ϖ, we shall obtain
[2960c] the corresponding terms of [2960, 2961, 2962] respectively. Substituting in [2960b] the values [2933a, 2958], we get,

$$H \,.\, (\tfrac{1}{3} - \mu^2) + H'''' \,.\, (1-\mu^2) \,.\, \cos.\, 2\varpi$$
[2960d] $$= \tfrac{25}{3}\alpha \,.\, \{h \,.\, (\tfrac{1}{3} - \mu^2) + h'''' \,.\, (1-\mu^2) \,.\, \cos.\, 2\varpi\} \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^3 \,;$$

which becomes identical by puttting

[2960e] $$H = \tfrac{25}{3}\alpha \,.\, h \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^3, \qquad H'''' = \tfrac{25}{3}\alpha \,.\, h'''' \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^3.$$

Momenta of inertia of a solid body about its principal axes. Ninth form.

$$A=\frac{8\pi}{15}\cdot S_0^1\cdot\rho\cdot d\cdot a^5-\tfrac{8}{27}\cdot\alpha\pi\cdot(h-\tfrac{1}{2}\varphi)\cdot S_0^1\cdot\rho\cdot d\cdot a^3-\tfrac{8}{9}\cdot\alpha\pi\cdot h''''\cdot S_0^1\cdot\rho\cdot d\cdot a^3;$$ [2960]

$$B=\frac{8\pi}{15}\cdot S_0^1\cdot\rho\cdot d\cdot a^5-\tfrac{8}{27}\cdot\alpha\pi\cdot(h-\tfrac{1}{2}\varphi)\cdot S_0^1\cdot\rho\cdot d\cdot a^3+\tfrac{8}{9}\cdot\alpha\pi\cdot h''''\cdot S_0^1\cdot\rho\cdot d\cdot a^3;$$ [2961]

$$C=\frac{8\pi}{15}\cdot S_0^1\cdot\rho\cdot d\cdot a^5+\tfrac{16}{27}\cdot\alpha\pi\cdot(h-\tfrac{1}{2}\varphi)\cdot S_0^1\cdot\rho\cdot d\cdot a^3.$$ [2962]

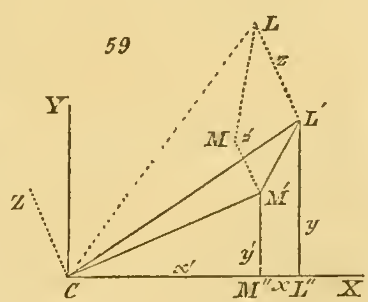
Symbols. L. [2963] x, y, z. [2964] x', y', z'. [2965]

3. We shall now investigate the values of dN, dN', dN'', which enter in the differential equations [2905—2907]. We shall put L *for the mass of a body which acts upon the earth ;* x, y, z, *the co-ordinates of its centre, referred to the centre of gravity of the earth,* $r_{,}=\sqrt{x^2+y^2+z^2}$; *also* x', y', z', *for the co-ordinates of a particle* dm *of the terrestrial spheroid; lastly we shall suppose,*

$$V=-L\cdot\frac{(xx'+yy'+zz')}{r_{,}^{3}}+\frac{L}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}.$$ V. [2966]

The attractive forces of the body L, upon the particle dm, resolved in lines parallel to the axes of x, y, z, and *in directions opposite to their origin*, being decreased by the same attractive forces, upon the centre of gravity of the earth, supposing it to be at rest, are $\left(\frac{dV}{dx'}\right)$, $\left(\frac{dV}{dy'}\right)$, $\left(\frac{dV}{dz'}\right)$, respectively.* These forces are what we have denoted by P, Q, R, in [212iv], therefore we shall have, by substitution in [225], Forces. [2966'] [2967]

If we now substitute these values of H, H'''', in [2936, 2937, 2938], we shall obtain the corresponding terms of A, B, C, [2960—2962], depending on h, h'''', respectively, in conformity to the method adopted in [2960c]. [2960f]



* (2027) In the annexed figure, C is the centre of the earth, taken as the origin of the rectangular axes CX, CY, CZ ; of which the last CZ is perpendicular to the plane of the figure. M is the place of the particle dm, whose co-ordinates, drawn parallel to these axes, are $CM''=x'$, $M''M'=y'$, $M'M=z'$. L is the place of the attracting body L, whose co-ordinates are x, y, z. We have also $CL=r_{,}$, and we shall put for brevity $LM=f$. Then by the principles of the orthographic projection [19b', &c.], we evidently have $(CL)^2=(CL'')^2+(L''L')^2+(L'L)^2$, [2967a] [2967b] [2967c]

Values of dN,

[2968] $$\frac{dN}{dt} = S.dm.\left\{x'.\left(\frac{dV}{dy'}\right) - y'.\left(\frac{dV}{dx'}\right)\right\};*$$

dN',

[2969] $$\frac{dN'}{dt} = S.dm.\left\{x'.\left(\frac{dV}{dz'}\right) - z'.\left(\frac{dV}{dx'}\right)\right\};$$

dN''.

[2970] $$\frac{dN''}{dt} = S.dm.\left\{y'.\left(\frac{dV}{dz'}\right) - z'.\left(\frac{dV}{dy'}\right)\right\}.$$

First form.

[2967d] or $r_{\prime}^2 = x^2 + y^2 + z^2$; also $(LM)^2 = f^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$ [392]. Now the attraction of the body L upon the particle dm, in the direction ML, is $\frac{L}{f^2}$; and if this be resolved in directions parallel to the axes of x, y, z, the part parallel to the

[2967e] axis x is $\frac{L}{f^2}\cdot\frac{L''M''}{LM} = \frac{L.(x-x')}{f^3}$ [394], *tending to increase the co-ordinate* x'. At the

[2967f] centre C, where $x'=0$, and f changes into $r_{\prime}$, this becomes $\frac{Lx}{r_{\prime}^3}$, representing the action of the body L upon the particle dm, at C, in the direction CX. Subtracting this from

[2967g] the preceding expression, we get $-\frac{Lx}{r_{\prime}^3} + \frac{L.(x-x')}{f^3}$, which represents the action of the body L upon the particle dm, resolved in a direction parallel to the axis of x, supposing the centre of gravity C to be at rest. If we take the partial differential of V [2966], relatively to x', we shall find it exactly equal to this part of the force of the body L, [2967g],

[2967h] resolved in a direction parallel to the axis of x'; so that it will be represented by $\left(\frac{dV}{dx'}\right)$, *tending to increase the co-ordinate* x'. In like manner, $\left(\frac{dV}{dy'}\right)$ represents the same force,

[2967i] resolved in the direction y'; and $\left(\frac{dV}{dz'}\right)$ the force in the direction z'; as is evident from the consideration that V [2966] is symmetrical in x, x'; y, y'; z, z'. Hence we have, for these three forces,

[2967k] $$\left(\frac{dV}{dx'}\right) = -\frac{Lx}{r_{\prime}^3} + \frac{L.(x-x')}{f^3}; \qquad \left(\frac{dV}{dy'}\right) = -\frac{Ly}{r_{\prime}^3} + \frac{L.(y-y')}{f^3};$$
$$\left(\frac{dV}{dz'}\right) = -\frac{Lz}{r_{\prime}^3} + \frac{L.(z-z')}{f^3};$$

which are similar to the forces P, Q, R, [212iv], respectively, as is observed above.

* (2027a) If we take the differentials of [225] relatively to t, divide them by dt, and

[2969a] substitute for P, Q, R, their values $\left(\frac{dV}{dx'}\right)$, $\left(\frac{dV}{dy'}\right)$, $\left(\frac{dV}{dz'}\right)$, [2967], we shall get [2968—2970].

Now by observing that we have,*

$$x'.\left(\frac{dV}{dy'}\right)-y'.\left(\frac{dV}{dx'}\right)=y.\left(\frac{dV}{dx}\right)-x.\left(\frac{dV}{dy}\right); \qquad [2971]$$

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* (2028) Multiplying the first of the equations [2967k] by $-y'$, the second by x', and adding the products, we get,

$$x'.\left(\frac{dV}{dy'}\right)-y'.\left(\frac{dV}{dx'}\right)=L.\left\{\frac{xy'-yx'}{r_{,}^{3}}+\frac{x'.(y-y')-y'.(x-x')}{f^3}\right\}$$

$$=L.\left\{\frac{yx'-xy'}{f^3}-\frac{(yx'-xy')}{r_{,}^{3}}\right\}=L.(yx'-xy').\left\{\frac{1}{f^3}-\frac{1}{r_{,}^{3}}\right\}; \qquad [2971a]$$

which represents the value of the first member of [2971]. We shall now investigate its second member; and in the first place we shall show that the terms depending on the differentials of $r_{,}^{3}$ [2966] vanish. For if we put for brevity $-L.(xx'+yy'+zz')=m$, and notice only the terms depending on the differential of $r_{,}^{3}$, neglecting for a moment the variableness of m, we shall have $V=-m\,r_{,}^{-3}$, or more generally $V=-m\,r_{,}^{n}$; [2971b]
whence $\left(\frac{dV}{dx}\right)=-nm.r_{,}^{n-1}.\left(\frac{dr_{,}}{dx}\right)=-nm.r_{,}^{n-1}.\frac{x}{r_{,}}=-nm.r_{,}^{n-2}.x$ [2965];
and in like manner $\left(\frac{dV}{dy}\right)=-nm.r_{,}^{n-2}y$; consequently

$$y.\left(\frac{dV}{dx}\right)-x.\left(\frac{dV}{dy}\right)=nm.r_{,}^{n-2}.(-xy+xy)=0. \qquad [2971c]$$

Therefore in finding the value of the second member of [2971], by means of V [2966], we may consider $r_{,}^{n}$, or $r_{,}^{-3}$, as a constant quantity, as was done in finding the first member in [2971a]. We may now observe, that if we change x, x', y, y', and their differentials, into $-x'$, $-x$, $-y'$, $-y$, and their differentials respectively, leaving $r_{,}$ unaltered, the expression V will remain identically the same. We may therefore make the same changes in [2971a], which was deduced from [2966] by the common processes of differentiation, and it will become, [2971c'] [2971d]

$$x.\left(\frac{dV}{dy}\right)-y.\left(\frac{dV}{dx}\right)=L.(y'x-x'y).\left\{\frac{1}{f^3}-\frac{1}{r_{,}^{3}}\right\}, \quad \text{or} \qquad [2971e]$$

$$y.\left(\frac{dV}{dx}\right)-x.\left(\frac{dV}{dy}\right)=L.(yx'-xy').\left\{\frac{1}{f^3}-\frac{1}{r_{,}^{3}}\right\}; \qquad [2971f]$$

which is exactly the same as the second member of [2971a]. Hence we get the equations [2971h]. If we change in this y, y', z, z', into z, z', y, y', respectively, which does not alter the value of V [2966], we get [2971i]. Changing in this last expression x, x', y, y', into y, y', x, x', respectively, we get [2971k], [2971g]

we shall obtain,

Values of dN, [2972]
$$\frac{dN}{dt} = S.dm.\left\{y.\left(\frac{dV}{dx}\right) - x.\left(\frac{dV}{dy}\right)\right\};$$

dN', [2973]
$$\frac{dN'}{dt} = S.dm.\left\{z.\left(\frac{dV}{dx}\right) - x.\left(\frac{dV}{dz}\right)\right\};$$

dN''. [2974]
$$\frac{dN''}{dt} = S.dm.\left\{z.\left(\frac{dV}{dy}\right) - y.\left(\frac{dV}{dz}\right)\right\}.$$

Second form.

[2974′] The co-ordinates x', y', z', being supposed very small in comparison with the distance $r_{\prime}$ of the body L from the centre of gravity of the earth, we may develop V, in a very converging series, according to the inverse powers of $r_{\prime}$. Hence we shall have very nearly,*

[2971h] $$x'.\left(\frac{dV}{dy'}\right) - y'.\left(\frac{dV}{dx'}\right) = y.\left(\frac{dV}{dx}\right) - x.\left(\frac{dV}{dy}\right) = L.(yx' - xy').\left\{\frac{1}{f^3} - \frac{1}{r_{\prime}^3}\right\};$$

[2971i] $$x'.\left(\frac{dV}{dz'}\right) - z'.\left(\frac{dV}{dx'}\right) = z.\left(\frac{dV}{dx}\right) - x.\left(\frac{dV}{dz}\right) = L.(zx' - xz').\left\{\frac{1}{f^3} - \frac{1}{r_{\prime}^3}\right\};$$

[2971k] $$y'.\left(\frac{dV}{dz'}\right) - z'.\left(\frac{dV}{dy'}\right) = z.\left(\frac{dV}{dy}\right) - y.\left(\frac{dV}{dz}\right) = L.(zy' - yz').\left\{\frac{1}{f^3} - \frac{1}{r_{\prime}^3}\right\}.$$

Substituting the second forms of these expressions in [2968—2970], we get [2972—2974]; and if the last forms [2971h—k] are used, we obtain the formulas [2971m—o]; in which

[2971l] $r_{\prime}^2 = x^2 + y^2 + z^2$, and $f^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$ [2967d].

[2971m] $$\frac{dN}{dt} = L.S.dm.(yx' - xy').\left\{\frac{1}{f^3} - \frac{1}{r_{\prime}^3}\right\};$$

[2971n] $$\frac{dN'}{dt} = L.S.dm.(zx' - xz').\left\{\frac{1}{f^3} - \frac{1}{r_{\prime}^3}\right\};$$

[2971o] $$\frac{dN''}{dt} = L.S.dm.(zy' - yz').\left\{\frac{1}{f^3} - \frac{1}{r_{\prime}^3}\right\}.$$

* (2029) Putting for brevity $xx' + yy' + zz' = r_{\prime}^2 n$, and neglecting the squares of x', y', z', n, we get from [2967d],

[2976a] $$f^2 = x^2 + y^2 + z^2 - 2.(xx' + yy' + zz') = r_{\prime}^2 - 2r_{\prime}^2 n = r_{\prime}^2.(1-2n);$$

[2976b] whence $f^{-3} = r_{\prime}^{-3}.(1+3n)$, and

Third form.

$$\frac{dN}{dt}=\frac{3L}{r_{,}^{5}}\cdot S\cdot dm\cdot(xx'+yy'+zz')\cdot(yx'-xy');$$ [2975]

$$\frac{dN'}{dt}=\frac{3L}{r_{,}^{5}}\cdot S\cdot dm\cdot(xx'+yy'+zz')\cdot(zx'-xz');$$ [2976]

$$\frac{dN''}{dt}=\frac{3L}{r_{,}^{5}}\cdot S\cdot dm\cdot(xx'+yy'+zz')\cdot(zy'-yz').$$ [2977]

We have seen in [260iv, &c.], that the values of p, q, r, are independent of the position of the plane of xy,* and if we take the equator of the earth [2977′]

$$\frac{1}{f^3}-\frac{1}{r_{,}^{3}}=\frac{1}{r_{,}^{3}}\cdot(1+3n)-\frac{1}{r_{,}^{3}}=3n\cdot\frac{1}{r_{,}^{3}}=3\cdot\frac{xx'+yy'+zz'}{r_{,}^{5}}.$$ [2976b]

Substituting this in [2971m—o], we obtain [2975—2977] respectively.

* (2030) The quantities p, q, r, determine the position of the *momentary* axis of rotation, relatively to the three principal axes x'', y'', z''. For by [259], the cosines of the angles formed by the *momentary* axis of rotation and the three principal axes x'', y'', z'', are represented respectively by [2977a]

$$\frac{q}{\surd(p^2+q^2+r^2)},\qquad \frac{r}{\surd(p^2+q^2+r^2)},\qquad \frac{p}{\surd(p^2+q^2+r^2)};$$ [2977b]

and it is very evident that these quantities, which have at any moment determinate values, must be wholly independent of the *arbitrary* situation of the plane of xy. [2977c]

The same result may be obtained from the consideration that p, q, r, *represent the angular velocities of rotation about the three principal axes*, as we shall soon see [2977g, &c.], and these velocities are evidently independent of the arbitrary situation of the plane xy. If we accent the letters in the formulas [231b, c], to prevent confusion in the use of the symbols, we shall find that the three angular motions $d\varphi'$, $d\psi'$, $d\omega'$, about the three axes z'', x'', y'', respectively, are equivalent to one angular motion [2977d]

The three angular velocities p, q, r, about the principal axes z'', x'', y'', are equivalent to one angular velocity s, about the momentary axis.

[2977e]

$$d\vartheta'=\surd(d\varphi'^2+d\psi'^2+d\omega'^2),$$

about a momentary axis, which forms with the axes z'', x'', y'', angles whose cosines are represented by $\frac{d\varphi'}{d\vartheta'}$, $\frac{d\psi'}{d\vartheta'}$, $\frac{d\omega'}{d\vartheta'}$, respectively. Now if we put $d\varphi'=p\,dt$, [2977f] $d\psi'=q\,dt$, $d\omega'=r\,dt$, and for brevity $d\vartheta'=dt\cdot\surd(p^2+q^2+r^2)=s\,dt$, we [2977g] shall find, that the *three angular velocities* p, q, r, *about the principal axes* z'', x'', y'',

[2978] for this plane, we shall have $\theta = 0$.* Moreover, if we take the first
[2979] principal axis for the axis of x, we shall have $\varphi = 0$.† Lastly we have, as
Momenta of inertia. in [229, 228],‡

[2980] $$S.dm.(y'^2+z'^2)=A, \qquad S.dm.(x'^2+z'^2)=B, \qquad S.dm.(x'^2+y'^2)=C;$$

[2981] $$S.x'y'.dm=0, \qquad S.x'z'.dm=0, \qquad S.y'z'.dm=0;$$

Fourth form of dN, therefore,§

[2982] $$\frac{dN}{dt}=\frac{3L}{r_{\prime}^5}.(B-A).xy;$$

dN',

[2983] $$\frac{dN'}{dt}=\frac{3L}{r_{\prime}^5}.(C-A).xz;$$

dN''.

[2984] $$\frac{dN''}{dt}=\frac{3L}{r_{\prime}^5}.(C-B).yz.$$

[2977h] *respectively, are equivalent to the single angular velocity* $s=\sqrt{(p^2+q^2+r^2)}$, *about the*
[2977i] *momentary axis of rotation; and that this momentary axis forms with the three principal*
[2977k] *axes* z'', x'', y'', *angles whose cosines are represented by* $\frac{p}{s}$, $\frac{q}{s}$, $\frac{r}{s}$, respectively.

[2978a] * (2031) This is evident, because the inclination of the equator to the plane of xy is equal to θ [2907g].

[2979a] † (2032) The first principal axis is CG [2907e], fig. 58, page 803; and CA is the axis of x [2907b]. When the axes CA, CB, CG, coincide, we shall have the angle $BCG=\varphi=0$ [2907f].

‡ (2033) Having supposed, in the preceding note, the axes CA, CG, to coincide,
[2980a] also the axes CH, CD; it is evident that the quantities x'', y'', z'', become x', y', z', respectively. Making these changes in [229, 228], we get [2980, 2981].

§ (2034) If we multiply the two factors of the expression [2975], we obtain,

[2984a] $$(xx'+yy'+zz').(yx'-xy')=xy.(x'^2-y'^2)+(y^2-x^2).x'y'+yz.x'z'-xz.y'z';$$

substituting this in the terms under the sign S in [2975], and bringing the terms x, y, z, from under that sign, because it affects only the quantities x', y', z', dm; the part under this sign in [2975] becomes,

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thus the equations [2905—2907], become,*

$$dp+\frac{(B-A)}{C}.qr.dt=\frac{3L.dt}{r_{,}^{5}}.\frac{(B-A)}{C}.xy;$$ [2985]

$$dq+\frac{(C-B)}{A}.rp.dt=\frac{3L.dt}{r_{,}^{5}}.\frac{(C-B)}{A}.yz; \qquad (F)$$ [2986]

$$dr+\frac{(A-C)}{B}.pq.dt=\frac{3L.dt}{r_{,}^{5}}.\frac{(A-C)}{B}.xz.$$ [2987]

Second form.

In these equations it is supposed that $r_{,}$ *is very great in comparison with the radius of the earth. This is the case relatively to the sun and moon; but it is remarkable that the equations [2985—2987] are very nearly correct, even when the attracting body is quite nigh to the earth, supposing the earth to be elliptical.* To prove this we shall observe that we have by [2907q, 2980a], [2987′]

The preceding equations are rendered more correct by means of the elliptical form of the earth.

$$x'=R.\sqrt{1-\mu^2}.\cos.\varpi, \qquad y'=R.\sqrt{1-\mu^2}.\sin.\varpi, \qquad z'=R.\mu.$$ [2988]

$xy.S.dm.(x'^2-y'^2)+(y^2-x^2).S.dm.x'y'+yz.S.dm.x'z'-xz.S.dm.y'z'$; [2984b]

which, by means of the equations [2981], is reduced to

$$xy.S.dm.(x'^2-y'^2).$$ [2984c]

Now $S.dm.(x'^2-y'^2)$ is evidently equal to the difference of the values of B, A, or $B-A$ [2980]; hence we have,

$$S.dm.(xx'+yy'+zz').(yx'-xy')=xy.(B-A).$$ [2984d]

Substituting this in [2975], we get [2982]; and in like manner we may deduce [2983, 2984] from [2976, 2977]. We may also obtain [2983, 2984] from [2982], by the method of derivation used in the preceding notes. For if we change y, z, y', z', into z, y, z', y', respectively, in [2975], it becomes as in [2976]; and by these changes A, B, [2980] become A, C, respectively; consequently [2982], which was derived from [2975], becomes as in [2983]. In like manner if we change x, y, x', y', into y, x, y', x', respectively, [2976] becomes as in [2977], and C, A, change into C, B; consequently [2983] becomes as in [2984]. [2984e] [2984f]

* (2035) Substituting in [2905, 2906, 2907], the values $\theta=0$, $\varphi=0$, [2978, 2979], [2985a] the second members of these equations become $\frac{dN}{C}$, $\frac{dN''}{A}$, $-\frac{dN'}{B}$, respectively; [2985b] and by using [2982—2984], we get the equations [2985—2987].

[2989] If we put ν and λ, for what the quantities μ, ϖ, corresponding to the particle dm, become relatively to the body L, we shall have,

Co-ordinates of the attracting body.

[2990] $$x = r_{\prime} \cdot \sqrt{1-\nu^2} \cdot \cos. \lambda, \qquad y = r_{\prime} \cdot \sqrt{1-\nu^2} \cdot \sin. \lambda, \qquad z = r_{\prime} \cdot \nu.$$

If we substitute these values in the function V [2966], and then develop it according to the powers of $\frac{R}{r_{\prime}}$, we shall have a series of this form,*

[2991] $$V = \frac{L}{r_{\prime}} + \frac{L.R^2}{r_{\prime}^3} \cdot U^{(2)} + \frac{L.R^3}{r_{\prime}^4} \cdot U^{(3)} + \&c. ;$$

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* (2036) If we substitute the values [2988, 2990] in the first member of [2990b], and reduce by means of [24] Int., we get [2990c]; substituting this and $r_{\prime}^2 = x^2 + y^2 + z^2$ [2965], $R^2 = x'^2 + y'^2 + z'^2$ [2988], in [2990e], we get [2990f], using for brevity

[2990a] $\delta = \sqrt{(1-\mu^2)} \cdot \sqrt{(1-\nu^2)} \cdot \cos. (\varpi - \lambda) + \mu \nu,$

[2990b] $$x x' + y y' + z z' = R r_{\prime} \cdot \{ \sqrt{(1-\mu^2)} \cdot \sqrt{(1-\nu^2)} \cdot [\cos. \varpi \cdot \cos. \lambda + \sin. \varpi \cdot \sin. \lambda] + \mu \nu \}$$

[2990c] $$= R r_{\prime} \cdot \{ \sqrt{(1-\mu^2)} \cdot \sqrt{(1-\nu^2)} \cdot \cos. (\varpi - \lambda) + \mu \nu \}$$

[2990d] $$= R r_{\prime} \cdot \delta ;$$

[2990e] $$(x-x')^2 + (y-y')^2 + (z-z')^2 = (x^2+y^2+z^2) - 2 \cdot (x x' + y y' + z z') + (x'^2 + y'^2 + z'^2)$$

[2990f] $$= r_{\prime}^2 - 2 r_{\prime} R \cdot \delta + R^2.$$

[2990g] Substituting [2990d, f] in [2966], we get $V = -\frac{LR}{r_{\prime}^2} \cdot \delta + \frac{L}{(r_{\prime}^2 - 2 r_{\prime} R \cdot \delta + R^2)^{\frac{1}{2}}}$.

[2990h] The last term of this expression is of the form [1626], changing s, r, ψ, S, $\cos. v$, $\cos. \theta$, into $r_{\prime}$, R, λ, L, μ, ν, respectively, by which means δ [1629] becomes as in [2990a]; and this term of [2990g] can be transformed into a series like [1627], equal to

$$\frac{L}{r_{\prime}} \cdot \left\{ P^{(0)} + \frac{R}{r_{\prime}} \cdot P^{(1)} + \frac{R^2}{r_{\prime}^2} \cdot P^{(2)} + \frac{R^3}{r_{\prime}^3} \cdot P^{(3)} + \&c. \right\}$$

[2990i] $$= \frac{L}{r_{\prime}} \cdot \left\{ 1 + \frac{R}{r_{\prime}} \cdot \delta + \frac{R^2}{r_{\prime}^2} \cdot P^{(2)} + \frac{R^3}{r_{\prime}^3} \cdot P^{(3)} + \&c. \right\} \qquad [1625f];$$

by which means the whole expression of V [2990g] becomes,

[2990k] $$V = \frac{L}{r_{\prime}} \cdot \left\{ 1 + \frac{R^2}{r_{\prime}^2} \cdot P^{(2)} + \frac{R^3}{r_{\prime}^3} \cdot P^{(3)} + \&c. \right\}.$$

Putting $P^{(i)} = U^{(i)}$, it becomes as in [2991], and the differential equation [1630] changes into [2992].

and it is easy to prove, as in [1630, &c], that the functions $U^{(2)}$, $U^{(3)}$, &c., are such that we have generally,

$$0=\left\{\frac{d\cdot\left\{(1-\mu^2)\cdot\left(\frac{dU^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddU^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i\cdot(i+1)\cdot U^{(i)}. \qquad [2992]$$

We shall now resume the equation [2972],

$$\frac{dN}{dt}=S\cdot dm\cdot\left\{y\cdot\left(\frac{dV}{dx}\right)-x\cdot\left(\frac{dV}{dy}\right)\right\}; \qquad [2993]$$

and we shall have,*

$$\begin{aligned} y\cdot\left(\frac{dV}{dx}\right)-x\cdot\left(\frac{dV}{dy}\right)=&\frac{L.R^2}{r_{\prime}^3}\cdot\left\{y\cdot\left(\frac{dU^{(2)}}{dx}\right)-x\cdot\left(\frac{dU^{(2)}}{dy}\right)\right\}\\ &+\frac{L.R^3}{r_{\prime}^4}\cdot\left\{y\cdot\left(\frac{dU^{(3)}}{dx}\right)-x\cdot\left(\frac{dU^{(3)}}{dy}\right)\right\}\\ &+\&c. \end{aligned} \qquad [2994]$$

The partial differentials of the second member of this equation are taken [2994'] relatively to variable quantities which are independent of μ and ϖ.† If we denote generally by $U^{(i)}$ the following function,

$$y\cdot\left(\frac{dU^{(i)}}{dx}\right)-x\cdot\left(\frac{dU^{(i)}}{dy}\right)=U'^{(i)}, \qquad [2995]$$

we shall have,

$$0=\left\{\frac{d\cdot\left\{(1-\mu^2)\cdot\left(\frac{dU'^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddU'^{(i)}}{d\varpi^2}\right)}{1-\mu^2}+i\cdot(i+1)\cdot U'^{(i)}; \qquad [2996]$$

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* (2037) In substituting V [2991] in the first member of [2994], we may consider $r_{\prime}^n$ as constant [2971c']; and $U^{(2)}$, $U^{(3)}$, $U^{(4)}$, &c., as the variable quantities; by which [2994a] means the expression evidently becomes as in the second member of [2994].

† (2038) The quantities x, y, z, depend on the situation of the body L [2964], and are wholly independent of μ, ϖ, which define the situation of the particle dm [2965]. Hence [2995a] if $U^{(i)}$ be a function of μ, ϖ, x, y, which satisfies the equation [2992], it will also satisfy it when multiplied by x or y, or by any function of x, y; or when its partial differential is [2995b] taken relatively to x or y. Therefore the expression $U'^{(i)}$ [2995] will satisfy the equation [2995c] [2996], which is similar to [2992].

so that this function $U'^{(i)}$ is of the same nature as $Y^{(i)}$, or $U^{(i)}$ [2995c]. The preceding expression of $\frac{dN}{dt}$ [2993] becomes, by what we have seen
[2997] in § 2, and by substituting for dm its value [2918] $R^2 dR \,.\, d\mu \,.\, d\varpi$, also for R its value $a + \alpha a \,.\, \{Y^{(1)} + Y^{(2)} + \&c.\}$ [2942],*

[2998]
$$\frac{dN}{dt} = \frac{\alpha L}{r_{\prime}^{3}} \,.\, S \,.\, \rho \,.\, d \,.\, (a^5 \,.\, Y^{(2)}) \,.\, U'^{(2)} \,.\, d\mu \,.\, d\varpi$$
$$+ \frac{\alpha L}{r_{\prime}^{4}} \,.\, S \,.\, \rho \,.\, d \,.\, (a^6 \,.\, Y^{(3)}) \,.\, U'^{(3)} \,.\, d\mu \,.\, d\varpi$$
$$+ \&c.;$$

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* (2039) Substituting $U'^{(i)}$ [2995] in [2994], and the resulting expression in [2993], using the characteristic Σ of finite integrals, we get [2997a]. This is reduced to the form [2997b], by the substitution of dm [2997], and integrating relatively to R;

[2997a] $$\frac{dN}{dt} = S.dm.\left\{\frac{L.R^2}{r_{\prime}^{3}} \,.\, U'^{(2)} + \frac{L.R^3}{r_{\prime}^{4}} \,.\, U'^{(3)} + \&c.\right\} = \Sigma \,.\, S \,.\, dm \,.\, \frac{L.R^{(i)}}{r_{\prime}^{i+1}} \,.\, U'^{(i)}$$

[2997b] $$= \Sigma \,.\, S \,.\, R^{i+2} dR \,.\, d\mu \,.\, d\varpi \,.\, \frac{L}{r_{\prime}^{i+1}} \,.\, U'^{(i)} = \Sigma \,.\, S \,.\, \frac{R^{i+3}}{i+3} \,.\, d\mu \,.\, d\varpi \,.\, \frac{L}{r_{\prime}^{i+1}} \,.\, U'^{(i)}.$$

[2997c] From [2942] we have $R^{i+3} = a^{i+3} + (i+3) \,.\, a^{i+3} \,.\, \alpha \,.\, \{Y^{(1)} + Y^{(2)} + Y^{(3)} + \&c.\}$; and if we substitute this in [2997b], we may neglect all the factors of $U'^{(i)}$, except those of the form $Y^{(i)}$, as is evident from [1476, 1476a]; and then it becomes

[2997d] $$\frac{dN}{dt} = \Sigma \,.\, S \,.\, a^{i+3} \,.\, \alpha\, Y^{(i)} \,.\, d\mu \,.\, d\varpi \,.\, \frac{L}{r_{\prime}^{i+1}} \,.\, U'^{(i)};$$

or as it may be written,

[2997e] $$\frac{dN}{dt} = \Sigma \,.\, \frac{\alpha L}{r_{\prime}^{i+1}} \,.\, S \,.\, (a^{i+3} \,.\, Y^{(i)}) \,.\, U'^{(i)} \,.\, d\mu \,.\, d\varpi;$$

in which $P^{(i)}$ or $U^{(i)}$ [2990i, k, &c.], and $U'^{(i)}$ [2995], are independent of R or a.
[2997f] Hence if the earth be composed of strata of different densities, the density being ρ throughout the whole stratum corresponding to the radius a, the term $(a^{i+3} \,.\, Y^{(i)})$, in the preceding expression [2997e], will become $S \,.\, \rho \,.\, d \,.\, (a^{i+3} \,.\, Y^{(i)})$; as is evident by the process
[2997g] used in [1503‴, &c.], the differential and integral being taken relatively to the quantity a. Substituting this in [2997e], we finally get, as in [2998],

[2997h] $$\frac{dN}{dt} = \Sigma \,.\, \frac{\alpha L}{r_{\prime}^{i+1}} \,.\, S \,.\, \rho \,.\, d \,.\, (a^{i+3} \,.\, Y^{(i)}) \,.\, U'^{(i)} \,.\, d\mu \,.\, d\varpi;$$

[2997i] observing that the least value of i is $i = 2$, as is evident from [2991, 2998].

the differentials $d\,.\,(a^5\,.\,Y^{(2)})$, $d\,.\,(a^6\,.\,Y^{(3)})$, &c., *referring to the variable quantity* a. Now the equation [1705] gives generally, at the surface of the earth, when i exceeds 2,*

$$S_0^1\,.\,\rho\,.\,d\,.\,(a^{i+3}\,.\,Y^{(i)}) = \left(\frac{2\,i+1}{3}\right).\,Y^{(i)}\,.\,S_0^1\,.\,\rho\,.\,d\,.\,a^3\,; \qquad [2999]$$

the integrals being taken from $a=0$ to $a=1$; and $Y^{(i)}$, in the second member of the equation, corresponds to the surface of the earth. Therefore we have, [3000]

$$\frac{\alpha L}{r_{\prime}^{4}}\,.\,S\,.\,\rho\,.\,d\,.\,(a^6\,.\,Y^{(3)})\,.\,U'^{(3)}\,.\,d\mu\,.\,d\varpi = \frac{7\,\alpha L}{3\,r_{\prime}^{4}}.S.Y^{(3)}.U'^{(3)}.d\mu.d\varpi.S.\rho.d.a^3. \qquad [3001]$$

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* (2040) If we notice only the permanent figure of the earth, as in [2951b—c], we may neglect S, S', &c., in [1632]; and then we shall have generally $Z^{(i)}=0$, when $i>2$. [2999a]
Substituting this in [1705], and neglecting the first integral, whose limits at the surface of the earth are $a=1$, $a=1$, [2951a], we get, by changing the sign f into S, as in [2907t],

$$0 = -\frac{4\,\pi}{3\,a}\,.\,Y^{(i)}\,.\,S_0^1\,.\,\rho\,.\,d\,.\,a^3 + \frac{4\,\pi}{(2\,i+1)\,.\,a^{i+1}}\,.\,S_0^1\,.\,\rho\,.\,d\,.\,(a^{i+3}\,.\,Y^{(i)}). \qquad [2999b]$$

The value $a=1$, corresponding to the surface, is to be substituted in the factors free from the sign of integration; then multiplying by $\frac{2i+1}{4\pi}$, we get [2999], in which $Y^{(i)}$, not included under the sign of integration, is its value at the surface, where $a=1$. Substituting [2999] in [2997h], we obtain

$$\frac{dN}{dt} = \Sigma\,.\,\frac{\alpha L}{r_{\prime}^{i+1}}\,.\,\frac{2i+1}{3}\,.\,S\,.\,Y^{(i)}\,.\,U'^{(i)}\,.\,d\mu\,.\,d\varpi\,.\,S_0^1\,.\,\rho\,.\,d\,.\,a^3\,; \qquad [2999c]$$

in which the integral $S.Y^{(i)}.U'^{(i)}.d\mu.d\varpi$ refers to the variable quantities μ, ϖ, and not to a. [2999d]
Now if the surface of the earth be elliptical, we shall have $Y^{(3)}=0$, $Y^{(4)}=0$, &c., [1502]; consequently the corresponding terms of [2999c] will vanish, and the whole expression will be reduced to its first term depending on $Y^{(2)}$, or $i=2$, which is of the order $\frac{1}{r_{\prime}^{3}}$. This term is the same as that retained in the second member of [2999e]
[2982], or [2985], because it is of the order $\frac{xy}{r_{\prime}^{5}}$, or $\frac{1}{r_{\prime}^{3}}$ [2990]. What we have [2999f]
here proved relatively to [2985] is also true for the equations [2986, 2987], as is evident from the consideration that they may be derived from each other, by changing y, y', into z, z', &c., as in [2967i, &c.]. [2999g]

If the figure of the earth be that of an ellipsoid, $Y^{(3)}=0$ [1502′, 1503*a*],

[3002] *and then the expression of* $\frac{dN}{dt}$ [2998] *is reduced to its first term; not only on account of the magnitude of* $r_{,}$, *but because the values of* $Y^{(3)}$, $Y^{(4)}$, &c. *vanish.* Although the elliptical figure does not exactly satisfy the measured

[3002′] degrees of the meridian, yet the agreement of the variations of gravity with that figure indicates that $Y^{(3)}$, $Y^{(4)}$, &c., are very small in comparison with $Y^{(2)}$; *we may therefore calculate the motions of the axis of the earth,*

[3003] *upon the supposition that it is an elliptical figure, without any sensible error.*

4. *We shall now refer the co-ordinates of the body* L *to a fixed plane, which we shall suppose to be that of the ecliptic at a given epoch;* X, Y, Z,

Axes X, Y, Z. being these new co-ordinates.

[3004] *The axis of* X is the line drawn from the centre of the earth to the moveable vernal equinox.

[3004′] *The axis of* Y is the line drawn from the centre of the earth to the moveable first point of the sign Cancer.

[3005] *The axis of* Z is that drawn from the same centre to the northern pole of the ecliptic.*

Co-ordinates of the attracting body.

We shall have, by [172, 173, 174],

[3006] $$x=X.\cos.\varphi+Y.\cos.\theta.\sin.\varphi-Z.\sin.\theta.\sin.\varphi;$$

[3007] $$y=Y.\cos.\theta.\cos.\varphi-X.\sin.\varphi-Z.\sin.\theta.\cos.\varphi;$$

[3008] $$z=Y.\sin.\theta+Z.\cos.\theta.$$

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* (2041) The plane of XY is supposed to be *fixed*, but the axes of X, Y, are

[3006*a*] moveable in that plane. The axis of X is the line drawn from the centre of the earth to the *moveable* vernal equinox; and the axis of Y is drawn through the same centre, perpendicularly to the axis of X, towards the *moveable* first point of Cancer. The axis of Z

[3006*b*] is drawn through the centre of the earth, perpendicularly to the plane of XY, towards the north pole of the fixed ecliptic. The axes x, y, are drawn through the centre of the earth,

[3006*c*] in the plane of the equator; the axis z is drawn through the same centre, perpendicularly to the plane xy, towards the north pole of the equator. For illustration, we may refer to the

[3006*d*] figure in page 112 of Vol. I; where the co-ordinates $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, x, y, z, correspond in the present notation to x, y, z, X, Y, Z, respectively. Making these changes in

[3006*e*] [172, 173, 174], and putting $\psi=0$, we get [3006, 3007, 3008] respectively. These

General equations of the motions of a solid body.

The differential equations [2985—2987] then become,*

$$dp+\frac{(B-A)}{C}.qr.dt=\frac{3L.dt.(B-A)}{2r_{\prime}^{5}.C}.\left\{\begin{array}{l}\{Y^{(2)}.\cos.^2\theta+Z^2.\sin.^2\theta-X^2-2YZ.\sin.\theta.\cos.\theta\}.\sin.2\varphi\\+\{2XY.\cos.\theta-2XZ.\sin.\theta\}.\cos.2\varphi\end{array}\right\}; \qquad [3009]$$

$$dq+\frac{(C-B)}{A}.rp.dt=\frac{3L.dt.(C-B)}{r_{\prime}^{5}.A}.\left\{\begin{array}{l}\{(Y^2-Z^2).\sin.\theta.\cos.\theta+YZ.(\cos.^2\theta-\sin.^2\theta)\}.\cos.\varphi\\-\{XY.\sin.\theta+XZ.\cos.\theta\}.\sin.\varphi\end{array}\right\}; \quad (G) \qquad [3010]$$

$$dr+\frac{(A-C)}{B}.pq.dt=\frac{3L.dt.(A-C)}{r_{\prime}^{5}.B}.\left\{\begin{array}{l}\{XY.\sin.\theta+XZ.\cos.\theta\}.\cos.\varphi\\+\{(Y^2-Z^2).\sin.\theta.\cos.\theta+YZ.(\cos.^2\theta-\sin.^2\theta)\}.\sin.\varphi\end{array}\right\}. \qquad [3011]$$

Third form.

We shall now integrate these equations. If the two momenta of inertia A and

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expressions are much abridged by the supposition of $\psi=0$, and we are enabled by this means to obtain more simple expressions of $d\psi$ [3036, 3041, 3090, &c.], than could otherwise be obtained. [3006f] This value of $d\psi$ represents the motion of the *moveable* equinox, [3006g] or axis CB, upon the *fixed* plane XY, in the time dt; its integral [3100] gives the value of ψ, or the motion of the equinoxes, upon the fixed plane XY, in the time t; [3006h] counting this motion from a point of this fixed plane corresponding to the situation of the equinox at the period taken for the epoch.

* (2042) Multiplying x, y, [3006, 3007], we get [3009b]; reducing it by means of [31, 32] Int., we find [3009c]; substituting this in [2985], we obtain [3009]. In like [3009a] manner, the product of y, z, [3007, 3008], is as in [3009d]; and by substitution in [2986], we get [3010]. Lastly, the product of x, z, [3006, 3008], is of the form [3009e]; hence [2987] becomes as in [3011].

$$xy=\{Y^2.\cos.^2\theta+Z^2.\sin.^2\theta-X^2-2YZ.\sin.\theta.\cos.\theta\}.\sin.\varphi.\cos.\varphi \qquad [3009b]$$
$$+(XY.\cos.\theta-XZ.\sin.\theta).(\cos.^2\varphi-\sin.^2\varphi)$$

$$=\{Y^2.\cos.^2\theta+Z^2.\sin.^2\theta-X^2-2YZ.\sin.\theta.\cos.\theta\}.\tfrac{1}{2}.\sin.2\varphi \qquad [3009c]$$
$$+(XY.\cos.\theta-XZ.\sin.\theta).\cos.2\varphi;$$

$$yz=\{(Y^2-Z^2).\sin.\theta.\cos.\theta+YZ.(\cos.^2\theta-\sin.^2\theta)\}.\cos.\varphi \qquad [3009d]$$
$$-\{XY.\sin.\theta+XZ.\cos.\theta\}.\sin.\varphi;$$

$$xz=\{(XY.\sin.\theta+XZ.\cos.\theta).\cos.\varphi+\{(Y^2-Z^2).\sin.\theta.\cos.\theta+YZ.(\cos.^2\theta-\sin.^2\theta)\}.\sin.\varphi. \qquad [3009e]$$

The equations [3009, 3011] are given, in Vol. V, [12180—12180″], in a general form, according to the method of Mr. Poisson; and the author has examined more minutely, in [3009f] that volume, several very small equations, which are neglected in this book.

[3011'] B are equal, which is the case if the earth be a spheroid of revolution,* the
[3012] first of these equations becomes $dp=0$, consequently $p=$ constant. If there be a small difference between these two momenta of inertia, the
[3012'] value of p will contain periodical equations; but they are insensible. For the momentary axis of rotation being but very little distant from the third principal axis, q and r are very small quantities;† and we can, without
[3013] sensible error, neglect the term $\frac{B-A}{C}\cdot rq.dt$ [3009]. The second member of the same equation may be developed in a series of sines and

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[3012a] * (2043) If the earth be composed of spheroidal strata of revolution, about the axis NS, fig. 58, page 803, its form will evidently be symmetrical relatively to the other two axes CG, CH; consequently the momenta of inertia about these two axes must be equal. The same result may also be obtained from the expressions of A, B, [2960, 2961]. For
[3012b] the earth being a spheroid of revolution, the radius R, for any given stratum, and for a given value of μ, must be the same for all values of ϖ; or in other words, R [2942] and $Y^{(2)}$
[3012c] [2958] must be independent of ϖ; therefore h'''' must vanish from [2958]. Substituting this value $h''''=0$, in [2960, 2961], they become equal, or $A=B$; hence we get,
[3012d] from [3009], $dp=0$, whose integral is $p=$ constant, as in [3012].

† (2044) The momentary axis of the earth is supposed to be very near to the *third*
[3013a] principal axis [2913]; hence it follows, that this momentary axis must be inclined to the first, or to the second axis, by an angle which is nearly equal to a right angle, consequently its cosine must be very small. Now these cosines being represented by $\frac{q}{\sqrt{(p^2+q^2+r^2)}}$,
[3013b] $\frac{r}{\sqrt{(p^2+q^2+r^2)}}$ [2977b]; q and r must be very small in comparison with p. Moreover, $\frac{B-A}{C}$ must be small, even when the earth is supposed not to be a spheroid of revolution,
[3013c] because it differs but little from a sphere; therefore $\frac{(B-A)}{C}\cdot qr$ must be extremely small, and may be neglected in [3009]. This is also evident, by referring to the values of p, q, r, deduced from astronomical observations. For $d\psi$ is of the same order as the precession of
[3013d] the equinoxes in the time dt, being about 50^s in a year. $d\theta$ is of the same order as the variation of the obliquity of the ecliptic, arising from the nutation, or from the secular decrement; the former being about $\pm 9^s$ in a semi-revolution of the moon's nodes, and the latter
[3013e] about half a sexagesimal second in a year; consequently $d\theta$ is much less than $d\psi$. Now $d\varphi$ is the diurnal motion of the earth in the time dt, being at the rate of about 360^d in
[3013f] one day, or $360^d\times 365$ in one year; therefore $d\varphi$ is to $d\psi$ in a ratio, which may be considered as of the same order as $360^d\times 365$ to 50^s, or as 9400000 to 1 nearly.

cosines of angles, varying with rapidity; since its terms are multiplied by the sine, or cosine, of 2φ; these terms must therefore be insensible [3014]
after integration.* We may suppose, in the two equations [3010, 3011],

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If we examine the first of the formulas [230], we shall evidently see that p is of the order $\frac{d\varphi}{dt}$; and from the two last of those formulas, q, r, are of the order $\frac{d\psi}{dt}$, $\frac{d\theta}{dt}$. Therefore [3013g]
q, r, may be considered as very small fractions of the order $\frac{1}{9400000}$, in comparison with p, and qr so extremely small that $qr\cdot\frac{B-A}{C}$ may be neglected in [3009]. [3013h]

* (2045) The values of X, Y, Z, which are computed in [3053—3055], are independent of the terms $\cos.2\varphi$, $\sin.2\varphi$, by which they are multiplied in [3009]; and [3015a]
it is evident, without any calculation, that this ought to be the case, because X, Y, Z, depend on the motion, or rather on the situation, of the attracting body L, whether it be the sun, or the moon; while the angle 2φ depends on the rotatory motion of the earth. This [3015b]
angle 2φ is nearly equal to $2nt$ [3024′], which varies rapidly, in comparison with the variations of X, Y, Z, θ; and we may suppose that these last quantities depend on the sines and cosines of angles of the form $it+\varepsilon$; i being very small in comparison with n. [3015c]
Then we may represent the second member of [3009] by terms depending on the angle $2.(nt+it+\varepsilon)$ [17—20] Int., making $dp=\Sigma.a.dt.\cos.2.(nt+it+\varepsilon)$; a [3015d]
being a very small coefficient, considered as a constant quantity, independent of t. The integral of this equation, relatively to t, gives $p=\Sigma.\frac{a}{2n+2i}.\sin.2.(nt+it+\varepsilon)$; in which the terms have the large divisor $2n+2i$, and are therefore not increased by [3015e]
integration. But they will be increased if dp contain terms of the form $a.dt.\cos.(it+\varepsilon)$; because the integration will introduce the very small divisor i, which may render such terms sensible. Now as no such small divisors are introduced, we may neglect the terms of the [3015f]
second member of [3009], and then we shall have $dp=0$, and $p=n$, as in [3015].

If we wish to carry on the approximation to a greater degree of accuracy, we may substitute in [3010, 3011] this value of $p=n$, and those of X, Y, Z, [3053, &c.]; and then [3015g]
by integration we shall obtain approximate values of q, r, to be substituted in the first member of [3009]. A new integration will give a corrected value of p; and by proceeding [3015h]
in this way we may obtain p, q, r, to any required degree of accuracy. Whoever wishes to examine this point more minutely, may consult an excellent paper of Mr. Poisson, in Vol. VII, page 199, of the "Mémoires de l' Académie Royale des Sciences de l' Institut de [3015i]
France." In this paper he demonstrates in a somewhat different manner the two formulas given by La Place [3120′, 3134]. He also proves that the integral $\int dt.\sqrt{(p^2+q^2+r^2)}$, [3015k]
or the angle described by each point of the earth, by its rotatory motion, increases in proportion to the time; *noticing terms of the second order of the disturbing forces*, which is [3015l]
necessary in this investigation, and neglecting the small equations of the motion, which are always insensible, as in [3133].

[3015] $p = n$; n being the mean angular rotatory velocity of the earth, about its third principal axis. *But as the discussion of the value of p is very important,*
[3015′] *because of its influence on the length of the day, we shall resume the subject, after having determined the values of q and r.*

We shall put for brevity,

P, P'.

[3016] $$P = \frac{3L}{r_{,}^{5}} \cdot \{(Y^2 - Z^2) \cdot \sin.\theta \cdot \cos.\theta + YZ \cdot (\cos.^2\theta - \sin.^2\theta)\};$$

[3017] $$P' = \frac{3L}{r_{,}^{5}} \cdot \{XY \cdot \sin.\theta + XZ \cdot \cos.\theta\};$$

General differential equations for q, r.

then the equations [3010, 3011] become*

[3018] $$dq + \frac{(C-B)}{A} \cdot rp \cdot dt = \frac{(C-B)}{A} \cdot dt \cdot \{P \cdot \cos.\varphi - P' \cdot \sin.\varphi\};$$

[3019] $$dr + \frac{(A-C)}{B} \cdot pq \cdot dt = \frac{(A-C)}{B} \cdot dt \cdot \{P' \cdot \cos.\varphi + P \cdot \sin.\varphi\}.$$

P and P' may be developed in sines and cosines of angles, increasing in
[3020] proportion to the time. If $k \cdot \cos.(it+\varepsilon)$ be any term of P, and $k' \cdot \sin.(it+\varepsilon)$ the corresponding term of P',† we shall have, by noticing only these terms,‡

[3017a] * (2046) Substituting [3016, 3017] in [3010, 3011], we get [3018, 3019].

† (2047) If a term of this form occur in P, but not in P', we shall have $k' = 0$; and
[3020a] in like manner we have $k = 0$, when the term vanishes from the expression of P, but not from P'.

[3021a] ‡ (2048) If we substitute $P = k \cdot \cos.(it+\varepsilon)$, $P' = k' \cdot \sin.(it+\varepsilon)$, [3020], in the first members of [3021b, d], and then reduce the expressions, by means of [17—20] Int., we finally obtain [3021c, e]. Substituting these in [3018, 3019], we get [3021, 3022], respectively:

[3021b] $$P \cdot \cos.\varphi - P' \cdot \sin.\varphi = k \cdot \cos.\varphi \cdot \cos.(it+\varepsilon) - k' \cdot \sin.\varphi \cdot \sin.(it+\varepsilon)$$

[3021c] $$= \tfrac{1}{2} \cdot (k+k') \cdot \cos.(\varphi + it + \varepsilon) + \tfrac{1}{2} \cdot (k-k') \cdot \cos.(\varphi - it - \varepsilon);$$

[3021d] $$P' \cdot \cos.\varphi + P \cdot \sin.\varphi = k' \cdot \cos.\varphi \cdot \sin.(it+\varepsilon) + k \cdot \sin.\varphi \cdot \cos.(it+\varepsilon)$$

[3021e] $$= \tfrac{1}{2} \cdot (k+k') \cdot \sin.(\varphi + it + \varepsilon) + \tfrac{1}{2} \cdot (k-k') \cdot \sin.(\varphi - it - \varepsilon).$$

$$dq+\frac{(C-B)}{A}.rp.dt=\frac{(C-B)}{2A}.dt.\{(k+k').\cos.(\varphi+it+\varepsilon)+(k-k').\cos.(\varphi-it-\varepsilon)\};$$ [3021]

$$dr+\frac{(A-C)}{B}.pq.dt=\frac{(A-C)}{2B}.dt.\{(k+k').\sin.(\varphi+it+\varepsilon)+(k-k').\sin.(\varphi-it-\varepsilon)\}.$$ [3022]

If we suppose in these equations,* General values of q, r.

$$q=M.\sin.(\varphi+it+\varepsilon)+N.\sin.(\varphi-it-\varepsilon);$$ [3023]

$$r=M'.\cos.(\varphi+it+\varepsilon)+N'.\cos.(\varphi-it-\varepsilon);$$ [3024]

we shall have, by observing that $d\varphi$ is nearly equal to $n\,dt$,† [3024']

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* (2049) The reason for assuming these forms of q, r, appears from a calculation similar to that in notes 174, 182, pages 184, 189, Vol. I. For by taking the differential of [3021], considering p as constant, and substituting the value of dr deduced from [3022], it may be put under the form $0=\frac{ddq}{dt^2}+a^2.q+Q$, a being a constant quantity, and Q a function of the form $m.\sin.(\varphi+it+\varepsilon)+m'.\sin.(\varphi-it-\varepsilon)$. This differential equation is of the same form as in [865], φ being equal to nt; and the value of q deduced from the solution [871], is evidently of the form [3023]. Substituting this value of q in [3021], we get r, of the form [3024]. [3023a] [3023b] [3023c]

† (2050) This value of $d\varphi$ evidently follows from the first of the equations [230], or from [3029], observing that $d\psi$ is extremely small in comparison with $d\varphi$ [3013f]. Now taking the differentials of [3023, 3024], and putting $d\varphi=n\,dt$ [3015, 3029], we get, by using, for brevity, $\varphi+it+\varepsilon=T$, $\varphi-it-\varepsilon=T'$, [3024a]

$$dq=dt.\{M.(n+i).\cos.T+N.(n-i).\cos.T'\};$$

$$dr=dt.\{-M'.(n+i).\sin.T-N'.(n-i).\sin.T'\}.$$ [3024b]

Substituting these values, and those of q, r, [3023, 3024], in [3021, 3022], multiplying by $\frac{2A}{dt}$, $\frac{2B}{dt}$, respectively, and then putting $p=n$, we obtain,

$$2A.\{M.(n+i).\cos.T+N.(n-i).\cos.T'\}+2.(C-B).n.\{M'.\cos.T+N'.\cos.T'\}$$
$$=(C-B).\{(k+k').\cos.T+(k-k').\cos.T'\};$$ [3024c]

$$2B.\{-M'.(n+i).\sin.T-N'.(n-i).\sin.T'\}+2.(A-C).n.\{M.\sin.T+N.\sin.T'\}$$
$$=(A-C).\{(k+k').\sin.T+(k-k').\sin.T'\}.$$ [3024d]

These equations must be identical; therefore the terms depending on the cosines and sines of T, T', must be separately equal to each other, in both members. Hence we get the four following equations,

[3025]
$$M = \frac{\left(\frac{k+k'}{2}\right).(C-B).\{n.(A+B-C)+iB\}}{(n+i)^2.AB-n^2.(A-C).(B-C)};$$

[3026]
$$M' = \frac{\left(\frac{k+k'}{2}\right).(C-A).\{n.(A+B-C)+iA\}}{(n+i)^2.AB-n^2.(A-C).(B-C)};$$

[3027]
$$N = \frac{\left(\frac{k-k'}{2}\right).(C-B).\{n.(A+B-C)-iB\}}{(n-i)^2.AB-n^2.(A-C).(B-C)};$$

[3028]
$$N' = \frac{\left(\frac{k-k'}{2}\right).(C-A).\{n.(A+B-C)-iA\}}{(n-i)^2.AB-n^2.(A-C).(B-C)}.$$

General equations to determine the angular velocities p, q, r, about the three principal axes.

We shall now resume the equations [230],

[3029]
$$d\varphi - d\psi.\cos.\theta = p\,dt;$$

[3030]
$$d\psi.\sin.\theta.\sin.\varphi - d\theta.\cos.\varphi = q\,dt;$$

[3031]
$$d\psi.\sin.\theta.\cos.\varphi + d\theta.\sin.\varphi = r\,dt.$$

[3024e]
$$2A.M.(n+i)+2.(C-B).n.M' = (C-B).(k+k');$$

[3024f]
$$-2B.M'.(n+i)+2.(A-C).n.M = (A-C).(k+k');$$

[3024g]
$$2A.N.(n-i)+2.(C-B).n.N' = (C-B).(k-k');$$

[3024h]
$$-2B.N'.(n-i)+2.(A-C).n.N = (A-C).(k-k').$$

Multiplying [3024e, f], by $(n+i).B$, $n.(C-B)$, respectively, and adding the products, the term multiplied by M' vanishes, and we get,

$$2M.\{(n+i)^2.AB+n^2.(A-C).(C-B)\}$$

[3024i]
$$=(k+k').(C-B).\{(n+i).B+n.(A-C)\};$$

from which we easily get M [3025]. Now multiplying [3024e, f] by $n.(A-C)$, $-(n+i).A$, respectively, and adding the products; the term multiplied by M vanishes in this sum, and we get,

$$2M'.\{(n+i)^2.AB-n^2.(A-C).(B-C)\}$$

[3024k]
$$=(k+k').(C-A).\{n.(B-C)+(n+i).A\};$$

These equations give,*

$$d\theta = r\,dt\,.\,\sin.\varphi - q\,dt\,.\,\cos.\varphi\,; \qquad [3032]$$

therefore we have,

$$\frac{d\theta}{dt} = \left(\frac{M'-M}{2}\right).\sin.(2\varphi+it+\varepsilon) + \left(\frac{N'-N}{2}\right).\sin.(2\varphi-it-\varepsilon)$$
$$+\frac{(N+N'-M-M')}{2}.\sin.(it+\varepsilon). \qquad [3033]$$

We may neglect the two first terms of this expression of $\frac{d\theta}{dt}$, because the coefficients are insensible,† and they do not increase by integration. But [3033′]

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from which we easily obtain M' [3026]. In like manner we may obtain N, N', from [3024g, h]; or more simply from the principle of derivation. For if we change M, M', k', i, into N, N', $-k'$, $-i$, respectively, the equations [3024e, f] will change into [3024l] [3024g, h] respectively. Making the same changes in [3025, 3026], which were deduced from [3024e, f], we get [3027, 3028].

* (2051) Multiplying [3030, 3031] by $-\cos.\varphi$, $\sin.\varphi$, respectively, adding the products, and putting $\cos.^2\varphi+\sin.^2\varphi=1$, we get [3032]. Dividing this by dt, and [3032a] substituting [3023, 3024], we get the following expression, which is reduced by means of [18, 19] Int.

$$\frac{d\theta}{dt} = M'.\sin.\varphi.\cos.(\varphi+it+\varepsilon) + N'.\sin.\varphi.\cos.(\varphi-it-\varepsilon)$$
$$-M.\cos.\varphi.\sin.(\varphi+it+\varepsilon) - N.\cos.\varphi.\sin.(\varphi-it-\varepsilon)\,;$$
$$=\tfrac{1}{2}M'.\{\sin.(2\varphi+it+\varepsilon)-\sin.(it+\varepsilon)\} + \tfrac{1}{2}N'.\{\sin.(2\varphi-it-\varepsilon)+\sin.(it+\varepsilon)\} \qquad [3032b]$$
$$+\tfrac{1}{2}M.\{-\sin.(2\varphi+it+\varepsilon)-\sin.(it+\varepsilon)\} + \tfrac{1}{2}N.\{-\sin.(2\varphi-it-\varepsilon)+\sin.(it+\varepsilon)\}.$$

This is easily reduced to the form [3033].

† (2052) The quantities P, P', [3016, 3017] are of the order of the disturbing forces of the sun and moon upon the different parts of the earth. For X, Y, Z, are of the order $r_{\prime}$ [3004, 2964]; therefore P, P', [3016, 3017] are of the order $\frac{L'}{r_{\prime}^{3}}$, or $\frac{g}{289\times(366{,}26)^2}$ [2300]; consequently k, k', [3020] are of the same order; g being the [3033a]

this is not the case with the third term, which may become sensible by
[3033‴] integration, if i be very small. In this case we may neglect i, in comparison with n, and we shall have very nearly,*

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gravity at the surface of the earth [2128$^{\text{xvii}}$]. These extremely small quantities k, k', are
[3033b] multiplied in [3025—3028] by $C-B$, or $C-A$, which is of the same order as the oblateness of the earth, or nearly $\frac{1}{300}$; so that the quantities M, M', N, N', corresponding to the sun's disturbing force, are of the order $\frac{g}{300\times 289\times(366{,}26)^2.n}$; the term n being introduced, because if we put $i=0$, in [3025—3028], all these formulas
[3033c] will contain n in the denominator [3034b]. Again, as $d\varphi$ is nearly equal to $n\,dt$ [3024′], the integration of [3033] introduces in the term $\frac{1}{2}.(M'-M).\sin.(2\varphi+it+\varepsilon)$, a divisor of the order $2n$, consequently that coefficient is of the order $\frac{g}{2\times 300\times 289\times(366{,}26)^2.n^2}$; but by [1594a] $\frac{n^2}{g}=\frac{1}{289}$, therefore this term is of the order $\frac{1}{2\times 300\times(366{,}26)^2}$, or $\frac{1}{80000000}$ nearly. Multiplying this by 206265^s, the radius in seconds, it becomes
[3033d] about $\frac{1}{400}$ of a second, which is wholly insensible; and as the lunar disturbing force [2706] is only three times that of the sun, it must also be insensible. Hence the first term of [3033] may be neglected; and as the second term is of the same order, it may also be neglected,
[3033e] and then the remaining term becomes $\frac{d\theta}{dt}=\frac{(N+N'-M-M')}{2}.\sin.(it+\varepsilon)$.

* (2053) Putting $i=0$ in [3025—3028], the denominator of these expressions
[3034a] becomes $n^2.\{AB-(A-C).(B-C)\}=n^2.C.(A+B-C)$, consequently the numerators and denominators of these expressions are divisible by $n.(A+B-C)$. Hence we get the equations [3034b], from which we easily obtain [3034c, d]. The first of the equations [3034d], being substituted in [3033e], gives [3034].

[3034b]
$$M=\frac{k+k'}{2nC}.(C-B);\qquad M'=\frac{k+k'}{2nC}.(C-A);$$
$$N=\frac{k-k'}{2nC}.(C-B);\qquad N'=\frac{k-k'}{2nC}.(C-A).$$

[3034c]
$$N-M=\frac{k'}{nC}.(B-C);\qquad N'-M'=\frac{k'}{nC}.(A-C);$$
$$M+N=\frac{k}{nC}.(C-B);\qquad M'+N'=\frac{k}{nC}.(C-A).$$

[3034d]
$$(N-M)+(N'-M')=N+N'-M-M'=\frac{k'}{nC}.(A+B-2C);$$
$$M+M'+N+N'=\frac{k}{nC}.(2C-A-B).$$

$$\frac{d\theta}{dt}=\left(\frac{A+B-2C}{2n.C}\right).k'.\sin.(it+\varepsilon).$$ [3034]

The preceding expressions of $q\,dt$, $r\,dt$, give,*

$$d\psi.\sin.\theta=r\,dt.\cos.\varphi+q\,dt.\sin.\varphi;$$ [3035]

hence we deduce,

$$\frac{d\psi}{dt}.\sin.\theta=\left(\frac{M'-M}{2}\right).\cos.(2\varphi+it+\varepsilon)+\left(\frac{N'-N}{2}\right).\cos.(2\varphi-it-\varepsilon)$$
$$+\frac{(M+M'+N+N')}{2}.\cos.(it+\varepsilon);$$ [3036]

neglecting the two first terms of this expression, which are always insensible, [3036′]
and supposing i to be quite small, we shall have very nearly,†

$$\frac{d\psi}{dt}.\sin.\theta=\left(\frac{2C-A-B}{2n.C}\right).k.\cos.(it+\varepsilon).$$ [3037]

If we denote by $\Sigma.k.\cos.(it+\varepsilon)$, the sum of the terms in which P is [3038]
developed [3020], and by $\Sigma.k'.\sin.(it+\varepsilon)$, the sum of the terms of the [3039]
development of P', Σ being the sign of finite integrals, we shall have,

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* (2054) Multiplying [3030, 3031] by $\sin.\varphi$, $\cos.\varphi$, respectively, adding the products, and putting $\sin.^2\varphi+\cos.^2\varphi=1$, we get [3035]. Dividing this by dt, substituting q, r, [3023, 3024], and reducing, we obtain [3036]. This may also be more easily obtained by [3035a] derivation from [3033]. For if we change φ, ε, M, M', into $\varphi+100°$, $\varepsilon+100°$, $-M$, $-M'$, respectively; the values of q, r, [3023, 3024] will remain unaltered, and $d\theta$ [3032] will become the same as $d\psi.\sin.\theta$ [3035]. Making the same changes in [3035b] [3033], we get,

$$\frac{d\psi}{dt}.\sin.\theta=\tfrac{1}{2}.(-M'+M).\sin.(2\varphi+it+\varepsilon+300°)$$
$$+\tfrac{1}{2}.(N'-N).\sin.(2\varphi-it-\varepsilon+100°)+\tfrac{1}{2}.(N+N'+M'+M).\sin.(it+\varepsilon+100°);$$ [3035c]

which is easily reduced to the form [3036].

† (2055) We may neglect the two terms in [3036] depending on the angles $2\varphi+it+\varepsilon$, $2\varphi-it-\varepsilon$, for the same reason that the similar terms in [3033] were neglected [3033d]; [3037a] then substituting in the last term the value of $M+M'+N+N'$ [3034d], we get [3037].

Differential expressions of $d\theta$, $d\psi$.

[3040] $$\frac{d\theta}{dt} = \left(\frac{A+B-2C}{2n.C}\right).\Sigma.k'.\sin.(it+\varepsilon);^*$$

[3041] $$\frac{d\psi}{dt}.\sin.\theta = \left(\frac{2C-A-B}{2n.C}\right).\Sigma.k.\cos.(it+\varepsilon). \qquad (H)$$

[3041′] Integrating these equations, without noticing the arbitrary constant quantities, we get the parts of θ and ψ, arising from the action of the body L. To
[3041″] obtain the complete values of these variable quantities, we must add to them the quantities depending on the state of the body at the origin of the motion. If we notice only this original motion, the two equations [3010, 3011] become,

[3042] $$dq + \frac{(C-B)}{A}.nr.dt = 0; \qquad dr + \frac{(A-C)}{B}.nq.dt = 0;$$

hence by integration,†

[3043] $$q = G.\sin.(\lambda t+\beta);$$

[3044] $$r = \frac{\lambda A}{n.(B-C)}.G.\cos.(\lambda t+\beta);$$

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* (2056) If $\frac{d\theta}{dt}$ [3034] contain a number of terms of the form $k'.\sin.(it+\varepsilon)$
[3040a] [3020, 3034], its complete value will be as in [3040]. In like manner, the sum of all the terms of the form [3037] becomes as in [3041].

† (2057) Neglecting the terms depending on the disturbing force L in [3010, 3011],
[3043a] they become as in [3042], which are precisely of the same form as the two last of the equations [278], changing p into n. If in the values of q, r, [279], which were deduced from [278], we change the constant quantities M, M', n, γ, [279], into G, G', λ, β, respectively, we shall get,

[3043b] $$q = G.\sin.(\lambda t+\beta), \qquad r = G'.\cos.(\lambda t+\beta).$$

Then from [280] we have, by changing as above n into λ, and p into n,

[3043c] $$\lambda = n.\left(\frac{(C-A).(C-B)}{AB}\right)^{\frac{1}{2}} \quad \text{and} \quad G' = -G.\left(\frac{A.(C-A)}{B.(C-B)}\right)^{\frac{1}{2}}.$$

These values of q, λ, agree with [3043, 3045] respectively. The preceding value of G' may evidently be put under the form,

[3043d] $$G' = -\frac{A}{n.(C-B)}.G.n.\left(\frac{(C-A).(C-B)}{AB}\right)^{\frac{1}{2}} = -\frac{A}{n.(C-B)}.G.\lambda = \frac{\lambda.A}{n.(B-C)}.G;$$

[3043e] hence r [3043b] becomes as in [3044].

G and β being two other arbitrary constant quantities, and

$$\lambda = n \cdot \sqrt{\frac{(C-A) \cdot (C-B)}{A\,B}}. \qquad [3045]$$

If we substitute these values of q and r in [3032],

$$\frac{d\theta}{dt} = r \cdot \sin.\varphi - q \cdot \cos.\varphi ; \qquad [3046]$$

we get by integration,*

$$\theta = h + \frac{\{n \cdot (B-C) - \lambda A\}}{2n \cdot (n+\lambda) \cdot (B-C)} \cdot G \cdot \cos.(\varphi + \lambda t + \beta)$$

$$- \left\{ \frac{\lambda A + n \cdot (B-C)}{2n \cdot (n-\lambda) \cdot (B-C)} \right\} \cdot G \cdot \cos.(\varphi - \lambda t - \beta) ; \qquad [3047]$$

h being a new arbitrary constant quantity. If the value of G be of a sensible magnitude, we shall be able to discover it, by the daily variations [3047′] of the latitude, or of the height of the pole.† But since no variation of this

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* (2058) Substituting q, r, [3043, 3044], in [3046]; then reducing by means of [18, 19] Int.; using for a moment, for brevity, $\varphi + \lambda t + \beta = T_{,}$, $\varphi - \lambda t - \beta = T_{,,}$, [3047a] we get,

$$\frac{d\theta}{dt} = \frac{\lambda A}{n \cdot (B-C)} \cdot G \cdot \cos.(\lambda t + \beta) \cdot \sin.\varphi - G \cdot \sin.(\lambda t + \beta) \cdot \cos.\varphi \qquad [3047b]$$

$$= \frac{\lambda A}{2n \cdot (B-C)} \cdot G \cdot \{\sin.T_{,} + \sin.T_{,,}\} + \tfrac{1}{2} G \cdot \{-\sin.T_{,} + \sin.T_{,,}\}$$

$$= -\frac{\{n \cdot (B-C) - \lambda A\}}{2n \cdot (B-C)} \cdot G \cdot \sin.T_{,} + \frac{\{\lambda A + n \cdot (B-C)\}}{2n \cdot (B-C)} \cdot G \cdot \sin.T_{,,}. \qquad [3047c]$$

Multiplying this last expression by dt, resubstituting the values of $T_{,}$, $T_{,,}$, [3047a], integrating, and observing that $d\varphi = n\,dt$ [3024′], nearly, we get θ [3047]; the constant [3047d] quantity h being added to complete the integral.

† (2059) The coefficient λ is very small in comparison with n [3045], therefore the variation of the angle λt, in the time t, must be small in comparison with that of φ [3024′]; consequently the expressions $\cos.(\varphi + \lambda t + \beta)$, $\cos.(\varphi - \lambda t - \beta)$, must pass nearly [3048a] through all their values from $+1$ to -1, in the course of a day, or rather in half a day; so that if G be of any sensible value, the quantity θ will have a diurnal variation. Now as [3048b]

We may neglect the parts of θ, ψ, depending on the original motion of the earth.

kind has been perceived, in the most accurate observations, it follows that G
[3048] is insensible; therefore *we may neglect the parts of* θ *and* ψ, *depending on the original motion of the earth.*

5. *We shall now resume the equations* [3040, 3041]. The first gives by integration, observing that the development of the function P' [3017] is
[3049] represented in [3020] by $P' = \Sigma \,.\, k' \,.\, \sin. (i\,t + \varepsilon)$,*

Obliquity of the ecliptic.

[3050]
$$\theta = h + \frac{(A + B - 2\,C)}{2\,n\,.\,C} \,.\, \textstyle\int P' dt.$$

The only bodies which have a sensible influence on the motions of the axis of the earth, are the sun and moon. We shall first consider the action of the
v, γ, Λ. *sun; and shall put* v *for its longitude, counted from the moveable vernal*
[3051] *equinox;* γ *the inclination of its orbit, upon the fixed plane; and* Λ *the*
[3052] *longitude of its ascending node; the angles* v *and* Λ *being referred to the moveable orbit of the sun,* we shall have,†

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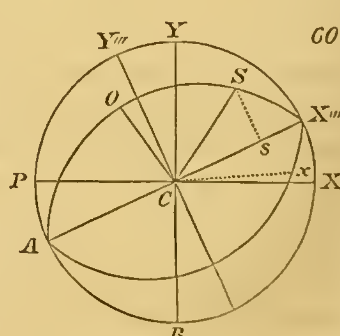
no such variation has been perceived, we may conclude that G is insensible. Hence
[3048c] the parts of q, r, [3043, 3044], depending on the original motion, may be neglected; consequently the similar part of $d\psi$, [3035, 3041], or of ψ, as well as that of θ [3032, 3040], may also be neglected.

* (2060) Substituting $P' = \Sigma \,.\, k' \,.\, \sin. (i\,t + \varepsilon)$ in [3040], it becomes

[3050a]
$$\frac{d\theta}{dt} = \frac{(A + B - 2\,C)}{2\,n\,.\,C} \,.\, P';$$

multiplying this by dt, integrating, and adding the constant quantity h, we get [3050].

[3053a] † (2061) In the annexed figure, C represents the centre of the earth; $X X_{,,,} Y Y_{,,,} P A B$ the fixed plane of XY; $X_{,,,} S A x$ the sun's orbit, or the moveable ecliptic, cutting the fixed plane in the line $A C X_{,,,}$. Then $C X$, $C Y$, are the axes of X, Y, respectively; and
[3053b] the axis of Z is drawn through C, perpendicularly to the fixed plane of $X Y$, towards the north pole of the fixed ecliptic; so that the rectangular co-ordinates of the sun at S, may be represented by X, Y, Z, and the radius
[3053c] $C S = r_{,}$. The co-ordinates X, Y, may be transformed into $x_{,,,}$, $y_{,,,}$, supposing $C X_{,,,}$ to be the axis of $x_{,,,}$,



$$X = r_{,} . \cos.^2 \tfrac{1}{2} \gamma . \cos. v + r_{,} . \sin.^2 \tfrac{1}{2} \gamma . \cos. (v - 2 \text{A}) ;$$ [3053]

$$Y = r_{,} . \cos.^2 \tfrac{1}{2} \gamma . \sin. v - r_{,} . \sin.^2 \tfrac{1}{2} \gamma . \sin. (v - 2 \text{A}) ;$$ [3054]

$$Z = r_{,} . \sin. \gamma . \sin. (v - \text{A}) ;$$ [3055]

hence we deduce,*

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and $C Y_{,,,}$ that of $y_{,,,}$; these axes being drawn perpendicularly to each other, in the fixed plane. Then if we make the angle $X_{,,,} C x = X_{,,,} C X = \text{A}$, we shall have in the [3053d]
present notation [3051, &c.] the angle $x C S = v$, $X_{,,,} C S = v - \text{A}$. Now if we let [3053e]
fall from S upon $C s X_{,,,}$ the perpendicular $S s$, we shall have,

$$C s = x_{,,,} = r_{,} . \cos. (v - \text{A}), \qquad S s = r_{,} . \sin. (v - \text{A}) ;$$ [3053f]

and it is evident that the co-ordinate Z, which represents the distance of the point S from the fixed plane, is equal to $S s$ multiplied by the sine of γ, the inclination of the planes $X_{,,,} Y_{,,,}$, $X_{,,,} S$; hence $Z = r_{,} . \sin. (v - \text{A}) . \sin. \gamma$ [3051], as in [3055]. Moreover, [3053g]
$y_{,,,}$ is equal to $S s$, multiplied by the cosine of the same angle of inclination, or

$$y_{,,,} = r_{,} . \sin. (v - \text{A}) . \cos. \gamma.$$ [3053h]

From these values of $x_{,,,}$, $y_{,,,}$, we may get those of X, Y, in the same manner as $x_{,,}$, $y_{,,}$, are found in [170]. For if we put $x_{,,} = X$, $y_{,,} = Y$, $\varphi = \text{A}$, we get

$$X = x_{,,,} . \cos. \text{A} - y_{,,,} . \sin. \text{A} ; \qquad Y = y_{,,,} . \cos. \text{A} + x_{,,,} . \sin. \text{A} ;$$ [3053i]

and by substituting $x_{,,,}$, $y_{,,,}$, [3053f, h], they become as in [3053k. &c.], which are successively reduced to the forms [3053l, m], by means of [20, 17, 6, 1] Int.

$$X = r_{,} . \{\cos. (v - \text{A}) . \cos. \text{A} - \sin. (v - \text{A}) . \sin. \text{A} . \cos. \gamma\}$$ [3053k]

$$= r_{,} . \{\tfrac{1}{2} . \cos. v + \tfrac{1}{2} . \cos. (v - 2 \text{A}) + [\tfrac{1}{2} . \cos. v - \tfrac{1}{2} . \cos. (v - 2 \text{A})] . \cos. \gamma\}$$

$$= r_{,} . \{(\tfrac{1}{2} + \tfrac{1}{2} . \cos. \gamma) . \cos. v + (\tfrac{1}{2} - \tfrac{1}{2} . \cos. \gamma) . \cos. (v - 2 \text{A})\}$$

$$= r_{,} . \{\cos.^2 \tfrac{1}{2} \gamma . \cos. v + \sin.^2 \tfrac{1}{2} \gamma . \cos. (v - 2 \text{A})\}.$$ [3053l]

In like manner we may obtain Y [3054] from [3053i]; but it is more easily derived from X, in the following manner. For if we change v, A, into $v - 100^{\circ}$, $\text{A} - 100^{\circ}$, [3053m]
respectively; $x_{,,,}$, $y_{,,,}$, [3053f, h], remain unaltered, while the value of X [3053i] changes into Y [3053i]; and by making the same changes in [3053], it becomes, by slight reductions, as in [3054].

* (2062) Putting for brevity $A' = r_{,} . \cos.^2 \tfrac{1}{2} \gamma$, $B' = r_{,} . \sin.^2 \tfrac{1}{2} \gamma$, $C' = r_{,} . \sin. \gamma$, [3056a]
we get from [3053—3055],

$$X = A' . \cos. v + B' . \cos. (v - 2 \text{A}), \quad Y = A' . \sin. v - B' . \sin. (v - 2 \text{A}), \quad Z = C' . \sin. (v - \text{A}).$$ [3056b]

[3056] $XY = \frac{1}{2} r_{\prime}^2 . \cos.^4 \frac{1}{2}\gamma . \sin. 2v + \frac{1}{4} r_{\prime}^2 . \sin.^2\gamma . \sin. 2\text{A} - \frac{1}{2} r_{\prime}^2 . \sin.^4 \frac{1}{2}\gamma . \sin. (2v - 4\text{A})$;

[3057] $$XZ = \frac{1}{2} r_{\prime}^2 . \sin. \gamma . \cos.^2 \frac{1}{2}\gamma . \sin. (2v - \text{A}) - \frac{1}{4} r_{\prime}^2 . \sin. 2\gamma . \sin. \text{A} + \frac{1}{2} r_{\prime}^2 . \sin. \gamma . \sin.^2 \frac{1}{2}\gamma . \sin. (2v - 3\text{A}).$$

On account of the extreme slowness of the motion of the equinoctial points,
[3057′] we may suppose dv equal to the angular motion of the sun, during the time dt, and we shall have, by § 19, 20, Book II, [1057],*

[3058] $$r_{\prime}^2 \, dv = a^2 m \, dt . \sqrt{1 - e^2};$$

[3059] *mt being the mean motion of the sun, a its mean distance from the earth,*
$m, a, e.$ *and e the ratio of the excentricity of its orbit to its mean distance.* Moreover we have, by neglecting the masses of the planets, in comparison with that
[3060] of the sun, $\frac{L}{a^3} = m^2$ [605′]; and the equation of the ellipsis gives,

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Substituting these in the first members of [3056d, g], we get the second members of these expressions; which, by successive reductions, become as in [3056f, i]; using formulas
[3056c] [31, 22, 19] Int.; observing also in [3056e, h], that $(\sin.\frac{1}{2}\gamma.\cos.\frac{1}{2}\gamma)^2 = (\frac{1}{2}.\sin.\gamma)^2 = \frac{1}{4}.\sin.^2\gamma$; and $\frac{1}{2} C' . (A' - B') = \frac{1}{2} r_{\prime}^2 . \sin.\gamma . (\cos.^2 \frac{1}{2}\gamma - \sin.^2 \frac{1}{2}\gamma) = \frac{1}{2} r_{\prime}^2 . \sin.\gamma . \cos.\gamma = \frac{1}{4} r_{\prime}^2 . \sin. 2\gamma$.

[3056d] $$XY = A'^2 . \sin. v . \cos. v + A'B' . \{\sin. v . \cos. (v - 2\text{A}) - \cos. v . \sin. (v - 2\text{A})\} - B'^2 . \sin. (v - 2\text{A}) . \cos. (v - 2\text{A})$$

$$= \frac{1}{2} A'^2 . \sin. 2v + A'B' . \sin. \{v - (v - 2\text{A})\} - \frac{1}{2} B'^2 . \sin. 2 . (v - 2\text{A})$$

[3056e] $$= \frac{1}{2} A'^2 . \sin. 2v + r_{\prime}^2 . (\sin. \tfrac{1}{2}\gamma . \cos. \tfrac{1}{2}\gamma)^2 . \sin. 2\text{A} - \frac{1}{2} B'^2 . \sin. 2 . (v - 2\text{A})$$

[3056f] $$= \frac{1}{2} A'^2 . \sin. 2v + \frac{1}{4} r_{\prime}^2 . \sin.^2\gamma . \sin. 2\text{A} - \frac{1}{2} B'^2 . \sin. 2 . (v - 2\text{A}),$$ as in [3056];

[3056g] $$XZ = A' C' . \cos. v . \sin. (v - \text{A}) + B' C' . \cos. (v - 2\text{A}) . \sin. (v - \text{A})$$

$$= \frac{1}{2} A' C' . \{\sin. (2v - \text{A}) - \sin. \text{A}\} + \frac{1}{2} B' C' . \{\sin. \text{A} + \sin. (2v - 3\text{A})\}$$

[3056h] $$= \frac{1}{2} A' C' . \sin. (2v - \text{A}) - \frac{1}{2} C' . (A' - B') . \sin. \text{A} + \frac{1}{2} B' C' . \sin. (2v - 3\text{A})$$

[3056i] $$= \frac{1}{2} A' C' . \sin. (2v - \text{A}) - \frac{1}{4} r_{\prime}^2 . \sin. 2\gamma . \sin. \text{A} + \frac{1}{2} B' C' . \sin. (2v - 3\text{A}),$$

as in [3057].

* (2063) We easily obtain [3058] from [585, 599, 605′], as in [1057], changing r, n, into $r_{\prime}$, m, respectively. Making the same change in [605′], we get $\mu = m^2 . a^3$; μ being the sum of the masses of the sun and earth; and by neglecting the last, on account of its
[3058a] smallness, in comparison with the mass L of the sun, we get $L = m^2 . a^3$ [3060].

$$\frac{a}{r_{\prime}} = \frac{1 + e \cdot \cos.(v - \Gamma)}{1 - e^2};^{*}$$ [3061]

Γ *being the longitude of the solar perigee:* therefore we shall have, relatively to the sun,† Γ. [3061′]

$$P'dt = \frac{3L \cdot dt}{r_{\prime}^5} \cdot \{XY \cdot \sin.\theta + XZ \cdot \cos.\theta\}$$ [3062]

$$= \frac{3m \cdot dv \cdot \{1 + e \cdot \cos.(v - \Gamma)\}}{(1 - e^2)^{\frac{3}{2}}} \cdot \left\{ \frac{XY}{r_{\prime}^2} \cdot \sin.\theta + \frac{XZ}{r_{\prime}^2} \cdot \cos.\theta \right\}.$$ [3062′]

If we substitute for $\frac{XY}{r_{\prime}^2}$, $\frac{XZ}{r_{\prime}^2}$, their preceding values, in terms of v, [3062″] we shall find, in the first place, that after having developed $P'dt$, in sines of the angle v, and its multiples, the terms depending on the longitude of the sun's perigee Γ‡ contain the angle v; therefore they cannot become sensible [3063]

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* (2064) Changing in [603] r, v, into $r_{\prime}$, $v - \Gamma$, respectively, in order to conform to [3061a] the present notation, we get [3061] by a slight reduction.

† (2065) Multiplying [3017] by dt, we get [3062], which may be put under the form $\frac{3L}{r_{\prime}} \cdot \frac{dt}{r_{\prime}^2} \cdot \left\{ \frac{XY}{r_{\prime}^2} \cdot \sin.\theta + \frac{XZ}{r_{\prime}^2} \cdot \cos.\theta \right\}$. But from [3058, 3058a, 3061] we have [3062a]

$$\frac{3L}{r_{\prime}} \cdot \frac{dt}{r_{\prime}^2} = \frac{3L}{r_{\prime}} \cdot \frac{dv}{a^2 m \cdot \sqrt{(1 - e^2)}} = \frac{3m^2 \cdot a^3}{r_{\prime}} \cdot \frac{dv}{a^2 m \cdot \sqrt{(1 - e^2)}} = \frac{3m \cdot dv}{\sqrt{(1 - e^2)}} \cdot \frac{a}{r_{\prime}}$$ [3062b]

$$= \frac{3m \cdot dv}{\sqrt{(1 - e^2)}} \cdot \frac{1 + e \cdot \cos.(v - \Gamma)}{1 - e^2};$$ [3062c]

substituting this in [3062a], we get [3062′].

‡ (2066) The quantity Γ is found, in [3062′] in connexion with the term $e \cdot \cos.(v - \Gamma)$ [3063a] only. This is multiplied by the factors XY, XZ, and when their values [3056, 3057] are substituted, it is multiplied by the terms

$$\sin.2v, \quad \sin.2\text{A}, \quad \sin.(2v - 4\text{A}), \quad \sin.(2v - \text{A}), \quad \sin.\text{A}, \quad \sin.(2v - 3\text{A}).$$ [3063b]

These products being reduced by means of [18, 19] Int., produce terms depending on sines of angles of the forms $\sin.(v - \Gamma_{\prime\prime})$, $\sin.(3v - \Gamma_{\prime\prime\prime})$; $\Gamma_{\prime\prime}$, $\Gamma_{\prime\prime\prime}$, being functions of A, Γ. [3063c] These last terms are multiplied by dv, in $P'dt$ [3062′]; and as dv is of the order mdt [3058], the integral of $P'dt$ will introduce the large divisor m in $\int P'dt$ [3050]. This divisor renders these terms insensible in comparison with those depending on the sines [3063d] of angles of the forms $\sin.\text{A}$, $\sin.\Gamma$; because these last quantities vary with extreme

by integration. But this is not the case with the terms depending on the
[3063'] longitude of the node; the function $\frac{XZ}{r_{\prime}^{2}}$ introduces in $P'dt$ the term

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slowness; and they may be represented generally by terms of the form $A_{\prime}.\sin. it$, i being
[3063e] a very small coefficient, which produces the divisor i in the integral $\int P'dt$. Now as no such terms are produced by the quantity $e.\cos.(v-\Gamma)$, it may be neglected; observing
[3063f] that in all these calculations, we neglect terms of the order of the square of the disturbing forces, and of the square of the oblateness of the earth. If we also neglect the square of e,
[3063g] the expression [3062'] becomes $P'dt = 3m.dv.\left\{\frac{XY}{r_{\prime}^{2}}.\sin.\theta + \frac{XZ}{r_{\prime}^{2}}.\cos.\theta\right\}$; in which we must substitute the values [3056, 3057]. If we put $\cos.^2\tfrac{1}{2}\gamma = 1-\sin.^2\tfrac{1}{2}\gamma$, in [3056], we obtain,

[3063h] $$\frac{XY}{r_{\prime}^{2}} = \tfrac{1}{2}.(1-2.\sin.^2\tfrac{1}{2}\gamma + \sin.^4\tfrac{1}{2}\gamma).\sin.2v + \tfrac{1}{4}.\sin.^2\gamma.\sin.2\text{A} - \tfrac{1}{2}.\sin.^4\tfrac{1}{2}\gamma.\sin.(2v-4\text{A}).$$

In the solar orbit γ is extremely small [3051], and in the lunar orbit $\gamma = 5^d$ nearly;
[3063i] hence $\sin.^2\tfrac{1}{2}\gamma < \tfrac{1}{500}$. Now as the terms depending on $\sin.2v$, $\cos.2v$, [3378—3380] do not much exceed 3″, or one sexagesimal second, we may neglect quantities of the order $\sin.^2\tfrac{1}{2}\gamma$, in the parts of [3063h] depending on the angles $2v$, $2v-4\text{A}$, which have
[3063k] great divisors in the integral, and we obtain $\frac{XY}{r_{\prime}^{2}} = \tfrac{1}{2}.\sin.2v + \tfrac{1}{4}.\sin.^2\gamma.\sin.2\text{A}$. In like manner, substituting $\cos.^2\tfrac{1}{2}\gamma = 1-\sin.^2\tfrac{1}{2}\gamma$ in [3057], and neglecting terms
[3063l] multiplied by $\sin.^2\tfrac{1}{2}\gamma$, we get $\frac{XZ}{r_{\prime}^{2}} = \tfrac{1}{2}.\sin.\gamma.\sin.(2v-\text{A}) - \tfrac{1}{4}.\sin.2\gamma.\sin.\text{A}$, Substituting [3063k, l] in [3063g], we get,

[3063m] $$\begin{aligned} P'dt &= 3m.dv.\sin.\theta.\{\tfrac{1}{2}.\sin.2v + \tfrac{1}{4}.\sin.^2\gamma.\sin.2\text{A}\} \\ &\quad + 3m.dv.\cos.\theta.\{\tfrac{1}{2}.\sin.\gamma.\sin.(2v-\text{A}) - \tfrac{1}{4}.\sin.2\gamma.\sin.\text{A}). \end{aligned}$$

We may neglect the term depending on $\sin.(2v-\text{A})$ in this expression; for by comparing its coefficient with that of $\sin.2v$ it is of the order $\sin.\gamma$, or $<\tfrac{1}{12}$, and as the terms depending on $\sin.2v$, $\cos.2v$, [3377—3380], are very small, it must be insensible; hence [3063m] becomes,

[3063n] $$\begin{aligned} P'dt = \tfrac{3}{2}m.\sin.\theta.(dv.\sin.2v) &- \tfrac{3}{4}m.\cos.\theta.(dv.\sin.2\gamma.\sin.\text{A}) \\ &+ \tfrac{3}{4}m.\sin.\theta.(dv.\sin.^2\gamma.\sin.2\text{A}). \end{aligned}$$

[3063o] The integral of this quantity is multiplied by $\frac{A+B-2C}{C}$ in the value of θ [3050]; and as this factor is of the same order as the ellipticity of the earth, or as it is usually called, of the order α, we may suppose θ to be constant in the preceding integration, neglecting quantities of the order α^2 [3063f], and then the integral becomes,

$-\frac{3\,m\,d\,v}{4}\,.\,\sin.\,2\gamma\,.\,\cos.\,\theta\,.\,\sin.\,\text{A}$ [3063n]; and on account of the slowness [3064]
of the variations of γ and A, this term may become very sensible in the value of θ. Thus we shall have very nearly, by observing that e and γ are extremely small, and retaining, among the terms multiplied by these quantities, only those which can increase considerably by integration, [3064′]

$$\int P'\,d\,t = -\tfrac{3}{4}\,m\,.\,\sin.\,\theta\,.\,\cos.\,2\,v - \tfrac{3}{2}\,m^2\,.\,\cos.\,\theta\,.\int\gamma\,d\,t\,.\,\sin.\,\text{A}. \qquad [3065]$$

$\gamma\,.\,\sin.\,\text{A}$ is the product of the inclination of the solar orbit, by the sine of [3066]
the longitude of its ascending node, counted from the moveable vernal equinox [3053d, g]. This inclination being very small, we may take for γ, either its sine or its tangent. Now we have seen in § 59, Book II, [3066′]
[1032, 1133, &c.], that if we put $\Gamma_{,}$ *equal to the longitude of the ascending* $\Gamma_{,}$.
node of this orbit, counted from the fixed equinox, according to the order of [3067]
the signs, $\text{tang.}\,\gamma\,.\,\sin.\,\Gamma_{,}$ will be given by a finite number of terms of the form $c\,.\,\sin.\,(g\,t+\beta)$,* and $\text{tang.}\,\gamma\,.\,\cos.\,\Gamma_{,}$ will be given by the same [3068] ψ
number of corresponding terms $c\,.\,\cos.\,(g\,t+\beta)$; moreover, ψ being the [3068′]

$$\int P'\,d\,t = -\tfrac{3}{4}\,m\,.\,\sin.\,\theta\,.\,\cos.\,2\,v - \tfrac{3}{4}\,m\,.\,\cos.\,\theta\,.\int\sin.\,2\,\gamma\,.\,\sin.\,\text{A}\,.\,d\,v$$
$$+\tfrac{3}{4}\,m\,.\,\sin.\,\theta\,.\int\sin.^2\gamma\,.\,\sin.\,2\,\text{A}\,.\,d\,v. \qquad [3063p]$$

If we continue to neglect quantities of the order γ^2, we may put $\sin.\,2\,\gamma = 2\,\gamma$, and neglect the last term; then it becomes as in [3065], observing that if we neglect terms [3063q]
depending on $e.\cos.(v-\Gamma)$, as in [3063a, &c.], we may put $d\,v = m\,d\,t$ [3058, 3061]. [3063r]
La Place, in this book, neglects the last term of [3063p] depending on the angle $2\,\text{A}$, and the similar term, depending on the angle $2\,\text{A}'$, corresponding to the attraction of the moon [3063s]
[3078]. This last term was introduced by Bessel in the value of θ [3089], and by Poisson in the value of ψ [3100], in the paper mentioned in [3015i]. Afterwards La Place resumed the subject, and in the fifth volume [12217—12230′], computed these terms of θ, ψ, depending on the angle $2\,\text{A}'$, &c. [3063t]

* (2067) Comparing the definitions of φ, γ, [1030′, 3051], and those of θ, $\Gamma_{,}$, [1030″, 3067], we get $\varphi=\gamma$, $\theta=\Gamma_{,}$; hence p, q, [1032] become $p=\text{tang.}\gamma.\sin.\Gamma_{,}$, [3068$a$]
$q=\text{tang.}\,\gamma\,.\,\cos.\,\Gamma_{,}$. Substituting these in the two first formulas of [1133], using the sign Σ of finite integrals, and changing N, $N_{,}$, &c., into c, c', &c., we get, as in [3068, &c.]

$$p=\text{tang.}\,\gamma\,.\,\sin.\,\Gamma_{,}=\Sigma\,.\,c\,.\,\sin.\,(g\,t+\beta), \qquad q=\text{tang.}\,\gamma\,.\,\cos.\,\Gamma_{,}=\Sigma\,.\,c\,.\,\cos.\,(g\,t+\beta). \qquad [3068b]$$

In the original work, the mark is not placed below Γ; we have introduced it, to distinguish $\Gamma_{,}$ from the longitude of the perigee [3061′].

retrograde motion of the equinoxes, counted from the fixed equinox, we
[3069] have $A = \Gamma_{\prime} + \psi$;* hence we obtain,†

[3070] $$\text{tang.}\,\gamma\,.\,\sin.\,A = \text{tang.}\,\gamma\,.\,\sin.\,\Gamma_{\prime}\,.\,\cos.\,\psi + \text{tang.}\,\gamma\,.\,\cos.\,\Gamma_{\prime}\,.\,\sin.\,\psi.$$

[3070′] Substituting $c\,.\,\sin.\,(g\,t+\beta)$ for $\text{tang.}\,\gamma\,.\,\sin.\,\Gamma_{\prime}$, and $c\,.\,\cos.\,(g\,t+\beta)$ for $\text{tang.}\,\gamma\,.\,\cos.\,\Gamma_{\prime}$, we get,

[3071] $$\text{tang.}\,\gamma\,.\,\sin.\,A = c\,.\,\sin.\,(g\,t+\beta+\psi).$$

Hence we see, that to obtain $\text{tang.}\,\gamma\,.\,\sin.\,A$, it is only necessary to
[3072] increase the angles of the terms of the expression of $\text{tang.}\,\gamma\,.\,\sin.\,\Gamma_{\prime}$
[3068*b*] by the quantity ψ. We may therefore, by neglecting quantities of
[3072′] the order c^2, substitute for ψ the mean motion of the equinoxes. Then
$\text{tang.}\,\gamma\,.\,\sin.\,A$ is composed of a finite number of terms of the form
[3073] $c\,.\,\sin.\,(f\,t+\beta)$,‡ which differ from the terms of the expression of
$\text{tang.}\,\gamma\,.\,\sin.\,\Gamma_{\prime}$ [3068*b*], only in the increase of the angles gt by the mean

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[3069*a*] * (2068) The symbol $\Gamma_{\prime}$ represents the longitude of the ascending node of the solar orbit upon the fixed plane, counted from the fixed equinox [3067]. Now by [3006*d*—*g*], ψ
[3069*b*] represents the precession of the equinoxes; therefore $\Gamma_{\prime}+\psi$ is the longitude of the ascending node, counted from the moveable equinox, and this is equal to A [3052].

[3070*a*] † (2069) Having $A=\Gamma_{\prime}+\psi$ [3069], we get from [21] Int.,

$$\sin.\,A = \sin.\,\Gamma_{\prime}\,.\,\cos.\,\psi + \cos.\,\Gamma_{\prime}\,.\,\sin.\,\psi\,;$$

multiplying this by $\text{tang.}\,\gamma$, we obtain [3070]; substituting [3068*b*], reducing by [21] Int., we find, as in [3071],

[3070*b*] $$\begin{aligned}\text{tang.}\,\gamma\,.\,\sin.\,A &= \Sigma\,.\,c\,.\,\{\sin.\,(g\,t+\beta)\,.\,\cos.\,\psi + \cos.\,(g\,t+\beta)\,.\,\sin.\,\psi\}\\ &= \Sigma\,.\,c\,.\,\sin.\,(g\,t+\beta+\psi).\end{aligned}$$

[3073*a*] ‡ (2070) Comparing $\frac{d\theta}{dt}$ [3033] with $\frac{d\psi}{dt}\,.\,\sin.\,\theta$ [3036], it appears that the variable parts of θ, ψ, are of the same order. This is also evident from the comparison of
[3073*b*] the numerical values of the variable terms of θ, ψ, [3377, 3379]. Therefore the variable part of ψ must be of the same order as γ [3051], or c [3070′]; so that if we substitute, in [3071], the mean value of ψ, we shall neglect terms of the order c^2, as in [3072′]. If we
[3073*c*] represent this mean value by $\psi=(f-g)\,.\,t$, the expression $\sin.\,(g\,t+\beta+\psi)$ will become $\sin.\,(ft+\beta)$, as in [3073], and then [3070*b*] becomes

[3073*d*] $$\text{tang.}\,\gamma\,.\,\sin.\,A = \Sigma\,.\,c\,.\,\sin.\,(f\,t+\beta).$$

motion of the equinoxes. We find in the same manner, that $\tang.\gamma.\cos.A$ is composed of a corresponding number of terms of the form $c.\cos.(ft+\beta)$; [3073′]
therefore by putting $\Sigma.c.\sin.(ft+\beta)$ for the sum of all the terms of [3074]
the value of $\tang.\gamma.\sin.A$, the expression of $\tang.\gamma.\cos.A$ becomes* [3074′]
$\Sigma.c.\cos.(ft+\beta)$; and these quantities will also be the expressions of [3075]
$\gamma.\sin.A$, and $\gamma.\cos.A$. This being premised, we shall have for the part of $fP'dt$ depending on the action of the sun,†

$$fP'dt=-\frac{3m}{4}.\sin.\theta.\cos.2v+\frac{3m^2}{2}.\cos.\theta.\Sigma.\frac{c}{f}.\cos.(ft+\beta). \qquad [3076]$$

We shall now consider the action of the moon; putting L' for its mass, and [3077]
a' for its mean distance from the earth. Moreover we shall suppose, for the moon, that m', v', Γ', e', A', γ', correspond respectively to the symbols we have named m, v, Γ, e, A, γ, for the sun; also [3078]

$$\frac{L'}{a'^3}=\lambda.m^2=\lambda.\frac{L}{a^3}; \qquad [3079]$$

we shall find, by the preceding analysis,

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* (2071) Substituting [3069] in the first member of [3075*b*] we get its second member. Developing this by means of [23] Int., using the values [3068*b*], reducing by [23] Int., we [3075*a*]
obtain the fourth expression [3075*c*]. Substituting in this $\psi=(f-g)t$ [3073*c*], we get [3075*d*], as in [3075],

$$\tang.\gamma\cos.A=\tang.\gamma.\cos.(\Gamma_{\prime}+\psi)=(\tang.\gamma.\cos.\Gamma_{\prime}).\cos.\psi-(\tang.\gamma.\sin.\Gamma_{\prime}).\sin.\psi \qquad [3075b]$$

$$=\Sigma.c.\{\cos.(gt+\beta).\cos.\psi-\sin.(gt+\beta).\sin.\psi\}=\Sigma.c.\cos.(gt+\beta+\psi) \qquad [3075c]$$

$$=\Sigma.c.\cos.(ft+\beta). \qquad [3075d]$$

If we neglect γ^3, as is done in [3066′], the expressions [3073*d*, 3075*d*] become, as in [3075],

$$\gamma.\sin.A=\Sigma.c.\sin.(ft+\beta); \qquad \gamma.\cos.A=\Sigma.c.\cos.(ft+\beta). \qquad [3075e]$$

† (2072) Substituting $\gamma.\sin.A$ [3075*e*] in [3065] we get

$$fP'dt=-\tfrac{3}{4}.m.\sin.\theta.\cos.2v-\tfrac{3}{2}m^2.\cos.\theta.\Sigma fdt.c.\sin.(ft+\beta); \text{ and since} \qquad [3076a]$$

$$\Sigma fdt.c.\sin.(ft+\beta)=-\Sigma\frac{c}{f}.\cos.(ft+\beta),$$ it becomes, as in [3076].

[3080] $$\int P'\,dt = -\frac{3\lambda m^2}{4m'}\,.\,\sin.\theta\,.\,\cos.2v' - \frac{3\lambda m^2}{2}\,.\,\cos.\theta\,.\int\gamma'\,dt\,.\,\sin.\Lambda'.\text{*}$$

The function $\frac{XY}{r^2}$ also introduces, in the integral $\int P'\,dt$, the term,†

[3081] $$\tfrac{3}{4}\lambda m^2\,.\,\sin.\theta\,.\int\gamma'^2\,dt\,.\,\sin.2\Lambda'.$$

[3081′] This term increases considerably by integration; but it is evident, that, notwithstanding this increase, it will yet be insensible; so that the only
[3081″] important terms produced in the value of the integral $\int P'\,dt$, and of course into the value of θ, by the action of the moon, are those we have noticed. Some astronomers have introduced in this value, a small equation
[3081‴] depending on the longitude of the perigee of the lunar orbit; but we see by the preceding analysis, that this equation does not exist. The mean motion of the moon's perigee being nearly double of the motion of her nodes, a term depending on the angle $2\Lambda' + \Gamma'$, might become sensible,

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* (2073) To compute the part of $\int P'\,dt$, depending upon the attraction of the moon,
[3080a] we must accent the letters m, v, &c. [3065] to conform to the notation [3078], and it will become,

[3080b] $$\int P'\,dt = -\tfrac{3}{4}m'\,.\,\sin.\theta\,.\,\cos.2v' - \tfrac{3}{2}\,.\,m'^2\,.\,\cos.\theta\int\gamma'\,dt\,.\,\sin.\Lambda'.$$

Accenting, in the same manner, the equation [3060], we get $m'^2 = \frac{L'}{a'^3} = \lambda m^2$ [3079],
[3080c]
and $m' = \frac{\lambda m^2}{m'}$. Substituting these values of m'^2, m' in [3080b], we get [3080].

† (2074) This corresponds to the last term of [3063p]. For by accenting the letters,
[3081a] as in [3078], it becomes $\frac{3}{4}m'\,.\,\sin.\theta\int\sin.^2\gamma'\,.\,\sin.2\Lambda'\,.\,dv'$; and $dv' = m'\,dt$, as in [3063r]. Substituting this value of dv', and that of $m'^2 = \lambda m^2$ [3080c]; changing
[3081b] also $\sin.\gamma'$ into γ', as in [3066′], it becomes, as in [3081]. This term was finally
[3081c] noticed by La Place, in the fifth volume [12223], as we have observed in [3063t]. If we substitute in [3081] the values [3086, &c.] $\gamma' = c'$, $\Lambda' = -f't - \beta'$, and put $\theta = h$,

[3081d] it becomes, $-\frac{3\lambda m^2}{4}\,.\,\sin.h\,.\,c'^2\,.\int dt\,.\,\sin.2(f't+\beta') = \frac{3\lambda m^2}{8f'}\,.\,\sin.h\,.\,c'^2\,.\,\cos.2(f't+\beta')$.

This produces, in θ [3050, 3089, 3101], the term,

[3081e] $$-\frac{3m^2}{4n}\,.\left(\frac{2C-A-B}{C}\right).\left\{\frac{\lambda c'^2}{4f'}\,.\,\sin.h\,.\,\cos.2(f't+\beta')\right\}.$$

although multiplied by $e'\gamma'^2$; but the preceding analysis shows, that no [3082]
such term exists in the integral $\int P'\,d\,t$.* [3082']

To estimate the function $\int \gamma'\,d\,t\,.\,\sin.\,\Lambda'$, we shall observe, that in all [3082'']
the changes which the position of the solar orbit suffers, the mean inclination
of the lunar orbit upon this plane remains always the same, as we shall [3083]
see in the theory of the moon.† Now by supposing this satellite to be

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* (2075) The mean motion of the moon in the time t, is $m't$ [3059, 3078]; and if we use the same notation as in [4817], we shall have $(1-c)\,.\,m't$ for the mean motion of [3082a]
the perigee; and $(1-g)\,.\,m't$ for the mean motion of the node. These quantities represent Γ', Λ' respectively, [3078, 3061', 3052], and if we substitute the values of c, g [5117], we have, by neglecting, for brevity, the constant parts of Γ', Λ', corresponding to $t=0$,

$$\Gamma' = (1-c)\,.\,m't = 0{,}00845199\,.\,m't; \qquad \Lambda' = (1-g)\,.\,m't = -\,0{,}00402175\,.\,m't; \quad [3082b]$$

$$2\,\Lambda' + \Gamma' = 0{,}00040849\,.\,m't = i\,m't; \qquad \text{putting,} \quad i = 0{,}00040849. \quad [3082c]$$

Now if $P'\,d\,t$ [3062'] contain a term of the form $m'\,d\,v'\,.\,A'\,.\,{}^{\sin.}_{\cos.}\,(2\,\Lambda' + \Gamma')$, and [3082d]

we substitute $d\,v' = m'\,d\,t$ [3063r], it becomes, $m'^2 dt\,.\,A'\,.\,{}^{\sin.}_{\cos.}\,i\,m'\,t$; whose integral [3082e]
produces, in $\int P'\,d\,t$, the term,

$$\mp \frac{m'.A'}{i}\,.\,{}^{\cos.}_{\sin.}\,i\,m'\,t = \mp \frac{m'.A'}{i}\,.\,{}^{\cos.}_{\sin.}\,(2\,\Lambda' + \Gamma'). \quad [3082f]$$

If we examine the value of $P'\,d\,t$ [3062', 3056, 3057], we shall find that the angle Γ' [3082g]
is always connected with the coefficient c', and $2\,\Lambda'$ with the coefficient γ'^2; therefore
A' will be of the order $e'\gamma'^2$, and the term of $\int P'\,d\,t$ [3082f], will be of the order [3082h]
$\frac{m'e'\gamma'^2}{i}$. Substituting the values $e' = 0{,}05487293$ [5120], $\gamma' = 0{,}0900807$ [5117],

and i [3082c], it becomes nearly equal to m', which is of the same order as the chief
term of [3080b], depending on the moon; namely, $-\frac{3}{4}m'\,.\,\sin.\,\theta\,.\,\cos.\,2\,v'$. This last [3082i]
expression produces, in [3377], a coefficient of the order $0'',3$; therefore a term of this kind would require to be noticed. But if we substitute the values of XY, XZ [3056, 3057] in [3062'], it will appear evident, by a very slight examination, that there is no
term having the factor $e\gamma^2$, or $e'\gamma'^2$, except it be multiplied by the sine or cosine [3082k]
of an angle containing either v, v', or its multiple, so that it contains no term of the form,

$e'\gamma'^2\,.\,{}^{\sin.}_{\cos.}\,(2\,\Lambda' + \Gamma')$, as is observed in [3082'].

† (2076) This is proved in Book VII, [4783—4804]. [3083a]

[3084] moved upon the plane of the solar orbit, we shall have $\gamma' = \gamma$, and $\text{A}' = \text{A}$; therefore, by noticing the variations of the solar orbit,* we have

[3085]
$$\int \gamma' dt \,.\, \sin.\, \text{A}' = -\Sigma\, \frac{c}{f} \,.\, \cos.\, (ft + \beta).$$

[3086] Moreover, if we put c' for the tangent of the mean inclination of the orbit
c', f', β'. of the moon, to that of the sun, and $-f't - \beta'$, for the longitude of the ascending node of the lunar orbit upon the sun's orbit, *counted from*
[3086'] *the moveable vernal equinox*, we shall have, for the terms depending on this inclination,†

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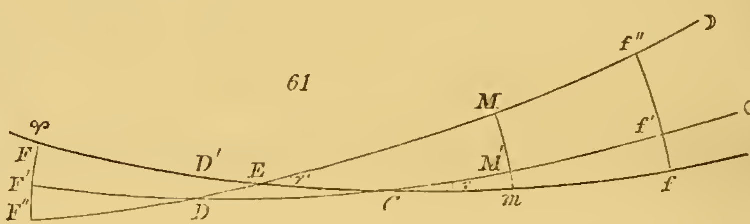
* (2077) Substituting, in $\gamma \,.\, \sin.\, \text{A}$ [3075c], the values of γ', A' [3084], it becomes
[3085a] $\gamma' \,.\, \sin.\, \text{A}' = \Sigma \,.\, c \,.\, \sin.\, (ft + \beta)$. Multiplying this by dt, and integrating, we get the value of $\int \gamma' dt \,.\, \sin.\, \text{A}'$ [3085], depending on the secular variations of the solar orbit.

† (2078) We shall suppose FCf to be an arch in the fixed plane, or fixed ecliptic,
[3087a] as it appears on the spherical surface of the heavens; $F'DCf'$ an



[3087b] arc of the sun's moveable orbit, $F''DEf''$ the moon's orbit; F' the first point of Aries,
[3087c] supposed to be *moveable*; F, F'', the corresponding points on the fixed plane, and on the lunar orbit. Then drawing the arc DD' perpendicular to FC, we shall have
[3087d] nearly $FCF' = \gamma$, $FC = \text{A}$ [3051, 3052] $FEF'' = \gamma'$, $FE = \text{A}'$ [3078]. If we neglect the powers of γ, γ', we may suppose the arch $FF'F''$ to be perpendicular to either of the arches CF, CF' or EF''; and in the triangles FCF', $F'DF'$, FEF'', we shall have nearly, $FF' = FCF' \,.\, \sin.\, FC$; $F'F'' = F'DF'' \,.\, \sin.\, F'D$,
[3087e] or $F'F'' = F'DF'' \,.\, \sin.\, FD'$; $FF'' = FEF'' \,.\, \sin.\, FE$, respectively; hence
[3087f] $FF'' = FF' + F'F''$, becomes $FEF'' \,.\, \sin.\, FE = FCF' \,.\, \sin.\, FC + F'DF'' \,.\, \sin.\, F'D$. Substituting in these equations, the preceding symbols, γ, A, γ', A'; also, for $F'DF''$, or rather for its tangent, the preceding value c' [3086], and $F'D = -f't - \beta'$ [3086],
[3087g] it becomes $\gamma' \,.\, \sin.\, \text{A}' = \gamma \,.\, \sin.\, \text{A} - c' \,.\, \sin.\, (f't + \beta')$. Multiplying this by dt, and integrating, we get,

[3087h]
$$\int \gamma' dt \,.\, \sin.\, \text{A}' = \frac{c'}{f'} \,.\, \cos.\, (f't + \beta') + \int \gamma\, dt \,.\, \sin.\, \text{A}.$$

Now we have shown in [3085, 3085a], that the quantity $\int \gamma\, dt \,.\, \sin.\, \text{A}$, or the part of $\int \gamma' dt \,.\, \sin.\, \text{A}'$, depending on the secular variation of the solar orbit, is $-\Sigma \,.\, \frac{c}{f} \,.\, \cos.\, (ft + \beta)$
[3087i] [3085]. Substituting this in [3087h], we get the whole value of $\int \gamma' dt \,.\, \sin.\, \text{A}'$, as in [3088].

$$\int \gamma'\, dt \,.\, \sin.\, \text{A}' = \frac{c'}{f'} \,.\, \cos.\, (f' t + \beta')\,; \qquad [3087]$$

connecting these two terms, we have, relatively to the moon,

$$\int \gamma'\, dt \,.\, \sin.\, \text{A}' = \frac{c'}{f'} \,.\, \cos.\, (f' t + \beta') - \Sigma \,.\, \frac{c}{f} \,.\, \cos.\, (ft + \beta)\,; \qquad [3088]$$

and we have, by the combined actions of the sun and moon,*

General expression of the obliquity of the ecliptic.

$$\theta = h + \frac{3m}{4n} \,.\, \left(\frac{2C - A - B}{C}\right) \,.\, \left\{ \begin{array}{l} \tfrac{1}{2} \sin.\, \theta \,.\, \{\cos.\, 2v + \frac{\lambda m}{m'} \,.\, \cos.\, 2v'\} \\ -(1+\lambda) \,.\, m \,.\, \cos.\, \theta \,.\, \Sigma \,.\, \frac{c}{f} \,.\, \cos.\, (ft + \beta) \\ + \frac{\lambda m c'}{f'} \,.\, \cos.\, \theta \,.\, \cos.\, (f' t + \beta') \end{array} \right\} . \qquad [3089]$$

6. *We shall now investigate the value of* ψ; and for this purpose we shall resume the equation [3041], putting it under the form,†

$$d\psi \,.\, \sin.\, \theta = \frac{(2C - A - B)}{2n \,.\, C} \,.\, P\, dt. \qquad [3090]$$

We have by the preceding article, relatively to the sun,‡

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* (2079) If we substitute the value of $\int \gamma'\, dt \,.\, \sin.\, \text{A}'$, [3088] in [3080], and then connect this part of $\int P' dt$ with that in [3076], we shall obtain,

$$\int P' dt = -\tfrac{3}{4} \,.\, m \,.\, \sin.\, \theta \,.\, \{\cos.\, 2v + \frac{\lambda m}{m'} \,.\, \cos.\, 2v'\} + \tfrac{3}{2} \,.\, m^2 \,.\, (1+\lambda) \,.\, \cos.\, \theta \,.\, \Sigma \,.\, \frac{c}{f} \,.\, \cos.\, (ft + \beta) \qquad [3089a]$$

$$-\tfrac{3}{2} \,.\, \lambda m^2 \,.\, \frac{c'}{f'} \,.\, \cos.\, \theta \,.\, \cos.\, (f' t + \beta')\,;$$

substituting this in [3050], it becomes, as in [3089]; to which we must add the term [3081*e*].

† (2080) Substituting $P = \Sigma \,.\, k \,.\, \cos.\, (it + \varepsilon)$ [3038], in [3041], we get [3090]. [3090*a*]

‡ (2081) If we square the value of Y [3054], and put, as in [1, 17] Int.

$$\sin.^2 v = \tfrac{1}{2} - \tfrac{1}{2} \cos.\, 2v, \qquad \sin.^2 (v - 2\text{A}) = \tfrac{1}{2} - \tfrac{1}{2} \,.\, \cos.\, (2v - 4\text{A})\,; \qquad [3091a]$$

$$2 \sin.\, v \sin.\, (v - 2\text{A}) = \cos.\, 2\text{A} - \cos.\, (2v - 2\text{A}),$$ it becomes

$$Y^2 = \tfrac{1}{2} \,.\, r_{\prime}^2 \,.\, \cos.^4 \tfrac{1}{2}\gamma \,.\, (1 - \cos.\, 2v) - r_{\prime}^2 \,.\, (\cos.\, \tfrac{1}{2}\gamma \,.\, \sin.\, \tfrac{1}{2} v)^2 \,.\, \{\cos.\, 2\text{A} - \cos.\, (2v - 2\text{A})\} \qquad [3091b]$$

$$+ \tfrac{1}{2} r'^2 \,.\, \sin.^4 \tfrac{1}{2}\gamma \,.\, \{1 - \cos.\, (2v - 4\text{A})\}.$$

This may be reduced by [31] Int., which gives $(\cos.\, \tfrac{1}{2}\gamma \,.\, \sin.\, \tfrac{1}{2}\gamma)^2 = (\tfrac{1}{2} \sin.\, \gamma)^2 = \tfrac{1}{4} \sin.^2 \gamma$. [3091*c*]

$$Y^2-Z^2=\tfrac{1}{2}r_{,}^2.\cos.^4\tfrac{1}{2}\gamma.(1-\cos.2v)-\tfrac{1}{4}.r_{,}^2.\sin.^2\gamma.\{\cos.2\text{A}-\cos.(2v-2\text{A})\}$$

[3091] $$-\tfrac{1}{2}.r_{,}^2.\sin.^2\gamma.\{1-\cos.(2v-2\text{A})\}+\tfrac{1}{2}r_{,}^2.\sin.^4\tfrac{1}{2}\gamma.\{1-\cos.(2v-4\text{A})\};$$

$$YZ=\tfrac{1}{4}r_{,}^2.\sin 2\gamma.\cos.\text{A}-\tfrac{1}{2}r_{,}^2.\sin.\gamma.\cos.^2\tfrac{1}{2}\gamma.\cos.(2v-\text{A})$$

[3092] $$+\tfrac{1}{2}r_{,}^2.\sin.\gamma.\sin.^2\tfrac{1}{2}\gamma.\cos.(2v-3\text{A}).$$

Therefore we shall find, by the analysis of the same article, neglecting
[3092′] the squares of e and γ, and the quantities which remain insensible after integration,*

Squaring the value of Z [3055], and substituting $\sin.^2(v-\text{A})=\tfrac{1}{2}-\tfrac{1}{2}\cos.2(v-\text{A})$
[3091d] [1] Int., we get, $Z^2=\tfrac{1}{2}r_{,}^2.\sin^2\gamma\,\{1-\cos.(2v-2\text{A})\}$. Subtracting this from Y^2 [3091b,c], we get [3091].

The value of YZ [3092] may be found by multiplying Y [3054] by Z [3055], and reducing, as in [3056g,h]. We may also derive it from the value of XZ [3057].
[3091e] For if we write $v-100^\circ$, and $\text{A}-100^\circ$, for v, and A respectively, the value of Z [3055] will be unchanged; but X [3053] will change into Y [3054]; consequently XZ [3057] will change into YZ. Now by making these changes in XZ [3057], the quantity $\sin.(2v-\text{A})$ becomes $\sin.(2v-\text{A}-100^\circ)=-\cos.(2v-\text{A})$; $\sin.\text{A}$
[3091f] becomes $-\cos.\text{A}$; $\sin.(2v-3\text{A})$ becomes $\sin.(2v-3\text{A}+100^\circ)=\cos.(2v-3\text{A})$; these changes being made in [3057], it becomes, as in [3092].

* (2082) Multiplying P [3016] by $dt=\frac{r_{,}^2.dv}{a^2m.\sqrt{1-ee}}$ [3058], we get Pdt [3093b].
[3093a] Substituting in this $\frac{L}{a^2mr_{,}}=m.\frac{a}{r_{,}}=m.\frac{\{1+e.\cos.(v-\Gamma)\}}{1-ee}$, which is easily deduced from [3058a, 3061], we get [3093c]

[3093b] $$P\,dt=\frac{3L\,dv}{a^2mr_{,}.\sqrt{1-ee}}.\left\{\frac{Y^2-Z^2}{r_{,}^2}.\sin.\theta.\cos.\theta+\frac{YZ}{r_{,}^2}.(\cos.^2\theta-\sin.^2\theta)\right\}$$

[3093c] $$=\frac{3m\,dv\{1+e.\cos.(v-\Gamma)\}}{(1-ee)^{\frac{3}{2}}}.\left\{\frac{Y^2-Z^2}{r_{,}^2}.\sin.\theta.\cos.\theta+\frac{YZ}{r_{,}^2}.(\cos.^2\theta-\sin.^2\theta)\right\}.$$

Now by the same method of reasoning as in [3062a, &c.], we shall find, that the factor
[3093d] $e.\cos.(v-\Gamma)$, when multiplied by the terms $\frac{Y^2-Z^2}{r_{,}^2}$, $\frac{YZ}{r_{,}^2}$ [3091, 3092], produces terms depending on the cosines of angles of the forms $v-\Gamma$, $v+\Gamma$, $3v-\Gamma$, &c.; all of which contain v, whose integrals will not have any terms with the small divisor i;

$$P\,.\,dt=\tfrac{3}{2}m^2\,.\,dt\,.\sin.\theta\,.\cos.\theta-\tfrac{3}{4}m\,.\sin.\theta\,.\cos.\theta\,.\,d\,.\sin.2v$$
$$+\tfrac{3}{2}m^2\,.\,\gamma\,dt\,.\cos.\text{A}\,.\,\{\cos.^2\theta-\sin.^2\theta\}. \qquad [3093]$$

In this we must substitute for $\gamma.\cos.\text{A}$, its value $\Sigma.c.\cos.(ft+\beta)$ [3075e]. [3094]

By the same analysis, we shall find, relatively to the moon,*

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consequently they may be neglected in Pdt; because they will not produce any sensible terms in ψ, deduced from the integral of [3090]. Therefore we may neglect this term, as we have done in computing $P'dt$ [3063f]; and by proceeding as in [3063f, &c.], we may also neglect quantities of the order of the square, or the product of the disturbing forces. Hence it appears, that if we neglect terms of the order e^2, we may change the factor, [3093e]

$\frac{3m\,dv.\{1+e.\cos.(v-\text{r})\}}{(1-ee)^{\frac{3}{2}}}$ [3093c], into $3m\,dv$; and we shall have, [3093f]

$$P\,dt=3m\,dv\,.\left\{\frac{Y^2-Z^2}{r_{\prime}^2}\,.\sin.\theta\,.\cos.\theta+\frac{YZ}{r_{\prime}^2}\,.(\cos.^2\theta-\sin.^2\theta)\right\}. \qquad [3093g]$$

Now if we substitute $\cos.^4\tfrac{1}{2}\gamma=(1-\sin.^2\tfrac{1}{2}\gamma)^2$ in [3091, 3092], and neglect terms of the order $\gamma^2(1-\cos.2v)$, $\gamma.\cos.(2v-2\text{A})$, and such terms of the order γ^2 as are neglected in [3063i, &c.), we shall obtain, [3093h]

$$\frac{Y^2-Z^2}{r_{\prime}^2}=\tfrac{1}{2}(1-\cos.2v)-\tfrac{1}{4}\sin.^2\gamma\,.\cos.2\text{A}\,; \qquad \frac{YZ}{r_{\prime}^2}=\tfrac{1}{4}\sin.2\gamma\,.\cos.\text{A}. \qquad [3093i]$$

Substituting these in [3093g], we get, successively, by putting $dv\,.\cos.2v=\tfrac{1}{2}d\,.\sin.2v$, and $\sin.2\gamma=2\gamma$ nearly;

$$P\,dt=3m\,dv\,.\left\{\begin{array}{l}[\tfrac{1}{2}(1-\cos.2v)-\tfrac{1}{4}\sin.^2\gamma\,.\cos.2\text{A}]\,.\sin.\theta\,.\cos.\theta\\+\tfrac{1}{4}\sin.2\gamma\,.\cos.\text{A}\,.(\cos.^2\theta-\sin.^2\theta)\end{array}\right\} \qquad [3093k]$$
$$=\tfrac{3}{2}m\,dv\,.\sin.\theta\,.\cos.\theta-\tfrac{3}{4}m\,.\sin.\theta\,.\cos.\theta\,.\,d\,.\sin.2v$$
$$+\tfrac{3}{2}m\,.\,dv\,.\gamma\,.\cos.\text{A}\,.(\cos.^2\theta-\sin.^2\theta) \qquad [3093l]$$
$$-\tfrac{3}{4}m\,dv\,.\sin.\theta\,.\cos.\theta\,.\sin.^2\gamma\,.\cos.2\text{A}.$$

We may put, as in [3063r], $dv=m\,dt$, and then the expression [3093l] becomes, as in [3093], with the addition of the last term, depending on 2A, namely, $-\tfrac{3}{4}m^2.dt.\sin.\theta.\cos.\theta.\sin.^2\gamma.\cos.2\text{A}$, which is neglected by La Place in this book; but the similar term depending on the moon's attraction, is calculated in the fifth volume [12228]. [3093m]

* (2083) Accenting the letters, m, v, γ, A, &c. [3093, 3093l, m], as in [3078], we get the following expression of the part of Pdt, depending upon the attraction of the moon;

$$P\,dt=\tfrac{3}{2}m'^2.dt.\sin.\theta.\cos.\theta-\tfrac{3}{4}m'.\sin.\theta.\cos.\theta.d.\sin.2v'+\tfrac{3}{2}m'^2.\gamma'dt.\cos.\text{A}'.\{\cos.^2\theta-\sin.^2\theta\} \qquad [3093n]$$
$$-\tfrac{3}{4}m'^2.dt.\sin.\theta.\cos.\theta.\sin.^2\gamma'.\cos.2\text{A}'.$$

[3095]
$$P\,dt = \frac{3\lambda . m^2 . dt}{2} . \sin.\theta . \cos.\theta - \frac{3\lambda . m^2}{4m'} . \sin.\theta . \cos.\theta . d . \sin. 2v'$$
$$+ \tfrac{3}{2}\lambda . m^2 . dt . \{\cos.^2\theta - \sin.^2\theta\} . \Sigma . c . \cos.(ft+\beta)$$
$$+ \tfrac{3}{2}\lambda . m^2 . dt . \{\cos.^2\theta - \sin.^2\theta\} . c' . \cos.(f't+\beta') ;$$

consequently we shall have,*

[3093o] Substituting $m'^2 = \lambda m^2$, $m' = \frac{\lambda m^2}{m'}$ [3080c], it becomes,

[3093p]
$$P\,dt = \tfrac{3}{2}\lambda m^2 . dt . \sin.\theta . \cos.\theta - \frac{3\lambda m^2}{4m'} . \sin.\theta . \cos.\theta . d . \sin. 2v'$$
$$+ \tfrac{3}{2}\lambda m^2 . \{\cos.^2\theta - \sin.^2\theta\} . \gamma' dt . \cos. \text{A}'$$
$$- \tfrac{3}{4}\lambda m^2 . dt . \sin.\theta . \cos.\theta . \sin.^2\gamma' . \cos. 2\text{A}'.$$

[3093q] If we take, in fig. 61, page 846, the arc $FCf = 100^\circ$, then draw the arc $ff'f''$ perpendicularly to FCf, it may also be considered as perpendicular to either of the arcs,
[3093r] $F'Cf'$, $F''Df''$, neglecting the powers of γ, γ', and we shall have, $ff'' = ff' + f'f''$.
[3093s] Now by the same method as in [3087e], we have, in the triangles fCf', $f'Df''$, fEf'', respectively, $ff' = fCf' . \sin. fC$; $f'f'' = f'Df'' . \sin. fD'$; $ff'' = fEf'' . \sin. fE$.
[3093t] Substituting these in [3093r], we get, $fEf'' . \sin. fE = fCf' . \sin. fC + f'Df'' . \sin. fD'$.
[3093u] If we use the symbols [3087d, g], we have, $fEf'' = \gamma'$, $fCf' = \gamma$, $f'Df''$ or its
[3093v] tangent $= c'$, $fE = 100^\circ - FE = 100^\circ - \text{A}'$, $fC = 100^\circ - FC = 100^\circ - \text{A}$, $fD' = 100^\circ - F'D = 100^\circ + f't + \beta'$. Substituting these values in the equation
[3093w] [3093t], we get, $\gamma' . \cos. \text{A}' = \gamma . \cos. \text{A} + c' . \cos.(f't+\beta')$. Multiplying this by dt; then substituting $\gamma . \cos. \text{A} = \Sigma . c . \cos.(ft+\beta)$ [3075c], we obtain,

[3093x]
$$\gamma' dt . \cos. \text{A}' = dt . \Sigma . c . \cos.(ft+\beta) + c' . dt . \cos.(f't+\beta').$$

Hence [3093p] becomes, as in [3095], with the addition of the term depending on
[3093y] $\cos. 2\text{A}'$; namely, $-\tfrac{3}{4}\lambda m^2 . dt . \sin.\theta . \cos.\theta . \sin.^2\gamma' . \cos. 2\text{A}'$. This term was afterwards noticed by the author, as we have already mentioned in [3093m].

* (2087) Substituting $\gamma . \cos. \text{A} = \Sigma . c . \cos.(ft+\beta)$ [3075e] in [3093], and then adding the two parts of $P\,dt$ [3093, 3095], we get the value of $P\,dt$, to be substituted in [3090]; then dividing by $dt . \sin.\theta$, we obtain [3096]; and if we use the value of l [3098], it becomes,

[3096a]
$$\frac{d\psi}{dt} = l . \left\{ \begin{array}{l} \dfrac{\cos.\theta}{\cos.h} - \dfrac{\cos.\theta}{2dt.\cos.h} . \left[d.\sin. 2v + \dfrac{\lambda m}{m'} . d . \sin. 2v' \right] . \dfrac{1}{m(1+\lambda)} \\ + \dfrac{(\cos.^2\theta - \sin.^2\theta)}{\sin.\theta . \cos.h} . \Sigma . c . \cos.(ft+\beta) + \dfrac{\lambda}{1+\lambda} . \dfrac{(\cos.^2\theta - \sin.^2\theta)}{\sin.\theta . \cos.h} . c' . \cos.(f't+\beta') \end{array} \right\} .$$

Differential equation of the Precession.

$$\frac{d\psi}{dt}=\frac{3m}{4n}\cdot\frac{(2C-A-B)}{C}\cdot\left\{\begin{array}{l}(1+\lambda).m.\cos.\theta-\frac{\cos.\theta}{2dt}.\left\{d.\sin.2v+\frac{\lambda m}{m'}.d.\sin.2v'\right\}\\ +(1+\lambda).m.\frac{\{\cos.^2\theta-\sin.^2\theta\}}{\sin.\theta}.\Sigma.c.\cos.(ft+\beta)\\ +\lambda m.\frac{\{\cos.^2\theta-\sin.^2\theta\}}{\sin.\theta}.c'.\cos.(f't+\beta')\end{array}\right\}. \quad [3096]$$

To integrate this equation, we shall observe, that the value of θ is not constant, and that its secular variations become sensible, by integration, in the first term of this expression of $\frac{d\psi}{dt}$.* Now the only variable part of the value of θ, which can, in the course of time, become somewhat large, is the following [3089]; [3096'] [3096'']

$$-\frac{3m^2}{4n}\cdot\left(\frac{2C-A-B}{C}\right).(1+\lambda).\cos.\theta.\Sigma.\frac{c}{f}.\cos.(ft+\beta). \quad [3097]$$

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* (2088) If we put l' for the term multiplied by $\frac{2C-A-B}{C}$ in the value of θ [3089], we shall have, $\theta=h+l'$; and by neglecting the square and higher powers of l', we shall find, from [61, 60] Int. $\cos.\theta=\cos.h-l'.\sin.h$; $\sin.\theta=\sin.h+l'.\cos.h$. Substituting these in [3096a], it produces terms, multiplied by ll', which are so small that they may be neglected in $d\psi$; but some of them, by integration, acquire a small divisor, which render them sensible in ψ. Now upon examination, it will easily be perceived, that the terms of l' [3097a], depending on the angles v, v', $f't$, corresponding to the motions of the sun, moon, and the moon's node, cannot produce such divisors. If we neglect such terms in the value of l' [3097a], we shall find, as in [3097], [3097a] [3097b] [3097c] [3097d]

$$l'=-\frac{3m^2}{4n}\cdot\left(\frac{2C-A-B}{C}\right).(1+\lambda).\cos.\theta.\Sigma.\frac{c}{f}.\cos.(ft+\beta); \quad [3097e]$$

and the quantities producing similar terms in $\frac{d\psi}{dt}$ [3096a], will be,

$$l.\left\{\frac{\cos.\theta}{\cos.h}+\frac{(\cos.^2\theta-\sin.^2\theta)}{\sin.\theta.\cos.h}.\Sigma.c.\cos.(ft+\beta)\right\}. \quad [3097f]$$

The last of these quantities is very small, as is evident from [3365, &c.]; so that we may substitute in it $\theta=h$, and neglect the correction depending on l', in the values of $\cos.\theta$, $\sin.\theta$ [3097b]. Therefore the only term of $\frac{d\psi}{dt}$, in which the variations of θ are required to be noticed, is the first; namely, $l.\frac{\cos.\theta}{\cos.h}$, as in [3096'']. If in this we substitute $\cos.\theta$ [3097b], and then l' [3097e], it becomes, [3097g] [3097h]

$$l-ll'.\text{tang}.h=l+l.\text{tang}.h.\left\{\frac{3m^2}{4n}\cdot\left(\frac{2C-A-B}{C}\right).(1+\lambda).\cos.\theta.\Sigma.\frac{c}{f}.\cos.(ft+\beta)\right\}; \quad [3097i]$$

putting $\theta=h$, and using l [3098], we obtain [3099].

This is therefore the only term necessary to be noticed; and if we put, for brevity,

[3098] $$l = \frac{3m^2}{4n} \cdot \left(\frac{2C - A - B}{C} \right) \cdot (1+\lambda) \cdot \cos. h,$$

[3098'] the first term of the expression of $\frac{d\psi}{dt}$ [3096*a*], will become, by neglecting quantities of the order c^2,

[3099] $$\frac{l \cdot \cos. \theta}{\cos. h} = l + l^2 \cdot \text{tang.}\, h \cdot \Sigma \cdot \frac{c}{f} \cdot \cos. (ft+\beta).$$

It is not necessary to notice the variableness of θ, in the other terms of the

[3099'] expression of $\frac{d\psi}{dt}$, which, by integration, gives,*

Precession of the equinoxes on the fixed ecliptic.

[3100] $$\psi = lt + \varepsilon + \Sigma . \left\{ \left(\frac{l}{f} - 1 \right) . \text{tang.}\, h + \cot. h \right\} . \frac{lc}{f} . \sin. (ft+\beta) - \frac{l}{2m(1+\lambda)} . \sin. 2v$$
$$- \frac{l\lambda}{2m' . (1+\lambda)} . \sin. 2v' + \frac{l\lambda}{(1+\lambda) . f'} . \frac{\{\cos.^2 h - \sin.^2 h\}}{\sin. h . \cos. h} . c' . \sin. (f't+\beta');$$

[3100'] ε being an arbitrary constant quantity.

The expression of θ of the preceding article, may be put under this form,

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* (2089) Substituting in [3096*a*], the value of its first term [3099], putting in the

[3100*a*] other terms $\theta = h$, multiplying by dt, and integrating, we get [3100]; observing that the terms affected with the sign Σ, appear at first under the form,

$$l^2 . \text{tang.}\, h . \Sigma . \frac{c}{f^2} . \sin. (ft+\beta) + l . \frac{(\cos.^2 h - \sin.^2 h)}{\cos. h . \sin. h} . \Sigma . \frac{c}{f} . \sin. (ft+\beta);$$

which, by substituting $\frac{\cos.^2 h - \sin.^2 h}{\cos. h . \sin. h} = \cot. h - \text{tang.}\, h,$ becomes,

[3100*b*]

$\Sigma . \left\{ \left(\frac{l}{f} - 1 \right) . \text{tang.}\, h + \cot. h \right\} . \frac{lc}{f} . \sin. (ft+\beta),$ as in [3100]. The term neglected in [3093*y*], being integrated as in [3081*d*, &c.], produces in $fP\,dt$, the term,

[3100*c*] $-\frac{3\lambda m^2}{8f'} . \sin. \theta . \cos. \theta . \sin.^2 \gamma' . \sin. 2(f't+\beta')$. Substituting in this, the values, $\theta = h$, $\sin. \gamma' = c'$ [3081*c*], the resulting expression will produce in ψ, [3090], the term,

[3100*d*] $$-\frac{(2C - A - B)}{2nCf' \sin. h} . \left\{ \tfrac{3}{8} \lambda m^2 . \sin. h . \cos. h . c'^2 . \sin. 2(f't+\beta') \right\}$$
$$= -\frac{l\lambda}{4(1+\lambda) . f'} . c'^2 . \sin. 2(f't+\beta') \text{ [3098]}.$$

This term of ψ must be added to ψ [3100], and to ψ' [3107].

$$\theta = h - \Sigma \cdot \frac{lc}{f} \cdot \cos.\,(ft+\beta) + \frac{l\lambda}{(1+\lambda)\cdot f'} \cdot c' \cdot \cos.\,(f't+\beta')^{*}$$

$$+ \frac{l \cdot \text{tang}.\, h}{2m \cdot (1+\lambda)} \cdot \left\{ \cos.\, 2v + \frac{m}{m'} \cdot \lambda \cdot \cos.\, 2v' \right\}.$$ Obliquity of the ecliptic referred to a fixed plane. [3101]

These values of ψ and θ, taken in connexion with that of $p=n$ [3015], furnish the necessary data for determining, at any time, the motion of the earth about its centre of gravity. [3101']

7. The expressions of ψ and θ correspond to the fixed plane. To obtain [3101''] their values, relative to the apparent ecliptic, we shall consider the spherical [3102] triangle, formed by the fixed ecliptic, the apparent ecliptic, and the equator. It is evident, that the difference of the two arcs, intercepted between the [3102'] equator and the ascending node of the solar orbit, in this triangle, is nearly equal to the product of cot. θ, by the inclination of the solar orbit to the [3103] fixed ecliptic, and by the sine of the longitude of its node.† This difference is therefore equal to $\cot.\,\theta \cdot \Sigma \cdot c \cdot \sin.\,(ft+\beta)$. Now if we put ψ' for the [3104]

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* (2090) This is deduced from [3089], by substituting, for the factor $\frac{3m}{4n} \cdot \left(\frac{2C-A-B}{C}\right)$, its value $\frac{l}{m(1+\lambda)\cdot\cos.\,h}$ [3098], and putting $\theta = h$, in the other terms. We must [3101a] also add to this expression the term neglected in [3081e].

† (2091) We shall suppose, in the annexed figure, that $DC'H$ is the equator, [3102a] $DCBAG$ the fixed ecliptic, $C'B'AF$ the apparent ecliptic; B, the point of the fixed ecliptic, corresponding to the origin of the angles $\Gamma_{\prime}$, ψ, and [3102b] B' the corresponding point on $C'B'AF$; so that we may consider BB' as perpendicular to AD. We shall also draw CC' perpendicularly to AD; then A is the ascending node of the solar orbit, and [3102c] DAC' the spherical triangle mentioned in [3102]. The angle DAC' being very small, the difference of the two arcs, DA, $C'A$, is expressed nearly by the small arc $CD = AD - AC'$.

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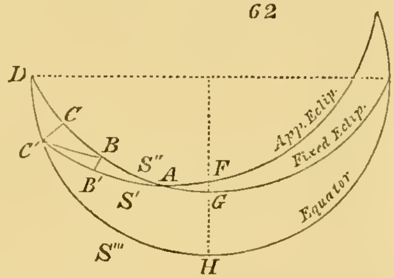

The arc CC' is nearly equal to the product of the inclination of the solar orbit DAC', by [3102d] the sine of AC' or AD, and as $CDC' = \theta$, we have very nearly, in the triangle CDC', $CD = CC' \cdot \cot.\,\theta$, or $AD - AC' = CC' \cdot \cot.\,\theta = (\text{Angle } DAC' \times \sin.\,AD) \times \cot.\,\theta$, [3102e] as in [3103]. Now $DAC' = \gamma$, $AD = \text{A}$ [3066', 3069]; therefore we have, by neglecting the powers of γ, as has been done throughout this calculation, $AD - AC' = \gamma \cdot \sin.\,\text{A} \cdot \cot.\,\theta$. [3102f] Substituting $\gamma \cdot \sin.\,\text{A}$ [3075e], it becomes, $AD - AC' = \cot.\,\theta \cdot \Sigma \cdot c \cdot \sin.\,(ft+\beta)$, as in [3104]. In all these calculations, terms of the order c^2 are neglected [3072'].

distance of the intersection of the apparent ecliptic and the equator from the invariable origin, from which we count the angle ψ, upon the fixed
[3105] plane, we shall have, very nearly, $\psi - \psi'$ for this difference;* therefore we shall find,

[3106] $$\psi - \psi' = \cot.\,\theta\,.\,\Sigma\,.\,c\,.\,\sin.\,(ft+\beta).$$

Hence we deduce,

Precession of the equinoxes referred to the moveable ecliptic.

[3107] $$\psi' = lt + \zeta + \Sigma\,.\left\{1 + \frac{l}{f}\,.\,\mathrm{tang.}^2 h\right\}.\left(\frac{l-f}{f}\right).\cot.\,h\,.\,c\,.\,\sin.\,(ft+\beta) + \frac{l\lambda}{(1+\lambda)\,.\,f'}\,.\,\frac{\{\cos.^2 h - \sin.^2 h)\}}{\sin.\,h\,.\,\cos.\,h}\,.\,c'\,.\,\sin.\,(f't+\beta') - \frac{l}{2m\,.\,(1+\lambda)}\,.\,\sin.\,2v - \frac{l\lambda}{2m'\,.\,(1+\lambda)}\,.\,\sin.\,2v'.$$

[3108] Then if we put θ' for the inclination of the apparent ecliptic to the equator, we easily find, by referring to the preceding spherical triangle, and observing that $\theta' - \theta$ is very small,†

[3109] $$\theta' - \theta = \Sigma\,.\,c\,.\,\cos.\,(ft+\beta)\,;$$

therefore we have,‡

Apparent obliquity of the ecliptic.

[3110] $$\theta' = h + \Sigma\,.\left(\frac{f-l}{f}\right).\,c\,.\,\cos.\,(ft+\beta) + \frac{l\lambda}{(1+\lambda)\,.\,f'}\,.\,c'\,.\,\cos.\,(f't+\beta') + \frac{l\,.\,\mathrm{tang.}\,h}{2m\,.\,(1+\lambda)}\,.\left\{\cos.\,2v + \frac{m}{m'}\,.\,\lambda\,.\,\cos.\,2v'\right\}.$$

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* (2092) In fig. 62, page 853, $\Gamma_{\prime} = BA$, or $B'A$ [3066′] nearly; and by [3068′, 3069],
[3106a] $\psi = BD$, $\Lambda = \Gamma_{\prime} + \psi = AD$; also from [3104], $\psi' = BC'$ or $B'C'$ nearly; hence
[3106b] $\psi - \psi' = BD - BC'$, which is evidently very nearly equal to CD or $AD - AC'$; therefore $\psi - \psi' = \cot.\,\theta\,.\,\Sigma\,.\,c\,.\,\sin.\,(ft+\beta)$ [3102f], as in [3106]. This gives, by
[3106c] changing θ into h, $\psi' = \psi - \cot.\,h\,.\,\Sigma\,.\,c\,.\,\sin.\,(ft+\beta)$; and by substituting the value of ψ [3100], we get [3107], as easily appears by connecting together the terms depending on c, observing that $\cot.\,h\,.\,\mathrm{tang.}\,h = 1$.

[3109a] † (2093) If we make $DH = DG = 100^{\circ}$, the arch GH will be equal to θ, the inclination of the equator to the *fixed* ecliptic [3101″]; and if this arch HG cuts the apparent ecliptic in F, the arch HF will be very nearly equal to θ', the arch DC' being extremely small;
[3109b] hence we have, $FG = HF - GH = \theta' - \theta$. But in the triangle GAF right-angled at G, we get very nearly, $FG = FAG\,.\,\sin.\,AG = \gamma\,.\,\sin.\,AG = \gamma\,.\,\sin.\,(100^{\circ} - AD)$;
[3109c] and as $AD = \Lambda$ [3106a], it becomes $FG = \gamma\,.\,\cos.\,\Lambda$, consequently $\theta' - \theta = \gamma\,.\,\cos.\,\Lambda$; and by substituting the value of $\gamma\,.\,\cos.\,\Lambda$ [3075c], we obtain [3109].

[3110a] ‡ (2094) From [3109] we get, $\theta' = \theta + \Sigma\,.\,c\,.\,\cos.\,(ft+\beta)$; and by substituting the value of θ [3101], and connecting the terms depending on c, we obtain [3110].

The part $\Sigma . \left(\frac{f-l}{f}\right) . c . \cos. (ft+\beta)$ *of this expression, gives the secular variation of the obliquity of the apparent ecliptic to the equator. If the earth be spherical, there will be no precession arising from the action of the sun and moon;* we shall then have $l=0$,* and the secular variation of the obliquity of the apparent ecliptic will be, $\Sigma . c . \cos. (ft+\beta)$. Hence we perceive, that the action of the sun and moon upon the terrestrial spheroid changes considerably the laws of this variation, which will almost vanish if the motion of precession, arising from this action, be very rapid in comparison with the motion of the solar orbit. For this last motion depends on the angles $(f-l)\,t$ [3113b],† in which the coefficient $(f-l)$ [3111] [3112] [3113] [3113']

If the earth be composed of spherical strata, there will be no precession arising from the action of the sun and moon.

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* (2095) The earth being supposed spherical, the length of the radius R', drawn from its centre of gravity to the surface [2908, 2919], is constant, or independent of μ, ϖ; hence the value of R'^5 [2923] must be reduced to its first term, $U^{(0)}$, making $U^{(1)}=0$, $U^{(2)}=0$, &c. Now $U^{(2)}$ being equal to 0, we must necessarily have, $H=0$, $H''''=0$, &c. [2934]; and by substituting these values in [2936—2938], we get the same expression for the values of A, B, C, namely, $\frac{8\pi}{15} . U^{(0)}$; hence $A=B=C$; therefore $2\,C-A-B=0$. Substituting this in [3098], we get, $l=0$; by which means the secular variation of the obliquity of the ecliptic [3111], is reduced to $\Sigma . c . \cos. (ft+\beta)$, as in [3112]. [3111a] [3111b] [3111b'] [3111c]

† (2096) This corresponds to [3073c], where we have put $(f-g) . t$, for the mean value of ψ, which is equal to $l\,t$ [3100]; hence $(f-g) . t = lt$, or $g=f-l$; and the angle $g\,t$, upon which the displacement of the solar orbit depends, becomes $(f-l) . t$, as in [3113']. If the earth be supposed spherical, we get, $l=0$ [3111b'], $f=g$, and the secular variation of the obliquity [3111] becomes, [3113a] [3113b]

$$\Sigma . \frac{f}{f} . c . \cos. (g\,t+\beta) = \Sigma . c . \cos. (g\,t+\beta), \qquad [3113c]$$

and this is to the general value [3111], corresponding to a spheroidal form, as quantities of the order $\frac{f}{f} . c$ are to those of the order $\left(\frac{f-l}{f}\right) . c$, that is, as $f:f-l$ or $f:g$; and as g is much less than f, the displacement of the solar orbit must be much less than if the earth be supposed of a spherical form. La Place estimates that the secular variation is decreased three quarters of its value from this cause. This calculation has no other difficulty than its great length. The method is to compute g, g_1, g_2, &c. [1100v], noticing the effect of all the planets; then the *actual* variation of the obliquity by formula [3111]. This is to be compared with the variation which would take place independently of the attractions of the sun and moon upon the spheroidal shell of the earth, by formula [3111c], to obtain the ratio mentioned in [3115]. [3113d] [3113e] [3113f]

[3114] is very small in comparison with l and f; so that the function

$$\Sigma\,.\left(\frac{f-l}{f}\right).\,c\,.\cos.\,(ft+\beta)$$

becomes nearly insensible. Upon the most probable suppositions relative

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La Place has published an article on this subject, in the *Connoisance des Tems* for 1827, in which he points out some corrections to be made in the method used by La Grange, Euler, and Schubert, in determining the variations of the obliquity of the ecliptic, and of

[3113g] the precession of the equinoxes; to reduce them to the forms given in [3107, 3110]. We shall here give some of the results of this calculation. If we retain only the secular

[3113h] terms of θ' [3110], and suppose its value to be h, at the commencement of the year 1801, when we shall suppose $t=0$, we shall get θ' [3113k]. If the earth be spherical, we have $l=0$, $f=g$ [3113b], and θ' becomes as in [3113l]. If we were to refer the angle $gt+\beta$,

[3113i] to the moveable equinox, by adding to this angle the mean precession of the equinoxes lt, it would become, $(gt+lt)+\beta=ft+\beta$ [3113a]; and [3113l] would change into the inaccurate formula [3113m], used by some astronomers.

[3113k] $$\theta'=h+\Sigma\,.\left(\frac{f-l}{f}\right).\,c\,.\{\cos.\,(ft+\beta)-\cos.\,\beta\}$$

[3113l] $$\theta'=h+\Sigma\,.\,c\,.\{\cos.\,(gt+\beta)-\cos.\,\beta\}$$

[3113m] $$\theta'=h+\Sigma\,.\,c\,.\{\cos.\,(ft+\beta)-\cos.\,(lt+\beta)\}.$$

This last expression represents the value of θ', noticing the mean precession of the equinoxes, arising from the oblate form of the earth, but neglecting its effect when combined with the

[3113n] secular change of the plane of the ecliptic. La Place remarks, that the last form [3113m] is used by La Grange and Schubert, instead of the first [3113k]; in consequence of this,

[3113o] they make the maximum variation of the obliquity of the ecliptic about four times too great. La Place gives the numbers of Schubert's calculation, taken from the third volume of the last edition of his astronomy, which are equivalent to those in the following expression [3113q]. The difference consists in the reduction of two terms of the form $a.\sin.gt+b.\cos.gt$ for

[3113p] each planet, into one of the form $\alpha.\sin.(gt+\mu)$, by putting $a=\alpha.\cos.\mu$, $b=\alpha.\sin.\mu$; whence $\tang.\mu=\frac{b}{a}$, $\alpha=a.\sec.\mu$, and formula [3113m] becomes;

[3113q] Apparent obliquity of the ecliptic, deduced from Schubert's formula.

$$\begin{aligned}
\theta'=h&+5680^{s}.\sin.\,(193^{d}\,18^{m}+t\,.\,50{,}1^{s}\quad)\\
&+\ \ 657\,.\sin.\,(216\ 05+t\,.\,24{,}5715)\\
&+5823\,.\sin.\,(347\ 21+t\,.\,31{,}4271)\\
&+\ \ 396\,.\sin.\,(\ 41\ 54+t\,.\,33{,}3851)\\
&+1578\,.\sin.\,(343\ 24+t\,.\,43{,}2280)\\
&+3469\,.\sin.\,(114\ 32+t\,.\,45{,}3674).
\end{aligned}$$

[3113r] The sum is $=17603=4^{d}\,53^{m}\,23^{s}=$ the maximum variation of the obliquity according to Schubert.

to the masses of the planets, the whole extent of the variation of the obliquity of the ecliptic is reduced, by the action of the sun and moon upon the terrestrial spheroid, to nearly one quarter of the value it would have independently of this action [3113w]; but this difference is not manifested till after two or three centuries. [3115] [3116]

The variation of the obliquity of the ecliptic is decreased by the action of the sun and moon, upon the spheroidal shell of the earth.

To prove this, we shall develop the function $\Sigma\,.\left(\frac{f-l}{f}\right).\,c\,.\cos.\,(ft+\beta)$, according to the powers of the time. It becomes, by neglecting the terms [3116']

From these coefficients, 5680^s, 657^s, &c., we may easily obtain their corrected values [3113u], by multiplying each of them by $\frac{f-l}{f}$ or $\frac{f-50^s{,}1}{f}$, using for f the coefficients of t, respectively; namely, $50^s{,}1$, $24^s{,}5715$, &c. To render these corrected coefficients positive, we have varied the angles of the preceding formula by $\pm 180^d$; therefore the corrected expression of the obliquity is, [3113s] [3113t]

$$
\begin{array}{rl}
 & \quad s \qquad\qquad\quad d \quad m \qquad\qquad s \\
\theta' = h + & 683\,.\sin.\,(\;\;36\;05 + t\,.\,24{,}5715) \\
+ & 3460\,.\sin.\,(167\;21 + t\,.\,31{,}4271) \\
+ & 198\,.\sin.\,(221\;54 + t\,.\,33{,}3851) \\
+ & 251\,.\sin.\,(163\;24 + t\,.\,43{,}2280) \\
+ & 362\,.\sin.\,(294\;32 + t\,.\,45{,}3674)
\end{array}
\qquad [3113u]
$$

Apparent obliquity of the ecliptic, by La Place.

The sum is $= 4954 = 1^d\,22^m\,34^s =$ the maximum variation of the obliquity, according to La Place. [3113v]

This formula makes the present annual decrement of the obliquity rather less than half a second. The maximum variation according to Schubert, is $4^d\,53^m\,23^s$ [3113r], which is nearly four times its true value, $1^d\,22^m\,34^s$ [3113v], corresponding to La Place's calculations. The angles in the formula [3113u], have different periods, depending on the coefficients of t, respectively; the first requires about fifty-three thousand years, the last about twenty-eight thousand years, to complete a revolution, or variation in the argument, of 360^d. The different rates, at which these angles increase, produce several periods of maxima and minima, besides those mentioned in [3113w]. [3113w] [3113x]

We may incidentally remark, that the formulas given by La Grange and Schubert, for the value of ψ', and for the length of the year, require a similar modification to that in [3113s, &c.]. La Place remarks, that the terms of the secular equation of ψ', given by Schubert, must be multiplied by $\frac{(f-l)}{f}.\left(1+\frac{l}{f}\,.\,\text{tang.}^2\,h\right)$, to obtain the corresponding terms of his formula [3107]. The numerical values of θ, θ', ψ, ψ', are treated of in this book in [3377—3380], and in Book VI, [4357—4360, 4614—4617]. [3113y] [3113z]

above the first power,*

[3117] $$\Sigma\,.\left(\frac{f-l}{f}\right).\,c\,.\,\cos.\,\beta-t\,.\,\Sigma\,.\,(f-l)\,.\,c\,.\,\sin.\,\beta.$$

The coefficient $f-l$ is, as we have seen [3113a], the same for the earth,
[3117'] supposing it to be spherical, as for the case where it differs from the sphere; *therefore the secular variation of the obliquity of the ecliptic is the same for these two cases, in times near the epoch.*

[3118] The function $\Sigma\,.\left\{1+\frac{l}{f}\,.\,\text{tang.}^2 h\right\}.\,(l-f)\,.\,\cot.\,h\,.\,c\,.\,\cos.\,(ft+\beta)$, in the expression of $\frac{d\psi'}{dt}$ [3107, 3118f], gives the diminution of the mean value of the year, by reducing it into time, *estimating the whole circumference*
[3118'] 400° *as one year.*† The decrease which takes place, by the motion of the

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[3116a] * (2097) Putting in [61] Int. $z=\beta$, $a=ft$, we get $\cos.(ft+\beta)=\cos.\beta-ft\,.\sin.\beta$, neglecting the square and higher powers of t, substituting this in the secular variation of the obliquity [3111], we get [3117]; and as $f-l=g$ [3113a], it becomes

[3116b] $$\Sigma\,.\left(\frac{f-l}{f}\right).\,c\,.\,\cos.\,\beta-t\,.\,\Sigma\,.\,g\,c\,.\,\sin.\,\beta.$$

Now g, c, β, are independent of the figure of the earth, therefore the term $-t\,.\,\Sigma\,.\,gc\,.\sin.\beta$, or the chief term of the secular variation, is independent of this figure; so that the effect
[3116c] of it can only be perceived in the terms which depend on t^2 and on the higher powers of t; and as the coefficients of these terms are very small, the effect of them will not be perceptible till many years after the epoch, as is observed in [3116].

[3118a] † (2098) The length of the year is ascertained by the time elapsed between two successive returns of the sun to the first point of Aries, determined by the intersection of the
[3118b] equator with the plane of the sun's moveable orbit; and this time must evidently be affected by the variations of ψ' [3104, 3107]. Now the general value of ψ' [3107], noticing only the first terms, and the secular equations, is

[3118c] $$\psi'=lt+\zeta+\Sigma\,.\left\{1+\frac{l}{f}\,.\,\text{tang.}^2 h\right\}.\left(\frac{l-f}{f}\right).\,\cot.\,h\,.\,c\,.\,\sin.\,(ft+\beta).$$

If we wish to deduce from this the annual precession $\delta\psi'$, we may increase the time t
[3118d] by one year, represented by T, and we shall obtain the value of $\psi'+\delta\psi'$; from which, subtracting ψ', we get $\delta\psi'$. Now it is evident, that if we neglect the square and higher powers of T, we shall obtain $\delta\psi'$, by taking the differential of ψ' relatively to the
[3118e] characteristic δ, and putting $\delta t=T$. Hence we obtain,

[3118f] $$\delta\psi'=l\,T+T\,.\,\Sigma\,.\left\{1+\frac{l}{f}\,.\,\text{tang.}^2 h\right\}.\,(l-f)\,.\,\cot.h\,.\,c\,.\,\cos.\,(ft+\beta)=l\,T+s\,T;$$

[3118g] s being put, for brevity, equal to the quantity [3118], under the sign Σ. Subtracting the value

ecliptic alone, neglecting the action of the sun and moon on the terrestrial spheroid, is,

The action of the sun and moon on the spheroidal

$$\Sigma . (l-f) . \cot. h . c . \cos. (ft+\beta) ;$$ [3119]

therefore *this action changes the value of the variation in the length of the year, and reduces it to nearly a quarter of the value it would have independently of this action.* [3119′]

shell of the earth, decreases the secular variation in the length of the year.

8. *We shall now consider the influence of this action upon the duration of the mean day. We shall first observe, that the momentary axis of rotation never varies from the third principal axis but by an insensible quantity.* [3119″] We have seen, in § 28 of the first book [259c], that the sine of the angle formed by these two axes is equal to $\frac{\sqrt{q^2+r^2}}{\sqrt{p^2+q^2+r^2}}$; now it is [3120] evident from what precedes, that q and r are insensible,* and that they have no perceptible influence on the values of θ and ψ, except by the integrations; therefore *we may suppose that the momentary axis of the earth coincides with its third principal axis, and that its poles of rotation always correspond very nearly to the same points of its surface.* [3120′]

The momentary axis of the earth corresponds nearly with the third principal axis.

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of $\delta\psi'$ from the whole circumference, we obtain the arc actually described in one year, $400^\circ - lT - sT$; and its mean value, $400^\circ - lT$, is the arc described in a *mean* year. Hence the *actual* year is *less* than the *mean* year, by the time which the sun takes to describe the arc sT, namely, $\frac{sT}{400^\circ - lT}$, or $\frac{sT}{400^\circ}$ nearly; and if we suppose [3118h] the whole circumference 400° to represent one year [3118′], this decrement of the year becomes equal to s [3118, 3118g]. Substituting $f-l=g$ [3113a] in [3118], we get [3118i] for this decrement, $-\Sigma . \left\{ 1+\frac{l}{f} . \text{tang.}^2 h \right\} . g . \cot. h . c . \cos. (ft+\beta)$. If we neglect [3118k] the action of the sun and moon on the spheroidal shell of the earth, which is the same as to suppose the earth spherical, we shall have $l=0$ [3113b], and then the preceding expression becomes as in [3119],

$$-\Sigma . g . \cot. h . c . \cos. (ft+\beta), \quad \text{or} \quad \Sigma . (l-f) . \cot. h . c . \cos. (ft+\beta) \ [3113a].$$ [3118l]

We may then compute the values of the two expressions [3118, 3119] as in [3113e, &c.]; this computation has no other difficulty than its length. La Place makes the ratio of these [3118m] quantities as 1 to 4 [3119′].

* (2099) Comparing [3023, 3024] with [3025—3028], we find that q, r, are of the order $C-A$, or $C-B$, multiplied by $k \pm k'$, all of which are very small. [3120a]

We shall now determine the rotatory velocity of the earth about its third
[3121] *principal axis.* It is evident that $p = \frac{d\varphi}{dt} - \frac{d\psi}{dt}\,.\,\cos.\theta$, expresses this
[3121'] velocity.* If in the equations [3009—3011], we suppose $A = B$, which
[3122] holds good when the earth is a spheroid of revolution,† the first of these equations gives $dp = 0$, consequently p is equal to a constant quantity n. But these equations being only approximative, relative to the action of the
[3123] body L, we shall proceed to prove that the equation $p = n$ is correct, when we notice all terms arising from this action.

[3123'] If, as in [2978], we take the plane of the equator for the plane of x and y, the equation [2905] will become,‡

[3124] $$dp = \frac{dN}{C};$$

and by [2972],

[3125] $$\frac{dN}{dt} = S.dm.\left\{ y.\left(\frac{dV}{dx}\right) - x.\left(\frac{dV}{dy}\right) \right\}.$$

Putting

[3126] $$V' = S.\frac{L\,dm}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}};$$

and observing that by the nature of the centre of gravity [254],§

[3121a] * (2100) We have seen, in [2977g], that p represents the rotatory velocity about the third principal axis, and by substituting its value [3029], we get [3121].

[3122a] † (2101) If the earth be a spheroid of revolution, we shall have $A = B$ [3011'], and then [3009] becomes $dp = 0$, whose integral is $p = n$.

[3124a] ‡ (2102) The equator of the earth being taken for the plane of xy, as in [2978], we get $\theta = 0$ [2907g]. Substituting this and $A = B$, in [2905], we get [3124].

[3127a] § (2103) Multiplying [2966] by dm, prefixing the sign of integration S, and using V' [3126], we get,

[3127b] $$S.V\,dm = -S.L\,dm.\frac{(xx'+yy'+zz')}{r_{,}^3} + S.\frac{L\,dm}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$
$$= -S.L\,dm.\frac{(xx'+yy'+zz')}{r_{,}^3} + V'.$$

This is to be substituted in the second member of [2972]. In the first term of [3127b], we may consider $r_{,}^3$ as a constant quantity [2971c', &c.]; consequently the partial

$$S.x'dm=0; \qquad S.y'dm=0; \qquad S.z'dm=0;$$ [3126']

we shall have [2966, 2971],

$$\frac{dN}{dt}=y.\left(\frac{dV'}{dx}\right)-x.\left(\frac{dV'}{dy}\right);$$ [3127]

V' being the product of L, by the sum of all the particles of the terrestrial spheroid, divided respectively by their distances from L.* It is evident that if the earth be considered as a spheroid of revolution, V' will be the same, when z and $\sqrt{x^2+y^2}$ are the same; therefore V' will be a function of these two quantities; hence we find $\frac{dN}{dt}=0$, consequently $dp=0$, or $p=n$. *Thus we have a very extensive case in which the rotatory motion of the earth about its third axis is rigorously uniform.* [3127'] [3128] [3128']

Extensive case in which the rotatory motion of the earth about its third axis is rigorously uniform.

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differential of this term, taken, relatively to x, and multiplied by y, produces in $S.y.\left(\frac{dV}{dx}\right).dm$, the quantity $-S.Ldm.\frac{yx'}{r_{,}^3}=-\frac{Ly}{r_{,}^3}.S.x'dm=0$ [3126']. In like manner, the partial differential of the same term, relative to y, produces, in $-S.x.\left(\frac{dV}{dy}\right)$, the quantity $S.Ldm.\frac{xy'}{r_{,}^3}=\frac{Lx}{r_{,}^3}.S.y'dm=0$ [3126']. Therefore we may neglect this first term of [3127b], and substitute, in [2972], the expression $S.Vdm=V'$; by which means [2972] becomes as in [3127]. [3127c] [3127d] [3127e]

* (2104) The distance of the particle dm of the earth, from the attracting body L, is $\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}$ [2967d]. The sum of all these particles, divided respectively by their corresponding distances, and multiplied by L, is evidently equal to V' [3126]. If the spheroid be formed by the revolution of a curve about the axis of z, V' will be a function of the ordinate z, and of the distance of the body L from that axis or $\sqrt{(x^2+y^2)}$ [2907n]; so that if we put $\sqrt{(x^2+y^2)}=x_{,}$, we shall have, V' equal to a function of $z, x_{,}$, which may be represented by $V'=\varphi(z,x_{,})$. Hence [3128a] [3128b] [3128c]

$$\left(\frac{dV'}{dx}\right)=\left(\frac{d.\varphi(z,x_{,})}{dx_{,}}\right).\left(\frac{dx_{,}}{dx}\right)=\left(\frac{d.\varphi(z,x_{,})}{dx_{,}}\right).\frac{x}{\sqrt{(x^2+y^2)}},$$

and [3128d]

$$\left(\frac{dV'}{dy}\right)=\left(\frac{d.\varphi(z,x_{,})}{dx_{,}}\right).\left(\frac{dx_{,}}{dy}\right)=\left(\frac{d.\varphi(z,x_{,})}{dx_{,}}\right).\frac{y}{\sqrt{(x^2+y^2)}}.$$

Substituting these in the second member of [3127], the terms mutually destroy each other, and we have $\frac{dN}{dt}=0$; hence [3124] becomes $dp=0$, whose integral is $p=n$, as in [3128]. [3128e]

In the general case, where the three principal momenta of inertia are
[3128''] unequal, the term $\left(\frac{B-A}{C}\right).q\,r\,d\,t,$ of the equation [3009] is insensible,
[3129] even after its double integration in the expression $\int p\,d\,t,$ which represents the rotatory motion of the earth during any period of time.* For we have
[3129'] seen, in §4, that the values of q and r do not contain small divisors, which are introduced into the values of θ and ψ by the integrations only.†

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* (2105) We have seen, in [3121a], that the angular motion about the third principal
[3129a] axis, in the time $d\,t,$ is $p\,d\,t;$ its integral, $\int p\,d\,t,$ will therefore represent the motion about that axis in the time $t.$ Now if, for brevity, we put the second member of the
[3129b] equation [3009] equal to $Q\,d\,t,$ we shall have, $d\,p+\left(\frac{B-A}{C}\right).q\,r\,.\,d\,t=Q\,d\,t.$ Its integral being taken, and the constant quantity n added to complete it, we get,
[3129c] $p+\left(\frac{B-A}{C}\right).\int q\,r\,d\,t=n+\int Q\,d\,t.$ Multiplying this by $d\,t,$ and again integrating, we
[3129d] obtain, $\int p\,d\,t=n\,t+\int d\,t\,.\int Q\,d\,t-\left(\frac{B-A}{C}\right).\int d\,t\,.\int q\,r\,d\,t\,;$ which contains the double integrals mentioned in [3128'', &c.].

† (2106) The quantities $\frac{d\,\theta}{d\,t},\ \frac{d\,\psi}{d\,t},$ given by the equations [3040, 3041], are of the
[3130a] order $\left(\frac{A+B-2C}{2n\,.\,C}\right).k\,;$ k being of the same order as P or P' [3020]. The values of $P,\ P'$ [3016, 3017], contain no terms having small divisors of the order $i,$ of the same
[3130b] magnitude, in other respects, as those before noticed; therefore $\frac{d\,\theta}{d\,t},\ \frac{d\,\psi}{d\,t},$ contain no
[3130c] terms having such small divisors. But the divisors of the order i are introduced in $\theta,\ \psi,$ by integrating [3040, 3041], as evidently appears from the expressions [3089, 3100], containing the very small divisor $f,$ which does not occur in $\frac{d\,\theta}{d\,t},$ or $\frac{d\,\psi}{d\,t}.$ Now as these
[3130d] small divisors are not found in $\frac{d\,\theta}{d\,t},\ \frac{d\,\psi}{d\,t},$ they will not appear in the values of $q,\ r,$ deduced from [3030, 3031]; namely,

[3130e] $$q=\frac{d\,\psi}{d\,t}\,.\,\sin.\,\theta\,.\,\sin.\,\varphi-\frac{d\,\theta}{d\,t}\,.\,\cos.\,\varphi\,;\qquad r=\frac{d\,\psi}{d\,t}\,.\,\sin.\,\theta\,.\,\cos.\,\varphi+\frac{d\,\theta}{d\,t}\,.\,\sin.\,\varphi.$$

[3130f] Therefore if we notice only the terms of $\frac{d\,\psi}{d\,t},\ \frac{d\,\theta}{d\,t},$ deduced from [3100, 3101], and depending on the secular equations, or on the angles of the form $(f\,t+\beta),$ these terms will be of the order $l\,c\,;$ consequently the similar terms of q and r [3130e] will be of
[3130g] the order $l\,c.$ Hence $\left(\frac{B-A}{C}\right).q\,r,$ which occurs in [3009], will be of the order

Therefore q and r are of the order $l\,c$, noticing only the very small angles depending on the secular variations of the earth's orbit; and the term $\left(\frac{B-A}{C}\right).q\,r$, is of the order $l^2c^2.\left(\frac{B-A}{C}\right)$ [3130h]. The double [3130] integration may introduce a divisor of the order l^2, and then it will become of the order $\left(\frac{B-A}{C}\right).c^2$; therefore it is insensible. [3131]

If in the second member of [3009], we substitute for θ, φ, ψ, their values, [3131'] given by a first approximation, it will suffice to notice only the terms of these values, which have very small divisors, which are of the form,* $\frac{H.lc}{f}.{}^{\text{sin.}}_{\text{cos.}}(f\,t+\beta)$, f being a very small coefficient of the same order as [3132] l [3130h]. But these terms, when substituted in the second member of [3009], are multiplied by the sine or cosine of $2\,\varphi$, and by l; so that after their double integration, in the expression of $fp\,d\,t$, they will remain [3133] insensible. Therefore we find, even in the case where the three momenta A, B, C, are unequal, that the rotatory motion of the earth can always be supposed uniform, or, in other words, p can always be considered as equal to a constant quantity n. [3134]

$l^2c^2.\left(\frac{B-A}{C}\right)$; and if this quantity be multiplied by a term of the form ${}^{\text{sin.}}_{\text{cos.}}(f\,t+\beta)$, its double integration in the expression [3129d] will introduce the divisor f^2, so that the term $\left(\frac{B-A}{C}\right).ff\,d\,t^2.q\,r$, will be of the order $\frac{l^2c^2}{f^2}.\left(\frac{B-A}{C}\right)$, or $c^2.\left(\frac{B-A}{C}\right)$, because l is [3130h] of the same order as $f=l+g$ [3113a, d], and such terms are neglected, as in [3072']. We may incidentally remark, that some of these terms may be less than what we have here stated, because the angle $f\,t+\beta$, becomes connected with $2\,\varphi$, by which [3130i] means the divisor f^2, introduced by the integration, is increased to $(2\,n)^2$ or $4\,n^2$ [3130k] nearly [3029, 3015].

* (2107) The terms of ψ, θ [3100, 3101], depending on the secular motions, are evidently of the form [3132]; and when these are substituted in $Q\,d\,t$ [3129b], or in the [3132a] second member of [3009], they are multiplied by quantities of the form ${}^{\text{sin.}}_{\text{cos.}}2\,\varphi$, producing terms depending on the angle $2\,\varphi\pm f\,t\pm\beta$. The double integral of these terms, in [3129d], will not therefore be increased by small divisors, as we have seen, in [3130k]. [3132b] This agrees with [3133].

Mean day. 9. *This is the place to discuss the variations of the day, which is called by*
[3135] *astronomers the mean day.* The mean sideral motion of the earth in its orbit is uniform, as we have shown in Book II, §54 [1051′, &c.]. We shall suppose that there is in this orbit a *second* sun, whose motion and epoch
[3135′] are the same as the mean motion and epoch of the real sun. Moreover, we shall suppose that there is in the plane of the equator a *third* sun, whose
[3136] motion is such that it coincides with the second sun whenever it passes the vernal equinox; its mean distance from this equinox being always equal to
Duration of the mean day. the mean longitude of the sun. *The interval between two successive returns of this third sun to the meridian, is what is called the mean day.* If the
[3137] motion of the equinox upon the apparent ecliptic were uniform, and the inclination of this ecliptic to the equator constant, the third sun would always move uniformly in the equator. But the secular variations in the
[3137′] motions of the equinoxes and in the obliquity of the ecliptic, produce in the motions of this third sun, small secular equations, which we shall now proceed to determine.

We have seen, in the preceding article, that the rotatory velocity of the
[3137″] earth may be supposed equal to a constant quantity n [3134], and that its momentary axis of rotation never varies from the third principal axis, but
s, v. by an insensible quantity [3120′]. Therefore if we put s for the angular velocity of the third sun, which is supposed to move in the plane of the
[3138] equator, and v for its distance from the vernal equinox, referred to the
[3139] *fixed* ecliptic, $n-s$ will be the angular velocity of the first principal axis of the earth, relatively to this sun; and we shall have,*

[3140] $$d\varphi - dv = (n-s)\,.\,dt.$$

Now we have in [3029, 3123],

[3141] $$d\varphi = n\,dt + d\psi\,.\,\cos.\,\theta;$$

hence we obtain,

[3142] $$dv = s\,dt + d\psi\,.\,\cos.\,\theta.$$

[3139a] * (2108) The distance of the *first* principal axis from the *fixed* equinox, is φ [2907f], and the distance of the third sun from the same equinox is v [3138]; therefore $\varphi - v$ is the
[3139b] distance of the *first* principal axis from the *third* sun. Its differential $d\varphi - dv$ represents the motion of that axis from the third sun, in the time dt. Now the angular velocity of this axis from the sun being $n-s$ [3139], its motion in the time dt is $(n-s)\,.\,dt$; putting
[3139c] this equal to the preceding expression [3139b], we get [3140]. Substituting the value of $d\varphi$ [3141], we obtain [3142].

We shall put v' for the angular distance of the third sun from the apparent equinox, or, in other words, from the intersection of the equator with the apparent ecliptic. Then it is evident from [3144b], that $v-v'$ is equal to $\frac{(\psi-\psi')}{\cos.\theta}$;* therefore it is equal to $\frac{\Sigma.c.\sin.(ft+\beta)}{\sin.\theta}$; hence we get [3143] [3144]

$$dv'=s\,dt+d\psi.\cos.\theta-dt.\frac{\Sigma.cf.\cos.(ft+\beta)}{\sin.\theta}. \quad [3145]$$

If gt be the sideral motion of the second sun upon the apparent ecliptic,† $g+\frac{d\psi'}{dt}$, will be its angular velocity, relatively to the apparent equinox. [3146] But we have from [3106],

$$\frac{d\psi'}{dt}=\frac{d\psi}{dt}-\cot.\theta.\Sigma.cf.\cos.(ft+\beta); \quad [3147]$$

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* (2109) We shall suppose, in fig. 62, page 853, that S', S'', S''', represent the places of the first, second, and third suns, respectively; then we have $S'''D=v$ [3138], $S'''C'=v'$ [3143]; hence $v-v'=S'''D-S'''C'=C'D$. But in the triangle [3144a] CDC', we have $C'D=\frac{CD}{\cos.CDC'}=\frac{CD}{\cos.\theta}=\frac{\psi-\psi'}{\cos.\theta}$ [3106b]. Substituting this in the [3144b] preceding equation, and using the value of $\psi-\psi'$ [3106c], we get $v-v'=\frac{\Sigma.c.\sin.(ft+\beta)}{\sin.\theta}$. Taking the differential, we obtain $dv'=dv-dt.\frac{\Sigma.cf.\cos.(ft+\beta)}{\sin.\theta}$; substituting dv [3144c] [3142], we get [3145].

† (2110) The distance of the moveable vernal equinox from the fixed point, taken for the origin of the longitudes, being represented by ψ' [3104], its differential $d\psi'$ will [3146a] represent the precession in the time dt. Adding this to the sideral motion of the second sun $g\,dt$, we get the whole motion, from the apparent equinox $g\,dt+d\psi'$; dividing this by the time dt, we get the angular velocity relatively to that equinox $g+\frac{d\psi'}{dt}$, as [3146b] in [3146]. Substituting the differential $d\psi'$ [3147], deduced from [3106], it becomes as in [3147]; which represents the angular velocity of the second sun, relatively to the apparent vernal equinox. Now the angular velocity of the second sun is equal to that of [3146c] the third, as is evident from the definitions in [3136]; and this last velocity is represented by $\frac{dv'}{dt}$ [3143]. Putting this equal to the expression [3148], and then substituting dv' [3145], we get,

$$s+\frac{d\psi}{dt}.\cos.\theta-\frac{\Sigma.cf.\cos.(ft+\beta)}{\sin.\theta}=g+\frac{d\psi}{dt}-\cot.\theta.\Sigma.cf.\cos.(ft+\beta); \quad [3146d]$$

hence we easily obtain s [3149], by some very slight reductions.

therefore this velocity is equal to,

[3148] $$g+\frac{d\psi}{dt}-\cot.\theta\,.\,\Sigma\,.\,cf\,.\cos.(ft+\beta);$$

and as this must be equal to $\frac{dv'}{dt}$ [3143], we shall have an equation, by means of which we may determine s, and we shall find,

[3149] $$s=g+(1-\cos.\theta)\,.\,\frac{d\psi}{dt}+\left(\frac{1-\cos.\theta}{\sin.\theta}\right).\,\Sigma\,.\,cf\,.\cos.(ft+\beta).$$

Substituting the preceding values of $d\psi$ and θ [3100, 3101], we obtain,*

[3150] $$\begin{aligned}s=g+l\,.\,(1-\cos.h)-\sin.h\,.\,\Sigma\,.\,\frac{l^2c}{f}\,.\cos.(ft+\beta)\\+(1-\cos.h)\,.\,\Sigma\,.\left\{\left(\frac{l^2}{f}-l\right).\,\mathrm{tang}.\,h+l\,.\cot.h\right\}.\,c\,.\cos.(ft+\beta)\\+\left(\frac{1-\cos.h}{\sin.h}\right).\,\Sigma\,.\,cf\,.\cos.(ft+\beta).\end{aligned}$$

[3150'] The time expressed in mean days is equal to $\int s\,dt$, therefore we have for the equation of this time,†

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* (2111) If we retain only the first terms of ψ, θ [3100, 3101], and those depending on the secular equations, we shall have,

[3150a] $$\frac{d\psi}{dt}=l+\Sigma\,.\left\{\left(\frac{l}{f}-1\right).\mathrm{tang}.h+\cot.h\right\}.\,lc\,.\cos.(ft+\beta),\qquad \theta=h-\Sigma.\frac{lc}{f}.\cos.(ft+\beta),$$

and from this last we get, by using, for $\cos.\theta$, a formula similar to [3097b];

[3150b] $$1-\cos.\theta=1-\cos.h-\sin.h\,.\,\Sigma\,.\,\frac{lc}{f}\,.\cos.(ft+\beta).$$

Multiplying this expression by $\frac{d\psi}{dt}$ [3150a], and neglecting terms of the order c^2, we obtain,

[3150c] $$\begin{aligned}(1-\cos.\theta)\,.\,\frac{d\psi}{dt}=l\,.\,(1-\cos.h)-\sin.h\,.\,\Sigma\,.\,\frac{l^2c}{f}\,.\cos.(ft+\beta)\\+(1-\cos.h)\,.\,\Sigma\,.\left\{\left(\frac{l^2}{f}-l\right).\,\mathrm{tang}.\,h+l\,.\cot.h\right\}.\,c\,.\cos.(ft+\beta).\end{aligned}$$

Substituting this in [3149], and in the last term of that expression, which is of the order cf, putting h for θ, we get [3150].

† (2112) Multiplying [3150] by dt, and taking the integral, we get, by putting, for brevity, $At=gt+lt\,(1-\cos.h)$,

[3151a] $$\begin{aligned}\int s\,dt=At-\sin.h\,.\,\Sigma\,.\,\frac{l^2c}{f^2}\,.\sin.(ft+\beta)\\+(1-\cos.h)\,.\,\Sigma\,.\left\{\left(\frac{l^2}{f^2}-\frac{l}{f}\right).\,\mathrm{tang}.\,h+\frac{l}{f}\,.\cot.h\right\}.\,c\,.\sin.(ft+\beta)\\+\left(\frac{1-\cos.h}{\sin.h}\right).\,\Sigma\,.\,c\,.\sin.(ft+\beta)+\text{constant};\end{aligned}$$

$$- \sin. h \,.\, \Sigma \,.\, \frac{l^2 c}{f^2} \,.\, \sin. (ft + \beta)$$

$$+ (1 - \cos. h) \,.\, \Sigma \,.\, \left\{ \left(\frac{l^2}{f^2} - \frac{l}{f} \right) . \operatorname{tang}. h + \frac{l}{f} \,.\, \cot. h \right\} . c \,.\, \sin. (ft + \beta) \qquad [3151]$$

$$+ \left(\frac{1 - \cos. h}{\sin. h} \right) . \Sigma \,.\, c \,.\, \sin. (ft + \beta).$$

Equation of the mean day.

This equation, reduced to time, estimating the whole circumference as equal to one day, amounts only to a few minutes in a period of several millions of years; and it is unnecessary for astronomers to notice it* [3152e]. [3151′] [3152]

At being the mean value of $\int s\,dt$. Now by [3139] $n - s$ is the angular velocity of the first principal axis, relatively to the third sun; therefore the angle described in the time t, is $\int dt\,(n-s) = nt - \int s\,dt$; and if we put this equal to the whole circumference 2π, which is described in one day, we shall get, by transposing $\int s\,dt$, and dividing by n, $t = \frac{2\pi + \int s\,dt}{n}$. This is the general value of the length of a day; and if we substitute the mean value of $\int s\,dt$ [3151b], we obtain the length of a mean day, which we shall represent by $t' = \frac{2\pi + At}{n}$. Subtracting this from the general value [3151c] we get the equation of a mean day equal to $\frac{\int s\,dt - At}{n}$. The numerator of this expression is the same as the function [3151], as is evident by substituting the value of $\int s\,dt$ [3151a]. Again, the preceding expression of a mean day t', being nearly equal to $\frac{2\pi}{n}$, we get $\frac{1}{n} = \frac{t'}{2\pi}$, consequently the preceding equation of the mean day is represented very nearly by $\frac{t'}{2\pi} . \left\{ \int s\,dt - At \right\}$; which is the same as to turn the function $\int s\,dt - At$ into time, supposing the whole circumference 2π, to be equal to a mean day t', as is observed in [3151′]. [3151b] [3151c] [3151d] [3151e] [3151f] [3151g]

* (2113) Any one who wishes to examine this calculation, may do it by the method indicated in [3113e, &c.]. It is here omitted on account of the length of the calculation. We may, however, show the smallness of this correction, by estimating roughly, by means of the expression [3366], the value of any one of the terms; as, for example, that multiplied by the first power of $\frac{l}{f}$ in [3151], which is [3152a]

$$(1 - \cos. h) . (\cot. h - \operatorname{tang}. h) \,.\, \Sigma \,.\, \frac{lc}{f} \,.\, \sin. (ft + \beta). \qquad [3152b]$$

Substituting h [3369], it becomes nearly $\frac{1}{7} . \Sigma . \frac{lc}{f} . \sin. (ft + \beta)$. Developing $\sin. (ft + \beta)$ according to the powers of ft, as in [3362c], we find that the term depending on t,

First method of investigating the effect of the tides, or the oscillations of the sea, upon the nucleus, supposing it to be wholly covered by the fluid.

10. *In the analysis of the preceding articles, the earth is supposed to be wholly solid; but it is covered, for the most part, by a fluid, whose oscillations*
[3152'] *may have some influence on the motions of the axis of the earth;* therefore it is important to examine into the extent of this influence, to ascertain whether the results we have obtained are varied by these oscillations. For this purpose we must determine the increments of the values of dN, dN', dN'' [2905, 2907], *arising from the action of the ocean on the nucleus*
[3152''] *or solid spheroid which it covers.* We find in the first book [225], that if P, Q, R, be the forces acting on the particle dm, of the terrestrial
[3152'''] spheroid, parallel to the axes of x', y', and z', and in *directions tending to increase the co-ordinates*,* we shall have,

General values of

[3153] dN,
$$\frac{dN}{dt} = S.\{Q.x' - P.y'\}.dm;$$

[3154] dN',
$$\frac{dN'}{dt} = S.\{R.x' - P.z'\}.dm;$$

[3155] dN''.
$$\frac{dN''}{dt} = S.\{R.y' - Q.z'\}.dm.$$

Pressure and attraction of the sea upon the nucleus.

We shall now see what quantities the action of the ocean introduces into these expressions. This fluid acts upon the terrestrial spheroid by its pressure and
[3156] *by its attraction. We shall consider these two effects separately. We shall suppose, for greater simplicity, that the plane of x', y' is the plane of the equator*, as is done in [2978].

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is $\frac{1}{7}.t.\Sigma.lc.\cos.\beta$; and if we put $t = iT$, it changes into $\frac{1}{7}.iT.\Sigma.lc.\cos.\beta$.
[3152c] If we now suppose l to be of the same order as f [3113d], we may consider $T.\Sigma.lc.\cos.\beta$ to be of the same order as $0'',24794$ [3366], and then the preceding term is of the order
[3152d] $\frac{1}{7}.i.0'',24794$, which is nearly $i.0'',04$. If i be equal to 1000 years, it becomes $i.0'',04 = 40''$; which corresponds to one centesimal second of time [3151']. This correction, small as it is, will be much decreased, if we take, for the unit of time, the mean
[3152e] length of the day at the assumed epoch 1750 [3357], corresponding to $t = 0$. By this means the term of the development depending on the first power of ft, may be combined with At [3151a]; and then the equation [3151] will depend on the smaller terms of the
[3152f] order f^2t, f^3t^3, &c. The subjects treated of in § 8, 9, are discusssed by Poisson in the paper [3015i]; noticing terms of the second order, he obtains the same general results.

* (2114) This agrees with note 116, page 144, Vol. I. The equations [3153—3155]
[3153a] are the same as [2968—2970], changing $\left(\frac{dV}{dx'}\right)$, $\left(\frac{dV}{dy'}\right)$, $\left(\frac{dV}{dz'}\right)$, into P, Q, R, respectively, as in [2967].

In the state of equilibrium, the pressure and the attraction of the ocean do not produce any motion in the axis of rotation of the earth; *therefore* [3156'] *it is only necessary to notice the stratum of water, which, by the attractions* [3156''] *of the sun and moon, is elevated above or depressed below the surface of equilibrium,* corresponding to the surface of the ocean, independently of these attractions. *We shall represent the thickness of this stratum by* αy, [3157] *and shall take for the unit of density that of the sea, and for the unit of distance the mean radius of the terrestrial spheroid. Thus we shall have to consider the action of an aqueous stratum, whose interior radius is unity,**

It is only necessary to notice the stratum elevated above or depressed below the stratum of equilibrium.

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* (2115) For illustration, we shall suppose that the nucleus, or solid part of the earth, is represented by $ABDF$, and that it is wholly covered by the ocean [3299], whose surface, in the state of equilibrium, is $A'B'D'F'$. This equilibrium is disturbed by the attractions of the sun and moon, which produce the various oscillations treated of in the [3158a] fourth book [2128^{ix}—2877]. In consequence of these oscillations, the ocean, at any moment, becomes of the form $A''B''D''F''$; some parts being elevated above the surface of equilibrium, as at D''; and other parts depressed below this surface, as at B'' [2128^{xiv}, &c.]. These elevations and depressions of the fluid, make its pressure on the nucleus different from what it is in the state of equilibrium [342^{iv}, &c.]; and by this [3158b] change in the position of the particles of the fluid, its attraction on the nucleus is varied. Now it is evident, that the combined mass of the earth and ocean, in its state of equilibrium $A'B'D'F'$, produces no rotatory motion [3156']; therefore it is only necessary to notice the effect of the pressure, and the attraction of the *stratum, corresponding to the difference of the two spheroids* $A'B'D'F'$, $A''B''D''F''$. We may suppose this [3158c] stratum to be composed of an infinite number of short and nearly vertical columns, as $D'D''$, situated above or below the surface of equilibrium $A'B'D'F'$ upon a radius, as $CD'D''$, [3158d] passing through the centre of the earth C; the action of any column falling below this surface, being considered as negative [2128^{xv}]. Putting $CD'=R'$ [2919], $D'D''=\alpha y$ [2128^{xiv}], we get $CD''=R'+\alpha y$; and in computing the action of the column αy, it will be sufficiently accurate to put $R'=1$; supposing the internal radius to be $CD'=1$, [3158e] the external $CD''=1+\alpha y$, as in [3158]. We have already found, in [342^{iv}—342^{v}], that the pressure of the column $D''D'$, at the point D', is communicated to the bottom of the ocean at D; where, according to the common principles of the pressure of fluids, it must act *in the direction* DP, *perpendicularly to the surface at the point* D, *with a force* [3158f]

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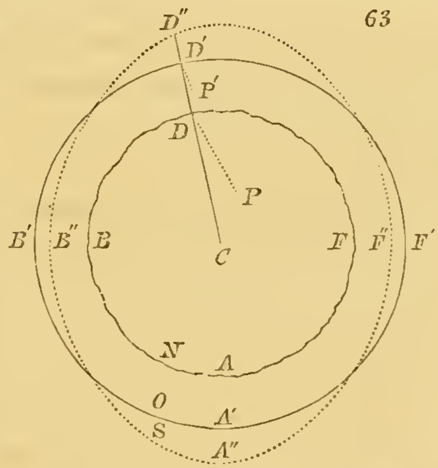

[3158] *and exterior radius* $1+\alpha y$ [3158e]. If we put g for gravity; the pressure of a column of this stratum will be the product of $\alpha g y$, by the base of the column; this will be, by § 36 of the first book [342^{iv}—342^{v}], the excess of the pressure, in the state of motion, above the pressure in the state of equilibrium.

[3159] We shall put R for the radius, drawn from the centre of gravity of the earth to the point of the surface of the nucleus, which this column presses
[3160] upon; μ for the cosine of the angle that the radius R makes with the axis of
[3161] rotation; and ϖ for the angle which the plane, drawn through this axis and the radius R, makes with the axis of x'; lastly, we shall suppose $u=0$ to
[3162] be the equation of the surface of the nucleus, which the sea covers; u being
Co-ordinates of a particle. a function of the co-ordinates x', y', z', which determine the position of the point of the nucleus mentioned in [3159], we shall have, as in [2988],

[3163] $$x'=R\,.\sqrt{1-\mu^2}\,.\cos.\varpi;$$
[3164] $$y'=R\,.\sqrt{1-\mu^2}\,.\sin.\varpi;$$
[3165] $$z'=R\,.\mu.$$

The base of the small column which we have just considered, may be
[3165'] supposed equal to $R^2.d\mu.d\varpi$;* therefore the pressure of this column is

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[3158g] *represented by* $\alpha p'=\alpha g y$ [$342'''$, 342^{iv}]; the density of the fluid being supposed equal to unity [3157]. *In the first method of investigation of the effect of the oscillations of the ocean* [3157—3298], *the sea is supposed to cover the whole nucleus, as is observed in* [3299];
[3158h] *the second method* [3299—3351] *is free from this restriction, and is conformable to the limited extent and depth of the ocean* [3299'].

* (2116) The expression of dm [2918] represents the magnitude of a rectangular
[3166a] parallelopiped, whose base is $R^2.d\mu.d\varpi$, and height dR. This base is perpendicular to the radius R, and its limits are determined by the values of $d\mu$, $d\varpi$. This is conformable to the calculation [1431b—d]. If the height dR be increased till it becomes equal to
[3166b] αy, the particle will become $\alpha y\,.R^2.d\mu.d\varpi$, and its pressure on the base will be $\alpha g y\,.R^2.d\mu.d\varpi$, as in [3166]. We shall now suppose, as in [3169, &c.], that the base on which this column of fluid rests, is not perpendicular to the radius R, but is inclined to it by a small angle $q_{\prime}$, of the same order as q [3169']. Then, without varying the values
[3166c] $d\mu$, $d\varpi$, the base, or part of the solid earth on which the pressure acts, will be increased in the ratio of 1 to $\cos. q_{\prime}$, by the common principles of orthographic projection; but this difference may be neglected, because it only produces terms of the order $q_{\prime}^2$ [44] Int., or q^2,
[3166d] which are rejected in [3169']. Hence the pressure of the column on the part of the solid nucleus, corresponding to the values of $d\mu$, $d\varpi$, becomes $\alpha g y\,.R^2.d\mu.d\varpi$, in a *direction perpendicular to the surface of the nucleus at that point*, as in [3166].

$\alpha g y . R^2 . d\mu . d\varpi$. This pressure is perpendicular to the surface of the spheroid [3158f]; by resolving it into three forces parallel to the axes of x', y', z', and tending to increase these co-ordinates, we shall have for these forces, by §3, Book I,* [3166]

Pressures.

$$-\frac{\alpha g y . R^2 . d\mu . d\varpi}{f} . \left(\frac{du}{dx'}\right); \qquad -\frac{\alpha g y . R^2 . d\mu . d\varpi}{f} . \left(\frac{du}{dy'}\right);$$

$$-\frac{\alpha g y . R^2 . d\mu . d\varpi}{f} . \left(\frac{du}{dz'}\right); \qquad [3167]$$

in which $f = \sqrt{\left(\frac{du}{dx'}\right)^2 + \left(\frac{du}{dy'}\right)^2 + \left(\frac{du}{dz'}\right)^2}$. The equation of the surface [3168]

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* (2117) If we accent R in the formula [19], to distinguish it from the radius R of the present notation, we shall find, that if several forces S, S', &c., in the directions s, s', &c., act upon a point, which is forced to move upon a surface, the resultant of all these forces will be a force $-R'$ in the direction r, perpendicular to this surface, as is evident from note 14, page 11, Vol. I. This force R' produces in the equation [19], the term $R'\delta r$, and by substituting $\delta r = N \delta u$ [19″], it becomes $R' . N \delta u$. Now changing, in formula [13], S into $-R'$ [3167c], and s into r, in order to conform to the present notation, we shall find, that the force $-R'$, in the direction r, resolved in a direction parallel to the axis of x, is $-R' . \left(\frac{\delta r}{\delta x}\right)$; and by substituting the preceding value of δr, it becomes $-R' . N . \left(\frac{du}{dx}\right)$; which may be reduced to [3167a] [3167b] [3167c] [3167d]

$$-\frac{R' . \left(\frac{\delta u}{\delta x}\right)}{\sqrt{\left\{\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta u}{\delta y}\right)^2 + \left(\frac{\delta u}{\delta z}\right)^2\right\}}}, \qquad [3167e]$$

by using the value of N [21]. Accenting the letters x, y, z, to conform to the notation [3152‴], and substituting f [3168], we get $-R' . \left(\frac{\delta r}{\delta x'}\right) = \frac{-R' . \left(\frac{\delta u}{\delta x'}\right)}{f}$. Substituting, in the second member of this expression, the value of the force or pressure $R' = \alpha g y . R^2 . d\mu . d\varpi$ [3166], we obtain $-R' . \left(\frac{\delta r}{\delta x'}\right) = \frac{-\alpha g y . R^2 . d\mu . d\varpi}{f} . \left(\frac{\delta u}{\delta x'}\right)$, being the same as the first formula [3167]. Changing x' into y', which may evidently be done, from the nature of the preceding demonstration, we shall get the value of $-R' . \left(\frac{\delta r}{\delta y'}\right)$, as in the second of the formulas [3167]; and by changing x' into z', we obtain the third of those formulas, corresponding to the pressures, in the directions parallel to the axes y', z', respectively. [3167f] [3167g] [3167h]

Equation of the surface of the solid nucleus.

of the solid spheroid, or nucleus, is of the form,*

[3169] $$x'^2+y'^2+z'^2=1+2q;$$

[3169'] q being a very small function of x', y', z', the square of which we shall neglect; therefore we have,

[3170] $$u=x'^2+y'^2+z'^2-1-2q.$$

This changes the expressions of the three preceding forces into,†

Forces of pressure on the nucleus.

$$-\frac{2\alpha g y\,.\,R^2\,.\,d\mu\,.\,d\varpi}{f}\,.\left\{x'-\left(\frac{dq}{dx'}\right)\right\};\quad -\frac{2\alpha g y\,.\,R^2\,.\,d\mu\,.\,d\varpi}{f}\,.\left\{y'-\left(\frac{dq}{dy'}\right)\right\};$$

[3171] $$-\frac{2\alpha g y\,.\,R^2\,.\,d\mu\,.\,d\varpi}{f}\,.\left\{z'-\left(\frac{dq}{dz'}\right)\right\}.$$

Hence we shall have, by noticing these forces only,‡

[3172] $$\frac{dN}{dt}=S\,.\,\frac{2\alpha g y\,.\,R^2\,.\,d\mu\,.\,d\varpi}{f}\,.\left\{x'\,.\left(\frac{dq}{dy'}\right)-y'\,.\left(\frac{dq}{dx'}\right)\right\};$$

[3173] $$\frac{dN'}{dt}=S\,.\,\frac{2\alpha g y\,.\,R^2\,.\,d\mu\,.\,d\varpi}{f}\,.\left\{x'\,.\left(\frac{dq}{dz'}\right)-z'\,.\left(\frac{dq}{dx'}\right)\right\};$$

[3174] $$\frac{dN''}{dt}=S\,.\,\frac{2\alpha g y\,.\,R^2\,.\,d\mu\,.\,d\varpi}{f}\,.\left\{y'\,.\left(\frac{dq}{dz'}\right)-z'\,.\left(\frac{dq}{dy'}\right)\right\}.$$

[3169a] * (2118) The equation of the surface of the spheroid $u=0$, given in [19', 1840], is afterwards reduced to the form [1849]; which, by accenting the letters x, y, z, and
[3169b] changing $\alpha u'$ into q, becomes $0=x'^2+y'^2+z'^2-1-2q$. Transposing the two last terms, we get [3169], which represents the equation of the surface of the nucleus. The equation of the surface of equilibrium is given in [3187].

† (2119) The partial differentials of u [3170], relatively to x', y', z', are

[3171a] $$\left(\frac{du}{dx'}\right)=2\,.\left\{x'-\left(\frac{dq}{dx'}\right)\right\};\quad \left(\frac{du}{dy'}\right)=2\,.\left\{y'-\left(\frac{dq}{dy'}\right)\right\};\quad \left(\frac{du}{dz'}\right)=2\,.\left\{z'-\left(\frac{dq}{dz'}\right)\right\};$$

substituting these in [3167], we get the three forces [3171].

‡ (2120) Putting the three forces [3171], respectively, equal to P, Q, R [3152''], we get

[3172a] $$Qx'-Py'=\frac{2\alpha g y\,.\,R^2.\,d\mu\,.\,d\varpi}{f}\,.\left\{x'\,.\left(\frac{dq}{dy'}\right)-y'\,.\left(\frac{dq}{dx'}\right)\right\};$$

[3172b] $$Rx'-Pz'=\frac{2\alpha g y\,.\,R^2.\,d\mu\,.\,d\varpi}{f}\,.\left\{x'\,.\left(\frac{dq}{dz'}\right)-z'\,.\left(\frac{dq}{dx'}\right)\right\};$$

[3172c] $$Ry'-Qz'=\frac{2\alpha g y\,.\,R^2.\,d\mu\,.\,d\varpi}{f}\,.\left\{y'\,.\left(\frac{dq}{dz'}\right)-z'\,.\left(\frac{dq}{dy'}\right)\right\};$$

and by substituting these values in [3153—3155], we obtain [3172—3174].

We shall now refer the partial differentials $\left(\frac{dq}{dx'}\right)$, $\left(\frac{dq}{dy'}\right)$, $\left(\frac{dq}{dz'}\right)$, to the variable quantities R, μ, ϖ. For this purpose, we have,*

$$R=\sqrt{x'^2+y'^2+z'^2}; \qquad \text{tang.}\,\varpi=\frac{y'}{x'}; \qquad \mu=\frac{z'}{R}; \qquad [3175]$$

whence we easily deduce,†

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* (2121) The formulas [3175] are easily proved, by substituting the values of x', y', z' [3163—3165], and reducing; putting $\cos.^2\varpi+\sin.^2\varpi=1$, $\sin.\varpi=\cos.\varpi\,.\,\text{tang.}\,\varpi$. [3175a]

† (2122) The value of R [3175] gives, by taking its partial differentials,

$$\left(\frac{dR}{dx'}\right)=\frac{x'}{\sqrt{(x'^2+y'^2+z'^2)}}=\frac{x'}{R}=\sqrt{(1-\mu^2)}\,.\,\cos.\varpi;$$
$$\left(\frac{dR}{dy'}\right)=\frac{y'}{R}=\sqrt{(1-\mu^2)}\,.\,\sin.\varpi; \qquad \left(\frac{dR}{dz'}\right)=\frac{z'}{R}=\mu. \qquad [3176a]$$

The differential of $\text{tang.}\,\varpi=\frac{y'}{x'}$ [3175], is $\frac{d\varpi}{\cos.^2\varpi}=\frac{x'dy'-y'dx'}{x'^2}$; and by substituting the values of x', y', [3163, 3164], we get,

$$\left(\frac{d\varpi}{dx'}\right)=-\cos.^2\varpi\,.\,\frac{y'}{x'^2}=\frac{-\sin.\varpi}{R.\sqrt{(1-\mu^2)}}; \quad \left(\frac{d\varpi}{dy'}\right)=\cos.^2\varpi\,.\,\frac{1}{x'}=\frac{\cos.\varpi}{R.\sqrt{(1-\mu^2)}}; \quad \left(\frac{d\varpi}{dz'}\right)=0. \qquad [3176b]$$

The differentials of μ, R [3175], are $d\mu=\frac{R\,dz'-z'\,dR}{R^2}$, $dR=\frac{x'dx'+y'dy'+z'dz'}{R}$; substituting this last in $d\mu$, we get,

$$d\mu=\frac{dz'(x'^2+y'^2+z'^2)-z'(x'dx'+y'dy'+z'dz')}{R^3}=\frac{(x'^2+y'^2).dz'-z'x'dx'-z'y'dy'}{R^3}. \qquad [3176c]$$

From this we obtain, by using [3163—3165],

$$\left(\frac{d\mu}{dx'}\right)=-\frac{z'x'}{R^3}=-\frac{\mu.\sqrt{(1-\mu^2)}.\cos.\varpi}{R};$$
$$\left(\frac{d\mu}{dy'}\right)=-\frac{z'y'}{R^3}=-\frac{\mu.\sqrt{(1-\mu^2)}.\sin.\varpi}{R}; \qquad \left(\frac{d\mu}{dz'}\right)=\frac{x'^2+y'^2}{R^3}=\frac{1-\mu^2}{R}. \qquad [3176d]$$

Now by considering q, in the first place, as a function of x', y', z', and then as a function of R, ϖ, μ, we get, as in [462],

$$\left(\frac{dq}{dx'}\right)=\left(\frac{dq}{dR}\right)\cdot\left(\frac{dR}{dx'}\right)+\left(\frac{dq}{d\varpi}\right)\cdot\left(\frac{d\varpi}{dx'}\right)+\left(\frac{dq}{d\mu}\right)\cdot\left(\frac{d\mu}{dx'}\right);$$
$$\left(\frac{dq}{dy'}\right)=\left(\frac{dq}{dR}\right)\cdot\left(\frac{dR}{dy'}\right)+\left(\frac{dq}{d\varpi}\right)\cdot\left(\frac{d\varpi}{dy'}\right)+\left(\frac{dq}{d\mu}\right)\cdot\left(\frac{d\mu}{dy'}\right); \qquad [3176e]$$
$$\left(\frac{dq}{dz'}\right)=\left(\frac{dq}{dR}\right)\cdot\left(\frac{dR}{dz'}\right)+\left(\frac{dq}{d\varpi}\right)\cdot\left(\frac{d\varpi}{dz'}\right)+\left(\frac{dq}{d\mu}\right)\cdot\left(\frac{d\mu}{dz'}\right).$$

[3176] $$\left(\frac{dq}{dx'}\right)=\sqrt{1-\mu^2}\,.\,\cos.\varpi\,.\left(\frac{dq}{dR}\right)-\frac{\sin.\varpi}{R.\sqrt{1-\mu^2}}\,.\left(\frac{dq}{d\varpi}\right)-\frac{\mu.\sqrt{1-\mu^2}\,.\,\cos.\varpi}{R}\,.\left(\frac{dq}{d\mu}\right);$$

[3177] $$\left(\frac{dq}{dy'}\right)=\sqrt{1-\mu^2}\,.\,\sin.\varpi\,.\left(\frac{dq}{dR}\right)+\frac{\cos.\varpi}{R.\sqrt{1-\mu^2}}\,.\left(\frac{dq}{d\varpi}\right)-\frac{\mu.\sqrt{1-\mu^2}\,.\,\sin.\varpi}{R}\,.\left(\frac{dq}{d\mu}\right);$$

[3178] $$\left(\frac{dq}{dz'}\right)=\mu\,.\left(\frac{dq}{dR}\right)+\frac{(1-\mu^2)}{R}\,.\left(\frac{dq}{d\mu}\right).$$

From these expressions, we easily obtain the following values of $\frac{dN}{dt}$, $\frac{dN'}{dt}$, $\frac{dN''}{dt}$; observing, that if we neglect, in [3172—3174], the

[3179] square of q, we may suppose* $R=1$ and $f=2$,

[3180] $$\frac{dN}{dt}=S\,.\,\alpha g y\,.\,d\mu\,.\,d\varpi\,.\left(\frac{dq}{d\varpi}\right);\dagger$$

Effect of the pressure of the column.

[3181] $$\frac{dN'}{dt}=S\,.\,\alpha g y\,.\,d\mu\,.\,d\varpi\,.\left\{\sqrt{1-\mu^2}\,.\,\cos.\varpi\,.\left(\frac{dq}{d\mu}\right)+\frac{\mu\,.\,\sin.\varpi}{\sqrt{1-\mu^2}}\,.\left(\frac{dq}{d\varpi}\right)\right\};$$

[3182] $$\frac{dN''}{dt}=S\,.\,\alpha g y\,.\,d\mu\,.\,d\varpi\,.\left\{\sqrt{1-\mu^2}\,.\,\sin.\varpi\,.\left(\frac{dq}{d\mu}\right)-\frac{\mu\,.\,\cos.\varpi}{\sqrt{1-\mu^2}}\,.\left(\frac{dq}{d\varpi}\right)\right\}.$$

Substituting the preceding values of $\left(\frac{dR}{dx'}\right)$, $\left(\frac{dR}{dy'}\right)$, &c., they become, as in

[3176f] [3176—3178], respectively. We may also observe, that if we write $\varpi-100^{\circ}$, for ϖ in [3163], this value of x' becomes like y' [3164], and the same change being made in [3176], it becomes, as in [3177].

[3179a] * (2123) Comparing the value of R [3175] with [3169], we get $R=\sqrt{(1+2q)}$, or by neglecting q^2, $R=1+q$; and as the formulas [3172—3174] are of the order q,

[3179b] we may, after substituting in them the values of x', y', z' [3163—3165], put $R=1$, neglecting q^2. In like manner, we may neglect, in f [3168], the terms of the order q; and then the values of $\left(\frac{du}{dx'}\right)$, $\left(\frac{du}{dy'}\right)$, $\left(\frac{du}{dz'}\right)$ [3170], are $2x'$, $2y'$, $2z'$, respectively;

[3179c] consequently f [3168] becomes $f=2\sqrt{(x'^2+y'^2+z'^2)}=2R=2$ [3175, 3179b], as above.

[3180a] † [2124] Substituting, in [3172—3174], the values $R=1$, $f=2$ [3179], and those of x', y', z' [3163—3165], we get

$$\frac{dN}{dt}=S\,.\,\alpha g y\,.\,d\mu\,.\,d\varpi\,.\sqrt{(1-\mu^2)}\,.\left\{\cos.\varpi\,.\left(\frac{dq}{dy'}\right)-\sin.\varpi\,.\left(\frac{dq}{dx'}\right)\right\};$$

[3180b] $$\frac{dN'}{dt}=S\,.\,\alpha g y\,.\,d\mu\,.\,d\varpi\,.\left\{\sqrt{(1-\mu^2)}\,.\,\cos.\varpi\,.\left(\frac{dq}{dz'}\right)-\mu\,.\left(\frac{dq}{dx'}\right)\right\};$$

$$\frac{dN''}{dt}=S\,.\,\alpha g y\,.\,d\mu\,.\,d\varpi\,.\left\{\sqrt{(1-\mu^2)}\,.\,\sin.\varpi\,.\left(\frac{dq}{dz'}\right)-\mu\,.\left(\frac{dq}{dy'}\right)\right\}.$$

We shall now determine the values of $\frac{dN}{dt}$, $\frac{dN'}{dt}$, and $\frac{dN''}{dt}$, *corresponding to the attraction of the aqueous stratum* $\alpha y \,.\, d\mu \,.\, d\varpi$ [3166*b*] *upon the solid terrestrial nucleus.* It is evident that if this nucleus, and the ocean which it covers, form a solid mass, there will be no motion in the mass, arising from the mutual attraction of all its particles; therefore the effect of the attraction of the aqueous stratum upon the ocean, added to the effect of its attraction upon the nucleus, is equal, and of a contrary sign, to the effect of the attraction of the whole earth upon this aqueous stratum. Hence it follows, that *the effect of the attraction of this stratum upon the nucleus, is equal to the sum of the effects of the attraction of the whole earth upon the stratum, and of the attraction of the stratum upon the ocean, this sum being taken with a contrary sign* [3184*d*].

Attraction of the aqueous stratum on the nucleus, [3182′] [3183] [3184] resolved into two other forces. [3184′]

Now multiplying [3176] by $-\sin.\varpi$, also [3177] by $\cos.\varpi$, and adding the two products, we find that the co-efficients of $\left(\frac{dq}{dR}\right)$, $\left(\frac{dq}{d\mu}\right)$, destroy each other, and by putting $\cos.^2\varpi + \sin.^2\varpi = 1$, in the co-efficient of $\left(\frac{dq}{d\varpi}\right)$, we obtain

$$\cos.\varpi \,.\left(\frac{dq}{dy'}\right) - \sin.\varpi \,.\left(\frac{dq}{dx'}\right) = \frac{1}{R.\sqrt{(1-\mu^2)}}\cdot\left(\frac{dq}{d\varpi}\right) = \frac{1}{\sqrt{(1-\mu^2)}}\cdot\left(\frac{dq}{d\varpi}\right). \qquad [3180c]$$

Substituting this in the first of the equations [3180*b*], we get [3180]. If we multiply [3178] by $\sqrt{(1-\mu^2)}\,.\cos.\varpi$, also [3176] by $-\mu$, and add the products, putting $R=1$ [3179], we shall find that the co-efficient of $\left(\frac{dq}{dR}\right)$ vanishes in the sum, and we get, by a slight reduction,

$$\sqrt{(1-\mu^2)}.\cos.\varpi.\left(\frac{dq}{dz'}\right) - \mu.\left(\frac{dq}{dx'}\right) = \left(\frac{dq}{d\mu}\right).\cos.\varpi.\left\{\left(1-\mu^2\right)^{\frac{3}{2}} + \mu^2.\sqrt{(1-\mu^2)}\right\} + \frac{\mu.\sin.\varpi}{\sqrt{(1-\mu^2)}}\cdot\left(\frac{dq}{d\varpi}\right) \qquad [3180d]$$

$$= \left(\frac{dq}{d\mu}\right).\cos.\varpi.\sqrt{(1-\mu^2)} + \frac{\mu.\sin.\varpi}{\sqrt{(1-\mu^2)}}\cdot\left(\frac{dq}{d\varpi}\right); \qquad [3180e]$$

hence the second of the equations [3180*b*] becomes, as in [3181]. Lastly, multiplying [3178] by $\sqrt{(1-\mu^2)}\,.\sin.\varpi$, and [3177] by $-\mu$; then adding the products, we find that the co-efficient of $\left(\frac{dq}{dR}\right)$ vanishes in the sum; that of $\left(\frac{dq}{d\varpi}\right)$ being reduced, as in [3180*d*, *e*], gives

$$\sqrt{(1-\mu^2)}\,.\sin.\varpi\,.\left(\frac{dq}{dz'}\right) - \mu\,.\left(\frac{dq}{dy'}\right) = \sqrt{(1-\mu^2)}\,.\sin.\varpi\,.\left(\frac{dq}{d\mu}\right) - \frac{\mu.\cos.\varpi}{\sqrt{(1-\mu^2)}}\cdot\left(\frac{dq}{d\varpi}\right). \qquad [3180f]$$

Substituting this in the last of the equations [3180*b*], we obtain [3182].

The resultant of the attraction of the whole earth upon the small column of the aqueous stratum $\alpha y \,.\, d\mu \,.\, d\varpi$, combined with the centrifugal force,
[3185] is perpendicular to the surface of equilibrium of the sea;* therefore we shall have the attraction of the whole earth upon this column, by supposing it to be acted upon by this resultant, and by the centrifugal force taken with a
[3185′] contrary sign [3184f]. The first of these two forces is the gravity g,
[3186] which must be multiplied by the mass of the particle $\alpha y \,.\, d\mu \,.\, d\varpi$ [3166b]; therefore, by supposing that the equation of the surface of equilibrium of the sea is

Equation of the surface of equilibrium.

[3187] $$x'^2 + y'^2 + z'^2 = 1 + 2q';$$

[3184a] * (2125) We shall put S for the *stratum* of the ocean, whose interior radius is 1, and exterior radius $1 + \alpha y$ [3158]; O for the whole ocean; $\mathcal{N}$ for the *nucleus*, or solid part of the earth; so that if the whole mass $\mathcal{N} + O$ be represented by E, we shall have
[3184b] $\mathcal{N} = E - O$; hence we get [3184c], which is successively reduced to the form [3184d], by observing that the attraction of S on $E = -$ attraction of E on S,

[3184c] Attraction of S on $\mathcal{N}$ = attraction of S on $(E - O)$ = attraction of S on E — attraction of S on O;

= — attraction of E on S — attraction of S on O;

[3184d] = — {attraction of E on S + attraction of S on O}, as in [3184, 3184′].

[3184e] Now we have the whole gravity of g on S = attraction of E on S + centrifugal force; always noticing the signs of these forces, as in [3152‴]; hence, by transposition, we find, as in [3185, &c.],

[3184f] Attraction of E on S = whole gravity g on S — centrifugal force of S;

substituting this in [3184d], we get,

[3184g] Attraction of S on $\mathcal{N}$ = — whole gravity g on S + centrifugal force of S — attraction of S on O.

By this means the attraction of the aqueous stratum S, on the solid nucleus $\mathcal{N}$, is reduced to the three different forces, given in the second member of [3184g]. The first of these
[3184h] forces, depending on the action of the whole gravity g, upon each particle of the stratum S, is computed in [3188], and the resulting terms of $\frac{d\mathcal{N}}{dt}$, $\frac{d\mathcal{N}'}{dt}$, $\frac{d\mathcal{N}''}{dt}$, being combined with those depending on the pressure of the stratum S [3180—3182], form the expressions
[3184i] [3191—3193]. The second term of [3184g] depending on the centrifugal force of each particle of the stratum S, is computed in [3199—3201]. The third term of [3184g], depending on the attraction of S on O, is computed in [3202—3211], and forms the expressions [3209—3211]. Combining the results of these three different expressions
[3184k] [3191—3193, 3199—3201, 3209—3211], we finally obtain the complete values of $\frac{d\mathcal{N}}{dt}$, $\frac{d\mathcal{N}'}{dt}$, $\frac{d\mathcal{N}''}{dt}$ [3212—3214], corresponding to the pressure, and attraction, of the stratum S, on the solid part of the earth.

we shall have, by what precedes, the parts of $\frac{dN}{dt}$, $\frac{dN'}{dt}$, $\frac{dN''}{dt}$, corresponding to this force, by changing q into q' in the preceding expressions of these quantities [3180—3182], and taking them with a contrary sign, as is observed in [3185', 3184g]. Now if we connect these terms with the preceding expressions [3180—3182], and put $\gamma = q' - q$ for the depth of the sea, which we shall suppose to be very small, we shall obtain the following values,*

[3188]

Effect of gravity.

[3189]

[3190]

$$\frac{dN}{dt} = -S.\alpha g y.d\mu.d\varpi.\left(\frac{d\gamma}{d\varpi}\right); \qquad [3191]$$

Effect of the pressure of the column and of the gravity g.

$$\frac{dN'}{dt} = -S.\alpha g y.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}.\cos.\varpi.\left(\frac{d\gamma}{d\mu}\right) + \frac{\mu.\sin.\varpi}{\sqrt{1-\mu^2}}.\left(\frac{d\gamma}{d\varpi}\right)\right\}; \qquad [3192]$$

$$\frac{dN''}{dt} = -S.\alpha g y.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}.\sin.\varpi.\left(\frac{d\gamma}{d\mu}\right) - \frac{\mu.\cos.\varpi}{\sqrt{1-\mu^2}}.\left(\frac{d\gamma}{d\varpi}\right)\right\}. \qquad [3193]$$

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* (2126) In [3166, &c.] the effect of the pressure of the small column of water, whose height is αy, is calculated upon the supposition that it acts upon the surface of the nucleus, or at the bottom of the sea. The equation of this surface is [3169]; that of the surface of the sea, in equilibrium, being [3187]. The radii, or the values of $\sqrt{(x'^2+y'^2+z'^2)}$, corresponding to these two surfaces, are, respectively, $\sqrt{(1+2q)}$, $\sqrt{(1+2q')}$ [3169, 3187]; or by neglecting q^2, q'^2, they become $1+q$ and $1+q'$; whose difference is $q'-q=\gamma$ [3190]. We may compute, in the above manner, the pressure of this column upon the surface of equilibrium. The only change necessary is that of using the equation of the surface of equilibrium [3187], instead of that of the nucleus [3169]; and this is done by merely accenting the symbol q; so that if we change q into q', in the values [3180—3182], we shall obtain the values of $\frac{dN}{dt}$, $\frac{dN'}{dt}$, $\frac{dN''}{dt}$, corresponding to the pressure of this column upon the surface of equilibrium. The signs of these expressions are to be changed, to obtain the values corresponding to the *first term* of the second member of [3184g]; therefore we must change q into $-q'$, in [3180—3182], to obtain the parts of $\frac{dN}{dt}$, $\frac{dN'}{dt}$, $\frac{dN''}{dt}$, depending on this first term. Hence this part of $\frac{dN}{dt}$ is $-S.\alpha g y.d\mu.d\varpi.\left(\frac{dq'}{d\varpi}\right)$; and by connecting it with the part computed in [3180], it becomes,

[3190a]

[3190b]

[3190c]

[3190d]

[3190e]

[3190f]

[3190g]

$$-S.\alpha g y.d\mu.d\varpi.\left\{\left(\frac{dq'}{d\varpi}\right)-\left(\frac{dq}{d\varpi}\right)\right\} = -S.\alpha g y.d\mu.d\varpi.\left(\frac{d.(q'-q)}{d\varpi}\right)$$
$$= -S.\alpha g y.d\mu.d\varpi.\left(\frac{d\gamma}{d\varpi}\right), \qquad [3190h]$$

as in [3191]. This is the same as [3180], changing q into $-\gamma$; and by making the

[3193'] *We must now consider the effect of the centrifugal force, taken with a contrary sign, and subtract it from these values*; or, in other words, we must add the
[3194] effect of the centrifugal force.* If we put n for the rotatory velocity of the earth, the centrifugal force of a particle of the column $\alpha y.d\mu.d\varpi$, will
[3195] be $n^2.\sqrt{1-\mu^2}$; and by multiplying it by the mass of the column, we shall
[3196] obtain $\alpha n^2 y.d\mu.d\varpi.\sqrt{1-\mu^2}$ for the whole force. This is directed according to the radius of the parallel of latitude of the earth; and by resolving it into two forces, the one parallel to the axis of x', the other parallel to the
[3197] axis of y', we shall get $\alpha n^2 y.d\mu.d\varpi.\sqrt{1-\mu^2}.\cos.\varpi$ for the first; and
[3198] $\alpha n^2 y.d\mu.d\varpi.\sqrt{1-\mu^2}.\sin.\varpi$ for the second. Hence we obtain for the parts of $\frac{dN}{dt}$, $\frac{dN'}{dt}$, $\frac{dN''}{dt}$, corresponding to the centrifugal force,

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[3190i] same change in [3181, 3182], we get [3192, 3193]. Hence it appears that the system of equations [3191—3193], comprises the effect of the pressures [3180—3182], and that of the gravity g, depending upon the first term of the second member of [3184g].

[3195a] * [2127] To compute the effect of the centrifugal force, corresponding to the *second term* of the second member of [3184g], we may observe, that the rotatory velocity at the distance 1 from the axis being n [3194], the centrifugal force at any distance from the axis is equal to n^2, multiplied by that distance [1569b]. Now the axis of revolution z', makes an angle with a particle of that column, whose cosine $=\mu$, sine $=\sqrt{(1-\mu^2)}$ [3160];
[3195b] and as its distance from the centre of the earth is $R=1$ nearly, the distance of the particle from the axis of z', or the radius of the parallel of latitude, is $\sqrt{(1-\mu^2)}$; consequently
[3195c] the centrifugal force of a single particle is $n^2.\sqrt{(1-\mu^2)}$, in the direction of the radius of the parallel of latitude. The forces P, Q, R [3152''], into which the preceding centrifugal force can be resolved, are evidently as in [3195d], observing that the force R is to be understood as a different symbol from the radius R [3195b];

[3195d] $$P=n^2.\sqrt{(1-\mu^2)}.\cos.\varpi; \qquad Q=n^2.\sqrt{(1-\mu^2)}.\sin.\varpi; \qquad R=0.$$

Substituting these in [3153—3155], we get,

[3195e] $$\frac{dN}{dt}=S.n^2.\sqrt{(1-\mu^2)}.\{x'.\sin.\varpi-y'.\cos.\varpi\}.dm.$$

[3195f] $$\frac{dN'}{dt}=-S.n^2.z'.\sqrt{(1-\mu^2)}.\cos.\varpi.dm;$$

[3195g] $$\frac{dN''}{dt}=-S.n^2.z'.\sqrt{(1-\mu^2)}.\sin.\varpi.dm.$$

Putting $R=1$ [3195b], in the expression of the particle dm, [3166b], we obtain
[3195h] $dm=\alpha y.d\mu.d\varpi$; substituting this and the values of x', y', z' [3163—3165], in [3195e—g], we get the expressions [3199—3201], corresponding to the centrifugal force, or second term of [3184g].

Effect of the centrifugal force.

$$\frac{dN}{dt} = 0\,;$$ [3199]

$$\frac{dN'}{dt} = -S\,.\,\alpha n^2 y\,.\,d\mu\,.\,d\varpi\,.\,\mu\,.\sqrt{1-\mu^2}\,.\,\cos.\,\varpi\,;$$ [3200]

$$\frac{dN''}{dt} = -S\,.\,\alpha n^2 y\,.\,d\mu\,.\,d\varpi\,.\,\mu\,.\sqrt{1-\mu^2}\,.\,\sin.\,\varpi.$$ [3201]

It now remains to determine the effect of the attraction of the aqueous stratum upon the ocean. For this purpose, we shall put αU for the sum of the particles of the stratum, divided by their respective distances from a particle of the ocean, whose co-ordinates are represented by x', y', z', or by the quantities R, μ, ϖ; then $\alpha.\left(\frac{dU}{dx'}\right)$, $\alpha.\left(\frac{dU}{dy'}\right)$, $\alpha.\left(\frac{dU}{dz'}\right)$, will be the attractions of the stratum upon this particle, parallel to these co-ordinates, and tending to increase them.* The mass of the particle is $dm = R^2 dR.d\mu.d\varpi$ [2918]; therefore we shall have for the parts of $\frac{dN}{dt}$, $\frac{dN'}{dt}$, $\frac{dN''}{dt}$, corresponding to the attraction of the aqueous stratum upon the ocean, [3202] [3203] [3204]

$$S\,.\,\alpha R^2 dR\,.\,d\mu\,.\,d\varpi\,.\left\{x'\,.\left(\frac{dU}{dy'}\right) - y'\left(\frac{dU}{dx'}\right)\right\};$$ [3205]

$$S\,.\,\alpha R^2 dR\,.\,d\mu\,.\,d\varpi\,.\left\{x'\,.\left(\frac{dU}{dz'}\right) - z'\left(\frac{dU}{dx'}\right)\right\};$$ [3206]

$$S\,.\,\alpha R^2 dR\,.\,d\mu\,.\,d\varpi\,.\left\{y'\,.\left(\frac{dU}{dz'}\right) - z'\left(\frac{dU}{dy'}\right)\right\}.$$ [3207]

To integrate these functions relatively to R, we shall observe, that the depth of the sea being supposed very small, we may put $R=1$, [3208]

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* (2128) This is evident from what has been proved in [1387a]; substituting the forces [3203] instead of P, Q, R, in [3153—3155], we get,

$$\frac{dN}{dt} = S\,.\,\alpha\,.\,dm\,.\left\{x'\,.\left(\frac{dU}{dy'}\right) - y'\,.\left(\frac{dU}{dx'}\right)\right\};$$ [3205a]

$$\frac{dN'}{dt} = S\,.\,\alpha\,.\,dm\,.\left\{x'\,.\left(\frac{dU}{dz'}\right) - z'\,.\left(\frac{dU}{dx'}\right)\right\};$$ [3205b]

$$\frac{dN''}{dt} = S\,.\,\alpha\,.\,dm\,.\left\{y'\,.\left(\frac{dU}{dz'}\right) - z'\,.\left(\frac{dU}{dy'}\right)\right\}.$$ [3205c]

Substituting dm [2918], they become as in [3205—3207].

[3208′] and $\int R^2 dR = \gamma$.* Moreover, if we change the partial differentials $\left(\frac{dU}{dx'}\right)$, $\left(\frac{dU}{dy'}\right)$, $\left(\frac{dU}{dz'}\right)$, into others relative to the variable quantities R, ϖ, μ, the preceding functions will become, by taking them with a contrary sign [3184′],†

Effect of the attraction of the aqueous stratum.

[3209] $$-S.\alpha\gamma.d\mu.d\varpi.\left(\frac{dU}{d\varpi}\right);$$

[3210] $$-S.\alpha\gamma.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}.\cos.\varpi.\left(\frac{dU}{d\mu}\right)+\frac{\mu.\sin.\varpi}{\sqrt{1-\mu^2}}.\left(\frac{dU}{d\varpi}\right)\right\};$$

[3211] $$-S.\alpha\gamma.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}.\sin.\varpi.\left(\frac{dU}{d\mu}\right)-\frac{\mu.\cos.\varpi}{\sqrt{1-\mu^2}}.\left(\frac{dU}{d\varpi}\right)\right\}.$$

If we add these values to the corresponding parts of $\frac{dN}{dt}$, $\frac{dN'}{dt}$, $\frac{dN''}{dt}$ [3191—3193, 3199—3201], we shall obtain *the whole expression of these quantities, corresponding to the attraction and pressure of the ocean on the solid terrestrial spheroid*,‡

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* (2129) The general value of R being put equal to $1+\gamma'$, we have, by neglecting
[3208a] γ'^2, $\int R^2 dR = \int(1+\gamma')^2.d\gamma' = \int d\gamma' = \gamma$, as above; the integral being taken from the bottom to the surface of the sea.

† (2130) Substituting, in [3205—3208], the value $\int R^2 dR = \gamma$ [3208], they become, respectively,

$$S.\alpha\gamma.d\mu.d\varpi.\left\{x'.\left(\frac{dU}{dy'}\right)-y'.\left(\frac{dU}{dx'}\right)\right\};$$

[3209a] $$S.\alpha\gamma.d\mu.d\varpi.\left\{x'.\left(\frac{dU}{dz'}\right)-z'.\left(\frac{dU}{dx'}\right)\right\};$$

$$S.\alpha\gamma.d\mu.d\varpi.\left\{y'.\left(\frac{dU}{dz'}\right)-z'.\left(\frac{dU}{dy'}\right)\right\}.$$

These may be derived from the formulas [3172—3174], by putting $R=1$, $f=2$ [3179],
[3209b] $gy=\gamma$, $q=U$; and if we make the same substitutions in [3180—3182], which were derived from [3172—3174]; they will give the corresponding values of the expression
[3209c] [3205—3207], as in [3209—3211]; the signs being changed, as in the *third term* of [3184g], which is negative.

‡ (2131) The sum of the parts of $\left(\frac{dN}{dt}\right)$ [3191, 3199, 3209], gives the whole
[3212a] value [3212]. The sum of the three parts of $\left(\frac{dN'}{dt}\right)$ [3192, 3200, 3210], gives
[3212b] its value [3213]. The three parts of $\left(\frac{dN''}{dt}\right)$ [3193, 3201, 3211], gives its value

$$\frac{dN}{dt} = -S.\alpha g y . d\mu . d\varpi . \left(\frac{d\gamma}{d\varpi}\right) - S.\alpha\gamma . d\mu . d\varpi . \left(\frac{dU}{d\varpi}\right);$$ [3212]

Complete values, corresponding to the attraction

$$\frac{dN'}{dt} = -S.\alpha g y . d\mu . d\varpi . \left\{\sqrt{1-\mu^2} . \cos.\varpi . \left(\frac{d\gamma}{d\mu}\right) + \frac{\mu . \sin.\varpi}{\sqrt{1-\mu^2}} . \left(\frac{d\gamma}{d\varpi}\right)\right\}$$

$$-S.\alpha\gamma . d\mu . d\varpi . \left\{\sqrt{1-\mu^2} . \cos.\varpi . \left(\frac{dU}{d\mu}\right) + \frac{\mu . \sin.\varpi}{\sqrt{1-\mu^2}} . \left(\frac{dU}{d\varpi}\right)\right\}$$ [3213]

and pressure of the ocean on the solid nucleus.

$$-S.\alpha n^2 y . d\mu . d\varpi \mu\sqrt{1-\mu^2} . \cos.\varpi;$$

First form.

$$\frac{dN''}{dt} = -S.\alpha g y . d\mu . d\varpi . \left\{\sqrt{1-\mu^2} . \sin.\varpi . \left(\frac{d\gamma}{d\mu}\right) - \frac{\mu . \cos.\varpi}{\sqrt{1-\mu^2}} . \left(\frac{d\gamma}{d\varpi}\right)\right\}$$

$$-S.\alpha\gamma . d\mu . d\varpi . \left\{\sqrt{1-\mu^2} . \sin.\varpi . \left(\frac{dU}{d\mu}\right) - \frac{\mu . \cos.\varpi}{\sqrt{1-\mu^2}} . \left(\frac{dU}{d\varpi}\right)\right\}$$ [3214]

$$-S.\alpha n^2 y . d\mu . d\varpi \mu\sqrt{1-\mu^2} . \sin.\varpi.$$

The preceding integrals must be taken from $\mu = -1$ to $\mu = 1$, and from $\varpi = 0$ to $\varpi =$ four right angles. Integrating relatively to ϖ, we have,* [3215]

$$S.\alpha g y . d\varpi . \left(\frac{d\gamma}{d\varpi}\right) = \alpha g y\gamma - S.\alpha g\gamma . d\varpi . \left(\frac{dy}{d\varpi}\right) + \text{constant};$$ [3216]

now it is evident, that at the two limits of the integral, where $\varpi = 0$, and $\varpi =$ four right angles, the function $\alpha g y\gamma$ is the same, since these two limits appertain to the same point of the surface of the spheroid; [3216′]

[3214]. These are the complete values, corresponding to the attraction and pressure of the ocean on the solid terrestrial spheroid; observing that the parts first computed [3180—3182] are included in [3191—3193], as appears in [3190g]. The limits of the integrals [3215] are similar to those in [1431d], and are so taken, as to include the whole surface of the earth. [3212c]

* (2132) The integration of the first member of [3216], by parts, relatively to ϖ [1716a], becomes as in its second member. This equation is easily proved to be correct, by taking its differential, relatively to ϖ, by which means it becomes identical. If we suppose the integral to vanish when $\varpi = 0$, and put y', γ', for the corresponding values of y, γ; the equation [3216] will become, at that point, $0 = \alpha g y'\gamma' +$ constant; or constant $= -\alpha g y'\gamma'$. Substituting this in [3216], we get, generally, [3216a] [3216b]

$$S.\alpha g y . d\varpi . \left(\frac{d\gamma}{d\varpi}\right) = -S.\alpha g y . d\varpi . \left(\frac{dy}{d\varpi}\right) + \alpha g y\gamma - \alpha g y'\gamma'.$$ [3216c]

At the other limit of the integral, we have $\varpi = 400°$, $y = y'$, and $\gamma = \gamma'$, consequently $\alpha g y\gamma - \alpha g y'\gamma' = 0$; substituting this in the preceding expression, it becomes, as in [3218].

[3217] therefore we have $\alpha g y \gamma + \text{constant} = 0$, consequently

[3218]
$$S.\alpha g y . d\varpi . \left(\frac{d\gamma}{d\varpi}\right) = -S.\alpha g \gamma . d\varpi . \left(\frac{dy}{d\varpi}\right).$$

Integrating relatively to μ, we have,*

[3219]
$$S.\alpha g y . d\mu . \left(\frac{d\gamma}{d\mu}\right) . \sqrt{1-\mu^2} . \sin.\varpi = \alpha g y \gamma . \sqrt{1-\mu^2} . \sin.\varpi + S.\alpha g y \gamma . \frac{\mu\, d\mu . \sin.\varpi}{\sqrt{1-\mu^2}}$$
$$- S.\alpha g \gamma . d\mu . \sqrt{1-\mu^2} . \left(\frac{dy}{d\mu}\right) . \sin.\varpi + \text{const.}$$

[3219′] The integral must be taken from $\mu = -1$ to $\mu = 1$; now y and γ are never infinite; and as the radical $\sqrt{1-\mu^2}$ is nothing at these limits, we have at the same limits,

[3220]
$$\alpha g y \gamma . \sqrt{1-\mu^2} . \sin.\varpi + \text{constant} = 0;$$

consequently

[3221]
$$S.\alpha g y . d\mu . d\varpi . \left(\frac{d\gamma}{d\mu}\right) . \sqrt{1-\mu^2} . \sin.\varpi = S.\alpha g y \gamma . d\varpi . \frac{\mu\, d\mu . \sin.\varpi}{\sqrt{1-\mu^2}}$$
$$- S.\alpha g \gamma . d\mu . d\varpi . \sqrt{1-\mu^2} . \sin.\varpi . \left(\frac{dy}{d\mu}\right).$$

We also find, by integrating relatively to ϖ,†

[3222]
$$S.\alpha g y . d\mu . d\varpi . \frac{\mu . \cos.\varpi}{\sqrt{1-\mu^2}} . \left(\frac{d\gamma}{d\varpi}\right) = -S.\alpha g \gamma . d\mu . d\varpi . \frac{\mu . \cos.\varpi}{\sqrt{1-\mu^2}} . \left(\frac{dy}{d\varpi}\right)$$
$$+ S.\alpha g y \gamma . d\varpi . \frac{\mu\, d\mu . \sin.\varpi}{\sqrt{1-\mu^2}};$$

[3219a] * (2133) The correctness of the equation [3219] is easily perceived by taking its differential relatively to μ; by which means it becomes identical. The constant quantity is found as in the preceding note, by means of [3220]. Substituting this, and taking the integral so as to correspond to the other limit of μ, and then multiplying by $d\varpi$, it becomes, as in [3221].

† (2134) In general, we have,

[3222a]
$$S.\alpha g y . d\varpi . \frac{\mu . \cos.\varpi}{\sqrt{(1-\mu^2)}} . \left(\frac{d\gamma}{d\varpi}\right) = \alpha g y \gamma . \frac{\mu . \cos.\varpi}{\sqrt{(1-\mu^2)}} + \text{constant} - S.\alpha g \gamma . d\varpi . \frac{\mu . \cos.\varpi}{\sqrt{(1-\mu^2)}} . \left(\frac{dy}{d\varpi}\right)$$
$$+ S.\alpha g y \gamma . d\varpi . \frac{\mu . \sin.\varpi}{\sqrt{(1-\mu^2)}},$$

as is easily perceived by taking its differential relatively to ϖ. At the limits of this integral,

[3222b] where $\varpi = 0$ and $\varpi = 400^\circ$, the terms $\alpha g y \gamma . \frac{\mu . \cos.\varpi}{\sqrt{(1-\mu^2)}} + \text{constant}$, must become 0, as in the two preceding notes, and then multiplying by $d\mu$, it becomes, as in [3222].

therefore we have,*

$$S.\alpha g y.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}.\sin.\varpi.\left(\frac{d\gamma}{d\mu}\right)-\frac{\mu.\cos.\varpi}{\sqrt{1-\mu^2}}.\left(\frac{d\gamma}{d\varpi}\right)\right\}$$
$$=-S.\alpha g\gamma.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}.\sin.\varpi.\left(\frac{dy}{d\mu}\right)-\frac{\mu.\cos.\varpi}{\sqrt{1-\mu^2}}.\left(\frac{dy}{d\varpi}\right)\right\}. \quad [3223]$$

In like manner we have,†

$$S.\alpha g y.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}.\cos.\varpi.\left(\frac{d\gamma}{d\mu}\right)+\frac{\mu.\sin.\varpi}{\sqrt{1-\mu^2}}.\left(\frac{d\gamma}{d\varpi}\right)\right\}$$
$$=-S.\alpha g\gamma.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}.\cos.\varpi.\left(\frac{dy}{d\mu}\right)+\frac{\mu.\sin.\varpi}{\sqrt{1-\mu^2}}.\left(\frac{dy}{d\varpi}\right)\right\}. \quad [3224]$$

Hence the preceding expressions of $\frac{dN}{dt}$, $\frac{dN'}{dt}$, $\frac{dN''}{dt}$, become,‡

Complete values for the whole pressure and attraction of the sea.

$$\frac{dN}{dt}=S.\alpha\gamma.d\mu.d\varpi.\left\{g.\left(\frac{dy}{d\varpi}\right)-\left(\frac{dU}{d\varpi}\right)\right\}; \quad [3225]$$

$$\frac{dN'}{dt}=S.\alpha\gamma.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}.\cos.\varpi.\left[g.\left(\frac{dy}{d\mu}\right)-\left(\frac{dU}{d\mu}\right)\right]+\frac{\mu.\sin.\varpi}{\sqrt{1-\mu^2}}.\left[g.\left(\frac{dy}{d\varpi}\right)-\left(\frac{dU}{d\varpi}\right)\right]\right\}$$
$$-S.\alpha n^2y.d\mu.d\varpi.\mu.\sqrt{1-\mu^2}.\cos.\varpi; \quad [3226]$$

$$\frac{dN''}{dt}=S.\alpha\gamma.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}.\sin.\varpi.\left[g.\left(\frac{dy}{d\mu}\right)-\left(\frac{dU}{d\mu}\right)\right]-\frac{\mu.\cos.\varpi}{\sqrt{1-\mu^2}}.\left[g.\left(\frac{dy}{d\varpi}\right)-\left(\frac{dU}{d\varpi}\right)\right]\right\}$$
$$-S.\alpha n^2y.d\mu.d\varpi.\mu.\sqrt{1-\mu^2}.\sin.\varpi. \quad [3227]$$

Second form.

11. *We shall now determine the influence of these quantities on the motion of the terrestrial spheroid about its centre of gravity.* For this purpose, we shall resume the equations [2905—2907]. If we neglect the very small quantities $\left(\frac{B-A}{C}\right).qr.dt$, $\left(\frac{C-B}{A}\right).rp.dt$, $\left(\frac{A-C}{B}\right).pq.dt$; and put also $\varphi=0$, $\theta=0$, because we have taken the principal axes for [3227′] [3228]

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* (2135) Subtracting [3222] from [3221], we get the equation [3223]. [3223a]

† (2136) This equation is found in the same manner as [3223]; or it may be derived from [3223], by changing the axis of x'' into that of y'', so as to put $100^{\circ}+\varpi$ for ϖ; by which means $\sin.\varpi$ changes into $\cos.\varpi$, $\cos.\varpi$ into $-\sin.\varpi$, and [3223] into [3224]. [3224a]

‡ (2137) Substituting [3218] in [3212], we get [3225]. Instead of the terms depending on γ [3213], we must use their values [3224], and we obtain [3226]. Lastly, substituting, in [3214], the terms depending on γ [3223], we get [3227]. [3225a]

[3229] the axes of x', y', z', [2978]; we shall have,*

[3230] $$dp = \frac{dN}{C}; \qquad dq = \frac{dN''}{A}; \qquad dr = -\frac{dN'}{B}.$$

We observe, in the first place, that the terms depending on very small angles, contained in dN, may, by integration, become very great in the
[3230'] value of p; it is therefore necessary to notice these terms.

We have found, in [3046, 3035], that

[3231] $$\frac{d\theta}{dt} = r.\sin.\varphi - q.\cos.\varphi;$$

[3232] $$\frac{d\psi}{dt}.\sin.\theta = r.\cos.\varphi + q.\sin.\varphi;$$

therefore if we put

[3233] $$\frac{d\theta}{dt} = x''; \qquad \frac{d\psi}{dt}.\sin.\theta = y'';$$

observing that $d\varphi$ is very nearly equal to $n\,dt$ [3024'], we shall obtain,†

[3234] $$dx'' = dr.\sin.\varphi - dq.\cos.\varphi + ny''.dt;$$

[3235] $$dy'' = dr.\cos.\varphi + dq.\sin.\varphi - nx''.dt.$$

If we substitute, for dq and dr, their values [3230], in which we may change A and B into C, we shall have,

[3236] $$dx'' = -\frac{dN'}{C}.\sin.\varphi - \frac{dN''}{C}.\cos.\varphi + ny''.dt;$$

[3237] $$dy'' = -\frac{dN'}{C}.\cos.\varphi + \frac{dN''}{C}.\sin.\varphi - nx''.dt.$$

[3230a] * (2138) Substituting $\theta = 0$, $\varphi = 0$ [2978], in [2905—2907], and neglecting the terms mentioned in [3228], we get [3230].

[3234a] † (2139) Substituting [3233] in [3231—3232], we get $x'' = r.\sin.\varphi - q.\cos.\varphi$; $y'' = r.\cos.\varphi + q.\sin.\varphi$; whose differentials are

[3234b] $$dx'' = dr.\sin.\varphi - dq.\cos.\varphi + d\varphi.(r.\cos.\varphi + q.\sin.\varphi);$$

[3234c] $$dy'' = dr.\cos.\varphi + dq.\sin.\varphi - d\varphi.(r.\sin.\varphi - q.\cos.\varphi).$$

The co-efficients of $d\varphi$, in these expressions, are equal to y'', $-x''$, respectively [3234a], and if we put, as in [3024'], $d\varphi = n\,dt$, we shall obtain [3234, 3235]. Substituting
[3234d] in these the values of dq, dr [3230], and changing A, B, into C, which may be done by neglecting terms of the order of the ellipticity of the earth, in comparison with the terms retained, we obtain the formulas [3236, 3237].

Let $H\,dt\,.\cos.(it+\varepsilon)$ be any term of $\frac{dN'\,.\sin.\varphi+dN''\,.\cos.\varphi}{C}$, and [3238]

$H'\,dt\,.\sin.(it+\varepsilon)$, the similar term of $\frac{dN'\,.\cos.\varphi-dN''\,.\sin.\varphi}{C}$;* then [3238']

the corresponding terms of x'' and y'' will be,†

$$x''=\left(\frac{nH'-iH}{i^2-n^2}\right).\sin.(it+\varepsilon); \qquad y''=\left(\frac{iH'-nH}{i^2-n^2}\right).\cos.(it+\varepsilon). \qquad [3239]$$

The terms depending on very small angles, or those in which i is very small, are hardly sensible in the values of x'', y''; but they become, by integration, very perceptible in the values of θ and ψ;‡ and we have seen, in [3040, 3041], that the precession and nutation depend on such quantities; it is therefore necessary to notice them. These terms are produced by those of dN' and dN'', which depend on angles differing but little from nt; for by multiplying them by $\sin.\varphi$, and $\cos.\varphi$, they produce quantities depending on very small angles;§ therefore we must pay particular attention to terms of this kind. [3239'] [3239''] [3239'''] [3239'''']

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* (2140) We may remark on these assumed forms as in [3020a]. [3238a]

† (2141) If we substitute, in [3236, 3237], the assumed values [3238], we get

$$dx''=-H\,dt\,.\cos.(it+\varepsilon)+ny''\,dt, \qquad dy''=-H'\,dt\,.\sin.(it+\varepsilon)-n\,x''\,dt. \qquad [3239a]$$

Taking the differential of the first of these equations; substituting the value of dy'', given by the second, and dividing by dt^2, we obtain,

$$\frac{d^2x''}{dt^2}+n^2x''+(nH'-iH)\,.\sin.(it+\varepsilon)=0. \qquad [3239b]$$

From this we get, as in [865, 871], $x''=\left(\frac{nH'-iH}{i^2-n^2}\right).\sin.(it+\varepsilon)$, as in [3239]. [3239c]

The value of dx'' deduced from this, being substituted in the first of the equations [3239a], gives

$$\left(\frac{nH'-iH}{i^2-n^2}\right).i\,dt\,.\cos.(it+\varepsilon)=-H\,dt\,.\cos.(it+\varepsilon)+n\,y''\,dt.$$

Dividing this by $n\,dt$, and reducing, we get y [3239]. [3239d]

‡ (2142) These values are deduced from [3231, 3232] or [3233], by integration, as in [3101, 3100]. [3239e]

§ (2143) If dN' contain a term of the form $L\,.\sin.(nt+vt+\varepsilon)$, it will produce, in dx'' [3236], a term depending on the product $\sin.\varphi\times\sin.(nt+vt+\varepsilon)$, or upon $\cos.(nt-\varphi+vt+\varepsilon)$ [17] Int. Now if we suppose v to be small, and nt nearly equal to φ [3024'], the coefficient of t, represented by i [3238], must be very small [3240a] [3240b]

[3239v] The terms in which i differs but little from n, become very great in the values of x'' and y'' [3239], because the divisor i^2-n^2 is then [3240] very small. These terms result from those of dN' and dN'', which contain very small angles,* and for that reason they must be noticed. They may also be produced by the terms of dN' and dN'', depending [3240'] on angles which differ but little from $2nt$; for if, for example, dN' contains the term $L\,dt\,.\,\sin.(2nt+vt+\varepsilon)$, v being very small, it will produce, in the function [3238] $\frac{dN'.\,\sin.\,\varphi+dN''.\,\cos.\,\varphi}{C}$, the term $\frac{L}{2C}.\,dt\,.\,\cos.\,(2nt-\varphi+vt+\varepsilon)$ [17] Int., and in the function $\frac{dN'.\,\cos.\,\varphi-dN''.\,\sin.\,\varphi}{C}$, the term $\frac{L}{2C}.\,dt\,.\,\sin.\,(2nt-\varphi+vt+\varepsilon)$ [18] Int. But in this case, H' being equal to H,† the corresponding

Terms depending on the [3241] angle $2nt$, may be neglected.

[3241']

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in this last expression $\cos.(nt-\varphi+vt+\varepsilon)$. The same term of dN' produces a similar expression in dy'' [3237]. Such terms in dN'', will produce like terms in [3236, 3237], [3240c] depending on very small angles. The results are the same, if dN' or dN'' contain terms of the form $L\,.\,\cos.\,(nt+vt+\varepsilon)$.

[3240d] * (2144) If dN' or dN'' contain a term of the form $L\,.\,\sin.\,(vt+\varepsilon)$, in which v is very small, it will produce, in dx'', dy'' [3236, 3237], terms having the factor $\sin.\,(vt+\varepsilon)\times\sin.\,\varphi$, or $\sin.\,(vt+\varepsilon)\times\cos.\,\varphi$, which, by reduction, as in the last [3240e] note, will produce terms depending on the sine or cosine of the angle $\varphi\pm vt\pm\varepsilon$. Now as φ is very nearly equal to nt [3024'], the factor of t, represented by i [3238], must be nearly equal to n, consequently the divisor i^2-n^2 [3239] will be extremely small.

[3241a] † (2145) Comparing [3238] with [3241], we get $H=\frac{L}{2C}$; $H'=\frac{L}{2C}$; hence $H'=H$. Substituting this in [3239], we get $x''=H.\left(\frac{n-i}{i^2-n^2}\right).\,\sin.\,(it+\varepsilon)=\frac{-H}{i+n}.\sin.\,(it+\varepsilon)$, and $y''=H.\left(\frac{i-n}{i^2-n^2}\right).\,\cos.\,(it+\varepsilon)=\frac{H}{i+n}.\,\cos.\,(it+\varepsilon)$; hence x'', y'', lose, [3241b] in this case, their small divisor $i-n$, and these parts become insensible, as in [3242]. This is not the case with the terms considered in the last note; for if $dN'=L.\sin.\,(vt+\varepsilon)$, [3241c] we shall obtain, in [3238], the terms $\frac{L}{2C}.\left\{\cos.\,(\varphi-vt-\varepsilon)-\cos.\,(\varphi+vt+\varepsilon)\right\}$, corresponding to $H\,dt\,.\,\cos.\,(it+\varepsilon)$; and in [3238'], the terms

$$\frac{L}{2C}.\left\{-\sin.\,(\varphi-vt-\varepsilon)+\sin.\,(\varphi+vt+\varepsilon)\right\},$$

corresponding to $H'\,dt\,.\,\sin.\,(it+\varepsilon)$. Now if we compare, in these two expressions, the

expressions of y'' and x'' lose their very small divisor $i-n$; therefore [3242]
they are insensible. We may also prove, in the same manner, that a [3242′]
term of dN'' of the form $L\,dt.\cos.(2nt+vt+\varepsilon)$, will produce,
in x'' and y'', none but insensible quantities; *therefore in the values of* Terms depending on the angles it, nt, must be noticed.
dN, dN', dN'', *it is only necessary to notice the terms depending on*
small angles [3250′, &c.], *and those in which the angles differ but little*
from nt [3253′, &c.]. [3243]

To analyze these different terms, we must use the differential equations of the motion of the sea. Taking into consideration a particle of its surface, determined in the state of equilibrium by the co-ordinates μ, ϖ, we shall suppose, that in the state of motion, it is elevated above the surface of equilibrium by the quantity αy; that its latitude is decreased by the [3244]
quantity αu, and that the angle ϖ is increased by αv.* We shall put, [3245]
also, v for the declination of the body L; Π for its right ascension; [3246]
$r_{\prime}$, for its distance from the centre of gravity of the earth; then supposing

$$\alpha f=\frac{3L}{2r_{\prime}^{3}}\cdot\left\{\cos.\theta\,.\,\sin.\mathrm{v}+\sin.\theta\,.\,\cos.\mathrm{v}\,.\,\cos.(\Pi-\varphi-\varpi)\right\}^{2}, \qquad [3247]$$

we shall have, from [2175—2176′], the three following equations,†

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co-efficient of the angle $\varphi-vt-\varepsilon$, or that of $\varphi+it+\varepsilon$, we shall find that they both give $H=-H'$. Substituting this in the co-efficients of the expressions of x'', y'' [3241d]
[3239], they both become $H'.\frac{i+n}{i^2-n^2}=\frac{H'}{i-n}$, containing the small divisor $i-n$.

* (2146) The notation here used is the same as in [2128^xi, 2128^xiv]. [3245a]

† (2147) The angle $nt+\varpi-\psi$, which occurs in [2192], is equal to the angular distance of the meridians passing through the body L, and the particle dm [2131c]. [3248a]
Now the right ascension of the first principal axis is equal to φ [2907f], and the meridian of the particle dm makes the angle ϖ with this axis [2907q]; hence the right ascension [3248b]
of the particle is $\varphi+\varpi$. Subtracting this from Π, the right ascension of the body L [3248c]
[3246], we obtain $\Pi-\varphi-\varpi$ for the angular distance from the meridian of the particle
dm. Substituting this in [2192], for $nt+\varpi-\psi$ [3248a], and changing r into $r_{\prime}$, we [3248d]
obtain the part of $\alpha V'$, corresponding to the disturbing force of the body L, on the particle
dm, equal to $\frac{3L}{2r_{\prime}^{3}}\cdot\left\{\left[\cos.\theta\,.\,\sin.\mathrm{v}+\sin.\theta\,.\,\cos.\mathrm{v}\,.\,\cos.(\Pi-\varphi-\varpi)\right]^{2}-\frac{1}{3}\right\}$; and by
using the abridged symbol [3247], it becomes $\alpha f-\frac{L}{2r_{\prime}^{3}}$. To obtain the complete [3248e]
value of $\alpha V'$ [2130′—2130‴], we must add the part αU [3202]; hence we obtain

[3248] $$y = \left(\frac{d\,.\,(\gamma u\,.\,\sqrt{1-\mu^2})}{d\mu}\right) - \left(\frac{d\,.\,(\gamma r)}{d\varpi}\right);$$

[3249] $$\left(\frac{ddu}{dt^2}\right) - 2n\,.\left(\frac{dv}{dt}\right).\mu\,.\sqrt{1-\mu^2} = \left\{ g\,.\left(\frac{dy}{d\mu}\right) - \left(\frac{dU}{d\mu}\right) - \left(\frac{df}{d\mu}\right) \right\}.\sqrt{1-\mu^2}; \qquad (I)$$

[3250] $$\left(\frac{ddv}{dt^2}\right) + 2n\,.\left(\frac{du}{dt}\right).\frac{\mu}{\sqrt{1-\mu^2}} = -\frac{\left\{ g\,.\left(\frac{dy}{d\varpi}\right) - \left(\frac{dU}{d\varpi}\right) - \left(\frac{df}{d\varpi}\right) \right\}}{1-\mu^2}.$$

[3250′] *If we consider only the angles which increase with extreme slowness*, that is to say, such as are independent of φ, it will be evident, that the part of f, corresponding to these angles, will be independent of ϖ; therefore the parts of y and U, relative to the same angles, will be independent of ϖ,* so that by considering only such terms, we shall have,

Investigation of terms depending on the small angle it.

[3251]

[3252] $$\left(\frac{df}{d\varpi}\right) = 0\,; \qquad \left(\frac{dy}{d\varpi}\right) = 0\,; \qquad \left(\frac{dU}{d\varpi}\right) = 0\,;$$

consequently,†

[3248f] $\alpha V' = \alpha U + \alpha f - \frac{L}{2r_{,}^3}$; and as $r_{,}$ is independent of μ, ϖ, the partial differentials corresponding to μ, ϖ, will be

[3248g] $$\alpha\,.\left(\frac{dV'}{d\mu}\right) = \alpha\,.\left(\frac{dU}{d\mu}\right) + \alpha\,.\left(\frac{df}{d\mu}\right); \qquad \alpha\,.\left(\frac{dV'}{d\varpi}\right) = \alpha\,.\left(\frac{dU}{d\varpi}\right) + \alpha\,.\left(\frac{df}{d\varpi}\right).$$

Dividing these by α, we get $\left(\frac{dV'}{d\mu}\right)$, $\left(\frac{dV'}{d\varpi}\right)$, and by substituting them in [2176, 2176′], we obtain [3249, 3250]. The equation [2175] is the same as [3248].

[3251a] * (2148) The principle made use of in [2191′, &c.], for satisfying the equation [2183], shows, that y and U must contain terms depending on the same angles as f, and no

[3251b] others; excepting such as arise from the initial state of the fluid, which must be neglected, for the reasons stated in [2191″]. Now ϖ is connected with φ, in the angle $\Pi - \varphi - \varpi$, which is found in the value of αf [3247]; it ought, therefore, to be

[3251c] connected with φ, in the values of U, y; so that if we notice terms depending on very small angles only, we may suppose that ϖ does not occur either in f, y, or U, and then the partial differentials of these quantities, relative to ϖ, will vanish, as in [3252].

[3254a] † (2149) Substituting [3252] in [3225], we get [3253]; and by making the same substitution in [3226, 3227], we obtain

[3254b] $$\frac{dN'}{dt} = S.\alpha\gamma.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}\,.\cos.\varpi.\left[g.\left(\frac{dy}{d\mu}\right) - \left(\frac{dU}{d\mu}\right)\right]\right\}$$
$$- S.\alpha n^2 y.d\mu.d\varpi.\mu.\sqrt{1-\mu^2}.\cos.\varpi\,;$$

[3254c] $$\frac{dN''}{dt} = S.\alpha\gamma.d\mu.d\varpi.\left\{\sqrt{1-\mu^2}\,.\sin.\varpi.\left[g.\left(\frac{dy}{d\mu}\right) - \left(\frac{dU}{d\mu}\right)\right]\right\}$$
$$- S.\alpha n^2 y.d\mu.d\varpi.\mu.\sqrt{1-\mu^2}.\sin.\varpi.$$

$$\frac{dN}{dt}=0;$$ [3253]

These expressions correspond to terms varying slowly.

$$\frac{dN'}{dt}.\sin.\varphi+\frac{dN''}{dt}.\cos.\varphi=S.\alpha\gamma.d\mu.d\varpi.\sqrt{1-\mu^2}.\sin.(\varphi+\varpi).\left\{g.\left(\frac{dy}{d\mu}\right)-\left(\frac{dU}{d\mu}\right)\right\};$$ [3254]

$$\frac{dN'}{dt}.\cos.\varphi-\frac{dN''}{dt}.\sin.\varphi=S.\alpha\gamma.d\mu.d\varpi.\sqrt{1-\mu^2}.\cos.(\varphi+\varpi).\left\{g.\left(\frac{dy}{d\mu}\right)-\left(\frac{dU}{d\mu}\right)\right\}.$$ [3255]

We have seen, in [2213], that if we notice only the terms increasing with extreme slowness, [2209‴], we may suppose nearly,*

$$0=g.\left(\frac{dy}{d\mu}\right)-\left(\frac{dU}{d\mu}\right)-\left(\frac{df}{d\mu}\right).$$ [3256]

This equation becomes the more correct, as these terms vary more slowly, in which case they have a greater influence on the motions of the axis of the earth. Hence we have, relatively to these terms,† [3256′]

Multiplying the first of these values by $\sin.\varphi$, the second by $\cos.\varphi$, adding the products, and putting $\cos.\varpi.\sin.\varphi+\sin.\varpi.\cos.\varphi=\sin.(\varphi+\varpi)$ [21] Int., we get the equation [3254], with the following additional term in the second member, [3254d]

$$-S.\alpha n^2 y.d\mu.d\varpi.\mu.\sqrt{(1-\mu^2)}.\sin.(\varphi+\varpi).$$

Now, by [3251], if we notice only angles increasing very slowly, y will be independent of ϖ, and then the integral of this term, relatively to ϖ, will be

$$S.\alpha n^2 y.d\mu.\mu.\sqrt{(1-\mu^2)}.\{\cos.(\varphi+\varpi)+\text{constant}\}.$$ [3254e]

The constant quantity is to be taken equal to $-\cos.\varphi$, so as to make the factor

$$\cos.(\varphi+\varpi)+\text{constant}=\cos.(\varphi+\varpi)-\cos.\varphi,$$ [3254f]

vanish at the first limit, where $\varpi=0$ [3215]. At the second limit, where $\varpi=400^\circ$ [3215], this factor becomes $\cos.\varphi-\cos.\varphi=0$; consequently the term [3254e] vanishes, and we obtain the equation [3254]. In like manner, multiplying [3254b, c] by $\cos.\varphi$, $-\sin\varphi$, respectively; adding the products, and putting [3254g]

$$\cos.\varpi.\cos.\varphi-\sin.\varpi.\sin.\varphi=\cos.(\varphi+\varpi)\ [23]\ \text{Int.},$$ [3254h]

we get [3255]; neglecting the term $-S.\alpha n^2 y.d\mu.d\varpi.\mu.\sqrt{(1-\mu^2)}.\cos.(\varphi+\varpi)$, for the same reason that we have neglected the term [3254e, &c.].

* (2150) Noticing only the terms depending on the angle it, in which i is extremely small [2209‴], we have $0=g.\left(\frac{dy}{d\mu}\right)-\left(\frac{dV'}{d\mu}\right)$ [2213]. Substituting $\left(\frac{dV'}{d\mu}\right)$ [3248g], we get [3256]. [3256a]

† (2151) From [3256], we get $g.\left(\frac{dy}{d\mu}\right)-\left(\frac{dU}{d\mu}\right)=\left(\frac{df}{d\mu}\right)$; substituting this in [3254, 3255], we get [3257, 3258], respectively. [3257a]

These expressions [3257] are nearly correct in the terms [3258] depending on angles which vary very slowly.

[3257] $$\frac{d\mathcal{N}'.\sin.\varphi + d\mathcal{N}''.\cos.\varphi}{dt} = S.\alpha\gamma.d\mu.d\varpi.\sqrt{1-\mu^2}.\sin.(\varphi+\varpi).\left(\frac{df}{d\mu}\right);$$

[3258] $$\frac{d\mathcal{N}'.\cos.\varphi - d\mathcal{N}''.\sin.\varphi}{dt} = S.\alpha\gamma.d\mu.d\varpi.\sqrt{1-\mu^2}.\cos.(\varphi+\varpi).\left(\frac{df}{d\mu}\right).$$

[3258′] *We shall now consider the parts of* $\frac{d\mathcal{N}'}{dt}$, $\frac{d\mathcal{N}''}{dt}$, *which depend on angles differing but very little from* nt. We have,*

Investigation of terms depending on angles differing but little from nt.

[3259] $$\frac{d\mathcal{N}'.\sin.\varphi + d\mathcal{N}''.\cos.\varphi}{dt} = S.\alpha\gamma.d\mu.d\varpi.\sqrt{1-\mu^2}.\sin.(\varphi+\varpi).\left\{g.\left(\frac{dy}{d\mu}\right)-\left(\frac{dU}{d\mu}\right)\right\}$$
$$-S.\alpha\gamma.d\mu.d\varpi.\frac{\mu}{\sqrt{1-\mu^2}}.\cos.(\varphi+\varpi).\left\{g.\left(\frac{dy}{d\varpi}\right)-\left(\frac{dU}{d\varpi}\right)\right\}$$
$$-S.\alpha n^2 y.d\mu.d\varpi.\mu.\sqrt{1-\mu^2}.\sin.(\varphi+\varpi).$$

The equation [3248] gives,†

[3260] $$S.\alpha n^2 y.d\mu.d\varpi.\mu.\sqrt{1-\mu^2}.\sin.(\varphi+\varpi)$$
$$= S.\alpha n^2.d\mu.d\varpi.\mu.\sqrt{1-\mu^2}.\sin.(\varphi+\varpi).\left\{\left(\frac{d.(\gamma u.\sqrt{1-\mu^2})}{d\mu}\right)-\left(\frac{d.(\gamma v)}{d\varpi}\right)\right\}.$$

Integrating from $\mu=-1$ to $\mu=1$, we get,‡

[3261] $$S.\mu d\mu.\sqrt{1-\mu^2}.\left(\frac{d.(\gamma u.\sqrt{1-\mu^2})}{d\mu}\right) = -S.\gamma u\, d\mu.(1-2\mu^2).$$

In like manner, by integrating from $\varpi=0$ to $\varpi=$ four right angles,

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[3259a] * (2152) Multiplying [3226] by $\sin.\varphi$, [3227] by $\cos.\varphi$; adding the products, and using $\sin.(\varphi+\varpi)$, $\cos.(\varphi+\varpi)$ [3254d, h], we get [3259].

[3260a] † (2153) Multiplying [3248] by $\alpha n^2.d\mu.d\varpi.\mu.\sqrt{(1-\mu^2)}.\sin.(\varphi+\varpi)$, and prefixing the characteristic S, we get [3260].

[3261a] ‡ (2154) Putting, for brevity, $W=\gamma u.\sqrt{(1-\mu^2)}$, in the first member of [3261], it becomes as in the first member of [3261b]. Integrating this by parts, it becomes as in the second member of [3261b], as is easily proved by taking the differential of both members. At the limits of the integral $\mu=-1$, $\mu=1$ [3215], we have $\mu.\sqrt{(1-\mu^2)}.W=0$, and the preceding expression becomes as in the first member of [3261c], which, by developing the terms $d.\{\mu.\sqrt{(1-\mu^2)}\}$, is reduced to the second form [3261c]. Resubstituting the value of W [3261a], we obtain [3261],

[3261b] $$S.\mu d\mu.\sqrt{(1-\mu^2)}.\left(\frac{dW}{d\mu}\right) = \mu.\sqrt{(1-\mu^2)}.W - S.W.\left\{\frac{d.\{\mu.\sqrt{(1-\mu^2)}\}}{d\mu}\right\}.d\mu.$$

[3261c] $$= -S.W.\left\{\frac{d.\{\mu.\sqrt{(1-\mu^2)}\}}{d\mu}\right\}d\mu = -S.W.\frac{1-2\mu^2}{\sqrt{(1-\mu^2)}}.d\mu.$$

we obtain,*

$$S.d\varpi.\sin.(\varphi+\varpi).\left(\frac{d.(\gamma v)}{d\varpi}\right)=-S.\gamma v.d\varpi.\cos.(\varphi+\varpi); \qquad [3262]$$

therefore,†

$$S.\alpha n^2 y.d\mu.d\varpi.\mu.\sqrt{1-\mu^2}.\sin.(\varphi+\varpi)=-S.\alpha n^2\gamma u.d\mu.d\varpi.(1-2\mu^2).\sin.(\varphi+\varpi)$$
$$+S.\alpha n^2\gamma v.d\mu.d\varpi.\mu.\sqrt{1-\mu^2}.\cos.(\varphi+\varpi). \qquad [3263]$$

We may suppose γu to be developed in a series of terms of the form $H.\cos.(it+s\varpi+\varepsilon)$; H being a function of μ only, and s an integral number, positive, negative, or zero, excluding fractional numbers,‡ because γu is the same at the two limits, where $\varpi=0$, or $\varpi=400^\circ$ [3215]. In like manner, γv may be developed in a similar series of terms of the form $M.\sin.(it+s\varpi+\varepsilon)$, M being a function of μ only. We shall put H', M', for the values of H, M, respectively, corresponding to the same angle it, and to the particular value $s=1$. The coefficient i [3264] [3264'] [3265] [3265'] [3266]

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* (2155) Integrating by parts, relatively to $d\varpi$, we get

$$S.d\varpi.\sin.(\varphi+\varpi).\left(\frac{d.(\gamma v)}{d\varpi}\right)=\sin.(\varphi+\varpi).\gamma v-S.\gamma v.d\varpi.\cos.(\varphi+\varpi), \qquad [3262a]$$

as is easily proved by differentiation. Now $\sin.(\varphi+\varpi)$ is the same at both limits of the integral; therefore we may neglect the term without the sign S, and then the preceding expression becomes as in [3262].

† (2156) Multiplying [3261] by $\alpha n^2.d\varpi.\sin.(\varphi+\varpi)$, also [3262] by $-\alpha n^2.\mu d\mu.\sqrt{(1-\mu^2)}$; and adding the products, we get

$$S.\alpha n^2.d\mu.d\varpi.\mu.\sqrt{(1-\mu^2)}.\sin.(\varphi+\varpi).\left\{\left(\frac{d.\{\gamma u.\sqrt{(1-\mu^2)}\}}{d\mu}\right)-\left(\frac{d.(\gamma v)}{d\varpi}\right)\right\}=-S.\alpha n^2\gamma u.d\mu.d\varpi.(1-2\mu^2).\sin.(\varphi+\varpi)$$
$$+S.\alpha n^2\gamma v.d\mu.d\varpi.\mu.\sqrt{(1-\mu^2)}.\cos.(\varphi+\varpi). \qquad [3263a]$$

Substituting this in [3260], we get [3263].

‡ (2157) We may suppose H to be a function of μ only; for if it contain a term of the form $h.\cos.(s'\varpi+\varepsilon')$, in which h is a function of μ, the expression $H.\cos.(it+s\varpi+\varepsilon)$ will become $h.\cos.(s'\varpi+\varepsilon').\cos.(it+s\varpi+\varepsilon)$, producing terms of the form $\frac{1}{2}h.\cos.\{it+(s\pm s').\varpi+\varepsilon\pm\varepsilon'\}$ [24] Int., similar to that proposed in [3264], $H.\cos.(it+s\varpi+\varepsilon)$, H being a function of μ, independent of ϖ. [3264a] Again, if we increase the angle ϖ by four right angles, we shall fall upon the same particle of fluid dm; therefore the developments of γu, γv, ought not to vary by such an increase in the value of ϖ; and as this will not be the case if s be a fraction, we must suppose s to be an integer, or zero, as in [3264]. [3264b] [3264c]

being supposed to differ but very little from n, the angle $it - \varphi + \varepsilon$, will increase with extreme slowness; therefore, by retaining only the terms depending on this angle, we shall find, that as the terms of γu and γv,
[3266'] in which s differs from unity, contain the angle ϖ, in the preceding integrals, they will vanish, when the integrals are taken within the proposed limits [3264'];* hence we have,

[3267]
$$S.\alpha n^2 y.d\mu.d\varpi.\mu.\sqrt{1-\mu^2}.\sin.(\varphi+\varpi)$$
$$=\alpha n^2\pi.\sin.(it+\varepsilon-\varphi).S.d\mu.\{(1-2\mu^2).H'+\mu.\sqrt{1-\mu^2}.M'\}.$$

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* (2158) Putting, as in [3264], $\gamma u = H.\cos.(it+s\varpi+\varepsilon)$, we shall get, by using [19] Int.,

[3265a]
$$S.\gamma u.d\varpi.\sin.(\varphi+\varpi) = -\tfrac{1}{2}S.H.\sin.(it+\varepsilon-\varphi+[s-1].\varpi).d\varpi$$
$$+\tfrac{1}{2}S.H.\sin.(it+\varepsilon+\varphi+[s+1].\varpi).d\varpi.$$

If we retain only the term containing the angle $it+\varepsilon-\varphi$, which varies with extreme slowness, it becomes $-\frac{1}{2}S.H.\sin.(it+\varepsilon-\varphi+[s-1].\varpi).d\varpi$; and when s differs from 1, its integral, relatively to ϖ, is

[3265b]
$$S.\gamma u.d\varpi.\sin.(\varphi+\varpi) = \frac{1}{2.(s-1)}.H.\cos.(it+\varepsilon-\varphi+[s-1].\varpi)+\text{constant}.$$

The constant quantity is to be determined, so that the integral may vanish at the first limit, where $\varpi=0$, and then by putting $\varpi=400^\circ$, at the other limit, the whole expression will also vanish, because $(s-1).\varpi$ becomes a multiple of 400°; therefore we have, in this case, for all integral values of s, excepting $s=1$, $S.\gamma u.d\varpi.\sin.(\varphi+\varpi)=0$.
[3265c] When $s=1$, the term of [3265a], depending on the angle $it+\varepsilon-\varphi$, becomes $-\frac{1}{2}S.H'.\sin.(it+\varepsilon-\varphi).d\varpi$ [3266]. Integrating this, relatively to ϖ, and observing that $\int_0^{2\pi}d\varpi=2\pi$ [1467b], it becomes $-\pi.H'.\sin.(it+s-\varphi)$. Hence we have, for the case of $s=1$,

[3265d]
$$S.\gamma u.d\varpi.\sin.(\varphi+\varpi) = -\pi.H'.\sin.(it+\varepsilon-\varphi).$$

In like manner, we may find the value of $S.\gamma v.d\varpi.\cos.(\varphi+\varpi)$; or we may derive it from [3265d], by the following process: if we change H, H', ε, φ, into M, M', $\varepsilon-100^\circ$, $\varphi+100^\circ$, respectively, the value of γu [3264], changes into γv [3265']; $\sin.(\varphi+\varpi)$ becomes $\cos.(\varphi+\varpi)$; and $\sin.(it+\varepsilon-\varphi)$ becomes $\sin.(it+\varepsilon-\varphi-200^\circ)=-\sin.(it+\varepsilon-\varphi)$; substituting these in [3265d], we get

[3265e]
$$S.\gamma v.d\varpi.\cos.(\varphi+\varpi) = \pi.M'.\sin.(it+\varepsilon-\varphi).$$

Multiplying [3265d] by $-\alpha n^2.d\mu.(1-2\mu^2)$, [3265e] by $\alpha n^2.d\mu.\mu.\sqrt{(1-\mu^2)}$, and adding the products, we get

[3265f]
$$-S.\alpha n^2\gamma u.d\mu.d\varpi.(1-2\mu^2).\sin.(\varphi+\varpi)+S.\alpha n^2\gamma v.d\mu.d\varpi.\mu.\sqrt{(1-\mu^2)}.\cos.(\varphi+\varpi)$$
$$=\alpha n^2\pi.\sin.(it+\varepsilon-\varphi).S.d\mu.\{(1-2\mu^2).H'+\mu.\sqrt{(1-\mu^2)}.M'\}.$$

The first member of this expression is the same as the second of [3263], and by substituting it, we get [3267].

If we multiply [3249] by $\alpha\gamma . d\mu . d\varpi . \sin . (\varphi + \varpi)$, also [3250] by

$$\alpha\gamma . d\mu . d\varpi . \mu . \sqrt{1-\mu^2} . \cos . (\varphi + \varpi),$$ [3268]

and add the products, we shall obtain,

$$S.\alpha\gamma.d\mu.d\varpi. \left\{ \begin{array}{c} \left(\frac{ddu}{dt^2}\right) . \sin . (\varphi+\varpi) + 2n\mu^2 . \left(\frac{du}{dt}\right) . \cos . (\varphi+\varpi) \\ + \left(\frac{ddv}{dt^2}\right) . \mu . \sqrt{1-\mu^2} . \cos . (\varphi+\varpi) - 2n . \left(\frac{dv}{dt}\right) . \mu . \sqrt{1-\mu^2} . \sin . (\varphi+\varpi) \end{array} \right\}$$ [3269]

$$= S . \alpha\gamma . d\mu . d\varpi . \sqrt{1-\mu^2} . \sin . (\varphi+\varpi) . \left\{ g . \left(\frac{dy}{d\mu}\right) - \left(\frac{dU}{d\mu}\right) - \left(\frac{df}{d\mu}\right) \right\}; \quad (O)$$

$$- S.\alpha\gamma.d\mu.d\varpi. \frac{\mu}{\sqrt{1-\mu^2}} . \cos . (\varphi+\varpi) . \left\{ g . \left(\frac{dy}{d\varpi}\right) - \left(\frac{dU}{d\varpi}\right) - \left(\frac{df}{d\varpi}\right) \right\}.$$

If we substitute for γu, γv, the sum of all the terms, relative to the angle it; observing that i differs but very little from n, the first member of this equation will become,* [3269']

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* (2159) From [3264—3265'], we have $\gamma u = H' . \cos . (it + \varpi + \varepsilon)$; [3269a] $\gamma v = M' . \sin . (it + \varpi + \varepsilon)$. In taking the differentials of these expressions, we may neglect that of γ, on account of its smallness in comparison with the differentials of u, v [337''', &c.]; and as H', M', are functions of μ only [3264, &c.], we shall have

$$\gamma . \left(\frac{du}{dt}\right) = - H' . i . \sin . (it + \varpi + \varepsilon); \qquad \gamma . \left(\frac{ddu}{dt^2}\right) = - H' . i^2 . \cos . (it + \varpi + \varepsilon).$$ [3269b]

Moreover, as i is nearly equal to n [3266], the preceding expressions become as in [3269c]; and in like manner, we obtain [3269d],

$$\gamma . \left(\frac{du}{dt}\right) = - H' . n . \sin . (it + \varpi + \varepsilon); \qquad \gamma . \left(\frac{ddu}{dt^2}\right) = - H' . n^2 . \cos . (it + \varpi + \varepsilon);$$ [3269c]

$$\gamma . \left(\frac{dv}{dt}\right) = M' . n . \cos . (it + \varpi + \varepsilon); \qquad \gamma . \left(\frac{ddv}{dt^2}\right) = - M' . n^2 . \sin . (it + \varpi + \varepsilon).$$ [3269d]

If we substitute these in [3269], and reduce them by means of [17—20] Int., retaining only the terms depending on the angle $it + \varepsilon - \varphi$, we may put

$$\gamma . \left(\frac{ddu}{dt^2}\right) . \sin . (\varphi + \varpi) = \tfrac{1}{2} H' . n^2 . \sin . (it + \varepsilon - \varphi);$$

$$\gamma . \left(\frac{du}{dt}\right) . \cos . (\varphi + \varpi) = - \tfrac{1}{2} H' . n . \sin . (it + \varepsilon - \varphi);$$ [3269e]

$$\gamma . \left(\frac{ddv}{dt^2}\right) . \cos . (\varphi + \varpi) = - \tfrac{1}{2} M' . n^2 . \sin . (it + \varepsilon - \varphi);$$

$$\gamma . \left(\frac{dv}{dt}\right) . \sin . (\varphi + \varpi) = - \tfrac{1}{2} M' . n . \sin . (it + \varepsilon - \varphi).$$

[3270] $$\alpha n^2 \pi \,.\, \sin.(it+\varepsilon-\varphi) \,.\, S \,.\, d\mu \,.\, \{(1-2\mu^2) \,.\, H' + \mu \,.\, \sqrt{1-\mu^2} \,.\, M'\};$$

[3270′] therefore we shall have, by noticing only the terms in which i is very nearly equal to n,

[3271]
$$S \,.\, \alpha n^2 y \,.\, d\mu \,.\, d\varpi \,.\, \mu \,.\, \sqrt{1-\mu^2} \,.\, \sin.(\varphi+\varpi)$$
$$= S \,.\, \alpha\gamma \,.\, d\mu \,.\, d\varpi \,.\, \sqrt{1-\mu^2} \,.\, \sin.(\varphi+\varpi) \,.\, \left\{ g \,.\, \left(\frac{dy}{d\mu}\right) - \left(\frac{dU}{d\mu}\right) - \left(\frac{df}{d\mu}\right) \right\}$$
$$- S \,.\, \alpha\gamma \,.\, d\mu \,.\, d\varpi \,.\, \frac{\mu}{\sqrt{1-\mu^2}} \,.\, \cos.(\varphi+\varpi) \,.\, \left\{ g \,.\, \left(\frac{dy}{d\varpi}\right) - \left(\frac{dU}{d\varpi}\right) - \left(\frac{df}{d\varpi}\right) \right\}.$$

Hence we deduce,*

These values are very nearly correct in the terms which depend on angles differing but little from nt,

[3272]
$$\frac{dN' \,.\, \sin.\varphi + dN'' \,.\, \cos.\varphi}{dt} = S \,.\, \alpha\gamma \,.\, d\mu \,.\, d\varpi \,.\, \sqrt{1-\mu^2} \,.\, \sin.(\varphi+\varpi) \,.\, \left(\frac{df}{d\mu}\right)$$
$$- S \,.\, \alpha\gamma \,.\, d\mu \,.\, d\varpi \,.\, \frac{\mu}{\sqrt{1-\mu^2}} \,.\, \cos.(\varphi+\varpi) \,.\, \left(\frac{df}{d\varpi}\right).$$

In like manner, we find,†

Multiplying these by $d\varpi$, and integrating from $\varpi = 0$ to $\varpi = 2\pi$, using $\int_0^{2\pi} . d\varpi = 2\pi$ [1467b], we obtain the following expressions, which are evidently the same as those in [3269e], multiplied by 2π;

[3269f]
$$S \,.\, \gamma \,.\, d\varpi \,.\, \left(\frac{ddu}{dt^2}\right) \,.\, \sin.(\varphi+\varpi) = n^2\pi \,.\, H' \,.\, \sin.(it+\varepsilon-\varphi);$$
$$S \,.\, \gamma \,.\, d\varpi \,.\, \left(\frac{du}{dt}\right) \,.\, \cos.(\varphi+\varpi) = -n\pi \,.\, H' \,.\, \sin.(it+\varepsilon-\varphi);$$
$$S \,.\, \gamma \,.\, d\varpi \,.\, \left(\frac{ddv}{dt^2}\right) \,.\, \cos.(\varphi+\varpi) = -n^2\pi \,.\, M' \,.\, \sin.(it+\varepsilon-\varphi);$$
$$S \,.\, \gamma \,.\, d\varpi \,.\, \left(\frac{dv}{dt}\right) \,.\, \sin.(\varphi+\varpi) = -n\pi \,.\, M' \,.\, \sin.(it+\varepsilon-\varphi).$$

[3269g] Now if we multiply these four last equations by $\alpha\, d\mu$; $\alpha\, d\mu \,.\, 2n\mu^2$; $\alpha\, d\mu \,.\, \mu \,.\, \sqrt{(1-\mu^2)}$; $-\alpha\, d\mu \,.\, 2n \,.\, \mu \,.\, \sqrt{(1-\mu^2)}$, respectively; then add the products, and prefix the sign of integration S, relatively to μ, we shall find, that the first member of the sum is equal to the first member of [3269]; and that the second member, connecting its terms, is equal to [3270], or to the second member of [3267]; therefore the first member of [3267] is equal to the second of [3269], as in [3271].

[3273a] * (2160) Substituting, in [3259], the value of the last term of its second member, given in [3271], neglecting the quantities which mutually destroy each other, we obtain [3272].

[3273b] † (2161) The formula [3273] may be computed by the method we have used [3259—3272]; or we may more simply derive it from [3272], in the following manner. If we review the calculation [3259—3272], we shall find, that the process will not be altered

$$\frac{dN'.\cos.\varphi - dN''.\sin.\varphi}{dt} = S.\alpha\gamma.d\mu.d\varpi.\sqrt{1-\mu^2}.\cos.(\varphi+\varpi).\left(\frac{df}{d\mu}\right)$$ [3273]

$$+S.\alpha\gamma.d\mu.d\varpi.\frac{\mu}{\sqrt{1-\mu^2}}.\sin.(\varphi+\varpi).\left(\frac{df}{d\varpi}\right).$$

and in those depending on the slowly varying angles it.

*These two equations hold good, when we notice those angles only which vary very slowly; and we have seen, in [3257, 3258], that the first term of the second member of each of them, contains also whatever relates to angles differing but very little from nt;** *therefore the formulas [3272, 3273], include all that relates to these two kinds of angles, which are the only ones that can have any sensible influence on the motions of the earth about its centre of gravity* [3243]. If we connect these equations with the equation $\frac{dN}{dt}=0$,† we shall have all that is necessary to determine the influence of the sea on these motions. [3273′] [3273″] [3274] [3274′]

We shall now observe, that these different equations are the same as if the sea form with the earth a solid mass. To prove this, we shall determine the values of dN, dN', dN'', relatively to the sea, in this hypothesis. The value of V [2966] is represented very nearly by [3274″]

These last equations are the same as if the sea form with the earth a solid mass.

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if we increase φ by 100°; or, in other words, change φ into $\varphi'=\varphi+100^\circ$. By this means $\sin.\varphi$, $\cos.\varphi$, $\sin.(\varphi+\varpi)$, $\cos.(\varphi+\varpi)$ become $\cos.\varphi$, $-\sin.\varphi$, $\cos.(\varphi+\varpi)$, $-\sin.(\varphi+\varpi)$, respectively; and [3272] changes into [3273].

* (2162) The terms of $\frac{dN'.\sin.\varphi+dN''.\cos.\varphi}{dt}$ and $\frac{dN'.\cos.\varphi-dN''.\sin.\varphi}{dt}$, computed in [3257, 3258], are of the same form as the terms of [3272, 3273], depending on $\left(\frac{df}{d\mu}\right)$; therefore these last formulas must include all the terms which have a perceptible influence on the motion of the axis of the earth, as in [3274]. [3273c]

† (2163) From the first of the equations [3230], we have $p=\int\frac{dN}{C}$; $\int p\,dt=\int\int\frac{dN}{C}.dt$. Now if we notice in dN, only the terms which depend on angles of the form nt, these expressions will not be increased by integration; because they will not be affected by the small divisor i or i^2; and we may, therefore, neglect them. Moreover, we have, for the terms varying with extreme slowness, $\frac{dN}{dt}=0$ [3253]. Hence this equation obtains for both these species of angles, as in [3274′]. [3274a]

[3275] $$V = \frac{L}{r_{\prime}} + R^2 \alpha f - \frac{L R^2}{2 r_{\prime}^3};\text{*}$$

which gives, by the substitution of $dm = R^2 dR \,.\, d\mu \,.\, d\varpi$ [2918], in [2968],†

[3276] $$\frac{dN}{dt} = S \,.\, \alpha R^4 dR \,.\, d\mu \,.\, d\varpi \,.\, \left\{ x' \,.\, \left(\frac{df}{dy'}\right) - y' \,.\, \left(\frac{df}{dx'}\right) \right\}.$$

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* (2164) Neglecting terms of the order $r_{\prime}^{-4}$ in [2990k], we have,

[3275a] $$V = \frac{L}{r_{\prime}} + \frac{L R^2}{r_{\prime}^3} \,.\, P^{(2)};$$

in which $P^{(2)}$ is the coefficient of $\frac{L R^2}{r_{\prime}^3}$, in the development of V [2990g], and is

[3275b] evidently represented by $P^{(2)} = \frac{3}{2}\delta^2 - \frac{1}{2}$; as in [1628]. In order to conform to the present notation, we must put, in the value of δ [2990a], $\mu = \cos.\theta$, $\sqrt{(1-\mu^2)} = \sin.\theta$

[3275c] [2128xii]; $\nu = \sin.\mathrm{v}$, $\sqrt{(1-\nu^2)} = \cos.\mathrm{v}$ [2989, 3246]; $\lambda = \Pi$ [2989, 3246]; and change ϖ into $\varphi + \varpi$; by which means $\cos.(\varpi - \lambda)$, or rather $\cos.(\lambda - \varpi)$, becomes $\cos.(\Pi - \varphi - \varpi)$. Hence $\delta = \cos.\theta \,.\, \sin.\mathrm{v} + \sin.\theta \,.\, \cos.\mathrm{v} \,.\, \cos.(\Pi - \varphi - \varpi)$; substituting

[3275d] this in $P^{(2)}$ [3275b], and the result in V [3275a], using [3247], it becomes as in [3275].

† (2165) Substituting the value of dm [2918] in [2968], we get

[3276a] $$\frac{dN}{dt} = S \,.\, R^2 dR \,.\, d\mu \,.\, d\varpi \,.\, \left\{ x' \,.\, \left(\frac{dV}{dy'}\right) - y' \,.\, \left(\frac{dV}{dx'}\right) \right\}.$$

Now $r_{\prime}$ [2964] is independent of x', y', z', so that we may consider $r_{\prime}$ as constant, in finding the partial differentials $\left(\frac{dV}{dy'}\right)$, $\left(\frac{dV}{dx'}\right)$. We may also consider R as constant,

[3276b] because its terms destroy each other in the function $x' \,.\, \left(\frac{dV}{dy'}\right) - y' \,.\, \left(\frac{dV}{dx'}\right)$. For if V be a function of R, represented by $\varphi(R)$, and we put $\left(\frac{d.\varphi(R)}{dR}\right) = \varphi'(R)$, we shall have

$$\left(\frac{dV}{dy'}\right) = \varphi'(R) \,.\, \left(\frac{dR}{dy'}\right), \qquad \left(\frac{dV}{dx'}\right) = \varphi'(R) \,.\, \left(\frac{dR}{dx'}\right);$$

hence

$$x' \,.\, \left(\frac{dV}{dy'}\right) - y' \,.\, \left(\frac{dV}{dx'}\right) = \varphi'(R) \,.\, \left\{ x' \,.\, \left(\frac{dR}{dy'}\right) - y' \,.\, \left(\frac{dR}{dx'}\right) \right\}.$$

But from $R = \sqrt{(x'^2 + y'^2 + z'^2)}$ [2990a], we obtain $\left(\frac{dR}{dx'}\right) = \frac{x'}{R}$, $\left(\frac{dR}{dy'}\right) = \frac{y'}{R}$;

[3276c] therefore $x' \,.\, \left(\frac{dR}{dy'}\right) - y' \,.\, \left(\frac{dR}{dx'}\right) = \frac{x'y' - y'x'}{R} = 0$, as in [3276b]. Hence it appears, that in substituting the value of V [3275] in [3276a], we may consider f as the only variable

[3276d] quantity; so that $\left(\frac{dV}{dx'}\right) = \alpha R^2 \,.\, \left(\frac{df}{dx'}\right)$, $\left(\frac{dV}{dy'}\right) = \alpha R^2 \,.\, \left(\frac{df}{dy'}\right)$, and [3276a] becomes as in [3276].

The depth of the sea being supposed very small, and the radius R very nearly equal to unity, we have, relatively to the sea,*

$$S \,.\, R^4 d\,R = \gamma; \qquad [3277]$$

consequently,

$$\frac{d\,N}{d\,t} = S \,.\, \alpha\gamma \,.\, d\,\mu \,.\, d\,\varpi \,.\, \left\{ x' \,.\, \left(\frac{d\,f}{d\,y'}\right) - y' \,.\, \left(\frac{d\,f}{d\,x'}\right) \right\}. \qquad [3278]$$

In like manner, we find,†

$$\frac{d\,N'}{d\,t} = S \,.\, \alpha\gamma \,.\, d\,\mu \,.\, d\,\varpi \,.\, \left\{ x' \,.\, \left(\frac{d\,f}{d\,z'}\right) - z' \,.\, \left(\frac{d\,f}{d\,x'}\right) \right\}; \qquad [3279]$$

$$\frac{d\,N''}{d\,t} = S \,.\, \alpha\gamma \,.\, d\,\mu \,.\, d\,\varpi \,.\, \left\{ y' \,.\, \left(\frac{d\,f}{d\,z'}\right) - z' \,.\, \left(\frac{d\,f}{d\,y'}\right) \right\}. \qquad [3280]$$

Transforming, by § 10 [3175, &c.], these partial differentials into others relative to the variable quantities, R, μ, ϖ, we shall have,‡

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* (2166) This integral is found as in [3208a], and by substituting it in [3276], we get [3278]. [3277a]

† (2167) We may deduce [3279, 3280] from [2969, 2970], in the same manner as [3278], from [2968]; or more simply by derivation; for if we change y, y', z, z', into z, z', y, y', respectively, the value of $\frac{dN}{dt}$ [2968], changes into $\frac{dN'}{dt}$ [2969]; because the value of V [2966] remains unaltered. Making the same changes in [3278], it becomes as in [3279]. In like manner, by changing x, x', y, y', into y, y', x, x', respectively, the value $\frac{dN'}{dt}$ [2969], becomes like $\frac{dN''}{dt}$ [2970]; and the same changes being made in [3279], it becomes as in [3280]. [3279a] [3279b]

‡ (2168) If we put the two expressions of $\frac{dN}{dt}$ [3172, 3180], equal to each other, after substituting in the first, the values $R=1$, $f=2$ [3179], and then dividing the whole by g, we shall get, [3281a]

$$S \,.\, \alpha y \,.\, d\,\mu \,.\, d\,\varpi \,.\, \left\{ x' \,.\, \left(\frac{dq}{dy'}\right) - y' \,.\, \left(\frac{dq}{dx'}\right) \right\} = S \,.\, \alpha y \,.\, d\,\mu \,.\, d\,\varpi \,.\, \left(\frac{dq}{d\varpi}\right). \qquad [3281b]$$

Now it is evident, that the second member of this equation is derived from the first, by merely changing the co-ordinates, as in [3175, &c.], and that the equation holds good whatever be the values of y, q. Putting $y=\gamma$, $q=f$, the first member of [3281b] becomes like the second of [3278], and the second member of [3281b] gives the equivalent expression of $\frac{dN}{dt}$ [3281]. The same operation being performed upon the [3281c] [3281d]

[3281] $$\frac{dN}{dt} = S\,.\,\alpha\gamma\,.\,d\mu\,.\,d\varpi\,.\left(\frac{df}{d\varpi}\right);$$

[3282] $$\frac{dN'}{dt} = S\,.\,\alpha\gamma\,.\,d\mu\,.\,d\varpi\,.\left\{\sqrt{1-\mu^2}\,.\,\cos.\varpi\,.\left(\frac{df}{d\mu}\right) + \frac{\mu\,.\,\sin.\varpi}{\sqrt{1-\mu^2}}\,.\left(\frac{df}{d\varpi}\right)\right\},$$

[3283] $$\frac{dN''}{dt} = S\,.\,\alpha\gamma\,.\,d\mu\,.\,d\varpi\,.\left\{\sqrt{1-\mu^2}\,.\,\sin.\varpi\,.\left(\frac{df}{d\mu}\right) - \frac{\mu\,.\cos.\varpi}{\sqrt{1-\mu^2}}\,.\left(\frac{df}{d\varpi}\right)\right\};$$

hence we obtain,*

[3284] $$\frac{dN'.\sin.\varphi + dN''.\cos.\varphi}{dt} = S\,.\,\alpha\gamma\,.\,d\mu\,.\,d\varpi\,.\sqrt{1-\mu^2}\,.\,\sin.(\varphi+\varpi)\,.\left(\frac{df}{d\mu}\right)$$

In these expressions the whole earth is supposed to be solid.

$$-S.\alpha\gamma.d\mu.d\varpi.\frac{\mu}{\sqrt{1-\mu^2}}\,.\,\cos.(\varphi+\varpi)\,.\left(\frac{df}{d\varpi}\right);$$

[3285] $$\frac{dN'.\cos.\varphi - dN''.\sin.\varphi}{dt} = S\,.\,\alpha\gamma\,.\,d\mu\,.\,d\varpi\,.\sqrt{1-\mu^2}\,.\,\cos.(\varphi+\varpi)\,.\left(\frac{df}{d\mu}\right)$$

$$+S.\alpha\gamma.d\mu.d\varpi.\frac{\mu}{\sqrt{1-\mu^2}}\,.\,\sin.(\varphi+\varpi)\,.\left(\frac{df}{d\varpi}\right);$$

[3286] and if we notice only terms increasing with extreme slowness, we have,† $\frac{dN}{dt}=0$. These equations are the same as those we have found in [3272, 3273, 3274′]‡ Hence we get the following remarkable theorem,

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equations [3173, 3181], produce the second members of the equations [3279, 3282], which must therefore be equal to each other. In like manner, from [3174, 3182], we obtain the transformation of [3280] into [3283].

[3284a] * (2169) Multiplying [3282] by $\sin.\varphi$, [3283] by $\cos.\varphi$; adding the products, and substituting the expressions $\sin.(\varphi+\varpi)$, $\cos.(\varphi+\varpi)$ [3254d, h], we get [3284]. Again, multiplying [3282] by $\cos.\varphi$, [3283] by $-\sin.\varphi$; adding the products, and making the same substitutions, we get [3285].

[3285a] † (2170) The quantity ϖ is connected with φ in the function αf [3247], it must therefore be connected with φ in the expression of $\left(\frac{df}{d\varpi}\right)$, as is evident from the form of the function αf [3247]; therefore, if we notice those terms only, which depend on small angles, we shall have, as in [3252], $\left(\frac{df}{d\varpi}\right)=0$. Substituting this in [3281], we get $\frac{dN}{dt}=0$ [3286].

‡ (2171) The expressions [3272, 3273, 3274′] correspond to a *fluid* state of the ocean; and they are respectively equal to [3284, 3285, 3286], which correspond to the supposition

La Place's theorem.

namely; that *the phenomena of the precession of the equinoxes, and the nutation of the earth's axis, are exactly the same as if the sea form a solid mass with the spheroid which it covers.* [3287]

The precession and nutation are the same as if the sea form a solid mass.

There exists, however, a case, mathematically possible, in which this theorem does not hold good; namely, where the nucleus of the earth, which is covered by the ocean, is supposed to be formed of spherical strata. It is evident, that there will then be no motion in the axis of rotation of the nucleus, arising from the attractions of the sun and moon, and from the attraction and pressure of the sea; since the resultant of all these forces passes through the centre of the nucleus. Let us now see what prevents the preceding analysis from including this case. [3288] [3288′]

Unimportant excepted case.

The parts of the expressions of αy, αu, and αv, which have an influence on the motions of the earth's axis, are those which depend on sines and cosines of angles of the form $it+\varpi$, in which i differs but little from n;* and we have seen, in [2240], that these parts correspond to the oscillations of the second kind. These oscillations can be determined, [3288″] [3289]

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of its being *solid*. Hence the values of H, H' [3238, 3238′], x'', y'' [3239], and $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$ [3233], must be the same in both cases. The value of dp [3230], becomes also in both cases, $dp=0$, [3274′, 3286]; therefore p is constant; and the motion of the earth's axis is the same as if the ocean were considered as solid. [3286a] [3286b]

This theorem, discovered by La Place, appears very remarkable, when we take into consideration the different manner in which the fluid acts in the two cases; for the ocean, during its oscillations in a fluid state, must operate upon the nucleus, by its *pressure* and its *attraction*, in a very different manner from what it would if the whole mass were to become of a solid form, corresponding to the state of equilibrium; and it seems wonderful that the whole effect on the rotatory motion of the earth, should be identically the same in circumstances which are so dissimilar. [3286c]

* (2172) The object, in this place, is merely to prove, that there is a case, in which some terms of the values of y, u, v, contain denominators of the order i^2-n^2, which vanish when $i=n$, and by this means they become infinite; making it necessary to retain terms of the order $i-n$, instead of neglecting them, as in [3269′, &c.]. This is proved to be the case in oscillations of the second kind [3295]; which is sufficient to account for the failure of the formulas in the case mentioned in [3288′]; and it then becomes necessary to notice the expressions of the oscillations of the first kind [2221, &c.], or those of the third kind [2298, &c.]. [3288a] [3288b]

[3289] in this case, by the article just mentioned; i being very little different from n, the expressions of y, u, and v, relative to the angle $it+\varpi$, are of the forms,*

[3290]
$$y=\frac{2lq.k.\mu.\sqrt{1-\mu^2}}{2lgq.\left(1-\frac{3}{5\rho}\right)-n^2}.\cos.(it+\varpi);$$

[3291]
$$u=\frac{-k}{2lgq.\left(1-\frac{3}{5\rho}\right)-n^2}.\cos.(it+\varpi);$$

[3292]
$$v=\frac{k}{2lgq.\left(1-\frac{3}{5\rho}\right)-n^2}.\frac{\mu}{\sqrt{1-\mu^2}}.\sin.(it+\varpi).$$

[3293] The depth of the sea is $l.(1-q\mu^2)$ [2196]; now we have, for the condition of equilibrium [1802],

[3294]
$$lq=\frac{5n^2\rho}{(10\rho-6).g};\dagger$$

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* (2173) By altering the epoch of the time t, in [2251], we may suppose the angle $it+\varpi-A$, of that expression to be put under the form $it+\varpi$. Substituting $\sin.\theta.\cos.\theta=\mu.\sqrt{(1-\mu^2)}$ [2128xii], we obtain the value of y [3290]. Comparing this
[3290a] with the assumed value [2178], we obtain $s=1$, $\varepsilon=0$; and the corresponding values of u, v [2178′, 2178″] become $u=b.\cos.(it+\varpi)$, $v=c.\sin.(it+\varpi)$. Substituting
[3290b] the values of b, c [2258, 2259], we get, [3291, 3292], respectively.

† (2174) The nucleus or solid part of the earth being spherical; the ellipticity of the surface of the ocean is equal to the difference between the depths of the sea at the pole and equator. Now these depths are $l-lq$, l, respectively [3293, 2128xii]; consequently the ellipticity is lq; and as this is represented by αh [1801′—1802], we shall have
[3294a] $\alpha h=lq$ equal to the second member of [1802]. This expression of lq may be much simplified, by observing that the ellipticity of the nucleus being nothing, we have,
[3294b] $\alpha h'=0$ [1800vi]; hence $-6\alpha h'.a'^5=0$. We have also $\int_0^{a'}\rho.d.(a^5h)=0$, because $\alpha h=0$, in all parts of the nucleus, from the centre to the surface, corresponding to the radius a' [1800v]. Lastly, if we suppose ρ to represent the uniform density of the
[3294c] nucleus, we shall have $\int_0^{a'}\rho.d.a^3=\rho a'^3$. Substituting these in the expression $\alpha h=lq$
[3294d] [1802, 3294a], we get $lq=\frac{5\alpha\varphi.\{1-a'^3+\rho a'^3\}}{4-10a'^3+10\rho a'^3}$. The depth of the sea being supposed very small, we have the radius of the nucleus a', very nearly equal to the radius of the surface of the fluid, which differs from unity by terms of the order α [1775]; and by

which renders the preceding expressions of $y, u, v,$ infinite; but as they cannot become infinite, unless we suppose $i-n=0$ [3239], it follows, that $y, u, v,$ are of the order $\frac{1}{i-n}$; therefore we must not put $i=n$, [3295]
in the equation [3270], as is done in [3269c, d], in taking the differentials of u and v, relatively to the time t; but we must notice, in these [3295′]
differentials, the factor $i-n$, by which these parts of u and v are multiplied. These terms, being divided by $i-n$, produce quantities, free from $i-n$, which render the parts of dN', dN'', relatively to [3296]
the attraction and pressure of the sea on the terrestrial spheroid, equal to nothing.*

We shall here observe, that in the preceding case, the oscillations of the [3296′]
sea, depending on the angle $it+\varpi$, are very great, when i differs but very little from n, and this condition obtains in the terms depending on the motion of the nodes of the lunar orbit. This motion is expressed [3297]
by $(i-n).t$; but a very small resistance, on the part of the terrestrial nucleus, is sufficient to decrease these oscillations considerably. The sea, by means of this resistance, acts horizontally on the spheroid, and by this [3298]
action it has an influence on the motion of its axis. We shall see, in the following article, that in this case, which is the case of nature, the preceding theorem holds good.

12. *In the preceding analysis, although it is very general, we have supposed that the sea wholly covers the terrestrial spheroid or nucleus, that it is of a* [3299]

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neglecting terms of the preceding expression of the order $\alpha\varphi l$, we may put $a'=1$, and then we obtain $lq=\frac{5\alpha\varphi.\rho}{10\rho-6}$. The quantity $\alpha\varphi$ represents the ratio of the centrifugal [3294e]
force to the gravity at the equator [1726′], and this is equal to $\frac{n^2}{g}$ [1594a, &c.]; substituting it in [3294e], we get [3294]. Multiplying this by

$$2g.\left(1-\frac{3}{5\rho}\right)=\frac{(10\rho-6).g}{5\rho}, \qquad \text{we get} \qquad 2lgq.\left(1-\frac{3}{5\rho}\right)=n^2; \qquad [3294f]$$

and by substitution, in the denominators, of [3290—3292], they become $n^2-n^2=0$; making $y, u, v,$ infinite.

* (2175) This must necessarily happen, for the reason mentioned in [3288], namely, [3295a]
that the resultant of all the forces passes through the centre of the spherical mass.

Second method of investigating the effect of the oscillations of the ocean, supposing that it does not wholly cover the earth.

regular depth, and suffers no resistance from the nucleus. But as these suppositions do not correspond with the actual state of the earth, it may be doubted, whether the preceding theorem is correct, relatively to the ocean. As this is a very important point in the theory of the motions of the
[3299′] earth, *we shall give the following general demonstration, whatever be the irregularities of the figure, the depth of the sea, or the resistance it suffers.* For this purpose, we shall recall to mind the principle of the preservation of areas, which has been demonstrated in Book I, Chap. V [167″].

Principle of the preservation of areas.

"*If we project upon a fixed plane, each particle of a system of bodies which react on each other, in any manner whatever, and then draw, from these*
[3299″] *projected positions, to a fixed point in that plane, lines which we shall call the radii vectores; the sum of the products of each particle, by the area described by its radius vector, will be proportional to the time* t; *so that*
[3300] *if we put this sum equal to* A, *we shall have* $A = h\,t$; *h being a constant quantity.*"

This principle has, in the present question, a great advantage, in being
[3300′] equally correct, even when the system is abruptly affected; as is the case with the sea, when its oscillations are suddenly changed by the friction and resistance of the shores.

If this system be subjected to the action of foreign bodies, A will no longer be proportional to the time t; and if we suppose the element $d\,t$,
[3300″] to be constant, the value of $d\,A$ will be variable. To determine its variation, we shall consider all the particles of the system as at rest and isolated. *We shall then compute the sum of all the products, formed by multiplying each particle by the area which its radius vector describes in the*
[3301] *time* $d\,t$, *by means of the foreign forces which act upon it; and this sum will be equal to* $d^2 A$;* for it follows, from the principle we have just mentioned, that the reaction of the different bodies of the system cannot produce any change in the value of $d^2 A$.

Investigation of $d^2 A$.

[3301a] * (2176) If $d\,A$ be the variation in the value A, in the time $d\,t$, its change during the next element of time will be, by the common principle of differentiation, $d\,A + d\,d\,A$; $d\,t$ being constant. The difference $d\,d\,A$, of these two values, represents the variation arising from the action of the particles upon each other, and the action of foreign bodies. But the effect of the first of these forces is nothing, by the principle [3300]; therefore $d\,d\,A$ represents the effect produced, in the time $d\,t$, by the action of foreign bodies only, as is observed in [3301].

This being premised, we shall suppose that a mass, of which a part is fluid, revolves about an axis, while it is acted upon by very small forces of the order α; its centre of gravity being at rest. If we suppose a fixed plane, passing through this centre, to be assumed for the plane of the projection; then draw through this point the radii vectores of the different particles; *the sum of the products of each particle, by the area which its radius vector describes, is the same, neglecting quantities of the order* α^2, *as if the mass were wholly solid.* To prove this, it is only necessary to show that the value of d^2A is the same, whether a part of the mass be supposed fluid, or the whole of it solid. Now we may observe, that at the expiration of any time whatever, the figure of the mass, and the manner in which it is presented to the action of foreign forces, cannot differ in the two hypotheses, but by quantities of the order α; and if we observe, also, that these forces are only of the order α, it must be evident, that the difference of the values of d^2A, in these two cases, must be of the order α^2; *therefore, by neglecting quantities of that order, we may suppose the corresponding values of* $\frac{dA}{dt}$ *to be the same in both hypotheses.* [3301′] [3302] [3302′] [3303] [3303′] [3304] [3304′]

The value of d^2A is the same, whether the earth be supposed wholly solid or covered by a fluid of small depth, neglecting quantities of the order α^2.

We shall now suppose, that the mass we have just mentioned is the earth. *In the first place, we shall consider the nucleus as a spheroid of revolution, differing but very little from a sphere, and covered by a fluid of small depth.* The attractions of the sun and moon excite oscillations in the fluid, and motions in the nucleus; but these oscillations and motions, by what has been said [3304′], are combined, in such a manner, that at the expiration of any time, the value of $\frac{dA}{dt}$ is the same as if the earth be wholly solid. *We shall now investigate the value of* $\frac{dA}{dt}$, *in this last supposition, that the earth is wholly solid.* [3305] [3306]

Case of the nucleus being a spheroid of revolution, covered by a fluid of small depth.

We shall suppose, that at the origin of the motion, θ is the inclination of the equator to a fixed plane, which we shall assume to be that of the ecliptic at a given epoch; ψ, the angle formed by the line of intersection of this plane with the equator, and by an invariable line drawn on this fixed ecliptic, through the centre of the earth; also, nt the rotatory motion of this body. It is evident, that all the changes which take place in the motion of the system, at the end of the time t, depend on the variations of θ, ψ, and n. We shall suppose, that, *at the end of the time* t, [3307] [3308] [3309]

Computation of dA, supposing the whole earth to be solid.

[3310] θ changes into $\theta+\alpha\delta\theta$, ψ into $\psi+\alpha\delta\psi$, and n into $n+\alpha\delta n$. We have already found,* that *the only terms necessary to be noticed, are,* FIRST, *Those which increase in proportion to the time;* SECOND. *Those*
[3311] *which are periodical, but multiplied by sines or cosines of angles increasing very slowly, and divided by the coefficients of the time t, in these angles.*
[3312] Therefore, by noticing only these terms, we may suppose θ, ψ, and n to be constant, in taking the differential of the function A.†

[3312'] We shall now suppose that the fixed plane, upon which the motions of the particles of the earth are projected, passes through its centre of gravity,
[3313] considered as at rest, and makes the angle γ with the fixed ecliptic‡

[3310a] * (2177) The most important of the variable terms of θ, ψ, n, depending on the precession, nutation, &c., have been computed in [3100, 3101, 3121, &c.]; and it has been shown, in the computation of these quantities, in several places of the preceding articles
[3310b] [3033a—e, 3063, &c.], that those parts may be neglected, which depend on rapidly varying angles, because they are not increased by integration.

† (2178) If the value of $A=ht$ [3300], be supposed to correspond to the earth,
[3312a] undisturbed, in any way whatever, by a foreign force, we shall have $\frac{dA}{dt}=h=$ a constant quantity, which may be considered as a function of the constant quantities θ, ψ, n, γ, β [3316]. In the case of nature, θ, ψ, n, contain variable terms, depending on slowly varying angles; but the differentials of these periodical quantities are rendered
[3312b] very small by the factors introduced by the differentiation; and in finding the partial differentials of the value of M [3321a—c], which takes the place of h [3312a, 3315], we may neglect the consideration of these very small terms, always rejecting quantities of the order α^2, as in [3317'], and using the theorem [2977h, &c.].

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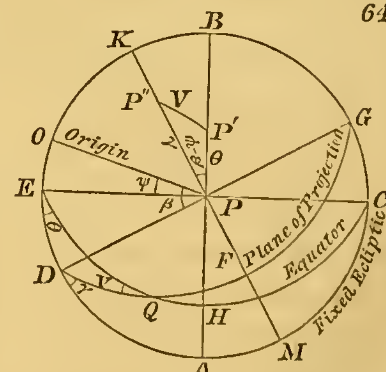

‡ (2179) To illustrate this, we shall refer to the
[3314a] annexed figure; in which $EACB$ represents the fixed ecliptic; PO the line taken as the origin of the angles β, ψ; EQC the equator; $DQFG$, the plane of projection on which the quantity A is
[3314b] computed; also $PP'P''$ the poles of the great circles $EACB$, EHC, $DQFG$, respectively. Then, by the above notation, we have the angle
[3314c] $OPD=\beta$, $OPE=\psi$, $EPD=\beta-\psi$; and as $EPD=BPK=P'PP''$, we have, in the
[3314d] spherical triangle $P'PP''$, the angle $P'PP''=\beta-\psi$;
[3314e] $PP'=$ angle $AEH=\theta$; $PP''=$ angle $FDM=\gamma$, $P'P''=$ the distance of the poles of the arcs QFG, QC, equal to the angle GQC, or V.

[3307]; moreover, that the intersection of these two planes forms the angle β with the invariable right line from which the angle ψ is counted. [3314]
We shall have, at the origin of the motion,

$$\frac{dA}{dt} = M, \qquad [3315]$$

M being a function of θ, ψ, n, and of the quantities γ, β, which [3316]
determine the position of the plane of projection. At the end of the time t, we shall obtain $\frac{dA}{dt}$; by changing, in M, the quantities θ, ψ, n, into [3317]
$\theta+\alpha\delta\theta$, $\psi+\alpha\delta\psi$, and $n+\alpha\delta n$. Putting, therefore, $\alpha.\delta.\frac{dA}{dt}$, for the variation of $\frac{dA}{dt}$ at the end of that time, we shall have, *by neglecting quantities of the order* α^2,* [3317′]

$$\alpha.\delta.\frac{dA}{dt} = \alpha.\delta\theta.\left(\frac{dM}{d\theta}\right) + \alpha.\delta\psi.\left(\frac{dM}{d\psi}\right) + \alpha.\delta n.\left(\frac{dM}{dn}\right). \qquad [3318]$$

We shall put C for the sum of the products of each particle of the earth, [3318′]
by the square of its distance from the axis of rotation; and V for the [3319]
inclination of the plane of projection to the terrestrial equator; it is evident that we shall have,† $M = \frac{1}{2}nC.\cos.V$. Now we have, [3320]

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* (2180) The expression [3318], is obtained by development, in the usual manner, as [3318a]
in [610—612], neglecting terms of the order α^2.

† (2181) The angular rotatory velocity of the earth about its axis, *at the origin* of the time t, is equal to n [3309]; hence the velocity of the particle dm, supposing it to be at the distance D from that axis, is nD, and the space described in the time dt, [3320a]
is $nD.dt$. Multiplying this by half the radius vector, or $\frac{1}{2}D$, we get $\frac{1}{2}nD^2dt$, which represents the area described by the particle in the first moment of time dt, [3320b]
projected upon the plane of the equator, neglecting terms of the order α^2. If we project it upon the plane DQG, which is inclined to the equator by the angle V [3319], it [3320c]
is evident, by the principles of the orthographic projection, that it will be decreased in the ratio of $\cos.V$ to 1; so that it will become $\frac{1}{2}nD^2.\cos.V.dt$. Multiplying this by dm, and prefixing the sign of integration S, it becomes $\frac{1}{2}ndt.\cos.V.S.D^2dm$; [3320d]
the sign S being supposed to include the whole mass of the earth. This is the quantity represented by $dA = Mdt$ [3315]. Hence $M = \frac{1}{2}n\cos.V.S.D^2dm$; but by [3319], $S.D^2.dm = C$, therefore $M = \frac{1}{2}nC.\cos.V$, as in [3320]. [3320e]

[3321] $$\cos. V = \cos. \gamma . \cos. \theta + \sin. \gamma . \sin. \theta . \cos. (\beta - \psi) ;^*$$

hence we find,

Value corresponding to the supposition that the whole mass is solid.

[3322] $$\alpha . \delta . \frac{dA}{dt} = \tfrac{1}{2} \alpha n C . \left\{ \begin{array}{l} \delta\theta . \{\sin. \gamma . \cos. \theta . \cos. (\beta - \psi) - \cos. \gamma . \sin. \theta\} \\ + \delta\psi . \sin. \gamma . \sin. \theta . \sin. (\beta - \psi) \end{array} \right\}$$
$$+ \tfrac{1}{2} \alpha C . \delta n . \{\cos. \gamma . \cos. \theta + \sin. \gamma . \sin. \theta . \cos. (\beta - \psi)\} ;$$

[3323] in which expression, we may, without sensible error, determine C [3318'], as if the earth were a sphere.† *We shall now investigate the expression of the same quantity, supposing the earth to be a solid spheroid, or nucleus, covered by a fluid of small depth.*

Computation, supposing the earth to be covered by a fluid.

[3324] We shall put $\delta\theta'$, $\delta\psi'$, $\delta n'$, for the variations of θ, ψ, n, relative to the nucleus, retaining in these variations only the terms proportional to the time, and those multiplied by sines or cosines of angles increasing very
[3324'] slowly, divided by the coefficients of the time, in these angles [3311]. It is evident, by what precedes, that there will be produced in the value of $\frac{dA}{dt}$, a variation which is very nearly equal to,‡

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* (2182) In the spherical triangle $PP'P''$, Fig. 64, page 904, we have, as in [63]
[3321a] Int., $\cos. P'P'' = \cos. PP' . \cos. PP'' + \sin. PP' . \sin. PP'' . \cos. P'PP''$. If we use the symbols [3314d, e], it becomes as in [3321]; substituting this in [3320], we get

[3321b] $$M = \tfrac{1}{2} n C . \{\cos. \gamma . \cos. \theta + \sin. \gamma . \sin. \theta . \cos. (\beta - \psi)\}.$$

The partial differentials of this expression are,

[3321c] $$\left(\frac{dM}{d\theta}\right) = \tfrac{1}{2} n C . \{ - \cos. \gamma . \sin. \theta + \sin. \gamma . \cos. \theta . \cos. (\beta - \psi)\} ;$$

[3321d] $$\left(\frac{dM}{d\psi}\right) = \tfrac{1}{2} n C . \sin. \gamma . \sin. \theta . \sin. (\beta - \psi) ;$$

[3321e] $$\left(\frac{dM}{dn}\right) = \tfrac{1}{2} C . \{ \cos. \gamma . \cos. \theta + \sin. \gamma . \sin. \theta . \cos. (\beta - \psi)\}.$$

Substituting these in [3318], it becomes as in [3322].

† (2183) For the quantities neglected in this case, are only of the order of the ellipticity
[3323a] of the earth, multiplied by the very small quantities $\delta\theta$, $\delta\psi$, or δn, in [3322, 3325], or by the much smaller quantities $\delta\theta' - \delta\theta$, $\delta\psi' - \delta\psi$, $\delta n' - \delta n$, in [3328].

‡ (2184) The expression [3325] is easily deduced from [3322], by merely accenting
[3326a] the quantities $\delta\theta$, $\delta\psi$, δn, in order to conform to the change of notation [3317, 3324].

$$\tfrac{1}{2}\,\alpha n C.\left\{\begin{array}{l}\delta\theta'.\{\sin.\gamma.\cos.\theta.\cos.(\beta-\psi)-\cos.\gamma.\sin.\theta\}\\ +\delta\psi'.\sin.\gamma.\sin.\theta.\sin.(\beta-\psi)\end{array}\right\}$$ [3325]

$$+\tfrac{1}{2}\,\alpha\,C.\delta n'.\{\cos.\gamma.\cos.\theta+\sin.\gamma.\sin.\theta.\cos.(\beta-\psi)\}.$$

As the depth of the sea is very small, we may still suppose, as in [3318′], that C represents the sum of the products, formed by multiplying each particle of the earth, by the square of its distance from the axis of rotation.* [3326] To obtain the whole variation of $\frac{dA}{dt}$, we must add to the preceding expression [3325], the part depending upon the motion of the fluid, which we shall denote by $\alpha\delta L$. Now we have seen, in [3306], that the whole variation of $\frac{dA}{dt}$ is equal to the expression computed in [3322], which corresponds to the supposition, that the fluid covering the earth, forms a solid mass with it. Hence we shall have, by putting these two expressions equal to each other,†

$\alpha\delta L$. [3327] [3327′]

Value corresponding to the supposition that the earth is a spheroid of revolution, covered by a mass of fluid of small depth.

$$0=\alpha n\,C.\left\{\begin{array}{l}(\delta\theta'-\delta\theta).\{\sin.\gamma.\cos.\theta.\cos.(\beta-\psi)-\cos.\gamma.\sin.\theta\}\\ +(\delta\psi'-\delta\psi).\sin.\gamma.\sin.\theta.\sin.(\beta-\psi)\end{array}\right\};\quad (q)$$ [3328]

$$+\alpha\,C.\{\delta n'-\delta n).\{\cos.\gamma.\cos.\theta+\sin.\gamma.\sin.\theta.\cos.(\beta-\psi)\}+2\alpha\delta L.$$

The only terms of the expression of $\alpha\delta L$, *necessary to be noticed, are,* FIRST. *Those which are proportional to the time;* SECOND. *The terms depending on the sines and cosines of angles increasing very slowly, and which are also divided by the coefficients of the time in those angles* [3311, 3310a]. In the calculation of these terms, we may neglect the [3329]

This computation, made for the nucleus, may be supposed also to include that part of the effect of the motion of the particles of the ocean, which corresponds to the variations $\delta n'$, $\delta\theta'$, $\delta\psi'$; so that C will depend, as in the former case, upon the combined mass of the nucleus and fluid. To obtain the complete value, we must add to the expression [3325], the quantity of $\alpha\delta L'$ [3327], which arises from the difference between the actual velocities of the particles of the fluid, and the parts which depend on the quantities $\delta n'$, $\delta\theta'$, $\delta\psi'$. [3326b]

* (2185) These neglected terms must evidently be very small, estimating C as in [3326b]. [3326c]

† (2186) Adding $\alpha\delta L$ to [3325], putting the sum equal to [3322], transposing the terms to the second member of the equation, and doubling the result, we get [3328]. [3327a]

variations of the motions of the terrestrial spheroid, because the influence of these variations upon the value of $\alpha \delta L$, bears the same proportion to
[3330] the variations themselves, as the ratio of the mass of the fluid to that of the nucleus.* We may also, in the calculation of the action of the sun and moon upon the sea, neglect the part of these attractions, whose resultant
[3331] passes through the centre of the nucleus; and which would tend *to keep the earth in equilibrium about its centre, if the sea should become solid*; for it is
[3331′] evident, that by means of this force, the variation of $\frac{dA}{dt}$, in the hypothesis of the sea being solid, will vanish;† and, by what has been said [3304′],
[3332] *the state of fluidity of the sea cannot influence this variation.* This part of the attraction produces in the ocean the oscillations of the first and of the third kinds,‡ which we have considered in §5, 6, 9, and 10, of Book IV. As to the other part of the attractions of the sun and moon, we have seen,
[3333] in §7, 8, of Book IV, that it produces the oscillations of the second kind, on which depends the difference of the two tides of the same day;§

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* (2187) The variations of the motions of the nucleus affect the ocean by the friction, by the change of action, from its different positions, &c. The whole effect of these causes,
[3330a] if we suppose it to communicate a similar variation to the fluid, will be decreased in the ratio of the mass of the fluid to the mass of the nucleus, as in [3330].

[3331a] † (2188) These forces tend, by hypothesis, towards the centre of the spheroid, which is supposed to be at rest; therefore no motion can be produced by these forces about any axis drawn through it.

‡ (2189) The forces depending on oscillations of the first kind, are derived from the function [2193], which does not contain ϖ, and must therefore be the same, in particles similarly situated, in every meridian of the earth. The forces depending on oscillations of the third kind, arising from the function [2195], are the same, in particles similarly situated, in opposite meridians, where the angle $2(nt+\varpi-\psi)$ is increased by 360^d or 0^d; therefore,
[3332a] in a fluid covering a solid spheroid of revolution [3305], the influence of particles, similarly situated, in opposite meridians, will mutually balance each other in the variation of $\frac{dA}{dt}$, depending upon these two oscillations, as in [3331, &c.].

§ (2190) Oscillations of the second kind depend on the function [2194], in which the
[3333a] sign of the factor $\cos.(nt+\varpi-\psi)$, changes, in particles similarly situated, in opposite meridians of a spheroid of revolution. This is different from what it is in the other two oscillations [3332a], and it is therefore separately discussed in [3333].

now, without being able to determine these oscillations for all hypotheses of the depth and density of the sea, we have seen, in the abovenamed articles, that the expressions of these oscillations contain neither terms proportional [3334]
to the time, nor sines or cosines of angles increasing very slowly, *divided by the coefficients of the time in these angles*.* Therefore, by putting x', y', z', [3335]
for the three rectangular co-ordinates that determine the position of a particle of the fluid, which we shall represent by dm, relatively to the plane of the projection; these co-ordinates, x', y', z', as well as their [3335']
differentials $\frac{dx'}{dt}$, $\frac{dy'}{dt}$, $\frac{dz'}{dt}$, will not contain any similar term; and this also holds good for the differential $\frac{x'dy'-y'dx'}{2dt}$, as well as for the [3336]
integral $S\,dm\,.\left(\frac{x'dy'-y'dx'}{2dt}\right)$, embracing the whole fluid mass. This integral represents the part of $\frac{dA}{dt}$, corresponding to the fluid;† hence it

---

* (2191) It appears from [2223, &c.], that the oscillations of the second kind depend on the sines and cosines of angles of the form $(it+\varpi-\psi)$, in which it differs from [3334a]
nt, by quantities of the order of the angular motions of the sun and moon [2223d']. If the equator, ecliptic, and plane of projection were fixed, the situation of a particle of the fluid, referred to the plane DQF, fig. 64, page 904, by means of the rectangular co-ordinates x', y', z', will contain the same angles, $it+\varpi-\psi$. Now the motions of these planes and angles are of the same order as the precession and nutation, or of the order δn, $\delta\theta$, $\delta\psi$, which are much smaller than the lunar and solar motions; therefore we may, notwithstanding these small motions, suppose x', y', z', and their differentials, to depend, as it regards this second oscillation, on the angles $it+\varpi-\psi$, in which it [3334b]
differs from nt by quantities of the same order as that of the motion of the sun or moon, in comparison with the rotatory motion of the earth; and that these terms have no small divisors.

† (2192) The quantity A [3300], represents the sum of the products, formed by multiplying each particle by the area it describes; therefore $\frac{dA}{dt}$ is its increment in the time dt, divided by dt. The area described by the particle dm, in the time dt [3337a]
is $\frac{1}{2}(x'dy'-y'dx')$ [167a, 3335']; multiplying this by $\frac{dm}{dt}$, and prefixing the sign S, corresponding to the whole fluid mass, we obtain the part of $\frac{dA}{dt}=\frac{1}{2}S\,dm\,.\left(\frac{x'dy'-y'dx'}{dt}\right)$, depending upon this oscillation of the fluid, as in [3336]. Now x', y', dx', dy', do [3337b]

[3337] follows, that its variation $\alpha\,\delta L$, contains no term of the nature of those now under consideration; we may therefore efface $2\alpha\,\delta L$ from the equation [3328], and then it becomes

$$
\begin{aligned}
0 = n\,.\,(\delta\theta' - \delta\theta)\,.\,\{\sin.\gamma\,.\,\cos.\theta\,.\,\cos.(\beta - \psi) - \cos.\gamma\,.\,\sin.\theta\} \\
+ n\,.\,\{\delta\psi' - \delta\psi\}\,.\,\sin.\gamma\,.\,\sin.\theta\,.\,\sin.(\beta - \psi) \\
+ (\delta n' - \delta n)\,.\,\{\cos.\gamma\,.\,\cos.\theta + \sin.\gamma\,.\,\sin.\theta\,.\,\cos.(\beta - \psi)\}
\end{aligned}
\qquad [3338]
$$

This equation holds good, whatever be the values of γ and β; we may
[3339] therefore suppose, in the first place,* $\beta = \psi$, and $\gamma = \theta$, which gives $0 = \delta n' - \delta n$; then the preceding equation becomes,

$$
\begin{aligned}
0 = (\delta\theta' - \delta\theta)\,.\,\{\sin.\gamma\,.\,\cos.\theta\,.\,\cos.(\beta - \psi) - \cos.\gamma\,.\,\sin.\theta\}. \\
+ (\delta\psi' - \delta\psi)\,.\,\sin.\gamma\,.\,\sin.\theta\,.\,\sin.(\beta - \psi).
\end{aligned}
\qquad [3340]
$$

[3341] Supposing, in this equation, $\gamma = 0$, we shall get $0 = \delta\theta' - \delta\theta$; consequently, also, $0 = \delta\psi' - \delta\psi$; therefore we shall have [3339, 3341],

$$\delta n' = \delta n; \qquad \delta\theta' = \delta\theta; \qquad \delta\psi' = \delta\psi. \qquad [3342]$$

---

not contain terms depending on slowly varying angles, *divided by the coefficients of the time in those angles*, when we restrict ourselves to the degree of approximation abovementioned.
[3337c] Hence these terms may be neglected [3329]; and we may therefore reject $2\alpha\,dL$ from [3328]; then dividing it by $\alpha\,C$, we get [3338].

* (2193) The plane of projection DQF, fig. 64, page 904, being arbitrary, we
[3339a] may take β, γ, at pleasure; and the equation [3338] must hold good, for all these values of β, γ. Now if we put $\beta = \psi$, we shall have $\cos.(\beta - \psi) = 1$, $\sin.(\beta - \psi) = 0$; hence [3338] becomes

$$
\begin{aligned}
0 &= n\,(\delta\theta' - \delta\theta)\,.\,(\sin.\gamma\,.\,\cos.\theta - \cos.\gamma\,.\,\sin.\theta) + (\delta n' - \delta n)\,.\,(\cos.\gamma\,.\,\cos.\theta + \sin.\gamma\,.\,\sin.\theta) \\
&= n\,(\delta\theta' - \delta\theta)\,.\,\sin.(\gamma - \theta) + (\delta n' - \delta n)\,.\,\cos.(\gamma - \theta).
\end{aligned}
\qquad [3339b]
$$

Putting $\gamma = \theta$, in the preceding equation, we get $0 = \delta n' - \delta n$, as in [3339]; hence [3338] becomes as in [3340]; which must be satisfied for all values of γ; and by putting $\gamma = 0$, we obtain $0 = (\delta\theta' - \delta\theta)\,.\,(-\sin\theta)$; hence $\delta\theta' - \delta\theta = 0$, as in [3341].
[3339c] Lastly, substituting $\delta\theta' - \delta\theta = 0$, in [3340], we get $(\delta\psi' - \delta\psi)\,.\,\sin.\gamma\,.\,\sin.\theta\,.\,\sin.(\beta - \psi)$, which is correct for all values of γ, β; therefore $\delta\psi' - \delta\psi = 0$, as in [3341]. Having proved, in [3342], that the values $\delta n'$, $\delta\theta'$, $\delta\psi'$, corresponding to a solid nucleus, covered by a fluid [3324], are respectively equal to δn, $\delta\theta$, $\delta\psi$, which correspond to a
[3339d] solid spheroid [3310], it is evident, that the variations of the motions of the earth must be the same in both hypotheses.

Hence it follows, that the variations of the motions of the terrestrial nucleus, covered by the fluid, are the same as if the sea form a solid mass with it. [3343]

Now it is easy to extend the preceding demonstration to the case of nature; in which the figure of the earth, as well as the depth of the sea, are very irregular, and the oscillations of the waters varied by a great number of obstacles; for all that is required to prove, is, that $\alpha \delta L$ contains then, [3344]
neither terms proportional to the time, nor the sines or cosines of angles increasing very slowly, divided by the coefficients of the time in these angles. Now if we recall to mind what has been said in § 15, Book IV [2372, &c.], we shall see that the expressions of the co-ordinates of the [3344′]
particles of the ocean do not contain similar terms. It is true that they depend on the elements of the orbit of the attracting body; and these elements, increasing slowly, introduce similar terms in the expressions of these co-ordinates, but without being divided by small coefficients.* It is [3345]
therefore generally correct, that *in whatever manner the waters of the ocean act upon the earth, either by their attraction, their pressure, their friction, or by the various resistances which they suffer, they communicate to the axis of the earth a motion which is very nearly equal to that it would acquire* [3346]
from the action of the sun and moon upon the sea, if it form a solid mass with the earth.

We have shown, in [3134], that the mean rotatory motion of the earth is uniform, supposing its body to be wholly solid; and we have just seen, [3347]

---

* (2194) The expression of αy [2400], does not contain terms depending on angles varying with extreme slowness, and divided by the coefficient of the time in those angles; observing that the declinations v, v′, which contain such angles, are not divided by small [3345a]
coefficients. Comparing αy [2400], with y [2372], we find that F, G, do not contain such divisors; therefore the part of $gy - V'$ [2373], depending on y, does not contain them. The other part of $gy - V'$, arising from the attraction of the bodies [3345b]
L, L', is independent of terms containing such divisors, as is evident from the forms of these expressions [2193—2195]. Hence F', G' [2373]; consequently H, K, P, Q [2382—2385], and u, v [2374, 2375], must be free from such terms. The co-ordinates of the particle dm depend on the quantities αy, αu, αv [2128^{xiv}, &c.]; therefore the [3345c]
coefficients of these co-ordinates, in the case treated of in [2372, &c.], must be free from terms divisible by the small coefficients here mentioned. We may extend the method of reasoning used in § 17—19, Book IV, to the case here treated of, and we shall finally [3345d]
obtain the same result as in [3345, 3346].

that the fluidity of the sea, and that of the atmosphere, cannot alter this
Trade winds. result. The motions excited in the atmosphere by the heat of the sun,
[3348] which causes the *trade winds*, might be supposed to decrease the rotation of the earth; because these winds blow between the tropics from east to west,* and their continual action on the sea, on the continents and mountains which they pass over, might have a tendency to decrease the rotatory motion. But the principle of preservation of areas [3300], proves
[3349] that the whole effect of the atmosphere on this motion must be insensible; for the solar heat, dilating the air equally in every direction, cannot alter the sum of the areas described by the radii vectores of each particle of the earth and atmosphere, multiplied respectively by their corresponding
[3350] particles; hence it follows, that the rotatory motion is not altered. *We are therefore assured, that while the trade winds decrease the rotatory velocity, the other motions of the atmosphere without the tropics accelerate it by the same quantity.*
[3351] *We may apply the same reasoning to earthquakes; and in general to every thing which agitates the earth below its surface.* The displacing of its parts is the only thing which can alter this motion. If, for example, a body placed at the pole, be transported to the equator, the sum
[3351′] of the areas must always remain the same; hence the rotatory motion of the earth will be decreased a little. But to render this perceptible, a great change must be supposed to be made in the constitution of the earth.

13. We shall now compare the preceding theory with observations; and
[3351″] shall investigate the consequences which result, relatively to the constitution of the terrestrial globe. If, in the expression of θ [3101], we reduce $\Sigma.\frac{lc}{f}.\cos.(ft+\beta)$, to a series, ascending according to the powers of t, we shall have, by retaining only the first power,†

[3352] $$\Sigma.\frac{lc}{f}.\cos.(ft+\beta)=\Sigma.\frac{lc}{f}.\cos.\beta-lt.\Sigma.c.\sin.\beta.$$

Fixed plane of
[3353] the ecliptic, Jan. 1, 1750. *We shall take for fixed plane, that of the ecliptic, at the beginning of the year* 1750, *where we shall fix the origin of the time t.* The square of

---

[3348a] * (2195) In the original work, this was erroneously written from west to east.

[3352a] † (2196) Multiplying the expression of $\cos.(ft+\beta)$ [3116a], by $\frac{lc}{f}$, and prefixing the sign Σ, we obtain [3352].

the inclination of the apparent ecliptic to this plane being, by § 5,*

$$\{\Sigma . c . \sin. (ft+\beta)\}^2 + \{\Sigma . c . \cos. (ft+\beta)\}^2, \qquad [3354]$$

we have,

$$\Sigma . c . \sin. \beta = 0 ; \qquad \Sigma . c . \cos. \beta = 0 ; \qquad [3355]$$

which gives, by neglecting the square of ft,†

$$\Sigma . \frac{lc}{f} . \cos. (ft+\beta) = \Sigma . \frac{lc}{f} . \cos. \beta. \qquad [3356]$$

Subtracting this term from h, we obtain the mean inclination of the equator to the ecliptic, at the beginning of the year 1750. Now h being arbitrary, [3357] we may suppose that it expresses this mean inclination; and then we must increase the value of h, by $\Sigma . \frac{lc}{f} . \cos. \beta$, in the other terms of the expression [3358] of θ; but on account of the smallness of these terms, we may dispense with this correction. Hence we have,‡

Obliquity.

$$\theta = h + \frac{l \lambda c'}{(1+\lambda) . f'} . \cos. (f't+\beta') + \frac{l . \text{tang.} h}{2m . (1+\lambda)} . \left\{ \cos. 2v + \frac{m}{m'} . \lambda . \cos. 2v' \right\} ; \qquad [3359]$$

and the value of θ' [3110], becomes,

---

* (2197) The sum of the squares of $\gamma . \sin. \text{A}$, $\gamma . \cos. \text{A}$ [3075e], putting $\sin.^2 \text{A} + \cos.^2 \text{A} = 1$, is $\gamma^2 = \{\Sigma . c . \sin. (ft+\beta)\}^2 + \{\Sigma . c . \cos. (ft+\beta)\}^2$; γ being [3354a] the inclination of the solar orbit to the fixed plane [3051]; as in [3354]. At the commencement of the time t [3353], we have $\gamma = 0$, $t = 0$; substituting these in [3354a], we get $0 = \{\Sigma . c . \sin. \beta\}^2 + \{\Sigma . c . \cos. \beta\}^2$; and as both the terms of the [3354b] second member are square numbers, their sum cannot vanish, unless each one is separately equal to nothing, as in [3355].

† (2198) Substituting the first equation [3355] in [3352], we get [3356]; hence the two first terms of θ [3101], become $h - \Sigma . \frac{lc}{f} . \cos. \beta$, which is the mean inclination [3356a] of the equator to the ecliptic at the epoch. Both these terms may be considered as being included in the arbitrary expression h, as is observed in [3357].

‡ (2199) If we change the arbitrary quantity h into $h + \Sigma . \frac{lc}{f} . \cos. \beta$, and then substitute the value of $\Sigma . \frac{lc}{f} . \cos. (ft+\beta)$ [3356] in [3101], it becomes as in [3359]; [3359a] neglecting the effect of this small change in the value of h, in the term tang. h.

Obliquity.

[3360] $$\theta' = h - t\,.\,\Sigma\,.\,cf\,.\,\sin.\beta + \frac{l\lambda c'}{(1+\lambda).f'}.\cos.(f't+\beta') + \frac{l\,.\,\text{tang}.h}{2m.(1+\lambda)}.\left\{\cos.2v + \frac{m}{m'}.\lambda.\cos.2v'\right\}.^{*}$$

[3361] Lastly, the values of ψ and ψ' [3100, 3107], become, by including in l all the terms by which t is multiplied,†

---

* (2200) Multiplying [3116a] by c, and prefixing the sign Σ, we get, successively,

[3360a] by using [3355], $\Sigma\,.\,c\,.\cos.(ft+\beta) = \Sigma\,.\,c\,.\cos.\beta - t\,.\,\Sigma\,.\,cf\,.\sin.\beta = -t\,.\,\Sigma\,.\,cf\,.\sin.\beta$.

[3360b] Substituting this in [3109], we get $\theta' = \theta - t\,.\,\Sigma\,.\,cf\,.\sin.\beta$, and then, from [3359], we obtain [3360].

† (2201) The terms under the sign Σ, in the expression of ψ [3100], are easily reduced to the form,

[3362a] $$\Sigma\,.\,\frac{l^2c}{f^2}.\sin.(ft+\beta).\,\text{tang}.h + (\cot.h - \text{tang}.h)\,.\,\Sigma\,.\,\frac{lc}{f}.\sin.(ft+\beta).$$

Now the equation [3352] holds good for all values of β, β' &c.; so that we may write $\beta - 90^d$ for β, &c.; and then, by using [3355], we get, successively,

[3362b] $$\Sigma\,.\,\frac{lc}{f}.\sin.(ft+\beta) = \Sigma\,.\,\frac{lc}{f}.\sin.\beta + lt\,.\,\Sigma\,.\,c\,.\cos.\beta = \Sigma\,.\,\frac{lc}{f}.\sin.\beta.$$

[3362c] If we decrease β by 90^d, in the equation [3116a], we get $\sin.(ft+\beta) = \sin.\beta + ft.\cos.\beta$; multiplying this by $\frac{l^2c}{f^2}$, and prefixing the sign Σ, we obtain,

[3362c'] $$\Sigma\,.\,\frac{l^2c}{f^2}.\sin.(ft+\beta) = \Sigma\,.\,\frac{l^2c}{f^2}.\sin.\beta + t\,.\,\Sigma\,.\,\frac{l^2c}{f}.\cos.\beta.$$

Substituting this and [3362b] in the preceding terms of ψ [3362a], they become,

[3362d] $$\left\{\Sigma\,.\,\frac{l^2c}{f^2}.\sin.\beta + t\,.\,\Sigma\,.\,\frac{l^2c}{f}.\cos.\beta\right\}.\,\text{tang}.h + (\cot.h - \text{tang}.h)\,.\,\Sigma\,.\,\frac{lc}{f}.\sin.\beta.$$

When these are substituted in [3100], we must put

[3362e] $$\zeta + \Sigma\,.\,\frac{l^2c}{f^2}.\sin.\beta\,.\,\text{tang}.h + (\cot.h - \text{tang}.h)\,.\,\Sigma\,.\,\frac{lc}{f}.\sin.\beta = 0,$$

because ψ is supposed to commence with t. Then the two first terms of [3100], namely, $lt+\zeta$, being combined with these under the sign Σ, become $lt + t\,.\,\Sigma\,.\,\frac{l^2c}{f}.\cos.\beta\,.\,\text{tang}.h$,

[3362f] which, by writing $l - \Sigma\,.\,\frac{l^2c}{f}.\cos.\beta\,.\,\text{tang}.h$ for l, becomes simply lt. Now if we neglect the effect of this change upon the other terms of ψ [3100], and substitute in it

[3362g] $$\frac{\cos.^2h - \sin.^2h}{\sin.h\,.\cos.h} = \frac{\cos.2h}{\frac{1}{2}\sin.2h} = 2\cot.2h \quad [32, 31], \text{ Int.},$$

we shall get [3362]. In the equation [3360a], we may change β into $\beta - 90^d$, &c.,

[3362h] as in [3362b], and we shall get $\Sigma\,.\,c\,.\sin.(ft+\beta) = t\,.\,\Sigma\,.\,cf\,.\cos.\beta$; hence [3106]

[3362i] becomes $\psi' = \psi - t\,.\cot.\theta\,.\,\Sigma\,.\,cf\,.\cos.\beta$. Substituting in this, for θ, its mean value h, [3359], and then using ψ [3362], we get [3363].

$$\psi = l\,t - \frac{l}{2\,m\,.\,(1+\lambda)}\,.\left\{\sin.\,2\,v + \frac{m}{m'}\,.\,\lambda\,.\,\sin.\,2\,v'\right\}$$
$$+ \frac{2\,l\,\lambda\,c'}{(1+\lambda)\,.\,f'}\,.\,\cot.\,2\,h\,.\,\sin.\,(f'\,t+\beta')\,;$$ [3362]

Precession.

$$\psi' = l\,t - t\,.\cot.\,h\,.\,\Sigma\,.\,c\,f\,.\cos.\,\beta - \frac{l}{2\,m\,.\,(1+\lambda)}\,.\left\{\sin.\,2\,v + \frac{m}{m'}\,.\,\lambda\,.\,\sin.\,2\,v'\right\}$$
$$+ \frac{2\,l\,\lambda\,c'}{(1+\lambda)\,.\,f'}\,.\,\cot.\,2\,h\,.\,\sin.\,(f'\,t+\beta').$$ [3363]

The term $-t\,.\,\Sigma\,.\,c\,f\,.\sin.\,\beta$ of the expression of θ' [3360], expresses the present secular decrease of the obliquity of the ecliptic. The exact quantity of this diminution has not yet been accurately ascertained. By taking a mean of the different observations, we may suppose this diminution to be $154'',3$ $[=50^s]$ in the present century; and if we put T for one Julian year, we may suppose [3364] T. [3364']

$$T\,.\,\Sigma\,.\,c\,f\,.\sin.\,\beta = 1'',543 \;\; [=0^s,5].$$ [3365]

This equation gives, by the theory of the planets, which we shall explain in the following book,*

$$T\,.\,\Sigma\,.\,c\,f\,.\cos.\,\beta = 0'',24794 \;\; [=0^s,08033].$$ [3366]

The annual precession of the equinoxes in the present century, is, by observation, nearly equal to $154'',63$ $[=50^s,1]$; therefore,† [3367]

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* (2202) Substituting for θ its mean value h, we get, from [3362i],

$$t\,.\cot.\,h\,.\,\Sigma\,.\,c\,f\,.\cos.\,\beta = \psi - \psi'\,;$$ [3366a]

and if we use the values of ψ, ψ' [4357, 4359], we obtain, successively, by means of the developments [3360a, 3362b], neglecting t^2, t^3, &c.,

$$t.\cot.h.\Sigma.c f.\cos.\beta = 42118'',3.\sin.(t.155'',542+95^\circ,2389) - 42000'',9.\cos.(t.100'',757) - 3146'',9.\sin.(t.43'',564)$$
$$= t\,.\left\{42118'',3\times155'',542\,.\cos.\,95^\circ,2389 - 3146'',9\times43'',564\right\}.\sin.\,1''.$$ [3366b]

The factor $\sin.\,1''$ is introduced in the second member, to render it homogeneous, instead of dividing by the radius, expressed in seconds. Substituting in this $t=T=1$, $h=26^\circ,0796$ [3369], and multiplying by $\operatorname{tang.} h$, we get nearly the same expression as in [3366]; which is liable to some degree of uncertainty, from not knowing accurately the values of the masses of the planets. [3366c]

† (2203) The part of ψ' [3363], containing the factor t, is $l\,t - t\,.\cot.\,h\,.\,\Sigma\,.\,c\,f\,.\cos.\,\beta$. Putting t equal to one Julian year T, it becomes $l\,T - \{T\,.\,\Sigma\,.\,c\,f\,.\cos.\,\beta\}\,.\cot.\,h = 154'',63$ [3367]; substituting [3366], we get [3368]; and then, by means of $h=26^\circ,0796$ [3369], we obtain $l\,T$ [3370]. [3367a] [3367b]

[3368] $$l\,T - 0'',24794\,.\,\cot.\,h = 154'',63.$$

[3369] The obliquity of the ecliptic in 1750, was observed to be $26^{\circ},0796$ $[= 23^{d}\,28^{m}\,17^{s},9]$, which is the value of h; hence we deduce,

[3370] $$l\,T = 155'',20 \ [=50^{s},2848].$$

[3370′] The mean inclination of the lunar orbit to the ecliptic, is $5^{\circ},7188$* $[= 5^{d}\,8^{m}\,48^{s},9]$, which gives,

[3371] $$c' = \text{tang.}\,5^{\circ},7188;$$

[3372] $f'\,T$ is the mean sideral† motion of the nodes of the lunar orbit in a Julian year, and by observation we have,

[3373] $$f'\,T = 215063'' \ [=19^{d}\,21^{m}\,20^{s},4].$$

[3374] A sideral year being $365^{days},256384$, we get,

[3375] $$m\,T = \frac{400^{\circ} \times 365^{days},25}{365^{days},256384} = 399^{\circ},9930 \ [= 359^{d}\,59^{m}\,37^{s},3].$$

[3376] Lastly, we have $m = m'\,.\,0,07480$, and the observations of the tides, give $\lambda = 3$ [2706]. This being premised, the preceding values of θ, θ' ψ, ψ', become‡

[3371a] * (2204) This inclination corresponds to the terms depending on the argument of latitude in Mason's corrected lunar tables, used by La Place in [5595]; but differs about a sexagesimal second from the value of γ, or c' [5117].

[3372a] † (2205) The sun's mean motion is 400° from the fixed equinox in one sideral year of $365^{d}\,6^{h}\,9^{m}11^{s} = 365^{days},25638$. Hence the motion in one Julian year, or $365^{days},25$, is $399^{\circ},9930$, as in [3375]; being the same as is used by the author in [4077]. He also puts, in [4835, 5117], $m = 0,0748013$ for the ratio of the mean motions of the sun

[3372b] and moon, which is represented in [3376], by $\frac{m}{m'}$. Hence the moon's mean motion in a

[3372c] Julian year, is $\frac{399^{\circ},9930}{0,0748013} = 5347^{\circ},407$. Multiplying this by $g - 1 = 0,00402175$ [4817, 5117], we get the mean sideral motion of the moon's nodes in a Julian year $f'\,T$, as in [3373] nearly. If we decrease this by the annual precession [3367], we shall obtain the annual motion, counted from the moveable equinox, as in [3086′]; if this be

[3372d] used instead of [3373], it will not affect the coefficients of $\sin.\,\text{A}'$, $\cos.\,\text{A}'$ [3377—3380], except by insensible quantities.

[3376a] ‡ (2206). The coefficients of the terms of [3377—3380], depending on the angles $2\,v'$, in the original work, are put equal to $0'',100$ $-0'',231$, instead of $0'',301$, $-0'',693$,

$$\theta = 26^\circ{,}0796 + 31''{,}036 \,.\, \cos. \mathrm{A}' + 1''{,}341 \,.\, \cos. 2\,v + 0''{,}301 \,.\, \cos. 2\,v'\,;$$ [3377]

Obliquity and precession.

$$\theta' = 26^\circ{,}0796 - i \,.\, 1''{,}543 + 31''{,}036 \,.\, \cos. \mathrm{A}' + 1''{,}341 \,.\, \cos. 2\,v + 0''{,}301 \,.\, \cos. 2\,v'\,;$$ [3378]

$$\psi = i \,.\, 155''{,}20 - 57''{,}998 \,.\, \sin. \mathrm{A}' - 3''{,}088 \,.\, \sin. 2\,v - 0''{,}693 \,.\, \sin. 2\,v'\,;$$ [3379]

$$\psi' = i \,.\, 154''{,}63 - 57''{,}998 \,.\, \sin. \mathrm{A}' - 3''{,}088 \,.\, \sin. 2\,v - 0''{,}693 \,.\, \sin. 2\,v'\,;$$ [3380]

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respectively. It is probable that this mistake was occasioned by omitting to multiply these terms by the factor $\lambda = 3$, which occurs in the formulas [3359—3363], from which [3377—3380] were computed, by using the numerical values [3370—3376]. These formulas are of such great importance in most astronomical theories, that we have given the calculation at full length, observing that from [3370, 3373, 3375], we have [3376b]

$$\frac{l}{f'} = \frac{l\,T}{f'\,T} = \frac{155{,}20}{215063}, \qquad \frac{l}{m} = \frac{l\,T}{m\,T} = \frac{155{,}20}{3999930}\,;$$ [3376c]

and $f'\,t + \beta' = -\mathrm{A}'$ [3086′, 3078, 3052],

$l\,T = 155''{,}20$ log.	2.1908917	$l\,T = 155''{,}20$ log.	2.1908917		
$f'\,T = 215063''$ log. co.	4.6674343	$m\,T = 3999930''$ log. co.	3.3979476		
$c' = 5^\circ{,}7188$ tang.	8.9545953	$\frac{1}{2.(1+\lambda)} = 0{,}125$ log.	9.0969100		[3376d]
radius in seconds, log.	5.8038801	radius in seconds, log.	5.8038801		
$\frac{\lambda}{1+\lambda} = 0{,}75$ log.	9.8750613	$\frac{l}{2\,m.(1+\lambda)} = 3''{,}088$ log.	0.4896294	... 0.4896294	
$\frac{l\,\lambda\,c'}{(1+\lambda).f'} = 31''{,}036$ log.	1.4918627	$h = 26^\circ{,}0796$ tang.	9.6377137		[3376e]
2 log.	0.3010300	$\frac{l.\mathrm{tang.}h}{2\,m.(1+\lambda)} = 1''{,}341$ log.	0.1273431		
$2\,h = 52^\circ{,}1592$ cot.	9.9705177	$\lambda.\frac{m}{m'} = 3.\frac{m}{m'} = 0{,}2244$ log.	9.3510229	... 9.3510229	
$\frac{2\,l\,\lambda\,c'}{(1+\lambda).f'}.\mathrm{cot.}2\,h = 57''998$ log.	1,7634104	$\frac{l.\mathrm{tang.}h}{2\,m.(1+\lambda)}.\frac{m}{m'}.\lambda = 0''{,}301$ log.	9.4783660		[3376f]
			$\frac{l}{2\,m.(1+\lambda)}.\frac{m}{m'}.\lambda = 0''{,}693$ log.	9.8406523	[3376g]

These coefficients being substituted in [3359—3363], give [3377—3380]. On account of the great importance of these formulas, we have reduced them to sexagesimals, as in the following table ;

$$\theta = 23^d\,28^m\,17^s{,}9 + 10^s{,}0556 \,.\, \cos. \mathrm{A}' + 0^s{,}4345 \,.\, \cos. 2\,v + 0^s{,}0975 \,.\, \cos. 2\,v'\,;$$ [3377a]

$$\theta' = 23^d\,28^m\,17^s{,}9 - i \,.\, 0^s{,}5 + 10^s{,}0556 \,.\, \cos. \mathrm{A}' + 0^s{,}4345 \,.\, \cos. 2\,v + 0^s{,}0975 \,.\, \cos. 2\,v'\,;$$ [3378a]

$$\psi = i \,.\, 50^s{,}2848 - 18^s{,}7914 \,.\, \sin. \mathrm{A}' - 1^s{,}0005 \,.\, \sin. 2\,v - 0^s{,}2245 \,.\, \sin. 2\,v'\,;$$ [3379a]

$$\psi' = i \,.\, 50^s{,}1001 - 18^s{,}7914 \,.\, \sin. \mathrm{A}' - 1^s{,}0005 \,.\, \sin. 2\,v - 0^s{,}2245 \,.\, \sin. 2\,v'.$$ [3380a]

The numerical values of the coefficients of the periodical terms of these formulas, require some small corrections, on account of the assumed value of $\lambda = 3$ [3376], which is [3380b]

[3381] *i being the number of Julian years elapsed from the beginning of* 1750, *and* A' *the longitude of the ascending node of the lunar orbit.*

υ, s. *If we put* υ *for the right ascension of a star, and* s *for its declination,*
[3382] *which must be supposed negative when it is south;* also $\delta\theta$, $\delta\theta'$, $\delta\psi$, $\delta\psi'$, $\delta\upsilon$, δs,

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found by all astronomers to be too great; but it is very difficult to ascertain its exact value by the observations we now have. The author successively reduces it to $\lambda = 2{,}566$
[3380b'] [4637] and to $\lambda = 2{,}35333$ [11905]; by using this last value, the chief term of the
[3380c] nutation is decreased from $10^s{,}0556$, to $9^s{,}4$; the other periodical terms requiring similar reductions. Poisson, in his memoir [3015i], adopts this last value of λ, as the most probable, and thence obtains the following expressions of these periodical parts of the obliquity and precession, which we shall represent by $\delta'\theta$, $\delta'\psi$, respectively, introducing the terms depending on $2A'$, formerly neglected [3081e, 3093y];

[3380d] Poisson's Formulas.
$$\delta'\theta = \quad 9^s{,}40041 \,.\, \cos. A' - 0^s{,}09167 \,.\, \cos. 2A' + 0^s{,}519 \,.\, \cos. 2\,v + 0^s{,}092 \,.\, \cos. 2\,v';$$
[3380e]
$$\delta'\psi = -17^s{,}56677 \,.\, \sin. A' + 0^s{,}84445 \,.\, \sin. 2\,A' - 1^s{,}196 \,.\, \sin. 2\,v - 0^s{,}211 \,.\, \sin. 2\,v'.$$

[3380e'] This value of $\lambda = 2{,}35333$ makes the mass of the moon equal to that of the earth divided by 74,946 [11906]. Dr. Brinkley, Bishop of Cloyne, finds, by means of 1618
[3380f] observations of ten different stars, that the chief term of the nutation is $9^s{,}25$. This is adopted by Mr. Baily, in his tables for determining the places of the fixed stars; and he
[3380g] finds the following values of $\delta'\theta$, $\delta'\psi$, in Vol. II, pp. xiv, xv, of the Memoirs of the Astronomical Society of London;

Brinkley and Baily's Formulas.
[3380h]
$$\delta'\theta = \quad 9^s{,}2500 \,.\, \cos. A' - 0^s{,}0903 \,.\, \cos. 2A' + 0^s{,}5447 \,.\, \cos. 2\,v + 0^s{,}0900 \,.\, \cos. 2\,v';$$
[3380i]
$$\delta'\psi = -17^s{,}2985 \,.\, \cos. A' + 0^s{,}2082 \,.\, \sin. 2\,A' - 1^s{,}2550 \,.\, \sin. 2\,v - 0^s{,}2074 \,.\, \sin. 2\,v'.$$

Lindeneau, by means of several hundred observations of the right ascension of the polar star, made by different astronomers, finds the chief term of the nutation to be only $8^s{,}97707$,
[3380k] corresponding to $\lambda = 2{,}03745$; this is used by Bessel, in his *Tabulæ Regiomontanæ*, published in 1830, page xv, where he gives,

[3380l] Bessel's Formulas.
$$\delta'\theta = \quad 8^s{,}97707 \,.\, \cos. A' - 0^s{,}08773 \,.\, \cos. 2\,A' + 0^s{,}57990 \,.\, \cos. 2\,v + 0^s{,}08738 \,.\, \cos. 2\,v';$$
[3380m]
$$\delta'\psi = -16^s{,}78332 \,.\, \sin. A' + 0^s{,}20209 \,.\, \sin. 2\,A' - 1^s{,}33589 \,.\, \sin. 2\,v - 0^s{,}20128 \,.\, \sin. 2\,v'.$$

These formulas will serve to show what degree of uncertainty remains in the observations of the nutation.

The author supposes, in [3378a], that the annual decrease of the obliquity of the ecliptic
[3380n] is $0^s{,}5$. This he increases to $0^s{,}521154$, in [4613], by changing the estimated values of the masses of Mars and Venus; and he also introduces, in the values of θ, &c., the
[3380o] terms depending on the second power of i or t. Poisson, by using different values of the masses of these planets, makes the annual decrement $0^s{,}45692$, and by noticing the

for the very small variations of θ, θ', ψ, ψ', υ, s, we shall find, by the differential formulas of spherical trigonometry,*

Motion in right ascension and in declination.

$$\delta s = \delta \psi \,.\, \sin.\theta \,.\, \cos.\upsilon + \delta\theta \,.\, \sin.\upsilon ; \qquad [3383]$$

$$\delta\upsilon = \delta\psi \,.\, \cos.\theta + \delta\psi \,.\, \sin.\theta \,.\, \text{tang}.s \,.\, \sin.\upsilon - \delta\theta \,.\, \text{tang}.s \,.\, \cos.\upsilon - \frac{i\,T}{\sin.h} \,.\, \Sigma \,.\, cf. \cos.\beta. \qquad [3384]$$

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second power of t, in the secular equations, he finds the following expressions, neglecting the periodical equations [3380d—m] ;

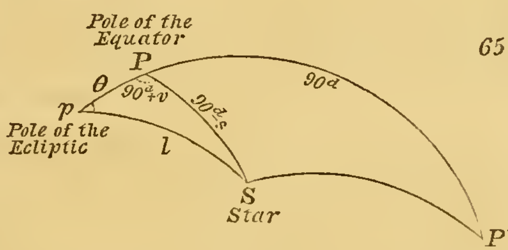
$$\begin{aligned}
\theta &= 23^d\, 28^m\, 18^s + t^2 \,.\, 0^s{,}00008001 ; \\
\theta' &= 23^d\, 28^m\, 18^s - t \,.\, 0^s{,}45692 - t^2 \,.\, 0^s{,}000002242 ; \\
\psi &= t \,.\, 50^s{,}37572 - t^2 \,.\, 0^s{,}00010905 ; \\
\psi' &= t \,.\, 50^s{,}22300 + t^2 \,.\, 0^s{,}00011637.
\end{aligned} \qquad [3380p]$$

Poisson's Formulas.

Bessel, in his *Tabulæ Regiomontanæ*, uses

$$\begin{aligned}
\theta &= 23^d\, 28^m\, 18^s + t^2 \,.\, 0^s{,}00000984233 ; \\
\theta' &= 23^d\, 28^m\, 18^s - t \,.\, 0^s{,}48368 - t^2 \,.\, 0^s{,}00000272295 ; \\
\psi &= t \,.\, 50^s{,}37572 - t^2 \,.\, 0^s{,}0001217945 ; \\
\psi' &= t \,.\, 50^s{,}21129 + t^2 \,.\, 0^s{,}0001221483.
\end{aligned} \qquad [3380q]$$

Bessel's Formulas.



65

* (2207) We shall, in the first place, compute the values of δs, $\delta\upsilon$, supposing the ecliptic to be fixed; and shall afterwards calculate the correction, depending on the displacement of the ecliptic. In the annexed figure 65, p is the pole of the fixed ecliptic in the year 1750; P the pole of the equator; S the place of the star; the meridian pP is continued to P', so as to make $PP' = 90^d$; and the points SP' are connected by the arc SP'. Putting the arc $pS = l$, we have $pP = \theta$, $pP' = 90^d + \theta$, $PS = 90^d - s$, angle $PpS = p$, angle $pPS = 90^d + \upsilon$; and in the spherical triangle PpS, we find [3383a] [3383b]

$$\sin.s = \cos.\theta \,.\, \cos.l + \sin.\theta \,.\, \sin.l \,.\, \cos.p \quad [1345^8]. \qquad [3383c]$$

Now the pole of the ecliptic and the place of the star being supposed to be fixed, the arc $pS = l$ is constant; hence the differential of the preceding equation, relative to the characteristic δ, becomes [3383d]

$$\delta s \,.\, \cos.s = -\delta p \,.\, \sin.\theta \,.\, \{\sin.l \,.\, \sin.p\} + \delta\theta \,.\, \{-\sin.\theta \,.\, \cos.l + \cos.\theta \,.\, \sin.l \,.\, \cos.p\}. \qquad [3383e]$$

This may be reduced, by observing that in the triangle SPp, we have

$$\sin.l \,.\, \sin.p = \cos.s \,.\, \cos.\upsilon \quad [1345^{15}]. \qquad [3383f]$$

Moreover, in the triangle SpP', we get

$$\cos.SP' = -\sin.\theta \,.\, \cos.l + \cos.\theta \,.\, \sin.l \,.\, \cos.p \quad [1345^8] ; \qquad [3383g]$$

By means of these formulas, we may reduce catalogues of stars from one epoch to another that is not far distant. For greater accuracy, it is best
[3385] to take the values of θ, υ, s, to correspond with the middle of the interval

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and by the same formula, we find, in the triangle SPP', where $\cos. PP' = 0$, $\sin. PP' = 1$, $\cos. SP' = \cos. s \,.\, \sin. \upsilon$. Putting these expressions of $\cos. SP'$ equal to each other, and substituting the resulting expression in [3383e], we get

[3383h] $$\delta s \,.\, \cos. s = -\delta p \,.\, \sin. \theta \,.\, \{\cos. s \,.\, \cos \upsilon\} + \delta \theta \,.\, \{\cos. s \,.\, \sin. \upsilon\}.$$

Dividing this by $\cos. s$, and substituting $\delta p = -\delta \psi$, because the precession $\delta \psi$,
[3383i] decreases the angle p, we get $\delta s = \delta \psi \,.\, \sin. \theta \,.\, \cos. \upsilon + \delta \theta \,.\, \sin \upsilon$, which is the same as in [3383]. This result may also be obtained very simply by a geometrical method,
[3383k] using the differential triangles.

[3383l] In the triangle pPS, we have $\cos. l = \cos. \theta \,.\, \sin. s - \sin \theta \,.\, \cos. s \,.\, \sin. \upsilon$ [1345[8]]. The differential of this expression, all being variable excepting l [3383d], is

[3383m] $$0 = \delta \theta \,.\, \{-\sin. \theta \,.\, \sin. s - \cos. \theta \,.\, \cos. s \,.\, \sin. \upsilon\} - \delta \upsilon \,.\, \sin. \theta \,.\, \cos. s \,.\, \cos. \upsilon + \delta s \,.\, \{\cos. \theta \,.\, \cos. s + \sin. \theta \,.\, \sin. s \,.\, \sin. \upsilon\}.$$

The term of this expression, depending on δs, becomes, by using the value of δs [3383i],

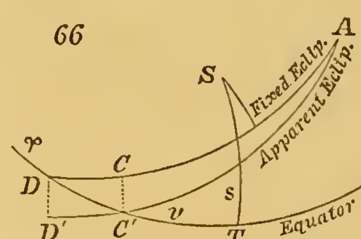
[3383n] $$\delta \theta \,.\, \{\cos. \theta \,.\, \cos. s \,.\, \sin. \upsilon + \sin. \theta \,.\, \sin. s \,.\, \sin.^2 \upsilon\} + \delta \psi \,.\, \sin. \theta \,.\, \cos. \upsilon \,.\, \{\cos. \theta \,.\, \cos. s + \sin. \theta \,.\, \sin. s \,.\, \sin. \upsilon\}.$$

Substituting this in [3383m], we find, that the second and third terms of the coefficient of $\delta \theta$ mutually destroy each other; the first and fourth terms become

[3383o] $$-\sin. \theta \,.\, \sin. s \,.\, (1 - \sin.^2 \upsilon) = -\sin. \theta \,.\, \sin. s \,.\, \cos.^2 \upsilon \,.$$

Then putting $\sin. s = \cos. s \,.\, \text{tang.}\, s$, and dividing by $\sin. \theta \,.\, \cos. s \,.\, \cos. \upsilon$, we get, as in [3384, 3383m], $\delta \upsilon = \delta \psi \,.\, \{\cos. \theta + \sin. \theta \,.\, \text{tang.}\, s \,.\, \sin. \upsilon\} - \delta \theta \,.\, \text{tang.}\, s \,.\, \cos. \upsilon$.

66



We shall now estimate the effect of the displacement of the ecliptic. For illustration, we shall refer to the annexed figure 66, which is similar to fig. 62, page
[3383p] 853. In the present figure, $DC'TH$ is the celestial equator, DCA the fixed ecliptic, $D'C'A$ the apparent ecliptic, noticing only the effect of the change of the plane of the ecliptic, now under consideration; ST an arc passing through the place of
[3383q] the star S perpendicularly to the equator. Then it is evident, that the declination of the star ST is not affected by this correction, because the plane of the equator remains unaltered; therefore the expression of δs [3383i, 3383] is complete. But the right ascension DT, is changed into $C'T$; so that it is decreased by the quantity DC', which represents the corresponding motion of the first point of Aries upon the equator. Now if we draw $C'C$ perpendicularly to the fixed ecliptic DC, we shall have the angle $C'DC = h$ nearly, and
[3383r] in the differential triangle $C'DC$, we find $C'D = \frac{DC}{\cos. h}$; consequently $\delta \upsilon = -\frac{DC}{\cos. h}$.

between the two epochs. The term $\frac{iT.\Sigma.cf.\cos.\beta}{\sin.h}$ is, by what precedes, equal to* $i.0'',62248=0'',20168$. These values of δs and $\delta \upsilon$, give, [3386]

$$\delta\theta=\frac{\delta s.\{\cos.\theta+\sin.\theta.\text{tang}.s.\sin.\upsilon\}-\{\delta\upsilon+i.0'',62248\}.\sin.\theta.\cos.\upsilon}{\cos.\theta.\sin.\upsilon+\sin.\theta.\text{tang}.s}; \quad [3387]$$

Change in the obliquity and in the precession.

$$\delta\psi=\frac{\delta s.\text{tang}.s.\cos.\upsilon+\{\delta\upsilon+i.0'',62248\}.\sin.\upsilon}{\cos.\theta.\sin.\upsilon+\sin.\theta.\text{tang}.s}. \quad [3388]$$

By the observed variations of the right ascensions and declinations of the stars, we may determine those of θ and ψ. It was thus that Bradley discovered the principal equations of θ, known by the name of *nutation*, which depends on the longitude of the node of the lunar orbit. His observations give $27'',778$ $[=9^s]$ for the coefficient of $\cos.\lambda'$, in the expression of θ [3377]. Maskelyne, by a more exact discussion of the [3389] [3390]

Nutation.

To find DC, we may observe, that the precession of the equinoxes, counted on the fixed plane of the ecliptic, is ψ [3362], and on the apparent ecliptic ψ' [3363]; their difference $\psi-\psi'=t.\cot.h.\Sigma.cf.\cos.\beta$, represents the value of the arc CD [3106b], or the decrement of the *precession* in the apparent orbit, which is equivalent to a *direct* motion of the equinoxes of $t.\cot.h.\Sigma.cf.\cos.\beta=iT.\cot.h.\Sigma.cf.\cos.\beta$ [3364′, 3381]. Substituting this for DC in $\delta\upsilon$ [3383r], we get $\delta\upsilon=-\frac{iT}{\sin.h}.\Sigma.cf.\cos.\beta$. Connecting this with $\delta\upsilon$ [3383o, &c.], we get the complete value of $\delta\upsilon$, as in [3384]. [3383r] [3383s] [3383t] [3383u]

* (2208) Dividing the expression [3366], by $\sin.h=\sin.26^{\circ},0796$ [3369], we get $0'',6225$, as in [3386] nearly; hence [3384] becomes

$$\delta\psi.\{\cos.\theta+\sin.\theta.\text{tang}.s.\sin.\upsilon\}-\delta\theta.\text{tang}.s.\cos.\upsilon=\delta\upsilon+i.0'',62248. \quad [3387a]$$

Multiplying this by $-\sin.\theta.\cos.\upsilon$, also [3383] by $\cos.\theta+\sin.\theta.\text{tang}.s.\sin.\upsilon$; then adding the products, the terms depending on $\delta\psi$ vanish, and we get the following expression, from which $\delta\theta$ [3387] is easily deduced;

$$\delta\theta.\{\cos.\theta.\sin.\upsilon+\sin.\theta.\text{tang}.s.(\sin.^2\upsilon+\cos.^2\upsilon)\}=\delta s.\{\cos.\theta+\sin.\theta.\text{tang}.s.\sin.\upsilon\}. -\{\delta\upsilon+i.0'',62248\}.\sin.\theta.\cos.\upsilon. \quad [3387b]$$

Again, multiplying [3387a] by $\sin.\upsilon$, also [3383] by $\text{tang}.s.\cos.\upsilon$; then adding the products, the terms multiplied by $\delta\theta$ vanish, and we get [3387c], from which [3388] is easily obtained;

$$\delta\psi.\{\cos.\theta.\sin.\upsilon+\sin.\theta.\text{tang}.s.(\sin.^2\upsilon+\cos.^2\upsilon)\}=\delta s.\text{tang}.s.\cos.\upsilon+\{\delta\upsilon+i.0'',62248\}.\sin.\upsilon. \quad [3387c]$$

[3391] same observations, estimates this coefficient to be equal to $29'',321$ [$=9'',5$], and we have found, by the theory $31'',036$ [3377]; the small difference is within the limits of the errors of observations, which agree as well as could be expected, with the law of universal gravitation. We may make these quantities exactly coincide, by decreasing a little the value of λ, which we

[3391′] have supposed equal to 3 [3376], and we may determine λ by these observations:* but the phenomena of the tides appear to me to give the value with greater accuracy, so that the coefficient in question cannot differ

[3392] but very little from $31'',036$ [3391a—d].

The retrograde motion of the equinoxes ψ, upon the fixed ecliptic, is

[3392′] produced by the retrograde motion of the pole of the earth upon a circle parallel to this fixed ecliptic. This last motion is represented by†

[3393] $$\psi \,.\, \sin. h = l t \,.\, \sin. h - \frac{l \lambda c'}{(1+\lambda) \,.\, f'} \,.\, \frac{\cos. 2h}{\cos. h} \,.\, \sin. A',$$

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* (2209) The coefficients $31'',036$, $57'',998$, computed in [3376c, f], contain the factor $\frac{\lambda}{1+\lambda} = \frac{3}{4}$, supposing $\lambda = 3$; so that they may be put under the forms

[3391a] $$31'',036 \times \tfrac{4}{3} \times \frac{\lambda}{1+\lambda} = 41'',381 \,.\, \frac{\lambda}{1+\lambda}, \qquad \text{and} \qquad 57'',998 \times \tfrac{4}{3} \times \frac{\lambda}{1+\lambda} = 77'',331 \,.\, \frac{\lambda}{1+\lambda}.$$

[3391b] Hence the term of θ, θ' [3377, 3378], depending on $\cos. A'$, is $41'',381 \,.\, \frac{\lambda}{1+\lambda} \,.\, \cos. A'$; and the term of ψ, ψ' [3379, 3380], depending on $\sin. A'$, is $77'',331 \,.\, \frac{\lambda}{1+\lambda} \,.\, \sin. A'$.

Now if we use Maskelyne's value, $29'',321 \,.\, \cos. A'$ [3391], we shall have

[3391c] $$29'',321 = 41'',381 \,.\, \frac{\lambda}{1+\lambda}; \qquad \text{hence} \qquad \lambda = \frac{29,321}{41,381 - 29,321} = \frac{29,321}{12,060} = 2,43.$$

The value $\lambda = 2,35333$ was finally assumed by the author, as we have observed in [3380b']. The expressions [3391b], in sexagesimals, are

[3391d] $$13',407 \,.\, \frac{\lambda}{1+\lambda} \,.\, \cos. A'; \qquad\qquad 25',055 \,.\, \frac{\lambda}{1+\lambda} \,.\, \sin. A'.$$

† (2210) If we neglect the secular equations depending on the change in the plane of the ecliptic, also the small terms depending on $2v$, $2v'$, and put, for a moment, for brevity,

[3393a] $$n = \frac{l \lambda c'}{(1+\lambda) \,.\, f'} = 31'',036 = 10',0556 \;[3377], \qquad f't + \beta' = -A' \;[3081c),$$

the expressions [3359, 3362] will become

[3393b] $$\theta = h + n \,.\, \cos. A', \qquad\qquad \psi = l t - 2n \,.\, \cot. 2h \,.\, \sin. A'.$$

The precession ψ represents the angular motion of the pole of the equator P, fig. 65,

noticing, according to the usual method of astronomers, only the greatest of the periodical equations of ψ. The term $\frac{l\lambda c'}{(1+\lambda)\cdot f'}\cdot\cos.\mathrm{A}'$, of [3394] the quantity θ*, indicates that the pole of the earth's axis has a motion in the direction of the circle of latitude passing through this pole. These two motions may be represented in the following manner. We may [3395] conceive the pole of the equator to be moved upon the circumference of a small ellipsis which is a tangent to the celestial sphere. The centre of this [3395'] ellipsis may be regarded as the mean pole of the equator, which describes uniformly, in each year, 155″,20 [=50^s,285] on the parallel to the fixed [3396] ecliptic, upon which it is situated. The transverse axis of this ellipsis is always a tangent to the circle of latitude, and subtends an angle of 62″,1 [=20^s,11] [3393f] in the plane of this great circle. The transverse [3396'] axis is to the conjugate axis, as the cosine of the obliquity of the ecliptic [3397] to the cosine of twice the obliquity [3393e]; therefore the conjugate axis subtends an angle of 46″,2 [=14^s,97] [3393h]. The apparent place of [3398] the pole of the equator upon this ellipsis, is determined in the following manner. Upon the plane of the ellipsis, a small circle is supposed to be described, having the same centre as the ellipsis, and a diameter equal to its transverse axis. A radius of this circle is supposed to revolve uniformly,

Geometrical method of considering the nutation.

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page 919, about the pole of the ecliptic p; hence the motion of the pole P, measured on the parallel to the ecliptic, passing through P, is evidently equal to $\psi\cdot\sin.\theta$, or $\psi\cdot\sin.h$ nearly. Substituting the value of ψ [3393b], and putting $\cot.2h=\frac{\cos.2h}{\sin.2h}=\frac{\cos.2h}{2\sin.h\cdot\cos.h}$, [3393$c$]

it becomes $\psi\cdot\sin.h=lt\cdot\sin.h-n\cdot\frac{\cos.2h}{\cos.h}\cdot\sin.\mathrm{A}'$, as in [3393]. Hence the [3393d]

maximum of the correction of θ [3393b], is n, and that of $\psi\cdot\sin.h$ [3393d], is $n\cdot\frac{\cos.2h}{\cos.h}$, which are to each other as $\cos.h$ to $\cos.2h$, as in [3397]. The nutation causes the [3393e] obliquity θ to change from $h+n$ to $h-n$ [3393b], which differ from each other by the quantity $2n=62'',072=20^s,11$ [3393a], as in [3396′] nearly. In like manner, [3393f] the part of $\psi\cdot\sin.h$ [3393d], depending on n, changes from $-n\cdot\frac{\cos.2h}{\cos.h}$ to $+n\cdot\frac{\cos.2h}{\cos.h}$; [3393$g$] the whole variation being $2n\cdot\frac{\cos.2h}{\cos.h}=46'',2=14^s,97$, as in [3398]. [3393$h$]

* (2211) This is the same as the term $n\cdot\cos.\mathrm{A}'$, in the value of θ [3393b], and it [3394a] indicates a motion of the pole P, fig. 65, page 919, in the direction of the circle of latitude pP.

with a retrograde motion, so as to coincide with the part of the transverse
[3399] axis nearest to the ecliptic, whenever the mean ascending node of the lunar orbit falls in the first point of Aries. Lastly, if from the extremity of this revolving radius, we let fall a perpendicular upon the transverse axis of the ellipsis, the first point where this perpendicular cuts the circumference
[3400] of the ellipsis, is the apparent place of the pole of the equator.*

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* (2212) This may be illustrated by referring to the annexed figure 67, in which C is
[3396a] the centre of the earth; HPE a spherical surface described about the centre C; P the pole of the fixed ecliptic; the dotted circle, $AEGH$, is the parallel to this ecliptic, described by the mean pole of the equator P', with a retrograde motion from P' towards
[3396b] A; P' is the centre of the small ellipsis $NB'SF'$ [3395'], which is drawn separately, on a large scale, in fig. 68, to avoid confusion. The transverse axis of this ellipsis, is $NP'S=62'',1=20',11$ [3396'], upon the circle of latitude $PNP'S$; and the conjugate
[3396c] axis $B'F'=46'',2=14',97$ [3398]. About the centre P', with the radius $P'S$, describe the circle $NBSF$; draw the radius $P'K$,
[3396d] making with $P'S$ the angle $SP'K=\text{A}'$; from K let fall, upon $P'S$, the perpendicular $KP''I$, cutting
[3396e] the ellipsis in P''; then P' will be the *mean* place of the pole of the ecliptic, and P'' its *apparent* place, deduced from the formulas [3393b], in which the
[3396f] most important equations only are noticed. For if we take, for a moment, the origin of the term t, at the instant when the mean pole of the equator is at E, fig.
[3396g] 67, and make the angle $EPQ=lt=155'',20.i$ [3364', 3379, 3381], $PP'=h=26^\circ,0796$; P' will be the mean place of the pole of the equator. The apparent distance of the pole of the equator from the pole of the ecliptic is represented by
[3396h] $\theta=h+n\,.\cos.\text{A}'$ [3393b], in which $n=31'',036$ [3393a]. Now by construction, in the triangle
[3396i] KIP', we have $P'K=n$, $IP'K=\text{A}'$; hence $P'I=P'K\,.\cos.IP'K=n\,.\cos.\text{A}'$; so that the apparent polar distance θ [3393b] is
[3396k] equal to $PP'+P'I=PI=PP''$ nearly. Again, in the right-angled triangle KIP',
[3396l] we have $KI=P'K\,.\sin.IP'K=n\,.\sin.\text{A}'$; then, by a property of the ellipsis [378t], and by construction [3397], we have,

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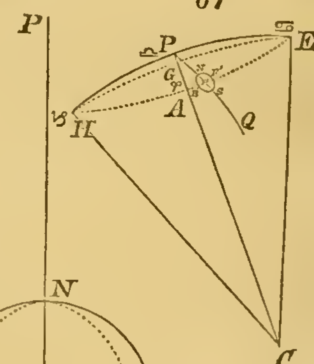
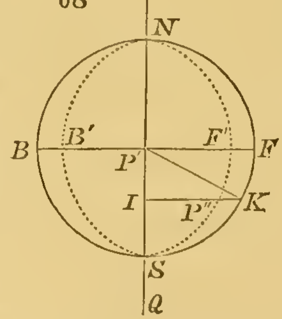


68



[3396m] $$\cos.h:\cos.2h::FP':F'P'::KI(=n\,.\sin.\text{A}'):IP''=\frac{n\,.\cos.2h}{\cos.h}\,.\sin.\text{A}'.$$

Hence IP'' represents the correction of the nutation [3393d], measured on the parallel of
[3396n] the ecliptic; consequently P'' is the place of the pole, corrected for the chief term of the nutation.

Heretofore astronomers have not noticed the inequalities depending on the angle $2v$; but on account of the precision of modern observations, they ought not to be neglected. [3401]

14. We shall now resume the value of l [3098], and for greater accuracy, we shall retain the squares of the excentricities and of the inclinations of the orbits; then we shall have,

$$lT = \tfrac{3}{4} \cdot \frac{m}{n} \cdot mT \cdot \cos. h \cdot \left(\frac{2C-A-B}{C}\right) \cdot \left\{ \frac{1}{(1-e^2)^{\frac{3}{2}}} + \lambda \cdot \frac{\{2\cos.^2\gamma - \sin.^2\gamma\}}{2(1-e'^2)^{\frac{3}{2}}} \right\};* \qquad [3402]$$

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* (2213) Dividing [3090] by $\sin.\theta$, we get $d\psi = \left(\frac{2C-A-B}{2n.C}\right) \cdot \frac{P\,dt}{\sin.\theta}$. In substituting the value of $P\,dt$ [3093c], we may make several reductions. *First.* We may neglect the term depending on e in the factor $1+e.\cos.(v-\Gamma)$, for the same reason as in [3093d—e]. *Second.* As the object of the present calculation is merely to find the terms of lt [3100, 3098] to a greater degree of accuracy, we may neglect, in Y^2-Z^2, YZ [3093c], the terms depending on the angles v, Λ, or their sums and differences, &c.; then the expression [3092] becomes $YZ=0$, and [3091] is as in [3402d, &c.], which by successive reductions, using [1, 6] Int., becomes as in [3402e]; [3402a] [3402b] [3402c]

$$Y^2-Z^2 = \tfrac{1}{2}r_{\prime}^2.\{\cos.^4\tfrac{1}{2}\gamma - \sin.^2\gamma + \sin.^4\tfrac{1}{2}\gamma\} = \tfrac{1}{2}r_{\prime}^2.\{(\tfrac{1}{2}+\tfrac{1}{2}.\cos.\gamma)^2 - \sin.^2\gamma + (\tfrac{1}{2}-\tfrac{1}{2}.\cos.\gamma)^2\} \qquad [3402d]$$

$$= \tfrac{1}{2}r_{\prime}^2.\{\tfrac{1}{2}+\tfrac{1}{2}.\cos.^2\gamma - \sin.^2\gamma\} = \tfrac{1}{2}r_{\prime}^2.\{\tfrac{1}{2}.(\cos.^2\gamma+\sin.^2\gamma) + \tfrac{1}{2}.\cos.^2\gamma - \sin.^2\gamma\}$$

$$= \tfrac{1}{2}r_{\prime}^2.\{\cos.^2\gamma - \tfrac{1}{2}.\sin.^2\gamma\} = \tfrac{1}{4}r_{\prime}^2.\{2\cos.^2\gamma - \sin.^2\gamma\}. \qquad [3402e]$$

Substituting [3402b, c, e] in [3093c], we get,

$$P\,dt = \frac{3m\,dv}{(1-e^2)^{\frac{3}{2}}} \cdot \tfrac{1}{4}.\{2\cos.^2\gamma - \sin.^2\gamma\}.\sin.\theta.\cos.\theta;$$

hence $d\psi$ [3402a] becomes

$$d\psi = \frac{(2C-A-B)}{2n.C} \cdot \frac{3m\,dv}{(1-e^2)^{\frac{3}{2}}} \cdot \frac{(2\cos.^2.\gamma - \sin.^2\gamma)}{4} . \cos.\theta. \qquad [3402f]$$

Integrating this, neglecting the secular variations of m, e, γ, θ, and changing θ into h, we get

$$\psi = \frac{3m}{4n} . v . \cos.h . \frac{(2C-A-B)}{C} . \frac{(2\cos.^2\gamma - \sin.^2\gamma)}{2.(1-e^2)^{\frac{3}{2}}}. \qquad [3402g]$$

This is the part of ψ depending on the sun, and by accenting the letters m, v, e, γ, as in [3078], we get the similar part of ψ depending on the moon,

$$\psi = \frac{3m'}{4n} . v' . \cos.h . \frac{(2C-A-B)}{C} . \frac{(2\cos.^2\gamma' - \sin.^2\gamma')}{2(1-e'^2)^{\frac{3}{2}}}. \qquad [3402h]$$

Substituting, in [3402g, h], the mean values of v, v', in one Julian year, namely, $v=mT$,

[3402′] e being the excentricity of the solar orbit, e' that of the lunar orbit, and γ the inclination of the lunar orbit to the ecliptic. We have, by observation,*

[3403] $$e = 0{,}016814\,; \qquad e' = 0{,}0550368\,;$$

[3404] $\frac{m}{n}$ is the ratio of the sideral day to the sideral year, and this ratio is equal to $0{,}00273033$;† hence we find,

[3405] $$l\,T = \left(\frac{2\,C - A - B}{C}\right).\{1 + \lambda\,.\,0{,}992010\}\,.\,7516'',30.‡$$

[3402i] $v' = m'\,T$ [3059, 3078, 3364′], and adding the two expressions together, we get the mean value of the precession in one year, represented by $l\,T$ [3100]. This sum becomes as in [3402]; observing, that as the solar orbit varies but little from the fixed ecliptic, we may neglect the square of γ [3051], and put $2\cos.^2\gamma - \sin.^2\gamma = 2$, in [3402g]; by which means this part becomes like the first term of [3402]. Moreover, by putting
[3402k] $m'^2 = \lambda\,m^2$ [3080c], and neglecting the accents on γ, it produces the second term of [3402], depending on λ.

[3403a] * (2214) This value of e is nearly the same as in [4080]; that of e' differs a little from [5117].

† (2215) The rotatory velocity being n [3015], the length of a sideral day will be
[3404a] $\frac{2\pi}{n}$, the whole circumference being 2π. Moreover, the mean sideral velocity of the sun is nearly equal to m [3059]; hence the length of a sideral year is nearly equal to $\frac{2\pi}{m}$; and as this is equal to 366,25638 sideral days [3374], we shall have

[3404b] $$\frac{2\pi}{n} : \frac{2\pi}{m} :: 1 : 366{,}25638, \quad \text{or} \quad m : n :: 1 : 366{,}25638 :: 0{,}00273033 : 1,$$

as in [3404].

‡ (2216) Using the value e, e' [3403], $\frac{m}{n}$ [3404], h [3369], $m\,T = 4000000''$,
[3405a] $\gamma = 5^{\circ},7188$ [3371], we get,

$$\frac{3m}{4n}\,.\,m\,T\,.\cos.h\,.(1-e^2)^{-\frac{3}{2}} = \tfrac{3}{4} \times 0{,}00273033 \times 4000000'' \times \cos.26^{\circ},0796\,.(1-e^2)^{-\frac{3}{2}} = 7516'',30\,;$$

[3405b] $$\left(\frac{1-e^2}{1-e'^2}\right)^{\frac{3}{2}}.\left(\frac{2\cos.^2\gamma - \sin.^2\gamma}{2}\right) = 0{,}992010\,;$$

substituting these in [3402], we obtain [3405]; and if we use $l\,T$, λ [3406], it becomes

[3405c] $$155'',20 = \left(\frac{2C - A - B}{C}\right).\{3{,}97603 + 2{,}97603\,.\,\beta\}\,.\,7516'',30\,;$$

which is easily reduced to the form [3407].

By the preceding article [3370], we may suppose, without any sensible error, that $l\,T = 155'',20$; therefore we shall have, by putting $\lambda = 3\,.\,(1+\beta)$, [3406]

$$\frac{(2\,C - A - B)}{C} = \frac{0{,}00519323}{1+\beta\,.\,0{,}748493}. \qquad [3407]$$

We have, very nearly, by § 2,*

$$\frac{2\,C - A - B}{C} = \frac{2\,\alpha.(h - \frac{1}{2}\,\varphi)\,.\,S_0^1\,.\,\rho\,.\,a^2\,d\,a}{S_0^1\,.\,\rho\,.\,a^4\,d\,a}. \qquad [3408]$$

It is remarkable, that the value of h'''' *of the same article* [2958], *does not enter into this equation; hence it follows, that the motions of the earth about its centre of gravity, are the same as if it were an ellipsoid of revolution, whose ellipticity is* $\alpha\,h$;† $\frac{1}{2}\,\alpha\,\varphi$ being equal to $\frac{1}{578}$ [1647a]. Comparing the two preceding expressions of $\frac{2\,C - A - B}{C}$, we get,‡ [3409] [3410]

$$\alpha\,h = 0{,}0017301 + \frac{0{,}00259661\,.\,S_0^1\,.\,\rho\,.\,a^4\,d\,a}{(1+\beta\,.\,0{,}748493)\,.\,S_0^1\,.\,\rho\,.\,a^2\,d\,a}. \qquad [3411]$$

Ellipticity of the earth.

In conformity to the laws of hydrostatics, we must suppose that the density of the strata of the terrestrial spheroid decreases from the centre to the

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* (2217) Using the values of A, B, C [2960—2962] we get

$$2\,C - A - B = \tfrac{48}{27}\,.\,\alpha\,\pi\,.\,(h - \tfrac{1}{2}\,\varphi)\,.\,S_0^1\,.\,\rho\,.\,d\,.\,a^3 = \tfrac{48}{9}\,.\,\alpha\,\pi\,.\,(h - \tfrac{1}{2}\,\varphi)\,.\,S_0^1\,.\,\rho\,.\,a^2\,d\,a. \qquad [3408a]$$

In dividing this by C [2962], and neglecting terms of the order α^2 in the result, we need only notice its first term, putting $C = \frac{8}{15}\,.\,\pi\,.\,S_0^1\,.\,\rho\,.\,a^5 = \frac{8}{3}\,.\,\pi\,.\,S_0^1\,.\,\rho\,.\,a^4\,d\,a$. [3408b] Substituting these in the first member of [3408], it becomes as in the second member of this formula.

† (2218) The radius of the ellipsoid [1503, 2958], is

$$a\,.\,\{1 + \alpha\,Y^{(2)}\} = a\,.\,\{1 + \alpha\,h\,.\,(\tfrac{1}{3} - \mu^2) + \alpha\,h''''.\,(1 - \mu^2)\,.\,\cos.\,2\,\varpi\};$$

and if the earth be a solid spheroid of revolution, ϖ must vanish, consequently $h'''' = 0$. [3409a] Then putting $a\,.\,(1 + \frac{1}{3}\,\alpha\,h) = a'$, and neglecting α^2, this radius becomes

$$a\,.\,(1 + \tfrac{1}{3}\,\alpha\,h) - a\,\alpha\,h\,.\,\mu^2 = a' - a'\,.\,\alpha\,h\,.\,\mu^2 = a'\,.\{1 - \alpha\,h\,.\,\mu^2\},$$

in which the ellipticity is evidently equal to $\alpha\,h$. [3409b]

‡ (2219) Putting the expressions [3407, 3408] equal to each other, and multiplying by $\frac{S_0^1.\rho\,.\,a^4\,d\,a}{2.\,S_0^1\,.\,\rho\,.\,a^2\,d\,a}$, we get $\alpha\,h - \frac{1}{2}\,\alpha\,\varphi = \frac{0{,}00259661\,.\,S_0^1\,.\,\rho\,.\,a^4\,d\,a}{(1+\beta.0{,}748493).\,S_0^1.\,\rho\,.\,a^2\,d\,a}$; substituting the value of $\alpha\,\varphi$ [3410], we easily obtain [3411]. [3410a]

[3412] surface; in which case $S_0^1 \cdot \rho \cdot a^4 d a$ is less than $\frac{3}{5} \cdot S_0^1 \cdot \rho a^2 d a$;* therefore, if we put $\beta = 0$, as is found by the observations on the
[3413] tides,† we shall find, that the value of αh is less than 0,0032881, or $\frac{1}{304}$. If the earth be an ellipsoid, its ellipticity will be expressed
[3414] by αh [3409b]; *therefore we cannot suppose the ellipticity to exceed* $\frac{1}{304}$.‡ *This fraction, and the former,* $\frac{1}{578}$ [3410], *are the limits of the ellipticity, which result from the precession and nutation of the earth's axis.*

[3415] We have seen, in the third book, [1800b], that $\alpha h = \frac{5}{4} \alpha \varphi$, in the hypothesis of the earth being homogeneous; hence we get, by means of the preceding equation, 3β very nearly equal to $-\frac{8}{5}$; consequently
[3416] $\lambda = \frac{7}{5}$.§ These differ too much from the value of λ, deduced from the

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* (2220) Putting $R = a$, in [278a], to conform to the present notation, we get, when

[3412a] ρ is constant, $\dfrac{S_0^1 \cdot \rho \cdot a^4 d a}{S_0^1 \cdot \rho \cdot a^2 d a} = \frac{3}{5}$; observing, that the value of R, or a, free from the sign of integration, is represented by $a = 1$ [2947′]. Now it has been shown, in [278b, &c.], that if the density of the spheroid be increased towards the centre, and decreased towards the surface, so that the whole mass may remain the same, the first member of [3412a] will

[3412b] be decreased, and we shall have $\dfrac{S_0^1 \cdot \rho \cdot a^4 d a}{S_0^1 \cdot \rho \cdot a^2 d a} < \frac{3}{5}$, as in [3412].

† (2221) Comparing [2706, 3079], we get $\lambda = 3$, and as this is put equal to $3(1+\beta)$
[3413a] [3406], we get $\beta = 0$; substituting this value of β in [3411], and using [3412b], we obtain, successively, as in [3413],

[3413b] $$\alpha h = 0{,}0017301 + 0{,}00259661 \times \frac{S_0^1 \cdot \rho \cdot a^4 d a}{S_0^1 \cdot \rho \cdot a^2 d a} < 0{,}0017301 + 0{,}00259661 \times \tfrac{3}{5} < 0{,}0032881 < \tfrac{1}{304}.$$

‡ (2222) This limit $\frac{1}{304}$, corresponds to the supposition of $\lambda = 3$ [3413a, b]; and it will
[3414a] be different if we use the value of $\lambda = 2{,}35333$ [3380b′], which was finally adopted by the author; for if we substitute $\lambda = 2{,}35333$, in [3406], we get $3 \cdot (1+\beta) = 2{,}35333$; hence $\beta = -0{,}21556$; and [3411], becomes, by using [3412b],

[3414b] $$\alpha h = 0{,}0017301 + 0{,}003096 \times \frac{S_0^1 \cdot \rho \cdot a^4 d a}{S_0^1 \cdot \rho \cdot a^2 d a} < 0{,}003587 < \tfrac{1}{279};$$

consequently the greatest ellipticity is $\frac{1}{279}$, instead of $\frac{1}{304}$. The least ellipticity evidently corresponds to the case, where the second term of [3411] vanishes, or $\dfrac{S_0^1 \cdot \rho \cdot a^4 d a}{S_0^1 \cdot \rho \cdot a^2 d a} = 0$; then $\alpha h = 0{,}0017301 = \frac{1}{578}$, as in [3414]. Hence the limits of the ellipticity,
[3414c] corresponding to this value of λ, are $\alpha h > \frac{1}{578} < \frac{1}{279}$.

§ (2223) If the earth be homogeneous, we shall have, as in [1800b, 3410],

$$\alpha h = \tfrac{5}{4} \alpha \varphi = \tfrac{5}{4} \cdot \tfrac{1}{289} = 0{,}0043252.$$

phenomena of the tides [3414*a*] to be admitted. Moreover, the nutation in this case is only $\frac{7}{9}$ of the preceding value [3391*a*],* or 24″,1; and this also varies too much from astronomical observations to be allowed; therefore, *these observations, and those of the tides, concur in proving, that we must reject the hypothesis of the homogeneity of the earth.* We have obtained the same result in the third book, by means of the observations of the length of a pendulum vibrating in a second; from which we have found $\frac{1}{321}$,† for the greatest limit of the value of αh. As this fraction is less than $\frac{1}{304}$ [3414], it follows, that the observations of the lengths of pendulums may be made to agree very well with those of nutation and precession, and with the observations of the phenomena of the tides. [3417] [3418] [3419] [3419′]

To obtain, in one view, the whole of the phenomena, which depend upon the figure of the earth, and to show the conformity with the theory of universal gravitation, we shall recapitulate the several results we have obtained, relative to the nature of the radii and the figure of the earth. Recapitulation. [3419″]

The expression of the radius of any spheroid, differing but little from a sphere, may be put under this form,‡

$$1+\alpha\,.\,\{Y^{(1)}+Y^{(2)}+Y^{(3)}+Y^{(4)}+\&c.\}. \qquad [3420]$$

If we fix the origin of the radius of the earth at its centre of gravity, we shall find, as in the third book, that the conditions of the equilibrium of the sea give $Y^{(1)}=0$ [1734‴], which reduces the expression of the radius [3420] to the following form, [3420′] [3421]

$$1+\alpha\,.\,\{Y^{(2)}+Y^{(3)}+Y^{(4)}+\&c.\}. \qquad [3422]$$

Substituting this and [3412*a*], in [3411], we get

$$0{,}0043252=0{,}0017301+\frac{0{,}00259661}{1+\beta\,.\,0{,}748493}\times\tfrac{3}{5}\,; \qquad [3416a]$$

hence we obtain $\beta=-0{,}534$, and [3406] becomes $\lambda=3\,.\,(1+\beta)=1{,}398=\frac{7}{5}$ nearly, as in [3416].

* (2224) The nutation is $41''381\,.\,\frac{\lambda}{1+\lambda}\,.\cos.\,A'$ [3391*b*]; substituting $\lambda=\frac{7}{5}$ [3416], it becomes $24'',1\,.\cos.\,A'$, as in [3417]. This is nearly $\frac{7}{9}$ of the value of $31'',036\,.\cos.\,A'$ [3377], corresponding to $\lambda=3$, as is observed in [3417]. [3417*a*]

† (2225) This is computed in [2044]; it falls between the two limits [3414] or [3414*c*]. [3419*a*]

‡ (2226) This agrees with [2942], putting, at the surface of the earth, $a=1$. [3420*a*]

[3422'] *The permanent state of the equilibrium of the sea requires that the axis of rotation of the earth should be one of its principal axes*, and for this reason it is necessary that $Y^{(2)}$ should be of the form [1763],*

[3423] $$Y^{(2)} = -h\,.\,(\mu^2 - \tfrac{1}{3}) + h''''.\,(1-\mu^2)\,.\,\cos.\,2\varpi;$$

h and h'''' being two arbitrary constant quantities, which can be determined by observation only, since they depend on the constitution or form of the terrestrial globe.

[3423'] *These results are the only ones which depend upon the permanent state of equilibrium of the earth. They are common to all the heavenly bodies, covered by a fluid in equilibrium.* The observations on the length of a pendulum vibrating in a second, give some light on the nature or relative magnitudes of the radii of the earth; for we have found from these observations, that the constant quantity αh is very nearly equal to†
[3424] 0,002978 [2048'], and that the constant quantity h'''' is insensible, in comparison with h. Moreover, the quantity, $Y^{(3)} + Y^{(4)} + \&c.$, is very
[3425] small in comparison with $Y^{(2)}$, and the first differential of $Y^{(3)} + Y^{(4)} + \&c.$, is small in comparison with the first differential of $Y^{(2)}$; *therefore, in calculating the radius of the earth, and that of its first differential, we may suppose, without any sensible error, this radius to be of the form,*

[3426] $$1 - 0{,}002978\,.\,(\mu^2 - \tfrac{1}{3}) = \text{the radius.}$$

The measures of the degrees of the meridian show, that this form of the radius must not be used in finding the second differentials of this radius,

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* (2227) It appears from [281^{iv}, 1745'''], that the earth cannot have a permanent axis
[3423a] of rotation, except about one of the principal axes; and then we shall have $Y^{(2)}$, as in [1763, 3423].

† (2228) The ellipticity of the earth, deduced from the observations on pendulums
[3424a] in [2048'], is $\frac{1}{335{,}78} = 0{,}002978$, supposing the earth to be an ellipsoid of revolution. The remarks relative to the smallness of h'''' in comparison with h, of $Y^{(3)} + Y^{(4)}$, &c., in comparison with $Y^{(2)}$, &c., correspond with [2056', &c.]. We may remark, that the numerical coefficient used by the author in [3424, 3426], namely, 0,002978, is too
[3424b] small; for by using the same observations, and correcting the errors of his calculations, we have found it to be 0,003178 [2054g]; other observations make it nearly equal to $\frac{1}{300}$ [2056z].

because it has been found, that the function $Y^{(3)}+Y^{(4)}+\&c.$, acquires a sensible value by a second differentiation [2056‴]. [3427]

The phenomena of the precession of the equinoxes, and the nutation of the axis of the earth, depend, as we have seen, upon $Y^{(2)}$.* *They do* [3428] *not determine the value of* αh, *but give the limits between which this value is comprised;* these limits are $\frac{1}{304}$ and $\frac{1}{578}$;† the preceding value [3424], [3428′] deduced from the observations on the force of gravity, falls within these limits. *The same phenomena indicate also a decrease in the densities of the* [3429] *strata of the terrestrial spheroid, from the centre to the surface,* without giving the precise law of the variation of density.‡ *This decrease of the density is also proved by the stability of the equilibrium of the sea* [2356″] *by the smallness of the action of the mountains upon a plumb-line, and, lastly, by the principles of hydrostatics,* which require that the densest parts should be nearest the centre, if the earth at its origin were in [3430] a fluid state.

Thus every phenomenon, depending on the figure of the earth, throws light upon the nature or magnitude of its radius; and we see that all these results agree with each other. *These observations are not, however,* [3430′] *sufficient to make known the interior constitution of the earth; but they indicate the most probable hypothesis of a density decreasing from the centre to the surface. Universal gravitation is therefore the true cause of all these phenomena; and if its effects are not so precisely verified in this case, as in the motions of the planets, it arises from the circumstance, that the inequalities of the attractive forces of the planets, depending on the small irregularities in their surfaces, and in their internal parts, disappear at great distances; so* [3431] *that we only perceive the simple phenomenon of the mutual attractions of these bodies towards their centres of gravity.*

Every phenomenon agrees with the principle of universal gravitation.

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* [2230] The values of A, B, C [2948—2950] depend on $Y^{(2)}$; and on these [3428a] quantities depend θ [3089], l [3098], and ψ [3100], &c.

† (2231) These limits correspond to [3414]; the first limit, $\frac{1}{304}$, must be changed [3429a] into $\frac{1}{279}$, as in [3414b].

‡ (2232) This follows from [3413, &c.], where it is shown, that if the gravity decrease from the centre to the surface, the ellipticity must fall between $\frac{1}{304}$ and $\frac{1}{578}$ [3413], or [3429a] rather between $\frac{1}{279}$ and $\frac{1}{578}$ [3414c].

Bouguer's hypothesis defective.

[3432] Bouguer's hypothesis, which we have examined in [1778, &c.], gives $\alpha h = 0,0054717$,* or $\frac{1}{183}$, which is too far from the limit $\frac{1}{304}$, to be admitted; so that the observations of the precession and nutation concur with the observations of the pendulum [1787″], in proving that we must reject this hypothesis.

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* (2233) Comparing $Y^{(2)}$, in Bouguer's hypothesis [1779], with that in [3423], we
[3432a] get $h'''=0$, $\alpha h = \alpha A = 0,0054717 = \frac{1}{183}$ [1785], as in [3432], which differs very much from the limit $\frac{1}{304}$ [3414], or $\frac{1}{279}$ [3414c].

CHAPTER II.

ON THE MOTION OF THE MOON ABOUT ITS CENTRE OF GRAVITY.

The moon's rotatory motion is equal to the motion of revolution.

15. *The moon, in revolving about the earth, keeps very nearly the same face towards us; which proves that the mean rotatory motion is exactly equal to the motion of revolution, and that the axis of rotation is nearly perpendicular to the plane of the ecliptic.* From observations on the motions of the spots on the moon's disk, Dominic Cassini made the remarkable discovery, that the *lunar equator is inclined about* $278'$ $[=2^d\,30^m]$ *to the plane of the ecliptic, and that the descending node of the lunar equator always coincides with the ascending node of the lunar orbit.* Tobias Mayer has since confirmed this [3433]
result by a great number of observations, which he made and discussed with all possible care, about the middle of the eighteenth century. *He, however, found the inclination to be less than Cassini had supposed it, making it only* $165'$ $[=1^d\,29^m]$.* He assures us, however, that the inclination has not [3434]
decreased since the time of that great astronomer; because he found, by the observations made in the time of Cassini, that it was the same then as at the time of his own observations, namely, $165'$. We shall now examine into the action of the earth and sun upon the lunar spheroid, in producing this result. [3434′]

The descending node of the lunar equator coincides with the ascending node of the lunar orbit.

16. We shall consider, in the first place, the action of the earth, and shall resume, for this purpose, the equations [3009—3011], which may evidently be applied to the moon, observing that L then represents the earth, $r_{\prime}$ its radius vector, drawn from the centre of the moon, supposing [3435]
this centre to be at rest; and X, Y, Z, the three co-ordinates of the earth, referred to a fixed ecliptic passing through the centre of the [3435′]

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* (2234) Nearly the same result has been obtained by later observations of Bouvard and Nicollet, who make the inclination $1^d\,28^m\,45^s$. [3432b]

[3435″] moon.* As the angle θ is very small, we shall neglect its square and its product by Z; we shall also neglect the product $\left(\frac{B-A}{C}\right).rq$, because
[3436] of the smallness of the factors $\left(\frac{B-A}{C}\right)$, r, and q; then the equations [3009—3011] become

[3437] $$dp=\frac{3L\,dt}{2r_{\prime}^{5}}.\left(\frac{B-A}{C}\right).\{(Y^2-X^2).\sin.2\varphi+2XY.\cos.2\varphi\};$$

General equations of the moon's motion.

[3438] $$dq+\left(\frac{C-B}{A}\right).rp.dt=\frac{3L\,dt}{r_{\prime}^{5}}.\left(\frac{C-B}{A}\right).\left\{\begin{array}{l}\{Y^2.\theta+YZ\}.\cos.\varphi\\-\{XY.\theta+XZ\}.\sin.\varphi\end{array}\right\};\quad(G')$$

[3439] $$dr+\left(\frac{A-C}{B}\right).pq.dt=\frac{3L\,dt}{r_{\prime}^{5}}.\left(\frac{A-C}{B}\right).\left\{\begin{array}{l}\{XY.\theta+XZ\}.\cos.\varphi\\+\{Y^2.\theta+YZ\}.\sin.\varphi\end{array}\right\}.$$

[3440] *If we put* v *for the apparent motion of the earth in longitude, seen from the*

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* (2235) The formulas [3009—3011] will give the moon's motion about its centre of
[3433a] gravity, noticing the action of the earth, by supposing A, B, C to represent the momenta of inertia of the moon about its three principal axes; putting also L for the mass of the earth, and X, Y, Z for its co-ordinates, referred to the centre of gravity of the moon
[3433b] as the origin; $r_{\prime}$ being the distance of the earth from that centre. The plane of XY is supposed to be drawn through the moon's centre of gravity, parallel to the fixed ecliptic. The axis of X is drawn, in this plane, from that centre towards the point of the heavens
[3433c] corresponding to the moveable descending node of the moon's equator [3001], or to the ascending node of the moon's orbit [3433]. The axis of Y is drawn from the same origin, in the same plane, in a direction towards a point of the heavens which is more
[3433d] advanced, according to the order of the signs, by 90^d, than that of the axis of X. The axis of Z is perpendicular to the plane XY, and directed towards the northern hemisphere. The first and second of the moon's principal axes are situated in the plane
[3433e] of the lunar equator; and the *third principal axis* is the axis of revolution. *The first principal axis, corresponding to the inertia* A [2907c, 2914], *forms, with the axis of* X, *the*
[3433f] *angle* φ [2907f]; *this principal axis is found, by observation, to be directed nearly towards the earth* [3440]; *so that* φ *is nearly equal to the angular distance* v [3440] *from the*
[3433g] *same axis* X; and $\varphi-v$ is very small. The quantity $\frac{Z}{r_{\prime}}$ is of the same order as the
[3433h] moon's latitude, which does not exceed $5^d\ 20^m$, and is generally much less; also θ is the inclination of the lunar equator to the fixed ecliptic [2907g], and $\theta_{\prime}$ its inclination to the variable ecliptic [3526]. These quantities being very small, we may neglect their squares
[3433i] and products, and put $\cos.\theta=1$, $\sin.\theta=\theta$; by these substitutions, the formulas [3009—3011] become as in [3437—3439], respectively, neglecting the small quantity depending on the factor qr, in the first of these equations, as in [3436].

moon, and counted from the descending node of the lunar equator; we shall [3440′]
have, by neglecting the square of the inclination of the lunar orbit to [3440″]
the ecliptic,*

$$X = r_{,} . \cos. v ; \qquad Y = r_{,} . \sin. v ; \qquad [3441]$$

hence [3437] becomes,†

$$d p = \frac{3 L d t}{2 r_{,}^3} . \left(\frac{B-A}{C}\right) . \sin. (2 v - 2 \varphi). \qquad [3442]$$

To integrate this equation, we shall observe, that if we put m *for the* m.
mean angular velocity of the earth about the moon, its mean motion will [3443]
be $\int m d t$, and we shall find,‡

$$v = \int m d t + \psi + H . \sin. \Pi + \&c. ; \qquad [3444]$$

$H . \sin. \Pi + \&c.$, *expressing the inequalities of* v, *arranged according to the*

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* (2236) If we put b for the angle formed by the axis of X and the line drawn from the moon's centre of gravity to the earth, we shall have, by the principles of orthographic [3441a] projection, $X = r_{,} . \cos. b$. Now if we put c for the latitude of the earth, as seen from the moon, and v the longitude [3440], we shall have, by spherics, as in [1345[27]], $\cos. b = \cos. v . \cos. c$. If we neglect c^2 [3440″], this becomes $\cos. b = \cos. v$; and by substitution, in the preceding value of X, we get $X = r_{,} . \cos. v$ [3441]. In like manner, by [3441b] changing the axis of X into that of Y, we have $Y = r_{,} . \cos. (90^d - v) = r_{,} . \sin. v$ [3441].

† (2237) The values X, Y [3441], give, by using [32, 31, 22], Int.,

$$Y^2 - X^2 = - r_{,}^2 . (\cos.^2 v - \sin.^2 v) = - r_{,}^2 . \cos. 2 v ; \quad 2 X Y = 2 r_{,}^2 . \sin. v . \cos. v = r_{,}^2 . \sin. 2 v. \qquad [3442a]$$

$$(Y^2 - X^2) . \sin. 2 \varphi + 2 X Y . \cos. 2 \varphi = r_{,}^2 . (-\cos. 2 v . \sin. 2 \varphi + \sin. 2 v . \cos. 2 \varphi)$$
$$= r_{,}^2 . \sin. (2 v - 2 \varphi).$$

Substituting this in [3437], we get [3442].

‡ (2238) The mean motion of the moon in longitude, in the time $d t$, is $m d t$, m being variable on account of the secular variations of the moon's motion; then in the time t, the angular motion is $\int m d t$; and if the equations of this motion be represented [3444a] by $H . \sin. \Pi + \&c.$ [5551], we shall have the angular motion of the moon, counted from the *fixed* equinox, equal to $\int m d t + H . \sin \Pi + \&c.$; and this represents also the angular motion of the earth, as seen from the moon. The retrograde motion of the nodes, in the [3444b] same time t, is represented by the angle ψ. Hence we have $\int m d t + \psi + H . \sin. \Pi + \&c.$, for the motion of the earth in the time t, counted, as in [3440′], from the *descending node of the lunar equator*, and this is assumed equal to v, in [3440]; hence we have, [3444c] $v = \int m d t + \psi + H . \sin. \Pi + \&c.$, as in [3444].

mean motion. Now putting

[3445] u.

$$u = \varphi - \psi - \int m\,dt,$$

we shall obtain,*

[3446]

$$2v - 2\varphi = -2u + 2H.\sin.\Pi + \&c.;$$

consequently,

[3447]

$$\sin.(2v - 2\varphi) = -\sin.2u + 2H.\cos.2u.\sin.\Pi + \&c.$$

If we neglect the square of θ, we shall have [3029],†

[3448]

$$p = \frac{d\varphi - d\psi}{dt};$$

hence,‡

[3449]

$$\frac{dp}{dt} = \frac{ddu}{dt^2} + \frac{dm}{dt};$$

therefore [3437] may be put under the following form,§

[3450]

$$\frac{ddu}{dt^2} + \frac{dm}{dt} = -\frac{3L}{2r_i^3}\left(\frac{B-A}{C}\right).\sin.2u + \frac{3L}{r_i^3}.\left(\frac{B-A}{C}\right).H.\cos.2u.\sin.\Pi + \&c.$$

It has been found, by observation, that the mean rotatory motion of the moon is equal to its mean motion of revolution about the earth; therefore

[3450'] u *is always very small* [3447d]; so that we may suppose $\sin.2u = 2u$,

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* (2239) The moon's first principal axis is found, by observation, to be directed nearly

[3447a] towards the earth [3433f]; therefore the angular distance of this axis from the *descending* node of the moon's equator, represented by φ [2907f], must be nearly equal to the earth's

[3447b] longitude, v [3440], seen from the moon; consequently $\varphi - v$ must be a small angle. If we substitute v [3444], in the first member of [3447c], and then u [3445], we get

[3447c]

$$v - \varphi = -(\varphi - \psi - \int m\,dt) + H.\sin.\Pi + \&c. = -u + H.\sin.\Pi + \&c.;$$

[3447d] consequently u must be small. Multiplying the preceding expression by 2, and then taking its sine, we get, as in [60], Int., the expression [3447].

[3448a] † (2240) Substituting $\cos.\theta = 1$ [3433i] in [3029], and dividing by dt, we get [3448].

‡ (2241) The differential of [3445] gives $d\varphi - d\psi = du + m\,dt$; substituting this

[3449a] in [3448], we get $p = \frac{du}{dt} + m$; its differential, divided by dt, is the same as in [3449].

§ (2242) Dividing [3442] by dt, then substituting the value of $\frac{dp}{dt}$ [3449], and

[3450a] that of $\sin.(2v - 2\varphi)$ [3447], we get [3450].

$\cos. 2u = 1$; moreover, we have very nearly,* $\frac{L}{r_{\prime}^3} = m^2$; therefore we shall obtain, [3451]

$$\frac{ddu}{dt^2} + 3m^2.\left(\frac{B-A}{C}\right).u = -\frac{dm}{dt} + 3m^2.\left(\frac{B-A}{C}\right).H.\sin.\Pi + \&c. \qquad [3452]$$

The value of $\frac{dm}{dt}$ depends on the secular equation of the moon, and we shall see, in the lunar theory, that if $m't$ be the mean sideral motion [3453] of the sun, and e' the excentricity of its orbit, we shall have,†

$$-\frac{dm}{dt} = \frac{3m'^2.e'de'}{m\,dt}; \qquad [3454]$$

therefore we shall have, very nearly, by integrating the preceding equation,

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* (2243) Rejecting terms of the order of the excentricity of the moon's orbit, we may put $r_{\prime}$ equal to the mean distance a of the moon from the earth, and then $\frac{L}{r_{\prime}^3}$ becomes $\frac{L}{a^3}$, which may be put equal to m^2, as is evident from [3060, 3435]. [3451a] Substituting this, and $\sin.2u = 2u$, $\cos.2u = 1$, in [3450], we obtain [3452], by transposing the second and third terms.

† (2244) If we retain, in the expression of the moon's mean motion [5095], the terms depending on the secular motion, we shall have $nt + \varepsilon = v + \frac{3}{2}m_{\prime}^2.f.(e'^2 - E'^2).dv$; [3452a] in which $m_{\prime}$ represents the ratio of the mean angular velocity of the sun in its apparent orbit, to that of the moon in her orbit [4835], this letter being accented to distinguish it from m of the present notation [3443]. [3452b] Taking the differential, and dividing by dt, we get $n = \frac{dv}{dt}.\{1 + \frac{3}{2}m_{\prime}^2.(e'^2 - E'^2)\}$. Dividing this by the coefficient of $\frac{dv}{dt}$, [3452c] neglecting the square and higher powers of $\frac{3}{2}m_{\prime}^2.(e'^2 - E'^2)$, on account of the smallness of the factors $m_{\prime}^2$, $(e'^2 - E'^2)$, we get $\frac{dv}{dt} = n - \frac{3}{2}n.m_{\prime}^2.(e'^2 - E'^2)$. The terms of [3452d] the second member of this expression, except the excentricity of the earth's orbit e', being considered as constant, and taking the differential, we obtain $d.\frac{dv}{dt} = -3n.m_{\prime}^2.e'de'$. Substituting, for $\frac{dv}{dt}$, its mean value m [3443], and dividing by dt, it becomes $\frac{dm}{dt} = -3n.m_{\prime}^2.\frac{e'de'}{dt}$. If we substitute, in the second member, for n, its value m [3452e] [3443, 3452a], also for $m_{\prime}$, its value $\frac{m'}{m}$ [3443, 3453], it becomes as in [3454].

[3455] and neglecting the quantity $\frac{m'^2 . d^2 . (e' d e')}{m^5 . d t^3 . \left(\frac{B-A}{C}\right)}$,*

General value of the libration in longitude. u.

[3456]
$$u = Q . \sin. \left\{ m t . \sqrt{3 . \left(\frac{B-A}{C}\right)} + F \right\} + \frac{m'^2 e' . \frac{d e'}{d t}}{m^3 . \left(\frac{B-A}{C}\right)}$$
$$- 3 m^2 . \left(\frac{B-A}{C}\right) . \frac{H . \sin. \Pi}{\left(\frac{d \Pi}{d t}\right)^2 - 3 m^2 . \left(\frac{B-A}{C}\right)} - \&c.;$$

[3456'] Q and F being two arbitrary constant quantities. We shall now examine the consequences which result from this integral.

[3457] We shall in the first place observe, that the term $\frac{m'^2 e' . \frac{d e'}{d t}}{m^3 . \left(\frac{B-A}{C}\right)}$ of this integral is insensible,† although divided by the small fraction $\frac{B-A}{C}$, on

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[3455a] * (2245) Substituting [3454] in [3452], and putting, for a moment, $a^2 = 3 m^2 . \left(\frac{B-A}{C}\right)$, we get

[3455b]
$$0 = \frac{d d u}{d t^2} + a^2 . u - \frac{3 m'^2 . e' d e'}{m d t} - a^2 . H . \sin. \Pi - \&c.$$

[3455c] Substituting $u = y + \frac{3 m'^2 . e' d e'}{m a^2 d t}$, and neglecting the term depending on the second differential of $e' d e'$, as in [3455, 3457f], it becomes

[3455d]
$$0 = \frac{d d y}{d t^2} + a^2 y - a^2 H . \sin. \Pi - \&c.$$

This is of the same form as [865a, 870'], putting $a K = - a^2 H$, $m t + \varepsilon = \Pi$, or $m = \left(\frac{d \Pi}{d t}\right)$; and the corresponding value of y [865b, 871], is

[3455e]
$$y = b . \sin. (a t + \varphi) - \frac{a^2 H . \sin. \Pi}{\left(\frac{d \Pi}{d t}\right)^2 - a^2} .$$

Substituting this in u [3455c], it becomes as in [3456], the constant quantities b, φ, being changed into Q, F, respectively, and a^2 into its value [3455a].

† (2246) To make a rough estimate of the value of this neglected term, we may use the value of $\frac{B-A}{C}$ [3578]; substituting in it the ratio of the mass of the earth to that of the

[3457a] moon $\lambda' = 75$ [3566,3380e']; hence we get $\frac{B-A}{C} = 0{,}000027$. The mean angular

account of the excessive slowness of the variation of the excentricity e'; therefore we may neglect this term. All the other terms of this value of u, [3458]
vary with much greater rapidity; but *this expression always remains very small, provided the coefficient* Q *be small.* Now the equation [3449],

$$\frac{dp}{dt} = \frac{ddu}{dt^2} + \frac{dm}{dt}, \qquad [3459]$$

gives

$$\int p\,dt = u + \int m\,dt;^* \qquad [3460]$$

and since $\int p\,dt$ *is the rotatory motion about the third principal axis* [3121], [3460′]
it follows, that the mean motions of rotation and revolution of the moon are perfectly equal to each other; so that the action of the earth, upon the lunar spheroid, makes the rotatory motion participate in the secular equations of the motion in the orbit. It is not necessary for this perfect equality, that [3461]
at the origin of the two motions they should be exactly equal, the probability of which is infinitely small. It is only necessary to suppose, at this origin,

It is not necessary that the rotatory motion, and the motion of revolution, should be equal at the origin of the motion.

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motion of the earth in its orbit, to that of the moon in its orbit, is represented by

$$\frac{m'}{m} = 0{,}0748\ [5117]; \quad \text{also} \quad e' = 0{,}0168\ [5117]. \qquad [3457b]$$

Hence the term mentioned in [3457], becomes nearly equal to

$$(0{,}0748)^3 \cdot \frac{0{,}0168}{0{,}000027} \cdot \frac{de'}{m'dt} = \tfrac{1}{4} \times \frac{de'}{m'dt}. \qquad [3457c]$$

In this we may substitute for $\frac{de'}{m'dt}$, the ratio of the annual variation of de', to the annual motion of the sun, which, by [4244], is expressed by $\frac{0'',187638}{2}$ to 360^d, or [3457d]
1 to 14000000; hence [3457c] becomes nearly $\frac{1}{4\times 14000000}$ or $\frac{1}{56000000}$. Multiplying [3457e]
this by the radius in seconds [1970h], it becomes a small fraction of a second, which is wholly insensible. The term neglected in [3455] must be much smaller than this, because [3457f]
the second differential of $e'de'$, introduces, as a factor, the square of the very small coefficient of t, which occurs in the expression of $e'de'$.

* (2247) Multiplying [3449a] by dt, and integrating, we get [3460]. Now $\int p\,dt$ represents the rotatory motion of the moon [3121, 3433e], $\int m\,dt$ the motion in the orbit [3460a]
[3443]; both of which increase indefinitely with the time, whilst u remains always small [3450′], depending on periodical equations only; therefore the mean values $\int m\,dt$, $\int p\,dt$, must be equal.

[3462] *where* $t=0$, *that the rotatory velocity* p *of the moon should be comprised between the limits,**

[3463] $$m+mQ\,.\sqrt{\frac{3\,.(B-A)}{C}}+\&c., \quad \text{and} \quad m-mQ\,.\sqrt{\frac{3\,.(B-A)}{C}}-\&c.;$$

which are arbitrary, because of the arbitrary quantity Q. These limits are

[3464] indeed very narrow, on account of the smallness of Q† and $\sqrt{\frac{3\,.(B-A)}{C}}$; but they enable us to avoid the improbable supposition, that the motions were so arranged at the origin, that forever afterwards the mean rotatory

[3464′] motion of the moon should be equal to the mean motion of revolution.

[3464″] *The value of* u *expresses the real libration of the moon in longitude, being the excess of the real rotatory motion above the mean motion in the orbit.* This expression contains, in the first place, the quantity

[3465] $Q\,.\,\text{sin.}\left\{mt\,.\sqrt{\frac{3\,.(B-A)}{C}}+F\right\}$; the limit Q being arbitrary; but as this term has not been perceived by observation, it must be quite small.

[3465′] Hence it follows, that $\sqrt{\frac{3\,.(B-A)}{C}}$ is a real number. For if it be imaginary, the preceding argument will change into an exponential quantity,

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* (2248) Substituting, in [3460], the value of u [3456], neglecting the term [3457], and using a^2 [3455a], we get

[3463a] $$\textstyle\int p\,dt=\int m\,dt+Q\,.\,\text{sin.}\,(m\,a\,t+F)-\frac{3\,H\,a^2.\,\text{sin.}\,\Pi}{\left(\frac{d\,\Pi}{d\,t}\right)^2-a^2}-\&c.$$

The differential of this, divided by dt, is $p=m+m\,a\,Q\,.\,\text{cos.}\,(m\,a\,t+F)-\&c.$; and as the limits of $\text{cos.}\,(m\,a\,t+F)$, are 1 and -1, the limits of p will be

[3463b] $m\pm m\,a\,Q\pm\&c.$; the greatest value being found by using the values $\pm$, which render all the terms positive; and the least, by using the opposite signs; therefore the extreme

[3463c] limits are as in [3463, 3463b]; and if p falls between these limits, at the origin of the motion, the equation [3463a] may always be satisfied.

† (2249) The quantity Q must be small, because the angle u [3456], which depends

[3464a] upon it, has not been discovered by observation; and $\frac{B-A}{C}$ is of the order $0{,}000027$ [3457a]; therefore the quantity $m\,Q\,.\sqrt{\frac{3.(B-A)}{C}}$ [3463], must be extremely small.

or an arc of a circle,* increasing indefinitely with the time, and augmenting the value of u indefinitely; which is contrary to observation. It is true, that if $B-A$ be negative, and Q nothing, there will not be, in u, [3466]
either arcs of a circle or exponential quantities;† but the slightest cause will produce them, and render the equilibrium unstable, which cannot be admitted. Therefore $B-A$ is a positive quantity; that is to say, the moon's momentum of inertia A, is less than that of B. The first of these [3467]
quantities, A, corresponds to the principal axis of the equator, directed towards the earth; because it relates to the first principal axis, which forms the angle φ with the line of the lunar equinoxes [3433f], whilst the [3467']
radius drawn from the centre of the moon to that of the earth, makes the angle v with the same equinox; and by what has been said, $\varphi-v$ *is a small angle* [3433g]. *Therefore the first principal axis of the lunar spheroid* [3468]
is always directed very nearly towards the earth. The radius of the lunar equator, in that direction, is lengthened by means of the attraction of the earth;‡ [3468']
consequently the momentum of inertia A, is less than the momentum of inertia B, corresponding to the second principal axis, situated in the equator. [3469]

The momentum of inertia A, corresponding to the first principal axis, directed towards the earth, is less than the momentum B, relative to the second axis, situated in the plane of the equator.

The duration of the period of the preceding argument [3465], is equal to a sideral month, divided by the coefficient $\sqrt{\frac{3.(B-A)}{C}}$; and as [3469']
the value of this coefficient is not accurately known, it is impossible to ascertain this time correctly. We shall hereafter show, that if the moon be homogeneous, this duration will not exceed seven years;§ and in the [3470]

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* (2250) As is explained in a similar case in note 179, Vol. I, page 187. [3465a]

† (2251) Because the whole expression [3465] would then vanish, on account of $Q=0$. [3466a]

‡ (2252) The attraction of the earth upon the moon, supposing it to be in a fluid state [3467a]
evidently lengthens the moon's first principal axis, directed towards the earth, and shortens, the second principal axis, situated in the plane of the equator; in the same manner as the [3467b]
attractions of the sun and moon upon the waters of the ocean, elevate the tides at high-water, and depress them at low-water. Hence it is evident, that the momentum of inertia A [2914], corresponding to the longer axis, must be less than the momentum B, [3467c]
corresponding to the second principal axis.

§ (2253) While the moon, by the mean motion, describes very nearly the arc mt, in its orbit [3443], the argument of the term u [3456], depending on Q, increases by [3469a]

case of nature, the difference of the momenta of inertia of the moon,
[3470′] relative to its three principal axes, is probably greater than in the case of homogeneity [3588, &c.]. This consideration makes it apparent, that
[3471] the term $\frac{m'^2 . e' d e'}{m^3 . \left(\frac{B-A}{C}\right) . dt}$, which we have neglected [3457], must be insensible.*

[3471′] *Among the terms of the expression of* u [3456], *those only are sensible which depend either on the equation of the moon's centre, which is great in itself; or on those which become great by having very small divisors,*
[3472] *corresponding to small values of* $\frac{d\Pi}{dt}$. The function $H . \sin. \Pi + \&c.$, is the sum of the periodical equations in the apparent motion of the moon [3444]; and if we suppose that $H . \sin. \Pi$, expresses the equation of
[3473] the centre, we shall have,† $H = 70005''$ $[6^d\, 18^m\, 2^s]$; and as Π *represents,*
[3474] *in this case, the mean anomaly of the moon,* we get,‡ $\left(\frac{d\Pi}{dt}\right)^2 = m^2 . 0{,}98317$;

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the quantity $mt . \sqrt{\left(\frac{3.(B-A)}{C}\right)}$; and as these quantities are to each other as 1 to $\sqrt{\left(\frac{3.(B-A)}{C}\right)}$, it follows, that the duration of the period of this term of u is equal to the time of a sideral revolution of the moon, $27^{days}{,}321$, divided by $\sqrt{\left(\frac{3.(B-A)}{C}\right)}$.
If we use the value of this divisor, deduced from [3457a], namely, 0,009 nearly, this period
[3469b] will exceed 3000 days, or about 8 years. If we use the value of the divisor 0,04, corresponding nearly to the observations of Mr. Nicollet [3483d], this period will be reduced
[3469c] to less than two years.

* (2254) We have seen, that the value of $\frac{B-A}{C}$, assumed in [3457a—f], renders
[3471a] the term in question insensible; and it is evident, that an increase of the value of $\frac{B-A}{C}$ [3471], decreases its magnitude.

† (2255) This differs a few seconds from the calculation of La Place [5220], and
[3473a] from the tables of Mason or Burg [5554]; but this does not sensibly affect the result [3478].

‡ (2256) The moon's mean motion being represented by mt [3443], that of the mean anomaly will be cmt [4817], using $c = 0{,}99154801$ [5117]; hence

[3474a] $$\left(\frac{d\Pi}{dt}\right)^2 = (cm)^2 = m^2 . 0{,}98317.$$

therefore we shall have, in the expression of u, the term

Term of the libration depending on the equation of the centre of the moon's orbit.

$$\frac{-3.\left(\frac{B-A}{C}\right).70005''.\sin.\Pi}{0,98317-3.\left(\frac{B-A}{C}\right)}\qquad\left[=-\frac{3.\left(\frac{B-A}{C}\right).22682^s.\sin.\Pi}{0,98317-3.\left(\frac{B-A}{C}\right)}\right].$$ [3475]

If the coefficient of $\sin.\Pi$ [3475], *be equal to* i *seconds, we shall have* [3476]

$$\left(\frac{B-A}{C}\right)=\frac{i.0,32772}{i-70005''}\qquad\left[=\frac{i.0,32772}{i-22682^s}\right].$$ [3477]

Limit of $B-A$, derived from the equation of the centre.

Since the term in question [3475] *has not been perceived by observation,* the quantity i cannot exceed $\pm 6000''$ [$=\pm 1944^s$], and $\left(\frac{B-A}{C}\right)$ *must be less than** 0,030721. [3478]

Among the terms of the expression of u [3456], *which have very small divisors, we find only the annual equation which can produce a sensible term in the value of* u. This equation is equal to $2064''.\sin.\Pi$† [$=669^s.\sin.\Pi$], [3479]

Substituting this, and H [3473], in the term of u [3456], depending on H, rejecting the factor m^2, which occurs in all the terms of the numerator and denominator, it becomes as in [3475]. Putting the coefficient of $\sin.\Pi$, in this expression, equal to i seconds, and multiplying by $\frac{0,98317}{3}-\left(\frac{B-A}{C}\right)$, we get

$$-\left(\frac{B-A}{C}\right).22682^s=i.0,32772-i.\left(\frac{B-A}{C}\right);$$ [3474b]

whence we easily get [3477].

* (2257) Substituting $i=\pm 1944^s$, in [3477], we get $\left(\frac{B-A}{C}\right)=\frac{\pm 1944^s\times 0,32772}{\pm 1944^s-22682^s}$. [3478a]

The lower sign gives $\frac{B-A}{C}=0,025870$, the upper sign $\frac{B-A}{C}=-0,030721$; so that it can have no positive value greater than 0,025870, nor a negative value exceeding 0,030721, independent of its sign. [3478b]

† (2258) If we examine the equations of the moon's mean longitude [5574—5579], observing that c and g are nearly equal to unity [5117], we shall find that the chief term depending on an angle of the form mv, in which m is small, is the term depending on $\sin.(c'mv-\varpi')$ [5575], whose coefficient is $2075'',71=672^s,53$; and in Mason's tables [5551], is $2063'',58=668^s,60$; being nearly the same as in [3479], putting Π equal to the sun's mean anomaly. Now as the motion of the apsides of the earth's orbit is very small, we may put $\frac{d\Pi}{dt}$ = the sun's mean angular motion $=0,0748.m$ [3457b]; [3479a] [3479b] [3479c]

Π being in this case the sun's mean anomaly; moreover, we have

[3480] $\frac{d\,\Pi}{d\,t} = m\,.\,0{,}0748$, consequently $\left(\frac{d\,\Pi}{d\,t}\right)^2 = m^2\,.\,0{,}005595$; therefore we shall have, in the expression of u, the following term,

Term of the libration in longitude depending on the annual equation of the moon's orbit.

[3481] $$-\frac{3\,.\left(\frac{B-A}{C}\right).\,2064''.\,\sin.\,\Pi}{0{,}005595-3\,.\left(\frac{B-A}{C}\right)} \qquad \left[= -\frac{3\,.\left(\frac{B-A}{C}\right).\,669^s.\,\sin.\,\Pi}{0{,}005595-3\,.\left(\frac{B-A}{C}\right)}\right].$$

[3481′] *If the coefficient of* $\sin.\,\Pi$, *in this expression, is equal to* i *seconds, we shall have*,*

[3482] $$\left(\frac{B-A}{C}\right) = \frac{i\,.\,0{,}001865}{i-2064''} \qquad \left[= \frac{i\,.\,0{,}001865}{i-669^s}\right].$$

[3479d] hence $\left(\frac{d\,\Pi}{d\,t}\right)^2 = 0{,}005595\,.\,m^2$ [3480]. Substituting this, and $H = 669^s$ [3479], in the term of u [3456], depending on H, rejecting m^2, which occurs in all the terms of the numerator and denominator, we get [3481].

* (2259) Putting the coefficient of $\sin.\,\Pi$ [3481] equal to i seconds, and multiplying

[3482a] by $\frac{0{,}005595}{3} - \left(\frac{B-A}{C}\right)$, we get $-\left(\frac{B-A}{C}\right).\,669^s = i\,.\,0{,}001865 - \left(\frac{B-A}{C}\right).\,i$, which is easily reduced to the form [3482]. If we put, successively, $i=0$, $i=-1944^s$,

[3482b] we obtain the two limits 0 and 0,0013876 [3483]. In like manner, the values

[3483a] corresponding to $i=0$, $i=669^s$, $i=1944^s$, are 0, $\mp\infty$, $+0{,}002843$, respectively, as in [3484]. Now $\frac{B-A}{C}$ being positive [3467], we need use only the positive values

[3483b] of these last limits, as in [3484]. We have altered the sign of i, also the signs of several terms in this chapter, to correct the typographical mistakes, some of which were noticed

[3483c] by the author in Chap. II, Book XIV; where he has treated of the libration in a somewhat different manner from that used in this place. He also gives the result of the calculation of Mr. Nicollet on the coefficient of the libration in longitude, depending on the annual equation, which he makes about $-289^s.\,\sin.$ (☉'s mean anomaly), corresponding to $i=-289^s$,

[3483d] in [3482]; consequently this function gives $\frac{B-A}{C} = 0{,}000563$ nearly, as in [12326]; i being negative, as the author supposes in [3485]. Substituting this in [3475], we get the value

[3483e] of this term of the libration of the moon in longitude equal to $-39^s.$(☽'s mean anomaly); so that by noticing only these two chief terms of the libration in longitude, we shall have

[3483f] $u = -289^s.\,\sin.$ (☉'s mean anomaly) $-\,39^s.\,\sin.$ (☽'s mean anomaly). These arcs are supposed to be viewed from the moon's centre; and when viewed from the earth, they fall short of two seconds; but being so very small, it is difficult to ascertain the terms of the

[3483g] libration with a very great degree of accuracy.

The term of u *must be very small, since it has not been perceived by* [3482′]
observation; therefore we shall suppose that i does not exceed $\pm 6000''$ Limit of $B-A$, derived from the annual equation.
$[=\pm 1944^s]$. *If* i *be negative, the two limits of* $\frac{B-A}{C}$ *will be* [3483]
0 *and* 0,0013876; *if* i *be positive, these limits will be* 0,0028430 *and* ∞. [3484]
We have just seen that $\frac{B-A}{C}$ *cannot exceed* 0,030721 [3478]; *and it is*
very probable that it is less than 0,002843, *in which case* i *will be* [3485]
negative [3483*d*].

17. *We shall now consider the equations* [3438, 3439]. The inclination θ [3485′]
of the lunar equator to the fixed ecliptic being supposed very small, we
shall transform the variable quantities q, r, into others that render the [3486]
integrations easier, as we have already done in a similar case, in [284, &c.].
For this purpose, we shall put

$$s=\theta \,.\, \sin.\, \varphi\,; \qquad s'=\theta \,.\, \cos.\, \varphi. \tag{3487}$$

From these we get,*

$$\frac{ds}{dt}=\frac{d\theta}{dt}\,.\,\sin.\,\varphi+\theta\,.\,\frac{d\varphi}{dt}\,.\,\cos.\,\varphi\,; \tag{3488}$$

$$\frac{ds'}{dt}=\frac{d\theta}{dt}\,.\,\cos.\,\varphi-\theta\,.\,\frac{d\varphi}{dt}\,.\,\sin.\,\varphi. \tag{3489}$$

If we neglect the square of θ, we shall obtain, from § 4,†

$$\frac{d\theta}{dt}=r\,.\,\sin.\,\varphi-q\,.\,\cos.\,\varphi\,; \tag{3490}$$

$$\theta\,.\,\frac{d\varphi}{dt}=p\,.\,\theta+r\,.\,\cos.\,\varphi+q\,.\,\sin.\,\varphi\,; \tag{3491}$$

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* (2260) The differentials of [3487] being divided by dt, give [3488, 3489]. [3488*a*]

† (2261) Dividing [3032] by dt, we obtain [3490]. If we multiply [3029] by θ,
we get, by neglecting the square of θ, $\theta\,.\,\frac{d\varphi}{dt}=p\,.\,\theta+\theta\,.\,\frac{d\psi}{dt}$. Dividing [3035] by dt, [3489*a*]
and putting θ for $\sin.\,\theta$, we find $\theta\,.\,\frac{d\psi}{dt}=r\,.\,\cos.\,\varphi+q\,.\,\sin.\,\varphi$; substituting this in the
preceding equation, it becomes as in [3491].

therefore we shall have,*

[3492] $$\frac{ds}{dt}=ps'+r; \qquad \frac{ds'}{dt}=-ps-q;$$

hence we deduce,†

[3493] $$\frac{dds}{dt^2}-p\cdot\frac{ds'}{dt}-s'\cdot\frac{dp}{dt}=\frac{dr}{dt};$$

[3494] $$\frac{dds'}{dt^2}+p\cdot\frac{ds}{dt}+s\cdot\frac{dp}{dt}=-\frac{dq}{dt}.$$

Substituting these values of $\frac{dr}{dt}$, $-\frac{dq}{dt}$, in [3438, 3439],‡ observing
[3495] that we may suppose $p=m$, in the products of p and its differential,

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* (2262) Substituting the values of $\frac{d\theta}{dt}$, $\theta\cdot\frac{d\varphi}{dt}$ [3490, 3491], in [3488, 3489], they become, by reduction, observing that the coefficient of q vanishes in the first of these equations, and the coefficient of r in the second,

[3492a] $$\frac{ds}{dt}=r\cdot(\sin.^2\varphi+\cos.^2\varphi)+p\theta\cdot\cos.\varphi; \qquad \frac{ds'}{dt}=-q\cdot(\cos.^2\varphi+\sin.^2\varphi)-p\theta\cdot\sin.\varphi;$$

putting $\cos.^2\varphi+\sin.^2\varphi=1$, and using [3487], we get [3492].

[3493a] † (2263) Transposing the terms ps', $-ps$, [3492], then taking the differentials, and dividing by dt, we obtain [3493, 3494].

‡ (2264) Substituting the values of $-\frac{dq}{dt}$, $\frac{dr}{dt}$ [3494, 3493], in [3438, 3439], we get,

[3495a] $$\frac{dds'}{dt^2}+p\cdot\frac{ds}{dt}+s\cdot\frac{dp}{dt}+\left(\frac{B-C}{A}\right)\cdot rp=\frac{3L}{r_{\prime}^5}\cdot\left(\frac{B-C}{A}\right)\cdot\{(Y^2.\theta+YZ).\cos.\varphi-(XY.\theta+XZ).\sin.\varphi\};$$

[3495b] $$\frac{dds}{dt^2}-p\cdot\frac{ds'}{dt}-s'\cdot\frac{dp}{dt}+\left(\frac{A-C}{B}\right)\cdot pq=\frac{3L}{r_{\prime}^5}\cdot\left(\frac{A-C}{B}\right)\cdot\{(XY.\theta+XZ).\cos.\varphi+(Y^2.\theta+YZ).\sin.\varphi\}.$$

[3495c] If we neglect the term $s\cdot\frac{dp}{dt}$, on account of its smallness, and put $p=m$, they become as in [3496, 3497]; observing that this neglected term is of the order s in
[3495d] comparison with the term depending on X, Y, Z [3496, 3497], as is evident from [3437].
[3495e] If we wish to examine more minutely these neglected terms, we may do it after computing the approximate values of s, s' [3530, 3531], and using them and dp [3459, 3454]. We may remark, that the terms between the braces, in the second members of [3495a, b], may be put under the forms $(Y\theta+Z)\cdot(Y\cdot\cos.\varphi-X\cdot\sin.\varphi)$, and
[3495f] $(Y\theta+Z)\cdot(X\cdot\cos.\varphi+Y\cdot\sin.\varphi)$, respectively, as is evident by reduction.

by the very small variable quantities s, s', or their differentials, we shall have, [3495']

Differential equations in s, s'. First form.

$$\frac{dds'}{dt^2}+m.\frac{ds}{dt}=\left(\frac{C-B}{A}\right).mr+\frac{3L}{r_{,}^5}.\left(\frac{B-C}{A}\right).\{(Y^2.\theta+YZ).\cos.\varphi-(XY.\theta+XZ).\sin.\varphi\}; \qquad [3496]$$

$$\frac{dds}{dt^2}-m.\frac{ds'}{dt}=\left(\frac{C-A}{B}\right).mq+\frac{3L}{r_{,}^5}.\left(\frac{A-C}{B}\right).\{(XY.\theta+XZ).\cos.\varphi+(Y^2.\theta+YZ).\sin.\varphi\}. \qquad [3497]$$

Now in [3492] we have,

$$r=\frac{ds}{dt}-ms'; \qquad q=-\frac{ds'}{dt}-ms; \qquad [3498]$$

and in [3441],

$$X=r_{,}.\cos.v; \qquad Y=r_{,}.\sin.v; \qquad [3499]$$

moreover, $v-\varphi$ is always a very small angle, as we have seen in [3447b]; therefore we may neglect its product by the quantities θ and Z; then the preceding differential equations become, by the substitution of m^2 for $\frac{L}{r_{,}^3}$ [3451],* [3499'] [3500]

Differential equations in s, s'. Second form.

$$\frac{dds'}{dt^2}+\left(\frac{A+B-C}{A}\right).m.\frac{ds}{dt}-m^2.\left(\frac{B-C}{A}\right).s'=0; \qquad [3501]$$

$$\frac{dds}{dt^2}-\left(\frac{A+B-C}{B}\right).m.\frac{ds'}{dt}-4m^2.\left(\frac{A-C}{B}\right).s=3m^2.\left(\frac{A-C}{B}\right).\frac{Z}{r_{,}}. \qquad [3502]$$

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* (2265) Substituting the values of X, Y [3499], in the first members of [3499a, b], and reducing by [22, 24], Int., we get the second members of these expressions,

$$Y.\cos.\varphi-X.\sin.\varphi=r_{,}.(\sin.v.\cos.\varphi-\cos.v.\sin.\varphi)=r_{,}.\sin.(v-\varphi); \qquad [3499a]$$

$$X.\cos.\varphi+Y.\sin.\varphi=r_{,}.(\cos.v.\cos.\varphi+\sin.v.\sin.\varphi)=r_{,}.\cos.(v-\varphi). \qquad [3499b]$$

Hence the factors [3495f] become

$$(Y\theta+Z).r_{,}.\sin.(v-\varphi) \qquad \text{and} \qquad (Y\theta+Z).r_{,}.\cos.(v-\varphi), \qquad [3499c]$$

which are to be substituted in the second members of [3495a, b], respectively. In making these substitutions, the author wholly neglects $\sin.(v-\varphi)$, on account of its smallness [3447b]; but Mr. Poisson has discovered, that the terms depending on $v-\varphi$, produce [3499d]
in s, s', small terms of the second order, relatively to the excentricity and inclination, and depending on the difference of longitude of the moon's perigee and node. These are computed by the author in Book XIV [12312, &c.], to which we shall refer for a more minute discussion of this subject; it being unnecessary to repeat the calculation in this place, [3499d']
since the general results are not affected by it. In following the method of the author, we

[3503] $\frac{Z}{r_{,}}$ is the latitude of the earth seen from the moon, above the fixed plane [3499g], which latitude is equal to that of the moon seen from the earth, but is of a contrary sign; therefore we shall have, by § 5,*

[3504] $$\frac{Z}{r_{,}} = c' \,.\, \sin.\,(mt + g't + \beta') + \Sigma \,.\, c \,.\, \sin.\,(mt - gt - \beta);$$

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may neglect the first of the factors [3499c], on account of the smallness of $v - \varphi$ [3447b], and $Y\theta + Z$. The second of these factors, neglecting the square of $v - \varphi$, is
[3499e] $(Y\theta + Z) \,.\, r_{,}$, which, by substituting $Y = r_{,} \,.\, \sin.\, v = r_{,} \,.\, \sin.\, \varphi$ nearly, becomes $(r_{,}\theta \,.\, \sin.\,\varphi + Z) \,.\, r_{,} = (r_{,}s + Z) \,.\, r_{,}$ [3487]. Hence the second member of [3495a], vanishes, as in [3501], and the second member of [3495b] becomes

[3499f] $$\frac{3L}{r_{,}^{5}} \,.\, \left(\frac{A-C}{B}\right) \,.\, (r_{,}s + Z) \,.\, r_{,} = \frac{3L}{r_{,}^{3}} \,.\, \left(\frac{A-C}{B}\right) \,.\, \left(s + \frac{Z}{r_{,}}\right) = 3m^2 \,.\, \left(\frac{A-C}{B}\right) \,.\, \left(s + \frac{Z}{r_{,}}\right).$$

Substituting these values, and those of r, q [3498], also $p = m$, in [3495a, b], neglecting dp [3495c], we get [3501, 3502]. The co-ordinate Z, is the perpendicular elevation
[3499g] of the earth above the fixed plane; $r_{,}$ is the distance of the moon and earth; therefore $\frac{Z}{r_{,}}$ must represent nearly the latitude of the earth, as seen from the moon [3503].

* (2266) We shall suppose, in fig. 61, page 846, that M is the place of the moon, at
[3503a] the time t, from the epoch; $MM'm$ the circle of latitude, drawn perpendicularly to Cm, and on account of the smallness of $M'm$, it may be considered as perpendicular to CM'.
[3503b] Then $Mm = MM' + M'm$ represents the latitude of the moon, seen from the earth, above the fixed plane; and by changing its sign, we get the latitude of the earth seen from
[3503c] the moon, $-MM' - M'm = \frac{Z}{r_{,}}$ [3503]. Adding the precession ψ to mt [3505],
[3503c′] we obtain $mt + \psi$, the mean longitude of the earth seen from the moon, counted from the *moveable equinox*; hence the moon's longitude, seen from the earth, and counted from
[3503d] the same equinox, is $mt + \psi - 180^d = \text{arc } Fm = \text{arc } F'M'$ nearly. Subtracting the arc
[3503e] $\Lambda = FC$ [3087d], we get $mt + \psi - \Lambda - 180^d = \text{arc } Cm = \text{arc } CM'$ nearly. Then in the triangle CmM', we have, by using γ [3087d], very nearly $\text{arc } M'm = \gamma \,.\, \sin.\, CM'$; substituting the preceding value of CM', we get, by successive reductions, using [22], Int., [3075e] and $\psi = (f - g) \,.\, t$ [3073c];

[3503f] $$\text{arc } M'm = \gamma \,.\, \sin.\,(mt + \psi - \Lambda - 180^d) = -\gamma \,.\, \sin.\,(mt + \psi - \Lambda)$$
$$= -\gamma \,.\, \cos.\,\Lambda \,.\, \sin.\,(mt + \psi) + \gamma \,.\, \sin.\,\Lambda \,.\, \cos.\,(mt + \psi)$$
$$= -\Sigma \,.\, c \,.\, \{\sin.\,(mt + \psi) \,.\, \cos.\,(ft + \beta) - \cos.\,(mt + \psi) \,.\, \sin.\,(ft + \beta)\}$$
[3503g] $$= -\Sigma \,.\, c \,.\, \sin.\,(mt + \psi - ft - \beta) = -\Sigma \,.\, c \,.\, \sin.\,(mt - gt - \beta).$$

[3503h] The mean longitude of the moon seen from the earth, counted from the fixed equinox, is $mt - 180^d$ [3505], and the longitude of the ascending node, upon the variable ecliptic,

in which mt *is the mean longitude of the earth, seen from the moon, relative* [3505] mt.
to a fixed equinox [3443], *and* $-g't-\beta'$ *the longitude of the ascending* [3505′]
node of the lunar orbit upon the moveable ecliptic, referred to the same $g't+\beta'$.
equinox. The functions $\Sigma.c.\sin.(gt+\beta)$, $\Sigma.c.\cos.(gt+\beta)$, *depend on the* [3506]
change in the plane of the moveable ecliptic [3072′—3074], *and the coefficient*
g *is extremely small in comparison with* m *and* g'.* We shall now put† [3507]

$$s=Q.\sin.(mt+g't+\beta');$$ [3508]

Assumed forms of

$$s'=Q'.\cos.(mt+g't+\beta');$$ [3509]

s, s'.

$-gt-\beta'$ [3505′]; subtracting this from the preceding expression, we get the moon's distance from this node, represented by the arc $D'm$, or DM'; hence

$$DM'=mt+g't+\beta'-180^d.$$ [3503*i*]

Now in the triangle $DM'M$, we have very nearly, by using c' [3086],

$$\text{arc } MM'=\text{tang.}\,MDM'.\sin.DM'=c'.\sin.(mt+g't+\beta'-180^d)$$
$$=-c'.\sin.(mt+g't+\beta').$$ [3503*k*]

Substituting this and [3503*g*], in [3503*c*], we get [3504].

* (2267) The quantity g is of the same order as the secular motion of the sun's apparent orbit [4339, 4244], which is only a few seconds in a year; and is therefore much smaller than the terms depending on the moon's mean motion mt, or on the motion of the node of the moon's orbit $g't$. [3507*a*]

† (2268) If we take the differential of [3502], and eliminate dds' by means of [3501], we shall get an equation containing d^3s, ds, s'; again, taking the differential, and eliminating ds' by means of [3502], we get an equation free from s', containing d^4s, d^2s, s, in a linear form, together with terms depending on the angles $mt+g't+\beta'$, $mt-gt-\beta$. [3508*a*] The integral of this equation will give s, in terms depending on the same angles, together with an arbitrary term, depending on an angle $lt+I$, similar to $at+\varphi$ [865*b*], and which holds good when Z is equal to nothing. [3508*b*] Substituting this value of s in [3502], we shall find that ds' and s' are composed of similar terms, depending on the same angles. The equations being linear, we may consider each of these angles separately, and if we substitute, in [3501, 3502], the assumed values [3508, 3509], and the corresponding term of $\frac{Z}{r_{\prime}}$ [3504], we shall get, by neglecting the factor $\sin.(mt+g't+\beta')$, or $\cos.(mt+g't+\beta')$, which is common to all the terms;

$$-(m+g')^2.Q'+\left(\frac{A+B-C}{A}\right).m.(m+g').Q-m^2.\left(\frac{B-C}{A}\right).Q'=0;$$ [3508*c*]

$$-(m+g')^2.Q+\left(\frac{A+B-C}{B}\right).m.(m+g').Q'-4m^2.\left(\frac{A-C}{B}\right).Q=3m^2.\left(\frac{A-C}{B}\right).c'.$$ [3508*d*]

The first of these equations gives Q' [3510].

for the parts of s and s', corresponding to the term $c'.\sin.(mt+g't+\beta')$, of the expression of $\frac{Z}{r_{,}}$, and we shall have,

[3510] Value of Q'.
$$Q'=\frac{m.(m+g').(A+B-C).Q}{(m+g')^2.A+m^2.(B-C)}.$$

If we suppose,

[3511] E.
$$E=m^2.(m+g')^2.\{(A+B-C)^2-4A.(A-C)-B.(B-C)\}$$
$$-(m+g')^4.AB-4m^4.(A-C).(B-C);$$

we shall get

[3512] Value of Q.
$$Q=\frac{3m^2.(A-C).c'}{E}.\{(m+g')^2.A+m^2.(B-C)\}.\text{*}$$

[3513] If we neglect the square of $\frac{g'}{m}$ and its product by $A-C$, $B-C$, and $A-B$, in the numerator and denominator of this expression of Q, we shall get †

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* (2269) Substituting Q' [3510], in [3508d], and multiplying by B, we get, without reduction,

[3512a] $$\left\{-(m+g')^2.B+\frac{m^2.(m+g')^2.(A-B-C)^2}{(m+g')^2.A+m^2.(B-C)}-4m^2.(A-C)\right\}.Q=3m^2.(A-C).c'.$$

Multiplying this by $(m+g')^2.A+m^2.(B-C)$, and connecting together the terms

[3512b] containing the factor $m^2.(m+g')^2$ in the first member, it becomes equal to EQ [3511], whence we easily obtain Q [3512].

† (2270) Developing E [3511], according to the powers of g', neglecting the square and higher powers of g', also terms of the order $g'.(A-C)$, &c., we get

[3513a] $$E=m^4.\{(A+B-C)^2-4A.(A-C)-B.(B-C)-AB-4.(A-C).(B-C)\}$$
$$+m^3g'.\{2.(A+B-C)^2-4AB\}.$$

Connecting the terms of the coefficient of m^4, which depend explicitly on $A-C$, it becomes

[3513b] $$(A+B-C)^2-4.(A-C).(A+B-C)-B.(A+B-C)=(A+B-C).\{(A+B-C)-4.(A-C)-B\}$$
$$=-3.(A+B-C).(A-C).$$

In the term depending on g' [3513a], we may put $A=B=C$, and it becomes $m^3g'.(2A^2-4A^2)=-2m^3g'.A^2$, or, as we may write, $-2m^3g'.A.(A+B-C)$; substituting this and [3513b], in [3513a], we get

[3513c] $$E=-3m^4.(A+B-C).(A-C)-2m^3g'.A.(A+B-C)$$
$$=-(A+B-C).m^3.\{3m.(A-C)+2Ag'\}.$$

Again, since the numerator of Q [3512] is multiplied by $(A-C)$, we may neglect in it the terms of the factor $(m+g')^2.A+m^2.(B-C)$, depending on g', by which means it

Value of Q.

$$Q = -\frac{3m.(A-C).c'}{3m.(A-C)+2Ag'};$$ [3514]

Second form.

and by supposing g' to be very small, we shall have,* $Q' = Q$. [3515]

The quantity g being insensible in comparison with $3m.\left(\frac{C-A}{2A}\right)$,† it follows, that the values of s, s', corresponding to $\frac{Z}{r_{\prime}}$, [3504], are [3516]

Parts of s, s', depending on Z, and on the secular equations.

$$s = -\frac{3m.(A-C).c'.\sin.(mt+g't+\beta')}{3m.(A-C)+2Ag'} - \Sigma.c.\sin.(mt-gt-\beta);$$ [3517]

$$s' = -\frac{3m.(A-C).c'.\cos.(mt+g't+\beta')}{3m.(A-C)+2Ag'} - \Sigma.c.\cos.(mt-gt-\beta).$$ [3518]

These expressions are not complete; since we must add to them the terms corresponding to Z equal to nothing [3508b]; now it is evident, that

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becomes $m^2.(A+B-C)$; substituting this and E [3513c], in [3512]; rejecting the factor $m^3.(A+B-C)$, which occurs in the numerator and denominator, it becomes as in [3514]. [3513d]

* (2271) Dividing the numerator and denominator of [3510], by m^2, and neglecting terms of the order $\frac{g'}{m}$, it becomes $Q' = \frac{(A+B-C).Q}{A+B-C} = Q$. [3515a]

† (2272) The quantities $B-A$, $C-A$, being nearly of the same order [3578], we may suppose $3m.\left(\frac{C-A}{2A}\right)$ [3464a] to be of the order $0,00004.m$; so that the ratio of g to $3m.\left(\frac{C-A}{2A}\right)$ will be of the order $\frac{g}{0,00004.m} = \frac{gt}{0,00004.mt}$; and if we take t equal to a year, we shall have, by using [3372c], [3517a]

$$0,00004.mt = 0,00004 \times 53474070'' = 2139'',$$ [3517b]

which is much greater than the values of g [4339, 4244, &c.]. Therefore we may neglect g, in the value of $Q = -\frac{3m.(A-C).C}{3m.(A-C)-2Ag}$ [3514], corresponding to the second term of Z [3504], and we shall get $Q = -c = Q'$ [3515]; thence we shall obtain $-\Sigma.c.\sin.(mt-gt-\beta)$, $-\Sigma.c.\cos.(mt-gt-\beta)$, for the resulting terms of s, s' [3508, 3509]. The terms of these expressions, depending on the angle $mt+g't+\beta'$, are easily deduced from [3508, 3509], by the substitution of the values of $Q = Q'$ [3514, 3515]. To obtain the complete values of s, s' [3517, 3518], depending on Z, we must add the terms mentioned in [3499d], and computed in [12320]. [3517c] [3517d]

if l and l' are put for the two positive values of $m+g'$ in the equation
[3519] $E=0$ [3511], we shall have nearly,

[3520] $$s = P \,.\, \sin.\, (l\,t+I) + P' \,.\, \sin.\, (l'\,t+I')\,;^*$$

Parts of s, s', independent of Z.

[3521] $$s' = P \,.\, \cos.\, (l\,t+I) + 2\,P' \,.\, \sqrt{\frac{A-C}{B-C}} \,.\, \cos.\, (l'\,t+I').$$

[3522] $$l = m - \tfrac{3}{2}\,m \,.\, \left(\frac{A-C}{A}\right); \qquad l' = 2\,m \,.\, \frac{\sqrt{(A-C)\,.\,(B-C)}}{A},$$

[3520a] * (2273) If we put $s=P\,.\,\sin.\,(l\,t+I)$, $s'=P_{,}\,.\,\cos.\,(l\,t+I)$, and substitute
[3520a'] them in [3501, 3502], we shall obtain two equations exactly similar to [3508c, d], changing $m+g'$, Q, Q', into l, P, $P_{,}$, respectively. From these equations, we get, as in
[3520b] [3512a, b], $E\,P = 3\,m^2 \,.\, (A-C)\,.\,c'\,.\,\{(l^2\,A + m^2\,.\,(B-C)\}$; E being the same as
[3520c] in [3511], changing $m+g'$ into l. Now to find the values of s, s', when $Z=0$, we must suppose c' equal to nothing, in [3520b], and we shall get $E\,P=0$; which can be satisfied, and leave P arbitrary, by putting $E=0$, as in [3519]. This is equivalent to a quadratic equation in l^2, and if we put, for brevity,

[3520d] $$(A+B-C)^2 - 4\,A\,.\,(A-C) - B\,.\,(B-C) = A\,B\,.\,h^2, \quad (A-C)\,.\,(B-C) = n\,.\,h^4\,.\,A\,B,$$

and divide the whole equation by $-A\,B$, after putting $m+g'=l$, we shall get $l^4 - m^2\,h^2\,l^2 + 4\,n\,m^4\,h^4 = 0$; whence

[3520e] $$l^2 = \tfrac{1}{2}\,m^2\,h^2 \pm \sqrt{(\tfrac{1}{4}\,m^4\,h^4 - 4\,n\,m^4\,h^4)} = m^2\,h^2\,.\,\{\tfrac{1}{2} \pm \sqrt{(\tfrac{1}{4} - 4\,n)}\}.$$

If we develop the radical according to the powers of n, and neglect n^2, on account
[3520e'] of its smallness, the two values represented by l^2, l'^2, will become $l^2 = m^2\,h^2\,(1-4\,n)$, $l'^2 = 4\,n\,.\,m^2\,h^2$. We shall soon see, that h is nearly equal to unity [3520h]; therefore n [3520d] must be so small, that we may neglect it, in the expression of l^2 [3520e']; then taking the square root of the values of l^2, l'^2, and retaining only the positive values, we
[3520f] get $l=m\,h$, $l'=2\,m\,h\,.\,\sqrt{n}$. In finding the value of h^2, from the first of the equations [3520d], we may neglect the second powers and products of $A-C$, $B-C$, &c.; then we shall have, successively,

[3520g] $$\begin{aligned}\{A+B-C\}^2 &= A^2 + 2\,A\,.\,(B-C) = (A\,B + A^2 - A\,B) + 2\,A\,.\,(B-C)\\ &= A\,B + A\,.\,(A-B) + 2\,A\,.\,(B-C).\end{aligned}$$

Substituting this in the first equation [3520d], and dividing by $A\,B$, we get the first expression of h^2 [3520h]; the second form is easily deduced from the first, by changing B into A, in the denominator of the terms

[3520h] $$h^2 = 1 + \left(\frac{A-B}{B}\right) + 2\left(\frac{B-C}{B}\right) - 4\,.\,\left(\frac{A-C}{B}\right) - \left(\frac{B-C}{A}\right) = 1 - 3\,.\,\left(\frac{A-C}{A}\right);$$

[3520i] the square root of this expression, gives $h = 1 - \tfrac{3}{2}\,.\,\left(\frac{A-C}{A}\right)$. As h differs so little

*P, P', I and I' being four arbitrary constant quantities. Connecting these values of s and s', with the preceding [3517, 3518], we shall have the complete values of s and s'.** [3523]

To prevent these values from increasing indefinitely, and to render the inclination of the lunar equator to the ecliptic nearly constant, as it is found to be by observation, it is necessary that the product $(A-C).(B-C)$, should be positive.† Now this condition is naturally satisfied; because the momentum of inertia C, of the moon, about the third principal axis, is greater than either of the momenta of inertia A, B, corresponding to the other two principal axes; as is evident from the consideration that the moon [3524] [3525]

The product of $A-C$ by $B-C$, must be positive.

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from unity, we may put $h=1$, in the second of the equations [3520d], and then dividing by AB, we get $n=\frac{(A-C).(B-C)}{AB}=\frac{(A-C).(B-C)}{A^2}$ nearly; substituting this value of n, and that of h [3520i], in [3520f], we get l, l' [3522]. With each of these values l, l', we have a term of s, as in [3520]. [3520k]

The corresponding term of s' [3521], may be obtained from [3510], by changing $m+g'$, Q, Q', into l, $P_{,}$, $P'_{,}$, as in [3520a'], by which means we get

$$P_{,}=\frac{ml.(A+B-C).P}{l^2A+m^2.(B-C)};\qquad P'_{,}=\frac{ml'.(A+B-C).P'}{l'^2A+m^2.(B-C)}. \qquad [3520l]$$

Substituting $l=mh$ [3520f] and $h=1$ [3520i], in this value of $P_{,}$, we get very nearly, $P_{,}=P$. In like manner, by the substitution of $l'=2mh.\sqrt{n}$ [3520f] in $P'_{,}$ [3520l], and neglecting terms of the order l'^2, we obtain [3520m]

$$P'_{,}=\frac{2m^2h\sqrt{n}.(A+B-C).P'}{m^2.(B-C)}=\frac{2h\sqrt{n}.(A+B-C).P'}{(B-C)}; \qquad [3520n]$$

now $A+B-C$ is nearly equal to A, and h nearly equal to unity; hence we obtain $P'_{,}=\frac{2\sqrt{n}.AP'}{(B-C)}=2P'.\sqrt{\left(\frac{A-C}{B-C}\right)}$ [3720k]. These values of $P_{,}$ $P'_{,}$ correspond to the expression of s' [3520a, 3521]. [3520o]

* (2274) To each of the quantities s, s', we must also add the term computed by Mr. Poisson [3499d], and by the author in the fourteenth book [12320]. [3523a]

† (2275) If $(A-C).(B-C)$ be negative, the value of l' [3522] will be imaginary, and the expressions of $\sin.(l't+I')$, $\cos.(l't+I')$, which occur in s, s' [3520, 3521], will change into arcs of circle, or exponential quantities [3465a]. In this case s, s', will increase indefinitely with the time, and the inclination $\theta=\sqrt{(s^2+s'^2)}$ [3487], will suffer similar variations; but this is found by observation not to be the case; therefore it must necessarily follow, that $(A-C).(B-C)$ is positive. [3524a] [3524b] [3524c]

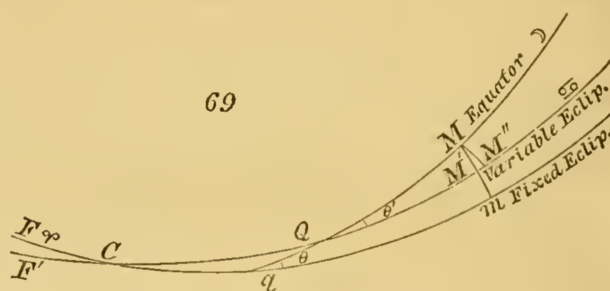
must be more flattened in the direction of the axis of rotation, than in any other direction.

$\theta_{,}$, $\varphi_{,}$, correspond to the variable ecliptic.

To refer the quantities s *and* s' *to the variable ecliptic, we shall put*
[3526] $\theta_{,}$ *for the inclination of the lunar equator to the variable ecliptic, and* $\varphi_{,}$ *for the angular distance of the first principal axis from the descending node of the lunar equator upon the same ecliptic; then it is evident that we shall have,**

[3527] $$\theta_{,} \,.\, \sin. \varphi_{,} - \Sigma \,.\, c \,.\, \sin. (m t - g t - \beta) = \theta \,.\, \sin. \varphi ;$$

[3528] $$\theta_{,} \,.\, \cos. \varphi_{,} - \Sigma \,.\, c \,.\, \cos. (m t - g t - \beta) = \theta \,.\, \cos. \varphi ;$$



* (2276) We shall suppose, in the annexed figure 69, that $F C q m$ is the arc of the celestial sphere, corresponding to the fixed ecliptic, $F' C Q M' M''$ the variable ecliptic,
[3526a] $q\, Q\, M$ the lunar equator; q or Q the *ascending* node of the lunar equator, corresponding respectively
[3526a′] to the fixed or to the variable ecliptic; M is the place in the celestial sphere, corresponding to the line drawn from the centre of the moon towards the earth, in the direction of the
[3526b] first principal axis [3447a]. From M let fall perpendicularly upon $C m$, $C M''$, the arcs $M m$, $M M''$; then, as the angle of inclination of the equator and ecliptic is small,
[3526c] we shall have very nearly $M M'' = M M'$, $M' m = M m - M M''$. Now we have
[3526d] the angle $M q m = \theta$, the angle $M Q M' = \theta_{,}$ [3133h]; and as φ [3447a] represents the angular distance of the moon's first principal axis from the *descending node* of the lunar equator upon the fixed ecliptic, we shall have its distance from the *ascending node* equal
[3526e] to $\varphi - 180^d = \text{arc}\, M q$; in like manner $\varphi_{,} - 180^d = \text{arc}\, M Q$ [3526]. In the
[3526f] spherical triangle $M M'' Q$, we have very nearly,

[3526g] $$M M'' = M Q M'' \,.\, \sin. M Q = \theta_{,} \,.\, \sin. (\varphi_{,} - 180^d) = - \theta_{,} \sin. \varphi_{,} ;$$

and in the triangle $M q m$,

[3526h] $$M m = M q m \,.\, \sin. M q = \theta \,.\, \sin. (\varphi - 180^d) = - \theta \,.\, \sin. \varphi ;$$

substituting these in the value of $M' m$ [3526c], we get

[3526i] $$M' m = M m - M M'' = - \theta \,.\, \sin. \varphi + \theta_{,} \,.\, \sin. \varphi_{,} , \quad \text{or} \quad \theta_{,} \,.\, \sin. \varphi_{,} - M' m = \theta \,.\, \sin. \varphi .$$

Now we have shown, in [3503g], that the latitude of the point M', fig. 61, page 846, of the variable ecliptic, referred to the true ecliptic, and corresponding to the place M of the
[3526k] *moon, as seen from the earth*, is $- \Sigma \,.\, c \,.\, \sin. (m t - g t - \beta)$; and by changing the sign, we get the latitude $M' m = \Sigma \,.\, c \,.\, \sin. (m t - g t - \beta)$, of the point M' of the variable
[3526l] ecliptic, in the annexed fig. 69, corresponding to the place M [3526a], of the *earth, seen*

therefore, by putting $s_{,} = \theta_{,} . \sin. \varphi_{,}$, $s_{,}' = \theta_{,} . \cos. \varphi_{,}$, we shall obtain,* [3529]

Values of $s_{,}$, $s_{,}'$, referred to the variable ecliptic.

$$s_{,} = P . \sin. (lt+I) + P' . \sin. (l't+I') - \frac{3m.(A-C).c'.\sin.(mt+g't+\beta')}{3m.(A-C)+2A.g'}; \qquad [3530]$$

$$s_{,}' = P . \cos. (lt+I) + 2P' . \sqrt{\frac{A-C}{B-C}} . \cos. (l't+I') - \frac{3m.(A-C).c'.\cos.(mt+g't+\beta')}{3m.(A-C)+2A.g'}. \qquad [3531]$$

Hence we see, that the motion of the lunar equator upon the plane of the apparent or variable ecliptic, is independent of the motion of this plane, so that the mean inclination of the lunar equator to this ecliptic, is always the same, notwithstanding the motion of the plane,† *because the attraction of the earth upon the spheroidal figure of the moon tends always to reduce the equator of this spheroid to the same degree of inclination.* [3532] [3533]

The inclination of the moon's equator to the variable ecliptic, is not affected by the secular motion of the plane of the ecliptic.

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from the moon, in the direction of the first principal axis. Substituting this in [3526*i*], we get [3527]. In like manner, we may proceed with the second principal axis; or we may obtain [3528] from [3527], by supposing the angles φ, $\varphi_{,}$, mt, to be increased by 90^d; [3526*m*]
since it is evident from [3447*b*], that the moon's second principal axis is nearly 90^d more advanced than the earth, in its apparent orbit about the moon. [3526*n*]

* (2277) Substituting s, s' [3487] and $s_{,}$, $s_{,}'$ [3529], in [3527, 3528], we get

$$s_{,} = s + \Sigma . c . \sin. (mt - gt - \beta); \qquad s_{,}' = s' + \Sigma . c . \cos. (mt - gt - \beta). \qquad [3530a]$$

If we substitute, in the first of these equations, the terms of s [3517, 3520], we shall obtain [3530]; also in the second equation, if we substitute the terms of s' [3518, 3521], we shall get [3531]. To each of the quantities s, s' [3530, 3531], we must add the term mentioned in [3523*a*]. It deserves particular notice, that in making these substitutions, the terms depending on $\Sigma . c . \sin. (mt - gt - \beta)$, $\Sigma . c . \cos. (mt - gt - \beta)$, or, in other words, the terms depending on the secular equations in the values of $s_{,}$, $s_{,}'$ [3530*a*], are destroyed by the similar terms of s, s' [3517, 3518], produced by the attraction upon the lunar spheroid, and having a contrary sign, so that s, $s_{,}'$ [3530, 3531], are independent of the terms depending on the secular motion of the ecliptic. [3530*b*] [3530*c*]

The values of $s_{,}$, $s_{,}'$, are independent of the secular motion of the ecliptic.

† (2278) Dividing the first of the equations [3529] by the second, we get $\text{tang}. \varphi_{,} = \frac{s'}{s_{,}'}$; [3533*a*]
substituting $s_{,}$, $s_{,}'$ [3530, 3531], we get [3534]. The sum of the squares of the equations [3529], gives $\theta_{,} = \sqrt{(s_{,}^2 + s_{,}'^2)}$. Neither of these values of $s_{,}$, $s_{,}'$, contains the terms [3533*b*]
$\Sigma . c . \sin. (mt - gt - \beta)$, $\Sigma . c . \cos. (mt - gt - \beta)$, depending on the secular motions of the ecliptic; hence it is evident, that *the inclination* $\theta_{,}$ *of the equator to the variable ecliptic and the motion* $\varphi_{,}$, *of the moon's first principal axis, counted from the intersection of the equator with the variable ecliptic, are independent of its secular motion, as is observed in* [3532]. [3533*c*]

The values of s and s' give

Expression of $\varphi_{,}$.

[3534] $$\text{tang.}\,\varphi_{,}=\frac{\left\{\begin{array}{l}3m.(A-C).c'.\sin.(mt+g't+\beta')\\ -\{3m.(A-C)+2Ag'\}.\{P.\sin.(lt+I)+P'.\sin.(l't+I')\}\end{array}\right\}}{\left\{\begin{array}{l}3m.(A-C).c'.\cos.(mt+g't+\beta')\\ -\{3m.(A-C)+2Ag'\}.\{P.\cos.(lt+I)+2P'.\sqrt{\frac{A-C}{B-C}}.\cos.(l't+I')\}\end{array}\right\}}.$$

[3534'] *If we suppose, in the first place, that P and P' are nothing, we shall have,**

[3535] $$\text{tang.}\,\varphi_{,}=\text{tang.}\,(mt+g't+\beta');$$

from which we get the two following values of $\varphi_{,}$,

[3536] $$\varphi_{,}=mt+g't+\beta';$$

[3537] $$\varphi_{,}=\pi+mt+g't+\beta';$$

π being the semi-circumference, or two right angles. To determine which of these two values really obtains in nature, we shall observe, that
[3538] $-g't-\beta'$ [3505'] is the longitude of the ascending node of the lunar orbit upon the apparent ecliptic; and we know by observation [3433], that this longitude is the same as that of the descending node of the lunar equator upon the same ecliptic. Now the rotatory motion of the moon being equal to its mean motion of revolution [3460'], and its first principal axis being directed nearly towards the earth [3468], we have, $\varphi_{,}$, increased by the longitude of the descending node of the lunar equator, equal
[3539] to mt; hence we get,†

[3540] $$\varphi_{,}=mt+g't+\beta'.$$

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* (2279) If, in the first approximation, we neglect the terms multiplied by P, P' [3534], also those mentioned in [3530*b*], and then divide the numerator and denominator
[3535*a*] by $3m.(A-C).c'.\cos.(mt+g't+\beta')$, we shall get [3535]; from which we may easily obtain [3536, 3537].

† (2280) The mean longitude of the earth, seen from the moon, is mt [3505];
[3535*b*] and $-g't-\beta'$ is the longitude of the *ascending node of the lunar orbit*, upon the apparent ecliptic [3505']; subtracting this from the preceding, we get $mt+g't+\beta'$, for the distance of the earth, seen from the moon, and counted *from the ascending node*
[3535*c*] *of the lunar orbit*, or from the *descending node of the lunar equator* [3433]. Now this represents very nearly the mean distance of the moon's first principal axis from the same
[3535*d*] descending node [3468], which is equal to $\varphi_{,}$ [3526]; hence we get [3540].

Therefore the first of the two preceding values [3536], is the only one which can be used; hence the equation $s_{,}=\theta_{,}.\sin.\varphi_{,}$ [3529], gives* Approximate value of $\theta_{,}$.

$$\theta_{,}=\frac{3m.(C-A).c'}{2Ag'-3m.(C-A)};$$ [3541]

from which we deduce, Approximate value of $C-A$.

$$\frac{C-A}{A}=\frac{2g'.\theta_{,}}{3m.(c'+\theta_{,})}.$$ [3542]

Mayer found, by his observations, $\theta_{,}=165'=1^{d}\,29^{m}$ [3434]; we also have [3371], [3543]

$$c=\text{tang.}\,5^{\circ},7188=\text{tang.}\,5^{d}\,8^{m}\,49^{s};\qquad g'=m.0,004019;\dagger$$ [3544]

therefore

$$\frac{C-A}{A}=0,000599.$$ [3545]

The preceding results hold good only when the arbitrary quantities P and P' vanish. We shall now examine the case in which these constant quantities, without being actually nothing, are extremely small. We have, from [30] Int., [3546]

$$\text{tang.}(\varphi_{,}-mt-g't-\beta')=\frac{\text{tang.}\,\varphi_{,}-\text{tang.}(mt+g't+\beta')}{1+\text{tang.}\,\varphi_{,}.\,\text{tang.}(mt+g't+\beta')}.$$ [3547]

Substituting, in the second member of this equation, the complete value of $\varphi_{,}$,‡

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* (2281) Substituting [3540], in $s_{,}$ [3529], we get $s_{,}=\theta_{,}.\sin.(mt+g't+\beta')$; [3541a] putting this equal to the expression [3530], and then supposing $P=0$, $P'=0$ [3534'], we find, that the whole will become divisible by $\sin.(mt+g't+\beta')$, and we shall obtain [3541]; from which we easily deduce the first approximate value of $\frac{C-A}{A}$ [3541b] [3542, 3545]. This value of $\theta_{,}$ [3541] is affected by the small periodical equations depending on the terms [3523a]. [3541c]

† (2282) This is nearly the same as in [3372c]. Substituting these values of c', g', and putting $\theta_{,}=\text{tang.}\,1^{d}\,29^{m}$ [3543], the expression [3542] becomes

$$\frac{C-A}{A}=\frac{2\times0,004019\times\text{tang.}\,1^{d}\,29^{m}}{3\,\text{tang.}\,5^{d}\,8^{m}\,49^{s}+3\,\text{tang.}\,1^{d}\,29^{m}}=0,000599,$$ [3545a]

as in [3545].

‡ (2283) Dividing the numerator and denominator of [3534], by $3m.(A-C)+2Ag'$, or its equal $2Ag'-3m.(C-A)$; substituting Q [3548], and putting $p=2.\sqrt{\left(\frac{A-C}{B-C}\right)}$, we get [3548a]

$$\text{tang.}\,\varphi_{,}=\frac{Q.\sin.(mt+g't+\beta')-P.\sin.(lt+I)-\;P'.\sin.(l't+I')}{Q.\cos.(mt+g't+\beta')-P.\cos.(lt+I)-pP'.\cos.(l't+I')};$$ [3548b]

which is to be substituted in [3547].

and putting, for brevity,

[3548] $$Q = \frac{-3m.(C-A).c'}{2Ag'-3m.(C-A)};$$

we shall have,*

Important equation.

[3549] $$\text{tang.}(\varphi_{,}-mt-g't-\beta')=\frac{\left\{\begin{array}{l} P.\sin.(mt+g't+\beta'-lt-I) \\ +P'.\left\{\sqrt{\frac{A-C}{B-C}}+\tfrac{1}{2}\right\}.\sin.(mt+g't+\beta'-l't-I') \\ +P'.\left\{\sqrt{\frac{A-C}{B-C}}-\tfrac{1}{2}\right\}.\sin.(mt+g't+\beta'+l't+I') \end{array}\right\}}{\left\{\begin{array}{l} Q-P.\cos.(mt+g't+\beta'-lt-I) \\ -P'.\left\{\sqrt{\frac{A-C}{B-C}}+\tfrac{1}{2}\right\}.\cos.(mt+g't+\beta'-l't-I') \\ -P'.\left\{\sqrt{\frac{A-C}{B-C}}-\tfrac{1}{2}\right\}.\cos.(mt+g't+\beta'+l't+I') \end{array}\right\}}.$$

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* (2284) In making the substitution of $\text{tang.}\varphi_{,}$ [3548b], it will be convenient to put, for a moment, the following abridged expressions of the numerator N, and denominator D;

[3549a] $$M=mt+g't+\beta'; \qquad L=lt+I; \qquad L'=l't+I';$$

[3549b] $$N=Q.\sin.M-P.\sin.L-P'.\sin.L'; \qquad D=Q.\cos.M-P.\cos.L-pP'.\cos.L';$$

[3549c] by which means we shall have $\text{tang.}\varphi_{,}=\frac{N}{D}$. Substituting this in [3547], then multiplying the numerator and denominator by $D.\cos.M$, we get

[3549d] $$\text{tang.}(\varphi_{,}-mt-g't-\beta')=\frac{\frac{N}{D}-\text{tang.}M}{1+\frac{N}{D}.\text{tang.}M}=\frac{N.\cos.M-D.\sin.M}{N.\sin.M+D.\cos.M};$$

in which we may compute, successively, the terms depending on Q, P, P', in the values

[3549e] of N, D [3549b]. *First.* Putting $N=Q.\sin.M$, $D=Q.\cos.M$, the numerator

[3549f] vanishes, and the denominator becomes $Q.(\sin.^2M+\cos.^2M)=Q$, as in [3549]. *Second.* Putting $N=-P.\sin.L$, $D=-P.\cos.L$ [3549b], we get, in the numerator,

[3549g] the term $P.(-\sin.L.\cos.M+\cos.L.\sin.M)=P.\sin.(M-L)$; and in the denominator, the term $-P.(\sin.L.\sin.M+\cos.L.\cos.M)=-P.\cos.(M-L)$ as in [3549]. *Third.* Putting $N=-P'.\sin.L'$, $D=-pP'.\cos.L'$ [3549b], and reducing, by means of [17—20] Int., we get, in the numerator, the term

[3549h] $$P'.\{-\sin.L'.\cos.M+p.\cos.L'.\sin.M\}=P'.\{(\tfrac{1}{2}p+\tfrac{1}{2}).\sin.(M-L')+(\tfrac{1}{2}p-\tfrac{1}{2}).\sin.(M+L')\};$$

and in the denominator, the term

$$-P'.\{\sin.L'.\sin.M+p.\cos.L'.\cos.M\}=-P'.\{(\tfrac{1}{2}p+\tfrac{1}{2}).\cos.(M-L')+(\tfrac{1}{2}p-\tfrac{1}{2}).\cos.(M+L')\},$$

as in [3549]. We may remark, that in the last expression of [3549d], we may deduce

[3549i] the terms of the numerator depending on P, P', from those of the denominator, by writing $M+90^d$, for M; and the same process may be used in abridging the calculation of [3549].

If the denominator of this expression be always of the same sign as Q, *and never vanish, the angle* $\varphi_{,}-mt-g't-\beta'$, *will always be less than a right angle, taken positively or negatively;* for it is evident, since the tangent [3550] of a right angle is infinite, that the denominator will vanish, if the angle $\varphi_{,}-mt-g't-\beta'$ become a right angle. *On the contrary, we see, that* [3550′] *if the sign of the denominator changes in passing through zero, the tangent of the proposed angle will become infinite, and the angle equal to a right angle. Now it is found by observation, that this never takes place;* hence it* [3551] *follows, that the preceding denominator is always of the same sign as* Q; *therefore* Q *is greater than* $P+2P'.\sqrt{\frac{A-C}{B-C}}$.† Moreover, the angle [3552] $\varphi_{,}-mt-g't-\beta'$ being found by observation [3551a], to be always [3553] very small, it follows, that the quantities P and P' are very small, in [3554] comparison with Q; and since the inclination of the lunar equator to the apparent ecliptic is equal to‡ $\sqrt{s_{,}^2+s_{,}'^2}$, *it follows, that this inclination must be very nearly constant, and equal to* Q. *Thus we see, that the phenomenon* [3555]

Inclinations of the moon's equator and orbit have a mutual dependence upon each other.

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* (2285) We have seen, in [3535b, d], that $\varphi_{,}$ is very nearly equal to $mt+g't+\beta'$; therefore their difference $\varphi_{,}-mt-g't-\beta'$, or its tangent, must always be very small. [3551a]

† (2286) The greatest possible values of the terms of the denominator of [3549], depending on P, P', are evidently found by putting the cosines of the angles, by which they are multiplied, equal to unity, and then they become,

$$-P-P'.\left\{\sqrt{\left(\frac{A-C}{B-C}\right)}+\tfrac{1}{2}\right\}-P'.\left\{\sqrt{\left(\frac{A-C}{B-C}\right)}-\tfrac{1}{2}\right\}=-P-2P'.\sqrt{\left(\frac{A-C}{B-C}\right)}; \quad [3552a]$$

so that if this quantity be less than Q, the denominator will always have the same sign as Q. This result will not be affected by taking into consideration the additional terms of s, s' [3523a]. It will merely introduce an additional term in the numerator and denominator of [3534, 3548b], also terms in the numerator and denominator of [3549]; but this will merely require a small increase in the limit of Q [3552], which must exceed the sum of all the other coefficients of the denominator of [3549], taken with the same sign. [3552b]

‡ (2287) The inclination of the lunar orbit to the apparent ecliptic, is $\theta_{,}=\sqrt{(s_{,}^2+s_{,}'^2)}$ [3554a] [3526, 3533b]. Now if we substitute in [3530, 3531], the abridged symbols [3548, 3548a, 3549a], we shall get

$$s_{,}=P.\sin.L+P'.\sin.L'-Q.\sin M; \quad s_{,}'=P.\cos.L+pP'.\cos.L'-Q.\cos.M. \quad [3554b]$$

If we neglect P, P', on account of their smallness in comparison with Q, we shall have $s_{,}=-Q.\sin.M$, $s_{,}'=-Q.\cos.M$; hence $\theta_{,}=\sqrt{(s_{,}^2+s_{,}'^2)}=Q$ [3554a], so that [3554c] $\theta_{,}$ is nearly equal to the constant quantity Q.

of the coincidence of the nodes of the lunar equator and orbit, and the constancy of the inclination of these two planes to each other, have a mutual
[3556] *relation to each other, depending on the theory of gravity; and the observations, which show that they take place simultaneously, confirm this theory in an admirable manner.*

We have observed in [3048], *relatively to the earth, that the arbitrary constant terms, depending on the initial state of its rotatory motion, are nothing, or at least that they are insensible,* by the most accurate observations.
[3557] We see, also, from what precedes, and from § 15, that the *same obtains in the moon,* and it is natural to suppose, that it extends to all the heavenly bodies;* for it is evident, that if the attractions of foreign bodies be excluded, the friction and resistance of the particles of the body, upon each other, must finally reduce it to a permanent form of equilibrium, which cannot
[3557'] subsist, except with a uniform rotatory motion about an invariable axis
[3558] [281iv]. Therefore we shall perceive, by observation, nothing more than the effect arising from foreign attractions.

We discover by observations merely the effect of foreign attractions.

Limits of A, B, C.

18. *We shall now examine into the results of the preceding investigation,*
[3559] *when applied to the figure of the moon.* Now A and B are less than C [3525]; moreover, B exceeds A [3467], and $\frac{B-A}{C}$ is comprised between the limits of 0 and 0,0013876 [3483]; lastly, we have found,
[3560] that $\frac{C-A}{A}$ is very nearly equal to 0,000599 [3545] *These are the results of observation, relative to the momenta of inertia A, B, C. We shall now compare them with the results of the theory of the figure of the lunar spheroid.*

Substituting for A, B, C, their values [2948—2950], we obtain†

[3557a] * (2288) This appears in [3554], where it is shown that the terms P, P', depending on the initial state of the moon, are very small or insensible, in comparison with Q, in the values of s, s', $s_{,}$, $s_{,}'$ [3520, &c., 3530, &c.].

† (2289) Subtracting A [2948] from B, C [2949, 2950], and using

$$\cos.^2\varpi - \sin.^2\varpi = \cos.2\varpi \quad [32] \text{ Int.},$$

we get

[3561a]
$$B-A = \alpha \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, (a^5\, Y^{(2)}) \,.\, d\mu \,.\, d\varpi \,.\, (1-\mu^2) \,.\, \cos.2\varpi;$$

$$C-A = \alpha \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, (a^5\, Y^{(2)}) \,.\, d\mu \,.\, d\varpi \,.\, \{(1-\mu^2) \,.\, \cos.^2\varpi - \mu^2\}.$$

Dividing these by $\frac{8}{15}\pi \,.\, S_0^1 \,.\, \rho \,.\, d \,.\, a^5$, which represents very nearly the value of A or C [2948, 2950], we get [3561, 3562].

General values of $B-A$, $C-A$, supposing the moon to be composed of strata varying in form and density from the centre to the surface.

$$\frac{B-A}{C}=\frac{15\alpha\,.\,S_0^1\,.\,\rho\,.\,d\,.\,(a^5\,.\,Y^{(2)})\,.\,d\mu\,.\,d\varpi\,.\,(1-\mu^2)\,.\,\cos.2\varpi}{8\pi\,.\,S_0^1\,.\,\rho\,.\,d\,.\,a^5}; \quad [3561]$$

$$\frac{C-A}{A}=\frac{15\alpha\,.\,S_0^1\,.\,\rho\,.\,d\,.\,(a^5\,.\,Y^{(2)})\,.\,d\mu\,.\,d\varpi\,.\{(1-\mu^2)\,.\,\cos.^2\varpi-\mu^2)\}}{8\pi\,.\,S_0^1\,.\,\rho\,.\,d\,.\,a^5}. \quad [3562]$$

The attraction of the earth upon the moon, has an influence on the moon's figure, and lengthens the axis directed towards the earth [3467a]. *If we suppose the moon to be covered by a fluid in equilibrium, and the earth to be situated in the plane of the lunar equator; then take the meridian which passes through the first and third principal axes of the moon, for the first meridian, or origin of the angle* ϖ, *and put for unity, the first semi-axis;* we shall find, as in [1705],* [3563] [3564] [3564′]

Additional supposition that the moon is covered by a fluid in equilibrium.

$$\tfrac{4}{5}\alpha\pi\,.\,S_0^1\,.\,\rho\,.\,d\,.\,(a^5\,.\,Y^{(2)})=\tfrac{4}{3}\alpha\pi\,.\,Y^{(2)}\,.\,S_0^1\,.\,\rho\,.\,d\,.\,a^3$$
$$+\tfrac{1}{2}g\,.\,(\mu^2-\tfrac{1}{3})-\frac{3L}{2r_{,}^3}\,.\{(1-\mu^2)\,.\,\cos.^2\varpi-\tfrac{1}{3}\}; \quad [3565]$$

g is the centrifugal force of a point of the lunar equator; this force, at the distance $r_{,}$ from the centre of the moon, is equal to $g\,r_{,}$ [1616i]; and g. [3565′]

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* (2290) Putting $i=2$ in [1705], then taking the integrals to correspond to the moon's surface, where $a=1$ [1702″, 3564′], the first integral will vanish; and we shall get, by substituting $a=1$ in all the terms, without the sign of integration, [3565a]

$$0=-\tfrac{4}{3}\pi\,.\,Y^{(2)}\,.\,S_0^1\,.\,\rho\,.\,d\,.\,a^3+\tfrac{4}{5}\pi\,.\,S_0^1\,.\,d\,.\,(a^5\,.\,Y^{(2)})+Z^{(2)}. \quad [3565b]$$

To obtain $\alpha Z^{(2)}$ from [1632], we must change S, s, into L, $r_{,}$, respectively, to conform to the present notation; and we shall get $\alpha Z^{(2)}=\frac{L}{r_{,}^3}\,.\,P^{(2)}-\tfrac{1}{2}g\,.\,(\mu^2-\tfrac{1}{3})$; [3565c] substituting this in [3565b], multiplied by α, we obtain

$$\tfrac{4}{5}\alpha\pi\,.\,S_0^1\,.\,\rho\,.\,d\,.\,(a^5\,.\,Y^{(2)})=\tfrac{4}{3}\alpha\pi\,.\,Y^{(2)}\,.\,S_0^1\,.\,\rho\,.\,d\,.\,a^3+\tfrac{1}{2}g\,.\,(\mu^2-\tfrac{1}{3})-\frac{L}{r_{,}^3}\,.\,P^{(2)}. \quad [3565d]$$

Putting $i=2$, in [1628], and using δ [1629], we get

$$P^{(2)}=\tfrac{3}{2}\,.\{\delta^2-\tfrac{1}{3}\}=\tfrac{3}{2}\,.\{[\cos.v\,.\,\cos.\theta+\sin.v\,.\,\sin.\theta\,.\,\cos.(\varpi-\psi)]^2-\tfrac{1}{3}\}. \quad [3565e]$$

Now the line s, or $r_{,}$, of the present notation, drawn from the moon to the earth [3435], being situated very nearly in the plane of the lunar equator, we shall have nearly $v=90^d$ [1620‴′]; moreover, as the earth is situated very nearly in the direction of the moon's first principal axis, from which ψ is counted [3564], we may also put $\psi=0$ in the preceding expression, and we shall obtain [3565f]

$$P^{(2)}=\tfrac{3}{2}\,.\{\sin.^2\theta\,.\,\cos.^2\varpi-\tfrac{1}{3}\}=\tfrac{3}{2}\,.\{(1-\mu^2)\,.\,\cos.^2\varpi-\tfrac{1}{3}\}\ [1616^{xxi}]. \quad [3565g]$$

Substituting this in [3565d], we get [3565].

since the rotatory motion of the moon is equal to its mean motion of

[3566] λ' revolution, we shall have very nearly,* $g\,r_{,}=\frac{L}{r_{,}^{2}}$. *We shall put* λ' *for*

[3566'] *the ratio of the mass L of the earth to that of the moon*, and we shall obtain,†

[3567] Mass of the earth.

$$L=\tfrac{4}{3}\pi\,.\,\lambda'\,.\,S_{0}^{1}\,.\,\rho\,.\,d\,.\,a^{3}.$$

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* [2291] The attractive force of the earth upon the moon, at the distance $r_{,}$ is $\frac{L}{r_{,}^{2}}$,

[3566a] and this is equal to the centrifugal force the moon will have, supposing it to revolve about the earth in the same time, in a circular orbit, whose radius is $r_{,}$; neglecting the mass of the moon, in comparison with that of the earth. Now the times of rotation and revolution being nearly equal, this must be nearly equal to the centrifugal force $g\,r_{,}$, corresponding to

[3566b] the rotatory motion; and from this we get $\frac{L}{r_{,}^{3}}=g$, as in [3566]. Substituting this, and $P^{(2)}$ [3565g], in [3565c], we get the following value of $\alpha Z^{(2)}$, which is used hereafter, observing that the second form of this expression is easily deduced from the first, by the substitution of $\cos.^{2}\varpi=\frac{1}{2}+\frac{1}{2}\cos.2\varpi$,

[3566c]

$$\begin{aligned}\alpha Z^{(2)}&=-\tfrac{1}{2}g\,.\,(\mu^{2}-\tfrac{1}{3})+\tfrac{3}{2}g\,.\,\{(1-\mu^{2})\,.\,\cos.^{2}\varpi-\tfrac{1}{3}\}\\&=\tfrac{5}{4}g\,.\,\{(\tfrac{1}{3}-\mu^{2})+\tfrac{3}{5}\,.\,(1-\mu^{2})\,.\,\cos.2\varpi\}.\end{aligned}$$

† (2292) The mass of the moon, supposing it to be composed of concentrical strata

[3567a] of variable densities, is represented by $\frac{4}{3}\pi\,.\,S_{0}^{1}\,.\,\rho\,.\,d\,.\,a^{3}$ [1811']; multiplying this by λ' [3566'], we get L, the mass of the earth [3567]; hence the value of g [3566b], becomes

[3567b] $g=\frac{4\pi\lambda'}{3r_{,}^{3}}\,.\,S_{0}^{1}\,.\,\rho\,.\,d\,.\,a^{3}$. Substituting these values of L, g, in [3565], we get

[3567c] $$\tfrac{4}{5}\alpha\pi\,.\,S_{0}^{1}\,.\,\rho\,.\,d\,.\,(a^{5}\,.\,Y^{(2)})=\left\{\tfrac{4}{3}\alpha\pi\,.\,Y^{(2)}+\frac{2\pi\lambda'}{3r_{,}^{3}}\,.\,(\mu^{2}-\tfrac{1}{3})-\frac{2\pi\lambda'}{r_{,}^{3}}\,.\,[(1-\mu^{2})\,.\,\cos.^{2}\varpi-\tfrac{1}{3}]\right\}.\,S_{0}^{1}\,.\,\rho\,.\,d\,.\,a^{3}.$$

Dividing this by $\frac{4}{5}\pi$, and using $Y^{(2)}$ [3567'], we obtain

[3567d]

$$\begin{aligned}\alpha\,.\,S_{0}^{1}\,.\,\rho\,.\,d\,.\,(a^{5}\,.\,Y^{(2)})&=\tfrac{5}{3}\,.\,\{\alpha h\,.\,(\tfrac{1}{3}-\mu^{2})+\alpha h''''.\,(1-\mu^{2})\,.\,\cos.2\varpi\}.\,S_{0}^{1}\,.\,\rho\,.\,d\,.\,a^{3}\\&\quad-\frac{\lambda'}{r_{,}^{3}}\,.\,\{\tfrac{5}{6}\,.\,(\tfrac{1}{3}-\mu^{2})+\tfrac{5}{2}\,.\,[(1-\mu^{2})\,.\,\cos.^{2}\varpi-\tfrac{1}{3}]\}\,.\,S_{0}^{1}\,.\,\rho\,.\,d\,.\,a^{3}.\end{aligned}$$

Substituting, in the last line of this expression, $\cos.^{2}\varpi=\frac{1}{2}+\frac{1}{2}\,.\,\cos.2\varpi$ [6] Int., we shall get for the factor of $\frac{\lambda'}{r_{,}^{3}}\,.\,S_{0}^{1}\,\rho\,.\,d\,.\,a^{3}$, the first of the following expressions [3567e], which, by successive reductions, becomes as in [3567f]; hence [3567d] becomes as in [3568];

[3567e] $$-\tfrac{5}{6}\,.\,(\tfrac{1}{3}-\mu^{2})-\tfrac{5}{4}\,.\,(1-\mu^{2})\,.\,(1+\cos.2\varpi)+\tfrac{5}{6}=-\tfrac{25}{36}+\tfrac{25}{12}\,.\,\mu^{2}-\tfrac{5}{4}\,.\,(1-\mu^{2})\,.\,\cos.2\varpi$$

[3567f] $$=\tfrac{5}{3}\,.\,\{-\tfrac{5}{4}\,.\,(\tfrac{1}{3}-\mu^{2})-\tfrac{3}{4}\,.\,(1-\mu^{2})\,.\,\cos.2\varpi\}.$$

This being premised, and observing that $Y^{(2)}$ [1763] is of the form

$$Y^{(2)}=h.(\tfrac{1}{3}-\mu^2)+h''''.(1-\mu^2).\cos.2\varpi;$$ [3567′]

we shall have,

$$\alpha.S_0^1.\rho.d.(a^5.Y^{(2)})=\tfrac{5}{3}.\left\{\alpha h-\frac{5\lambda'}{4r_{\prime}^3}\right\}.(\tfrac{1}{3}-\mu^2).S_0^1.\rho.d.a^3 \qquad (i)$$

$$+\tfrac{5}{3}.\left\{\alpha h''''-\frac{3\lambda'}{4r_{\prime}^3}\right\}.(1-\mu^2).\cos.2\varpi.S_0^1.\rho.d.a^3;$$ [3568]

hence we shall find,*

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* (2293) If we substitute [3568] in [3561, 3562], the integrals relative to μ, ϖ, will become of the form of the first member of [1548e], and we may make use of this formula in finding these integrals. In this case, the terms depending on $\sin.n\varpi$, vanish, so that we may neglect $A^{(n)}$, $A'^{(n)}$, and as $i=2$, we shall have $\lambda_0=\mu^2-\frac{1}{3}$, $\lambda_2=1-\mu^2$ [1548h], $\gamma_0=\frac{9}{4}$ [1521], $\gamma_2=\frac{3}{4}$ [1520]; hence $\frac{4}{(2i+1).\gamma_0}=\frac{16}{45}$, $\frac{4}{(2i+1).\gamma_2}=\frac{16}{15}$; substituting these in [1548f, g, e], they become, respectively, [3569a] [3569b] [3569c]

$$Y^{(2)}=B^{(0)}.(\mu^2-\tfrac{1}{3})+B^{(2)}.(1-\mu^2).\cos.2\varpi;$$ [3569d]

$$Z^{(2)}=B'^{(0)}.(\mu^2-\tfrac{1}{3})+B'^{(2)}.(1-\mu^2).\cos.2\varpi;$$ [3569d′] Integral formulas.

$$\int_{-1}^{1}\int_0^{2\pi}Y^{(2)}.Z^{(2)}.d\mu.d\varpi=\frac{16\pi}{45}.B^{(0)}.B'^{(0)}+\frac{16\pi}{15}.B^{(2)}.B'^{(2)}.$$ [3569e]

In applying this to the integral, in the second member of [3561], we may put $B'^{(0)}=0$, $B'^{(2)}=1$, $Z^{(2)}=(1-\mu^2).\cos.2\varpi$ [3569d′]; then $Y^{(2)}$ will be represented by the second member of [3561], divided by this value of $Z^{(2)}$, using [3568]. Hence we get [3569f]

$$B^{(0)}=-\frac{15}{8\pi.S_0^1.\rho.d.a^5}\times\tfrac{5}{3}.\left\{\alpha h-\frac{5\lambda'}{4r_{\prime}^3}\right\}.S_0^1.\rho.d.a^3;$$ [3569g]

$$B^{(2)}=\frac{15}{8\pi.S_0^1.\rho.d.a^5}\times\tfrac{5}{3}.\left\{\alpha h''''-\frac{3\lambda'}{4r_{\prime}^3}\right\}.S_0^1.\rho.d.a^3.$$ [3569g′]

In this case, the integral [3569e] is reduced, by means of [3569f], to the single term $\frac{16\pi}{15}.B^{(2)}$, which, by the substitution of $B^{(2)}$ [3569g′], is easily reduced to the form [3569]. [3569h]

In finding the integral [3562], we must substitute $\cos.^2\varpi=\frac{1}{2}+\frac{1}{2}.\cos.2\varpi$ [6] Int., in the last factor of the expression, after putting it equal to $Z^{(2)}$, and we shall get

$$Z^{(2)}=(1-\mu^2).\cos.^2\varpi-\mu^2=\tfrac{1}{2}.(1-\mu^2)+\tfrac{1}{2}.(1-\mu^2).\cos.2\varpi-\mu^2$$
$$=-\tfrac{3}{2}.(\mu^2-\tfrac{1}{3})+\tfrac{1}{2}.(1-\mu^2).\cos.2\varpi.$$ [3569i]

Comparing this with [3569d′], we get $B'^{(0)}=-\frac{3}{2}$, $B'^{(2)}=\frac{1}{2}$. Hence the integral [3569e], becomes equal to $-\frac{8\pi}{15}.B^{(0)}+\frac{8\pi}{15}.B^{(2)}=\frac{8\pi}{15}.\{-B^{(0)}+B^{(2)}\}$; and as the values of $B^{(0)}$, $B^{(2)}$, are the same as in the former case [3569g, g′], we shall get, by substitution, and a slight reduction, the same expression as in [3570]. [3569k]

Values corres- [3569] ponding to the supposition [3570] that the moon is covered by a fluid in equilibrium.

$$\frac{B-A}{C}=\tfrac{10}{3}\cdot\left\{\alpha h''''-\frac{3\lambda'}{4r_{\prime}^{3}}\right\}\cdot\frac{S_0^1\cdot\rho\cdot d\cdot a^3}{S_0^1\cdot\rho\cdot d\cdot a^5};$$

$$\frac{C-A}{A}=\tfrac{5}{3}\cdot\left\{\alpha h+\alpha h''''-\frac{2\lambda'}{r_{\prime}^{3}}\right\}\cdot\frac{S_0^1\cdot\rho\cdot d\cdot a^3}{S_0^1\cdot\rho\cdot d\cdot a^5}.$$

In case the moon is homogeneous, the equation [3568] gives,*

[3571] $$\alpha Y^{(2)}=\tfrac{5}{3}\cdot\left(\alpha h-\frac{5\lambda'}{4r_{\prime}^{3}}\right)\cdot(\tfrac{1}{3}-\mu^2)+\tfrac{5}{3}\cdot\left(\alpha h''''-\frac{3\lambda'}{4r_{\prime}^{3}}\right)\cdot(1-\mu^2)\cdot\cos.2\varpi;$$

Calculation, supposing the moon to be homogeneous.

comparing this with the following value, deduced from [3567′],

[3572] $$\alpha Y^{(2)}=\alpha h\cdot(\tfrac{1}{3}-\mu^2)+\alpha h''''\cdot(1-\mu^2)\cdot\cos.2\varpi;$$

we obtain

[3573] $$\alpha h=\frac{25\lambda'}{8r_{\prime}^{3}};\qquad \alpha h''''=\frac{15\lambda'}{8r_{\prime}^{3}}=\tfrac{3}{5}\alpha h.$$

[3573′] We may here observe, that $\alpha h+\alpha h''''$ *expresses the excess of the first principal axis, directed towards the earth, above the polar semi-axis;*† *and*

[3574] *that* $\alpha h-\alpha h''''$ *denotes the excess of the second principal semi-axis above the polar semi-axis. In the case of homogeneity, these differences are* $\frac{40\lambda'}{8r_{\prime}^{3}}$ *and*

[3575] $\frac{10\lambda'}{8r_{\prime}^{3}}$; *therefore the first is four times the second.* In the same case we have,‡

Differences of the principal semi-axes of the moon.

Values, supposing the moon to be homogeneous. First form.

[3576] $$\frac{B-A}{C}=\frac{15\lambda'}{4r_{\prime}^{3}};\qquad \frac{C-A}{A}=\frac{5\lambda'}{r_{\prime}^{3}};$$

[3572a] * (2294) When $\rho=1$, we have $S_0^1\cdot\rho\cdot d\cdot(a^5\cdot Y^{(2)})=Y^{(2)}$, $S_0^1\cdot\rho\cdot d\cdot a^3=1$; hence [3568] becomes as in [3571]. If we compare, respectively, the coefficients of $\frac{1}{3}-\mu^2$, $(1-\mu^2)\cdot\cos.2\varpi$, in [3571, 3572], we get

[3572b] $$\alpha h=\tfrac{5}{3}\cdot\left(\alpha h-\frac{5\lambda'}{4r_{\prime}^{3}}\right);\qquad \alpha h''''=\tfrac{5}{3}\cdot\left(\alpha h''''-\frac{3\lambda'}{4r_{\prime}^{3}}\right);$$

which are easily reduced to the form [3573].

† (2295) The general expression of the moon's radius [1724, 1720″], putting $a=1$, and supposing $\alpha Y^{(0)}$ to be included in a, is

[3573a] $$1+\alpha Y^{(2)}=1+\alpha h\cdot(\tfrac{1}{3}-\mu^2)+\alpha h''''\cdot(1-\mu^2)\cdot\cos.2\varpi\ [3572].$$

The polar semi-axis corresponds to $\mu=1$, and is $1-\frac{2}{3}\alpha h$. The equatorial semi-axis,

[3573b] directed towards the earth, corresponds to $\mu=0$, $\varpi=0$ [3564′], and is $1+\frac{1}{3}\alpha h+\alpha h''''$.

[3573c] Lastly, the other equatorial semi-axis, corresponding to $\mu=0$, $\varpi=90^{d}$, is $1+\frac{1}{3}\alpha h-\alpha h''''$. The differences of these semi-axes are as in [3573′, 3574]; and if we substitute the values [3573], they become as in [3575].

‡ (2296) The density being constant, or $\rho=1$, we have $\int_0^1\rho\cdot d\cdot a^3=1$; $\int_0^1\rho\cdot d\cdot a^5=1$;

[3576a] substituting these, and [3573], in [3569, 3570], we get [3576].

$\frac{1}{r_{\prime}}$ is the apparent semi-diameter of the moon; its actual length being taken for unity [3564′]; and since by observations, this semi-diameter is equal to 2912″ [= $15^m\ 43^s,5$], we may suppose $\frac{1}{r_{\prime}} = \sin. 2912''$; hence we get* [3577]

$$\frac{B-A}{C} = 0{,}0000003618\,.\,\lambda'; \qquad \frac{C-A}{A} = 0{,}0000004824\,.\,\lambda';$$ [3578]

Second form.

the conditions [3559] that A and B are less than C, also B greater than A, are therefore satisfied. We have seen, in the fourth book, that the phenomena of the tides give nearly† $\lambda' = 59$, and then the condition [3579]
that $\frac{B-A}{C}$ is less than 0,0013876 [3560], is fulfilled; but the condition that $\frac{C-A}{A}$ is nearly equal to 0,000599 [3560], is very far from being so; and even by supposing $\lambda' = 1000$, it will not be satisfied. *Hence it follows, that the moon is not homogeneous, and that it is of a different figure from what it would assume if the mass were wholly fluid.* [3580]

The moon is not homogeneous, nor of the same form as if it were fluid.

If the moon be formed of strata of different densities, originally fluid, and retaining the figures of equilibrium which they must then have assumed, it will follow, from § 30, Book III, that the radius of the lunar spheroid will be, as in the case of homogeneity, of the form,‡ [3580′]

Calculation, supposing the moon to be formed of concentrical strata of different densities, corresponding to the original state of equilibrium.

$$1 + \alpha h\,.\,(\tfrac{1}{3} - \mu^2) + \tfrac{3}{5}\,\alpha h\,.\,(1 - \mu^2)\,.\,\cos. 2\varpi;$$ [3581]

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* (2297) The value $\frac{1}{r_{\prime}} = \sin. 2920''$, is used in computing [3578] from [3576]; so that the numbers [3578] ought to be decreased a little, to conform to the value $\sin. 2912''$, assumed in [3577]; but this difference is not of much importance. [3578a]

† (2298) The calculation of λ' is made in [4321], from the expression [2706], derived from the observations of the tides. [3579a]

‡ (2299) If we substitute the value of $Z^{(2)}$ [1724′] in [1725], and put, for brevity, the coefficient of $U^{(2)}$ equal to $-a^5 H^{-1}$, H being a function of a, we shall get

$$-\tfrac{4}{5}\pi a^5\,.\,S_a^1\,\rho\,.\,dh + \tfrac{4}{3}\pi a^2 h\,.\,S_0^a\,\rho\,.\,d\,.\,a^3 - \tfrac{4}{5}\pi\,.\,S_0^a\,\rho\,.\,d\,.\,(a^5 h) = a^5 H^{-1}; \qquad U^{(2)} = HZ^{(2)};$$ [3581a]

therefore $Y^{(2)} = hHZ^{(2)}$ [1716]. Hence the expression of the radius [1724] becomes, by including $a\alpha\,Y^{(0)}$ in the value of a, and neglecting α^2, $a + \alpha a h H Z^{(2)}$. This result [3581b]

Excess of the equatorial semi-axis. and then, as in the case of homogeneity, *the excess of the principal semi-axis, directed towards the earth, above the semi-polar axis, is four times as great as*
[3582] *that of the excess of the second principal semi-axis, above the polar semi-axis* [3575]. The equation [3568] gives,*

[3583] $$\alpha h - \tfrac{3}{5} \cdot \frac{\alpha \cdot S_0^1 \rho \cdot d \cdot (a^5 h)}{S_0^1 \rho \cdot d \cdot a^3} = \frac{5 \lambda'}{4 r_{\prime}^3}.$$

[3583'] We have seen, in [1717", &c.], that *the values of h increase from the centre to the surface, while the densities* [1709''''] *diminish*, so that we may suppose at the surface,†

[3584] $$S_0^1 \rho \cdot d \cdot (a^5 h) = (1 - q) \cdot h \cdot S_0^1 \rho \cdot d \cdot a^5;$$

[3584'] q. q *being positive;* hence we shall obtain,‡

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of the calculation [1709'''—1731] for the earth, will correspond to the moon, by substituting the value of $\alpha Z^{(2)}$, relative to the moon [3566c], by which means the radius becomes

[3581c] $$a + \tfrac{5}{4} \alpha\, a g h H . \{(\tfrac{1}{3} - \mu^2) + \tfrac{3}{5} . (1 - \mu^2) . \cos. 2 \varpi\};$$

and by changing the arbitrary symbol, $\tfrac{5}{4} \alpha g h H$, into αh, it becomes

[3581d] $$a . \{1 + \alpha h . (\tfrac{1}{3} - \mu^2) + \tfrac{3}{5} \alpha h . (1 - \mu^2) . \cos. 2 \varpi\};$$

being of the same form as in [3581], when $a = 1$.

* (2300) If we substitute, in the first member of [3568], the value

[3583a] $$\alpha Y^{(2)} = \alpha h . \{(\tfrac{1}{3} - \mu^2) + \tfrac{3}{5} . (1 - \mu^2) . \cos. 2 \varpi\} \quad [3581];$$

[3583b] and in the second member $h'''' = \tfrac{3}{5} h$ [3581d, 3572], the whole will become divisible by the factor $(\tfrac{1}{3} - \mu^2) + \tfrac{3}{5} . (1 - \mu^2) . \cos. 2 \varpi$, which is not affected with the sign of integration S, because this sign refers to a, and the quantities depending upon it. The
[3583c] quotient of the divisor is $\alpha . S_0^1 \rho . d . (a^5 h) = \tfrac{5}{3} . \left\{ \alpha h - \frac{5 \lambda'}{4 r_{\prime}^3} \right\} . S_0^1 \rho . d . a^3$. Dividing this by $\tfrac{5}{3} . S_0^1 \rho . d . a^3$, we get [3583].

† (2301) The value of h in the second member of [3584] corresponds to the surface, and for distinction, we may denote it by h_1, as in [1721d]. Then we shall have

[3584a] $h < h_1$; $a^5 h < h_1 . a^5$; $d . (a^5 h) < h_1 . d . a^5$, and $S_0^1 \rho . (a^5 h) < h_1 . S_0^1 \rho . d . a^5$,

which is satisfied by the equation [3584], supposing q to be a positive quantity.

‡ (2302) Substituting [3584] in [3583], we get $\alpha h_1 - \tfrac{3}{5} . \frac{(1-q) . \alpha h_1 . S_0^1 \rho . d . a^5}{S_0^1 \rho . d . a^3} = \frac{5 \lambda'}{4 r_{\prime}^3}$.

[3585a] Dividing this by the coefficient of αh_1, we get the value of αh_1 [3585], corresponding to the surface.

$$\alpha h = \frac{\frac{5}{4} \cdot \frac{\lambda'}{r_{\prime}^3}}{1 - \frac{3}{5} \cdot (1-q) \cdot \frac{S_0^1 \rho \cdot d \cdot a^5}{S_0^1 \rho \cdot d \cdot a^3}} . \qquad [3585]$$

Now we have $h'''' = \frac{3}{5} h$ [3583b]; therefore we shall find,*

Values, corresponding [3586] to the supposition that the strata are of the same form [3587] as if the moon were fluid at the origin.

$$\frac{B-A}{C} = \frac{\frac{3}{2} \cdot \frac{\lambda'}{r_{\prime}^3} \cdot (1-q) \cdot S_0^1 \rho \cdot d \cdot a^3}{S_0^1 \rho \cdot d \cdot a^3 - \frac{3}{5} \cdot (1-q) \cdot S_0^1 \rho \cdot d \cdot a^5} ; \qquad [3586]$$

$$\frac{C-A}{A} = \frac{\frac{2\lambda'}{r_{\prime}^3} \cdot (1-q) \cdot S_0^1 \rho \cdot d \cdot a^3}{S_0^1 \rho \cdot d \cdot a^3 - \frac{3}{5} \cdot (1-q) \cdot S_0^1 \rho \cdot d \cdot a^5} . \qquad [3587]$$

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* (2303) The investigation of these equations will serve as another example of the use of the formula [3569e]. Substituting the value of $\alpha Y^{(2)}$ [3583a], in $S_0^1 \cdot \rho \cdot d \cdot (a^5 Y^{(2)})$, we may bring the terms depending on μ, ϖ, from under the sign S, and it will become as in [3586b]; then, by using successively the values of $B'^{(0)}$, $B'^{(2)}$ [3586a], it changes into the value of $Z^{(2)}$ [3569d', 3586b'];

$$B'^{(0)} = -\alpha \cdot S_0^1 \rho \cdot d \cdot (a^5 h) ; \qquad B'^{(2)} = \tfrac{3}{5} \cdot \alpha \cdot S_0^1 \rho \cdot d \cdot (a^5 h) ; \qquad [3586a]$$

$$\alpha \cdot S_0^1 \rho \cdot d \cdot (a^5 Y^{(2)}) = \alpha \cdot \{(\tfrac{1}{3} - \mu^2) + \tfrac{3}{5} \cdot (1-\mu^2) \cdot \cos. 2\varpi\} \cdot S_0^1 \rho . d \cdot (a^5 h) \qquad [3586b]$$

$$= B'^{(0)} \cdot (\mu^2 - \tfrac{1}{3}) + B'^{(2)} \cdot (1-\mu^2) \cdot \cos. 2\varpi = Z^{(2)}. \qquad [3586b']$$

Substituting, in [3569e], these values of $B'^{(0)}$, $B'^{(2)}$, $Z^{(2)}$, we get the following expression,

$$\int_{-1}^{1} \int_0^{2\pi} Y^{(2)} . Z^{(2)} . d\mu . d\varpi = -\frac{16\pi}{45} . B^{(0)} . \alpha . S_0^1 \rho . d . (a^5 h) + \frac{16\pi}{25} \cdot B^{(2)} \cdot \alpha \cdot S_0^1 \rho \cdot d \cdot (a^5 h) ; \qquad [3586c]$$

which may be used in finding the integrals of [3561, 3562], relatively to μ, ϖ. If we put, for brevity, $F = \frac{15}{8\pi . S_0^1 \rho . d . a^5}$, and use the expression [3586b'], the equations [3586d] [3561, 3562] will become,

$$\frac{B-A}{C} = F . \int_{-1}^{1} \int_0^{2\pi} \{(1-\mu^2) \cdot \cos. 2\varpi\} . Z^{(2)} . d\mu . d\varpi ; \qquad [3586e]$$

$$\frac{C-A}{A} = F . \int_{-1}^{1} \int_0^{2\pi} \{(1-\mu^2) \cdot \cos.^2 \varpi - \mu^2\} . Z^{(2)} . d\mu . d\varpi . \qquad [3586f]$$

Comparing the second member of [3586e] with the first of [3586c], we get

$$Y^{(2)} = F . (1-\mu^2) \cdot \cos. 2\varpi ;$$

hence we have, from the equation [3569d], $B^{(0)} = 0$, $B^{(2)} = F$; and the integral [3586e,c], [3586g] becomes as in the first expression [3586h]; which is easily reduced to the third form [3586h], by the substitution of F [3586d], and the integral [3584];

$$\frac{B-A}{C} = \frac{16\pi}{25} \cdot F . \alpha . S_0^1 \rho \cdot d \cdot (a^5 h) = \frac{6\alpha . S_0^1 \rho . d . (a^5 h)}{5 S_0^1 \rho . d . a^5} = \tfrac{6}{5} \cdot (1-q) \cdot \alpha h_1 . \qquad [3586h]$$

[3588] *It is evident, that the case of homogeneity is that in which the value of* $\frac{C-A}{A}$ *is the greatest, since the densities decrease from the centre to the surface,*
[3589] *and* $S_0^1 \rho \,.\, d \,.\, a^3$ *is greater than* $S_0^1 \rho \,.\, d \,.\, a^5$.* *Now we have just seen* [3580], *that the moon, being homogeneous, the value of* $\frac{C-A}{A}$ *is considerably less than by observation; therefore the moon has not*
[3590] *the figure of equilibrium, which it would have assumed, if it had been originally fluid.*

The moon has not the figure which corresponds to the original state of equilibrium.

Now, substituting the value of αh_1 [3585]; also multiplying the numerator and denominator by $S_0^1 \rho \,.\, d \,.\, a^3$, we get [3586]. In like manner, we may find the integral of [3586f]; for by substituting $\cos.^2 \varpi = \frac{1}{2} + \frac{1}{2} \,.\, \cos. 2\varpi$, in the factor $F \,.\, \{(1-\mu^2) \,.\, \cos.^2 \varpi - \mu^2\}$,
[3586i] it becomes $-\frac{3}{2} F \,.\, (\mu^2 - \frac{1}{3}) + \frac{1}{2} F \,.\, (1-\mu^2) \,.\, \cos. 2\varpi$; comparing this with $Y^{(2)}$ [3569d],
[3586k] we get $B^{(0)} = -\frac{3}{2} F$, $B^{(2)} = \frac{1}{2} F$; hence the integral [3586c, f] becomes as in the first expression [3586l], which is reduced, successively, by the substitution of F [3586d] and of the integral [3584];

[3586l]
$$\frac{C-A}{A} = \frac{8\pi}{15} \,.\, F \,.\, \alpha \,.\, S_0^1 \rho \,.\, d \,.\, (a^5 h) + \frac{8\pi}{25} \,.\, F \,.\, \alpha \,.\, S_0^1 \rho \,.\, d \,.\, (a^5 h)$$
$$= \frac{64\pi}{75} \,.\, F \,.\, \alpha \,.\, S_0^1 \rho \,.\, d \,.\, (a^5 h) = \frac{8 S_0^1 \rho . d . (a^5 h) \alpha}{5 S_0^1 \rho . d . a^5} = \frac{8}{5} \,.\, (1-q) \,.\, \alpha h_1.$$

Substituting the value of αh_1 [3585], also multiplying the numerator and denominator by $S_0^1 \,.\, \rho \,.\, d \,.\, a^3$, we get [3587].

* (2304) If the moon be homogeneous, and the density $\rho = 1$, the integral
[3589a] $S_0^1 \rho \,.\, d \,.\, (a^5 h)$ will become h_1, and $S_0^1 \rho \,.\, d \,.\, a^5$ will be equal to unity. Substituting these in [3584], we get $h_1 = (1-q) \,.\, h_1$, or $1 = 1-q$; hence we have for the case of homogeneity $q=0$. If the densities increase towards the centre, we shall have, by substituting $S_0^1 \rho \,.\, a^4 da = \frac{1}{5} \,.\, S_0^1 \rho \,.\, d \,.\, a^5$; $S_0^1 \rho \,.\, a^2 da = \frac{1}{3} \,.\, S_0^1 d \,.\, a^3$ in [3412b];
[3589b] $\frac{\frac{1}{5} . S_0^1 \rho . d . a^5}{\frac{1}{3} . S_0^1 \rho . d . a^3} < \frac{3}{5}$; so that we may put $S_0^1 \rho \,.\, d \,.\, a^5 = (1-q') \,.\, S_0^1 \rho \,.\, d \,.\, a^3$, q' *being a positive quantity.* Substituting this in [3587], then dividing the numerator and denominator by $(1-q) \,.\, S_0^1 \rho \,.\, d \,.\, a^3$, developing also the fraction, whose denominator is $1-q$, in an infinite series, according to the powers of q, we get

[3589c]
$$\frac{C-A}{A} = \frac{2\lambda'}{r_{\prime}^3} \,.\, \frac{1}{\frac{1}{1-q} - \frac{3}{5} \,.\, (1-q')} = \frac{1}{\frac{2}{5} + \frac{3}{5} q' + q + q^2 + \&c.}.$$

All the terms of the denominator of this last expression are *positive*, therefore the greatest
[3589d] value of $\frac{C-A}{A}$ must correspond with $q=0$, or the case of homogeneity.

We may imagine an infinite number of hypotheses, in which the momenta of inertia A, B, C, satisfy the preceding conditions. There is no doubt [3591] *that the high mountains, and the other inequalities, which we perceive on the moon's surface, have a great influence in the differences of these momenta of inertia,* because the oblateness of the lunar spheroid is very minute, and [3591′] the whole mass is but of small magnitude.

19. *It now remains to consider the influence of the sun's action upon the motions of the lunar equator;* but without entering upon a minute discussion of this action, *it is easy to prove that it is insensible.* For as S denotes the [3592] mass of the sun, and r'' its mean distance from the moon, or from the earth, this action is of the order $\frac{S}{r''^3}$ [2192]; therefore this force is to that of the earth upon the moon, as $\frac{S}{r''^3}$ to $\frac{L}{r_{,}^3}$. Now the theory of central [3592′] forces gives this ratio equal to that of the square of the time of the sideral revolution of the moon, to the square of the time of the sideral revolution of the earth,* which is nearly equal to $\frac{1}{178}$. *Hence we see that the sun's action on the lunar spheroid, may be neglected, in comparison with the action* [3593] *of the earth upon the same body.*

The sun's influence on the moon's figure is insensible.

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* (2305) If in [709], we change M, m', a, h, into S, L, r'', $r_{,}$, respectively, and neglect the mass of the moon p, in comparison with that of the earth m', it will become $\frac{L}{S}=\frac{r_{,}^3}{r''^3}\cdot\frac{T^2}{\mathrm{T}^2}$, from which we easily get $\frac{S}{r''^3}:\frac{L}{r_{,}^3}::\mathrm{T}^2:T^2$, T being the time of the [3592a] sun's sideral revolution, and T that of the moon's, as in [3592′, &c.].

CHAPTER III.

ON THE MOTIONS OF SATURN'S RINGS ABOUT THEIR CENTRES OF GRAVITY.

20. In treating of the figure of Saturn's rings [2070—2116], we have
[3593'] seen, that *each ring is a solid body, whose centre of figure coincides nearly with the centre of Saturn; but that the centre of gravity of the ring can be, and must be, at some distance from the centre of Saturn* [2116'''']. *This centre of gravity revolves about the planet in the same time as the ring; and it is evident, that the ring revolves about its centre of gravity in the same time as about Saturn* [2116'''']. *The action of the sun and the satellites upon these rings, must produce in their planes, a motion of precession similar to that*
[3594] *of the equator of the earth. This action being different for each of the rings, it would seem that their motions must be different, and that finally the rings*
[3594'] *would cease to be in nearly the same plane; which appears to be contrary to observation.* For, although two centuries have not yet elapsed since the discovery of the rings, yet if they were not compelled to move in the same
[3594''] plane, we must suppose that they were discovered precisely at the epoch when their planes coincided, which is very improbable. *Therefore it is probable, that there is a cause which retains them all in one fixed or moveable plane.* But what is this cause? *To investigate it, is the object of the following analysis.*

The solid rings which revolve about Saturn like satellites, have a motion of precession.

21. We may here use the equations [2905—2907], and for this purpose,
[3594'''] we must ascertain the values of dN, dN', dN'', relative to the action of Saturn on the ring, and the action of a distant body L. We shall
[3594''''] first consider the action of Saturn, and shall put V *for the sum of all the particles of this planet, divided by their respective distances from any*
V, r', μ.
[3595] *particle* dm *of the ring;* r' *for the radius drawn from this particle to the*
[3596] *centre of Saturn, and* μ *for the cosine of the angle, which this radius makes*

with the axis of rotation of this planet. We shall represent the radius of the spheroid of Saturn by the following expression ;*

$$1+\alpha\,Y^{(2)}+\alpha\,Y^{(3)}+\alpha\,Y^{(4)}+\&c. = \text{radius of Saturn};$$ [3597] Radius.

and shall put $\alpha\varphi'$, *for the ratio of the centrifugal force to gravity, at its equator; the mass of Saturn being taken for unity.* Then we shall have, as in [1811, &c.],† [3598] $\alpha\varphi'$, [3598′]

Value of V, corresponding to Saturn. First form.

$$V=\frac{1}{r'}+\frac{\alpha.\{Y^{(2)}+\frac{1}{2}\varphi'.(\mu^2-\frac{1}{3})\}}{r'^3}+\frac{\alpha\,Y^{(3)}}{r'^4}+\frac{\alpha\,Y^{(4)}}{r'^5},\ \&c.$$ [3599]

This value of V will be reduced to nearly its two first terms, if r' be somewhat large in comparison with the *radius of the spheroid of Saturn, taken for the unit of distance.* Moreover, if this planet be a spheroid of revolution,‡ as is natural to suppose, we shall have $Y^{(3)}=0$, $Y^{(4)}=0$, &c. ; which renders the reduction of V to its two first terms correct ; therefore we may suppose, [3600] [3601]

$$V=\frac{1}{r'}+\frac{\alpha.\{Y^{(2)}+\frac{1}{2}\varphi'.(\mu^2-\frac{1}{3})\}}{r'^3}.$$ [3602] Second form.

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* (2306) This is of the same general form as in [3422], $Y^{(0)}$, $Y^{(1)}$, being neglected, as in [1498′, 1498″] ; upon the supposition, that $Y^{(0)}$ is included in the constant term of the radius [1480′, 1704″] ; and that the origin of the radius is at the centre of gravity of the spheroid [1483vi, 1745]. [3597a]

† (2307) The general expression of the mass M of Saturn, may be put, as in [1811′], $M=\frac{4}{3}\pi.S_0^1\rho.d.a^3$; the densities of the strata of which it is composed being variable. This is taken for unity [3598′] ; hence $M=\frac{4}{3}\pi.S_0^1\rho.d.a^3=1$; substituting this in [1811], and neglecting $Z^{(3)}$, $Z^{(4)}$, &c., as in [1720″], we get [3599a]

$$V=\frac{1}{r}+\frac{\alpha}{r^3}.\left\{Y^{(2)}+\frac{Y^{(3)}}{r}+\frac{Y^{(4)}}{r^2}+\&c.\right\}-\frac{\alpha}{r^3}.Z^{(2)}.$$ [3599b]

Now if we substitute the preceding value of M in $\alpha Z^{(2)}$ [1793], we shall get

$$\alpha Z^{(2)}=-\tfrac{1}{2}\alpha\varphi.(\mu^2-\tfrac{1}{3});$$ [3599c]

hence V [3599b] becomes as in [3599], the quantities φ, r, being accented, to conform to the present notation [1726′, 3595, &c.].

‡ (2308) The author here supposes the planet to be an *ellipsoid* of revolution, as in the case mentioned in [1787‴—1792] ; where the planet is composed of elliptical strata, of different densities. For then we shall have $Y^{(3)}=0$, $Y^{(4)}=0$, &c. [1792], as in [3601], and the function [3599] will be reduced to the form [3602]. [3600a]

The function $Y^{(2)}$ is reduced, as we have seen in [2958], to the following form,

[3603] $$Y^{(2)}=h\,.\,(\tfrac{1}{3}-\mu^2)+h''''\,.\,(1-\mu^2)\,.\,\cos.\,2\varpi.$$

[3604] If Saturn be a solid of revolution, h'''' will vanish.* Moreover, in case the quantity h'''' is comparable with h, it is easy to prove, that its influence on the motions of the ring is insensible; because of the rapidity of the
[3605] earth's rotatory motion. Therefore we shall suppose $h''''=0$, consequently

[3606] Third form of V.
$$V=\frac{1}{r'}-\frac{(\alpha h-\frac{1}{2}\alpha\varphi')\,.\,(\mu^2-\frac{1}{3})}{r'^3};$$

[3606′] Oblateness of Saturn. *αh being evidently the oblateness of Saturn* [3604d].

x', y', z'. Now x', y', z', being the co-ordinates of the particle dm, relative to the
[3607] centre of gravity of the ring, we have, as in [2968—2970],†

General values of

[3608] dN,
$$\frac{dN}{dt}=S\,.\,dm\,.\left\{x'\,.\left(\frac{dV}{dy'}\right)-y'\,.\left(\frac{dV}{dx'}\right)\right\};$$

[3609] dN',
$$\frac{dN'}{dt}=S\,.\,dm\,.\left\{x'\,.\left(\frac{dV}{dz'}\right)-z'\,.\left(\frac{dV}{dx'}\right)\right\};$$

[3610] dN''.
$$\frac{dN''}{dt}=S\,.\,dm\,.\left\{y'\,.\left(\frac{dV}{dz'}\right)-z'\,.\left(\frac{dV}{dy'}\right)\right\}.$$

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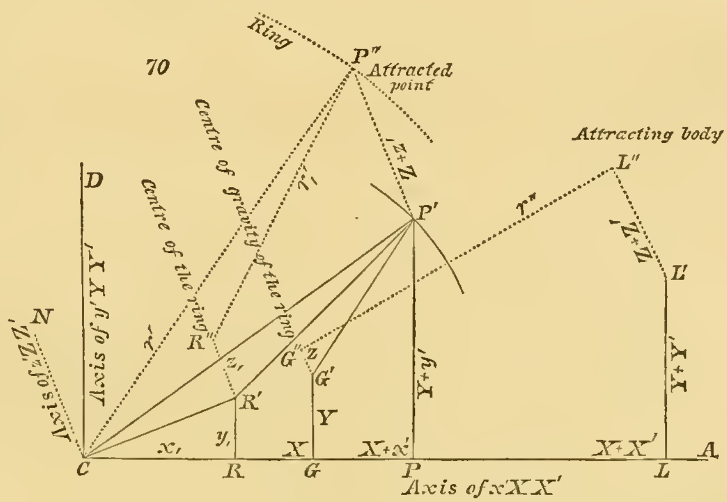
* (2309) If Saturn be a solid of revolution, ϖ will not occur in the radius of the spheroid $1+\alpha Y^{(2)}$ [3597, 3603]; therefore we must have $h''''=0$; and this radius
[3604a] will become $1+\alpha h\,.\,(\frac{1}{3}-\mu^2)$. Moreover, if h'''' be finite, its effect on the rings will be very much decreased, in consequence of the rapid motion of Saturn about its axis, by which
[3604b] means any protuberant part of the planet, corresponding to h'''', is made to act in *every different direction* upon the ring, during the revolutions of the planet about its axis; and its effect in one situation, is nearly balanced by the contrary effect in another position; so that,
[3604c] on the whole, the effect must be nearly insensible, and we may put $h''''=0$. In this case, the expression of the radius [3604a] becomes $1-\frac{2}{3}\alpha h$, at the pole, where $\mu=1$;
[3604d] and at the equator, where $\mu=0$, it becomes $1+\frac{1}{3}\alpha h$; the difference of these two expressions is αh, representing the oblateness of Saturn, as in [3606′].

† (2310) For the purpose of illustrating the notation here used, we shall refer to the annexed figure 70. The plane of the figure represents the equator of Saturn, taken as
[3608a] the plane of x', y' [3614]; C is the centre of gravity of Saturn; CA is the axis of x', or X', *parallel to the direction of the line of intersection of the equator and the plane of the ring* [3637]; CD, perpendicular to CA, is the axis of y', or Y'; CN, perpendicular
[3608a'] to the plane of the figure, is the axis of revolution of Saturn, or the axis of z', or Z'.

To determine V, *we shall neglect the thickness of the ring, and shall* [3611]
consider it as a circular arc of unequal density, in the different parts of its
circumference, its centre being near to the centre of Saturn. Putting $X, Y, Z.$
$X, Y, Z,$ *for the co-ordinates of the centre of gravity of the ring, referred* [3612]
to the centre of Saturn; $X+x'$, $Y+y'$, $Z+z'$, *will be the co-ordinates* $x', y', z'.$
of the particle dm, *referred to the same centre. If we take the equator of* [3613]

These axes are similarly situated to those in figure 58, page 803, supposing the parts A, B, to coincide, as in [3006*f*]. The axes of X'', Y'', Z'', [3631], are parallel to the [3608*b*]
three principal axes of the ring, corresponding to CG, CH, CN, respectively, in figure 58 [2907*c*—*e*]. R'' is the centre of figure of the ring; G'' its centre of gravity; P'' the place of the attracted particle dm; L'' the place of the attracting body L. The dotted lines $L'L''$, $R'R''$, $G'G''$, $P'P''$, CN, are perpendicular to the plane of the figure, or parallel to the axis of revolution of Saturn, taken as the axis of Z. [3608*b'*]



LL', RR', GG', PP', CD, are perpendicular to CA. Then, by the above notation, we have,

$$CR = x_{,}, \qquad RR' = y_{,}, \qquad R'R'' = z_{,}; \tag{3608c}$$
$$CG = X, \qquad GG' = Y, \qquad G'G'' = Z; \tag{3608d}$$
$$CP = X + x', \qquad PP' = Y + y', \qquad P'P'' = Z + z'; \tag{3608e}$$
$$CL = X + X', \qquad LL' = Y + Y', \qquad L'L'' = Z + Z'; \tag{3608f}$$
$$CP'' = r', \qquad R''P'' = r'_{,}, \qquad G''L'' = r''. \tag{3608g}$$

Now the formulas [2968—2970] give the values of $\frac{dN}{dt}$, $\frac{dN'}{dt}$, $\frac{dN''}{dt}$, corresponding to the attraction of a foreign body L, upon a particle dm of the earth, whose co-ordinates are x', y', z' [2965], referred to the centre of gravity of the earth; and it is evident, that [3608*h*]
the same formulas may be used, as in [3608—3610], to represent the action of Saturn upon a particle dm of the ring, whose co-ordinates are x', y', z' [3607], referred to the [3608*i*]
centre of gravity of the ring.

[3614] *Saturn for the plane of* x', y', which we shall at first suppose to be invariable, we shall have,*

Distance of a point of the ring from Saturn's centre.

[3615] $$r' = \sqrt{\{(X+x')^2+(Y+y')^2+(Z+z')^2\}}; \qquad \mu = \frac{Z+z'}{r'}.$$

$x_{,}, y_{,}, z_{,}$.

We shall put $x_{,}$, $y_{,}$, $z_{,}$, *for the co-ordinates of the centre of the circumference*
[3616] *of the ring, referred to the centre of Saturn;* these co-ordinates being supposed so small, that we may neglect the squares and their products by α; we shall have,†

[3617] $$r' = r'_{,} + \frac{x_{,}.(X+x')+y_{,}.(Y+y')+z_{,}.(Z+z')}{r'_{,}};$$

---

* (2311) From the principles of the orthographic projection, we have

[3615a] $$CP'' = \sqrt{\{CP^2+PP'^2+P'P''^2\}},$$

and by using the symbols [3608e, g], we get r' [3615]. From [3596], we have,

[3615b] $$\mu = \cos. NCP'' = \sin. P''CP' = \frac{P'P''}{CP''} = \frac{Z+z'}{r'} \quad [3608e, g].$$

† (2312) Subtracting the rectangular co-ordinates of the centre of the ring R'' [3608c], from the corresponding co-ordinates of the attracted point P'' [3608e], C being the origin of these co-ordinates; we get the rectangular co-ordinates of the attracted point P'',
[3617a] *referred to the centre of the ring, as their origin,* namely, $X+x'-x_{,}$, $Y+y'-y_{,}$, $Z+z'-z_{,}$. The square root of the sum of the squares of these co-ordinates, gives, in like manner as in [3615a], the distance between the points $R''P''$, or the radius of the ring $r'_{,}$ [3608g]; hence we get the first value of $r'_{,}$ [3617c]. Developing this expression
[3617b] according to the powers of $x_{,}$, $y_{,}$, $z_{,}$, neglecting terms of the second and higher orders of these quantities, it becomes as in [3617d]; substituting r' [3615], we finally obtain [3617e], which is the same as in [3617];

[3617c] $$r'_{,} = \{(X+x'-x_{,})^2+(Y+y'-y_{,})^2+(Z+z'-z_{,})^2\}^{\frac{1}{2}};$$

[3617d] $$= \{(X+x')^2+(Y+y')^2+(Z+z')^2-2x_{,}.(X+x')-2y_{,}.(Y+y')-2z_{,}.(Z+z')\}^{\frac{1}{2}};$$

$$= \{r'^2-2x_{,}.(X+x')-2y_{,}.(Y+y')-2z'.(Z+z')\}^{\frac{1}{2}};$$

[3617e] $$= r' - \frac{1}{r'}.\{x_{,}.(X+x')+y_{,}.(Y+y')+z_{,}.(Z+z')\}.$$

Now as r' differs from $r'_{,}$ only by quantities of the order $x_{,}$, we may change the divisor r' into $r'_{,}$, in the terms of the second member, and then, by transposition, we get r' [3617]. Substituting this in [3606], neglecting $x_{,}^2$, $y_{,}^2$, $z_{,}^2$, $\alpha x_{,}$, $\alpha y_{,}$, $\alpha z_{,}$, we obtain

[3617f] $$V = \frac{1}{r'_{,}} - \left\{ \frac{x_{,}.(X+x')+y_{,}.(Y+y')+z_{,}.(Z+z')}{r'^3_{,}} + \frac{(\alpha h-\frac{1}{2}\alpha\varphi').(\mu^2-\frac{1}{3})}{r'^3_{,}} \right\}.$$

$r_{\prime}'$ being the radius of the circumference of the ring. If we then observe, that by the nature of the centre of gravity of the ring, we have [3618]

$$\int x'\,dm = 0; \qquad \int y'\,dm = 0; \qquad \int z'\,dm = 0 \qquad [3126']; \tag{3619}$$

we shall obtain,*

Values produced by the action of Saturn. First form.

$$\frac{dN}{dt} = 0; \tag{3620}$$

$$\frac{dN'}{dt} = -\frac{2\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}'^{5}}.\int x'z'.dm; \tag{3621}$$

$$\frac{dN''}{dt} = -\frac{2\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}'^{5}}.\int y'z'.dm. \tag{3622}$$

We shall now suppose that the inclination θ, *of the plane of the ring to the plane of the equator, is very small; so that we may neglect its square*, which [3623] θ.

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* (2313) The partial differentials of the value of μ [3615], give, by neglecting terms of the order z', in [3619a], or $\alpha z'^2$, in [3619b];

$$\left(\frac{d\mu}{dx'}\right)=0; \qquad \left(\frac{d\mu}{dy'}\right)=0; \qquad \left(\frac{d\mu}{dz'}\right)=\frac{1}{r'}. \tag{3619a}$$

In finding the values of $\left(\frac{dV}{dx'}\right)$, $\left(\frac{dV}{dy'}\right)$, $\left(\frac{dV}{dz'}\right)$, from [3617f], we must consider $r_{\prime}'$ as a constant quantity, because, by hypothesis, the ring is a circle, whose radius is $r_{\prime}'$; hence we obtain,

$$\left(\frac{dV}{dx'}\right)=-\frac{x_{\prime}}{r_{\prime}'^{3}}; \qquad \left(\frac{dV}{dy'}\right)=-\frac{y_{\prime}}{r_{\prime}'^{3}}; \qquad \left(\frac{dV}{dz'}\right)=-\frac{z_{\prime}}{r_{\prime}'^{3}}-\frac{2.(\alpha h-\frac{1}{2}\alpha\varphi').\mu}{r_{\prime}'^{4}}. \tag{3619b}$$

In substituting these values in the formulas [3608—3610], we may bring the terms $x_{\prime}$, $y_{\prime}$, $z_{\prime}$, $r_{\prime}'$, h, φ', from under the sign S, because they are the same for all the particles dm; and we shall get,

$$\frac{dN}{dt} = -\frac{y_{\prime}}{r_{\prime}'^{3}}.Sx'dm+\frac{x_{\prime}}{r_{\prime}'^{3}}.Sy'dm; \tag{3619c}$$

$$\frac{dN'}{dt} = -\frac{z_{\prime}}{r_{\prime}'^{3}}.Sx'dm+\frac{x_{\prime}}{r_{\prime}'^{3}}.Sz'dm-\frac{2.(\alpha h-\frac{1}{2}\alpha\varphi')}{r_{\prime}'^{4}}.Sx'\mu.dm; \tag{3619d}$$

$$\frac{dN''}{dt} = -\frac{z_{\prime}}{r_{\prime}'^{3}}.Sy'dm+\frac{y_{\prime}}{r_{\prime}'^{3}}.Sz'dm-\frac{2.(\alpha h-\frac{1}{2}\alpha\varphi')}{r_{\prime}'^{4}}.Sy'\mu.dm. \tag{3619e}$$

All the terms of these expressions vanish, except those containing μ, by using the equations [3619]. In substituting the value μ [3615], in $Sx'\mu.dm$, $Sy'\mu.dm$, we may bring [3619f] the term Z from under the sign S, and then this term Z will be multiplied by $Sx'dm$, $Sy'dm$, which are equal to nothing [3619]; therefore we may neglect Z in this value of μ, and put simply $\mu=\frac{z'}{r_{\prime}'}$, in [3619d—e], neglecting all the other terms; by this [3619g] means we obtain [3620—3622]

[3624] is the same as to suppose $\sin.\theta = \theta$, $\cos.\theta = 1$; we shall then take
[3624′] for the axis of x', the intersection of the plane of the ring with the equator of Saturn. This being premised, the values of x', y', z', Book I, § 26 [227], will become,*

[3625] $$x' = x''.\cos.\varphi - y''.\sin.\varphi;$$

[3626] $$y' = x''.\sin.\varphi + y''.\cos.\varphi + z''.\theta;$$

[3627] $$z' = z'' - y''.\theta.\cos.\varphi - x''.\theta.\sin.\varphi.$$

Hence we obtain, from the same article,†

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* (2314) We may, for the sake of illustration, refer, as in [3608b], to figure 58, page 803, supposing ADP to be the plane of Saturn's equator, BOP the plane of the
[3625a] ring; CG the *first* principal axis of the ring, or the axis of x''; CH the *second* principal
[3625b] axis, or axis of y''; CN the *third* principal axis, or axis of z''. The general values of x', y', z', in terms of x'', y'', z'', are given in [227]; but they are much simplified
[3625c] in the present case, by putting $\sin\theta = \theta$, $\cos.\theta = 1$ [3624]; observing, moreover, that
[3625d] the angle $ACB = \psi = 0$, $\sin.\psi = 0$, $\cos.\psi = 1$; because CA, the axis of x', is supposed, in [3624′], to coincide with CB, parallel to the line of intersection of the ring
[3625e] and equator. These values of θ, ψ, being substituted in [227], we get [3625—3627].

† (2315) Substituting the values of x', y', z' [3625—3627], in $\int x'z'.dm$, $\int y'z'.dm$,
[3628a] we may neglect the products $x''y''$, $x''z''$, $y''z''$, as is evident from [228]; and we shall get, by means of the values of A, B, C, $2s$ [246c, b], the following expressions;

[3628b]
$$\begin{aligned} S\,x'z'.dm &= -\theta.\sin.\varphi.\cos.\varphi.S\,x''^2.dm + \theta.\sin.\varphi.\cos.\varphi.S\,y''^2.dm \\ &= \theta.\sin.\varphi.\cos.\varphi.\{S\,y''^2.dm - S\,x''^2.dm\} \\ &= \theta.\sin.\varphi.\cos.\varphi.\{(s-B)-(s-A)\} = \theta.\sin\varphi.\cos.\varphi.\{A-B\} \end{aligned}$$

[3628c]
$$= \tfrac{1}{2}\theta.\sin.2\varphi.(A-B);$$

[3628d]
$$\begin{aligned} S\,y'z'.dm &= -\theta.\sin.^2\varphi.S\,x''^2.dm - \theta.\cos.^2\varphi.S\,y''^2.dm + \theta.S\,z''^2.dm \\ &= -\theta.\sin.^2\varphi.(s-A) - \theta.\cos.^2\varphi.(s-B) + \theta.(s-C) \\ &= s\theta.(-\sin.^2\varphi - \cos.^2\varphi + 1) + A\theta.\sin.^2\varphi + B\theta.\cos.^2\varphi - C\theta \\ &= \theta.\{A.\sin.^2\varphi + B.\cos.^2\varphi - C\} \\ &= \theta.\{\tfrac{1}{2}A.(1-\cos.2\varphi) + \tfrac{1}{2}B.(1+\cos.2\varphi) - C\} \end{aligned}$$

[3628e]
$$= \theta.\{\tfrac{1}{2}.(A+B-2C) + \tfrac{1}{2}.(B-A).\cos.2\varphi\}.$$

Substituting [3628c, e] in [3620—3622], we get [3628—3630], respectively. We may
[3628f] observe, as in [2914], that A, B, C, represent the momenta of inertia of the ring, relative to its *first*, *second*, and *third* principal axes, respectively.

Values produced by the action of Saturn. Second form.

$$\frac{dN}{dt} = 0;$$ [3628]

$$\frac{dN'}{dt} = \frac{\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}^{\prime 5}}.(B-A).\theta.\sin.2\varphi;$$ [3629]

$$\frac{dN''}{dt} = -\frac{\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}^{\prime 5}}.(A+B-2C).\theta - \frac{\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}^{\prime 5}}.(B-A).\theta.\cos.2\varphi.$$ [3630]

We shall now investigate the values of $\frac{dN}{dt}$, $\frac{dN'}{dt}$, $\frac{dN''}{dt}$, *relatively to the action of any body situated at a considerable distance from the ring. We shall put* X'', Y'', Z'', *for the three co-ordinates of the attracting body, referred to the centre of gravity of the ring, and drawn parallel to its three principal axes*; r'' for the distance of the body from this centre; then we shall have, as in [2982—2984], by referring dN, dN', dN'' to the same co-ordinates,* [3631] X'', Y'', Z''. [3632] [3632']

General values depending on the action of a distant body. First form

$$\frac{dN}{dt} = \frac{3L}{r''^5}.(B-A).X''Y'';$$ [3633]

$$\frac{dN'}{dt} = \frac{3L}{r''^5}.(C-A).X''Z'';$$ [3634]

$$\frac{dN''}{dt} = \frac{3L}{r''^5}.(C-B).Y''Z''.$$ [3635]

We may, without any sensible error, in these expressions, suppose the origin of the co-ordinates X'', Y'', Z'', and of the radius r'', to be in the centre of Saturn. Putting X', Y', Z', for the co-ordinates of the body L, referred to the plane of the equator of Saturn, *the axis of* X' *being the line of intersection of the plane of the equator and that of the ring* [3608a]; we shall have, between X', Y', Z', X'', Y'', Z'', the same relations as we have found between x', y', z', x'', y'', z'' [235]. Hence we deduce,† [3636] X', Y', Z'. [3637]

$$X'' = X'.\cos.\varphi + Y'.\sin.\varphi - \theta.Z'.\sin.\varphi;$$ [3638]

$$Y'' = Y'.\cos.\varphi - X'.\sin.\varphi - \theta.Z'.\cos.\varphi;$$ [3639]

$$Z'' = Z' + \theta.Y';$$ [3640]

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* (2316) These are the same as the formulas [2982—2984], changing the co-ordinates x, y, z, of the attracting body, referred to the centre of gravity of the earth and its three principal axes [2964, 2978, &c.], into the co-ordinates X'', Y'', Z'' [3632], referred to the centre of gravity and the three principal axes of the ring. [3633a]

† (2317) Changing, in [235], x', y', z', into X', Y', Z'; also x'', y'', z'', into X'', Y'', Z'', respectively; and using the values of θ, ψ [3625c, d], we get [3638—3640]. [3638a]

consequently,*

[3641] $X''Y''=\frac{1}{2}.(Y'^2-X'^2).\sin.2\varphi+X'Y'.\cos.2\varphi-\theta Z'.\{Y'.\sin.2\varphi+X'.\cos.2\varphi\};$

[3642] $X''Z''=Y'Z'.\sin.\varphi+X'Z'.\cos.\varphi+\theta.\{(Y'^2-Z'^2).\sin.\varphi+X'Y'.\cos.\varphi\};$

[3643] $Y''Z''=Y'Z'.\cos.\varphi-X'Z'.\sin.\varphi+\theta.\{(Y'^2-Z'^2).\cos.\varphi-X'Y'.\sin.\varphi\}.$

[3644] *We shall put* v *for the angle which the radius* r'' *makes with the line*
[3645] *of intersection of the orbit of* L, *and the equator of Saturn;* ψ *for the angle which this intersection forms with the intersection of the plane of the*
[3646] *ring and the equator; also* θ' *for the inclination of the orbit of* L, *to the plane of the equator of Saturn;* then we shall have,†

[3647] $$X'=r''.\cos.v.\cos.\psi-r''.\sin.v.\cos.\theta'.\sin.\psi;$$

[3648] $$Y'=r''.\sin.v.\cos.\theta'.\cos.\psi+r''.\cos.v.\sin.\psi;$$

[3649] $$Z'=r''.\sin.v.\sin.\theta'.$$

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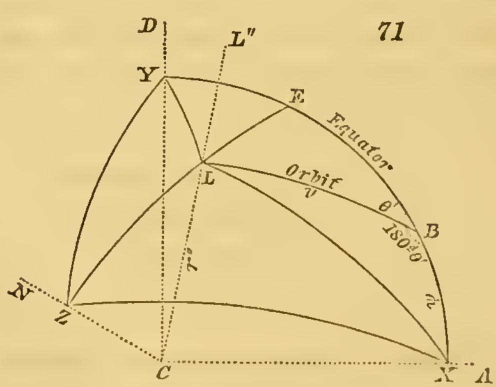
* (2318) Multiplying the values of X'', Y'' [3638], and reducing, by means of [31, 32] Int., we get, successively, the expressions [3641a], as in [3641]; always neglecting terms of the order θ^2;

[3641a] $$X''Y''=(Y'^2-X'^2).\sin.\varphi.\cos.\varphi+X'Y'.(\cos.^2\varphi-\sin.^2\varphi)-\theta Z'.\{2Y'.\sin.\varphi.\cos.\varphi+X'.(\cos.^2\varphi-\sin.^2\varphi)\}$$
$$=\tfrac{1}{2}.(Y'^2-X'^2).\sin.2\varphi+X'Y'.\cos.2\varphi-\theta Z'.\{Y'.\sin.2\varphi+X'.\cos.2\varphi\}.$$

Multiplying X'', Y'' [3638, 3639] by Z'' [3640], neglecting θ^2, we get [3642, 3643].

† (2319) About the centre C, figure 70, page 973, with a radius equal to unity, describe, as in the annexed figure 71, a
[3647a] spherical surface $YEBXZL$, intersecting the axes of X, Y, Z, in the points X, Y, Z, respectively; the line CL'', in the point L; and the orbit of the body L, in the arc BL. The arc ZL being continued, intersects XY perpendicularly in E; then we have, in the
[3647b] present notation, arc $XB=\psi$, $BY=90^d-\psi$, arc $BL=v$ [3644, &c.]; angle $LBY=\theta'$, angle $LBX=180^d-\theta'$. In the oblique-angled triangles XBL, YBL, and in the right-angled triangle BEL, we have, from [1345[8], 1345[28]], the following expressions;



[3647c] $$\cos.XL=\cos.XB.\cos.BL+\sin.XB.\sin.BL.\cos.XBL;$$

[3647d] $$\cos.YL=\cos.YB.\cos.BL+\sin.YB.\sin.BL.\cos.YBL;$$

[3647e] $$\cos.ZL=\sin.LE=\sin.BL.\sin.YBL;$$

If we neglect the terms depending on the sines and cosines of the angle v and its multiples,* also those which depend on the angle 2φ, because all these terms become insensible by these integrations, we shall have,† [3650]

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in which we must substitute the symbols [3647b], and we shall get the values of $\cos. XL$, $\cos. YL$, $\cos. ZL$, to be substituted in X', Y', Z' [3647g], which are easily derived from the common principles of the orthographic projection; observing that the origin of these co-ordinates may be considered as in the centre of Saturn [3636, &c.], or that we may neglect X, Y, Z, in comparison with X', Y', Z'; [3647f]

$$X' = r''.\cos. XL;\qquad Y' = r''.\cos. YL;\qquad Z' = r''.\cos. ZL. \qquad [3647g]$$

This substitution having been made, these values become as in [3647—3649], respectively.

* (2320) The values [3647—3649] are to be substituted in [3641—3643], to obtain $X''Y''$, $X''Z''$, $Y''Z''$; from which we get dN, dN', dN'' [3633—3635]; and thence the corresponding terms of the second members of [2905—2907], or of [3660—3662]; finally, these terms of [3660—3662], reappear in the second members of [3670, 3671]. These last equations are linear in s, s', and are solved as in [871]; for if we put, for brevity, $\alpha K = \frac{3L}{2r''^3}.\sin.\theta'.\cos.\theta'$, and substitute, in [3670, 3671], the value of $\varphi - \psi$ [3673]; we shall get for the terms of s, s', depending on this value of K, the same expressions as in [871]; namely, [3650a] [3650b]

$$s = \frac{\alpha K}{p^2 - \varepsilon^2}.\sin.(pt + I), \qquad s' = \frac{\alpha K}{p^2 - \varepsilon^2}.\cos.(pt + I), \qquad [3650c]$$

as in [3674, 3675]. These terms are expressly given by the author, because they are very much increased, by reason of the smallness of the divisor $p^2 - \varepsilon^2$ [3668, &c.]; and terms, which have not this small divisor, are neglected. Thus, if the angle depend on 2φ, instead of $\varphi - \psi$, the coefficient p [3673] will change into $2p$ nearly; and the divisor will become $4p^2 - \varepsilon^2$ nearly; which is so large, that the resulting term, depending on the angle 2φ, will be insensible. In like manner, if the motion of the body L be slow, in comparison with the angular motion of the ring, the angle v, or $2v$, will be of the form $p't + I'$, p' being small; hence the divisor $p'^2 - \varepsilon^2$, will be nearly equal to $-\varepsilon^2$; and as this is not small, the term will not be increased by the integration; therefore we may neglect the terms containing v and $2v$, as in [3650]. [3650d] [3650e] [3650f]

† (2321) Since $\sin.v.\cos.v = \frac{1}{2}.\sin.2v$, $\sin.^2 v = \frac{1}{2} - \frac{1}{2}.\cos.2v$, $\cos.^2 v = \frac{1}{2} + \frac{1}{2}.\cos.2v$, we may, by neglecting $2v$, as in [3650], put $\sin.v.\cos.v = 0$, $\sin.2v = \frac{1}{2}$, $\cos.^2 v = \frac{1}{2}$, $\cos.^2 v - \sin.^2 v = 0$, in the products of the co-ordinates [3647—3649]; hence we get, by using [1, 6, 31, 32] Int., [3651a]

$$X'Y' = -\tfrac{1}{2}r''^2.\cos.^2\theta'.\sin.\psi.\cos.\psi + \tfrac{1}{2}r''^2.\sin.\psi.\cos.\psi$$
$$= \tfrac{1}{2}r''^2.\sin.\psi.\cos.\psi.\{1 - \cos.^2\theta'\} = \tfrac{1}{4}r''^2.\sin.2\psi.\sin.^2\theta'; \qquad [3651b]$$

[3651] $X''\,Y''=0;$

[3652] $X''\,Z''=\tfrac{1}{2}\,r''^2.\sin.\theta'.\cos.\theta'.\sin.(\varphi-\psi)+\tfrac{1}{2}\,r''^2.\theta.\{\cos.^2\theta'-\tfrac{1}{2}\sin.^2\theta'\}.\sin.\varphi$
$-\tfrac{1}{4}\,r''^2.\theta.\sin.^2\theta'.\sin.(\varphi-2\psi);$

[3653] $Y''\,Z''=\tfrac{1}{2}\,r''^2.\sin.\theta'.\cos.\theta'.\cos.(\varphi-\psi)+\tfrac{1}{2}\,r''^2.\theta.\{\cos.^2\theta'-\tfrac{1}{2}\sin.^2\theta'\}.\cos.\varphi$
$-\tfrac{1}{4}\,r''^2.\theta.\sin.^2\theta'.\cos.(\varphi-2\psi).$

Therefore we shall have, by the action of the body L,*

[3651c] $Y'Z'=\tfrac{1}{2}\,r''^2.\sin.\theta'.\cos.\theta'.\cos.\psi;$

[3651d] $X'Z'=-\tfrac{1}{2}\,r''^2.\sin.\theta'.\cos.\theta'.\sin.\psi;$

$Y'^2-Z'^2=\tfrac{1}{2}\,r''^2.\{\cos.^2\theta'.\cos.^2\psi+\sin.^2\psi-\sin.^2\theta'\}$
$=\tfrac{1}{4}\,r''^2.\{\cos.^2\theta'.(1+\cos.2\psi)+(1-\cos.2\psi)-2\sin.^2\theta'\}$
$=\tfrac{1}{4}\,r''^2.\{\cos.^2\theta'+1-2\sin.^2\theta'\}+\tfrac{1}{4}\,r''^2.\{\cos.^2\theta'-1\}.\cos.2\psi$

[3651e] $=\tfrac{1}{2}\,r''^2.\{\cos.^2\theta'-\tfrac{1}{2}\sin.^2\theta'\}-\tfrac{1}{4}\,r''^2.\sin.^2\theta'.\cos.2\psi;$

$Y'^2-X'^2=\tfrac{1}{2}\,r''^2.\{\cos.^2\theta'.(\cos.^2\psi-\sin.^2\psi)-(\cos.^2\psi-\sin.^2\psi)\}$

[3651f] $=\tfrac{1}{2}\,r''^2.\cos.2\psi.\{\cos.^2\theta'-1\}=-\tfrac{1}{2}\,r''^2.\cos.2\psi.\sin.^2\theta'.$

These values being substituted in [3641—3643], produce the expressions [3651—3653]. For every term of [3641] contains 2φ, which is to be neglected [3650], and then it

[3651g] becomes as in [3651]. Again, the two first terms of [3642] are found by taking the sum of the products, formed by multiplying [3651c, d], by $\sin.\varphi$, $\cos.\varphi$, respectively, and substituting $\sin.\varphi.\cos.\psi-\cos.\varphi.\sin.\psi=\sin.(\varphi-\psi)$, by which means it becomes

[3651h] $\tfrac{1}{2}\,r''^2.\sin.\theta'.\cos.\theta'.\sin.(\varphi-\psi)$, as in the first term of [3652]. The third term of [3642] is $\theta.(Y'^2-Z'^2).\sin.\varphi$; and if we substitute in this the value [3651e], we shall obtain, for the part independent of 2ψ, the expression $\tfrac{1}{2}\,r''^2\theta\{\cos.^2\theta'-\tfrac{1}{2}\sin.^2\theta'\}.\sin.\varphi$, as in the

[3651i] second term of [3652]. Moreover, the part of this last expression depending on 2ψ, is $-\tfrac{1}{4}\,r''^2.\theta.\sin.^2\theta'.\sin.\varphi.\cos.2\psi$; adding this to the last term of [3642], namely, $\theta.X'Y'.\cos.\varphi=\tfrac{1}{4}\,r''^2.\theta.\sin.^2\theta'.\cos.\varphi.\sin.2\psi$ [3651b], and putting

[3651k] $$-\sin.\varphi.\cos.2\psi+\cos.\varphi.\sin.2\psi=-\sin.(\varphi-2\psi),$$

it becomes as in the last term of [3652].

In like manner, we may find $Y''\,Z''$; or more simply by derivation from [3652]; for if we write $\varphi+90^d$, for φ, in [3642], it changes this expression of $X''\,Z''$, into

[3651l] $Y''\,Z''$ [3643]; observing that this does not alter the values of X', Y', Z' [3647—3649], which do not contain φ; and by making the same change in [3652], we get [3653].

[3654a] * (2322) Substituting the values [3651—3653] in [3633—3635], we get [3654—3656].

Values

$$\frac{d\mathcal{N}}{dt} = 0\,;$$ [3654]

depending on the action of a

$$\frac{d\mathcal{N}'}{dt} = \frac{3L}{2r''^3}\,.\,(C-A)\,.\left\{\begin{array}{l}\sin.\theta'.\cos.\theta'.\sin.(\varphi-\psi)+\{\cos.^2\theta'-\tfrac{1}{2}\sin.^2\theta'\}.\theta.\sin.\varphi\\ -\tfrac{1}{2}\theta.\sin.^2\theta'.\sin.(\varphi-2\psi)\end{array}\right\};$$ [3655]

distant body.

$$\frac{d\mathcal{N}''}{dt} = \frac{3L}{2r''^3}\,.\,(C-B)\,.\left\{\begin{array}{l}\sin.\theta'.\cos.\theta'.\cos.(\varphi-\psi)+\{\cos.^2\theta'-\tfrac{1}{2}\sin.^2\theta'\}.\theta.\cos.\varphi\\ -\tfrac{1}{2}\theta.\sin.^2\theta'.\cos.(\varphi-2\psi)\end{array}\right\}.$$ [3656]

Second form.

We may here observe, that the values of $\frac{d\mathcal{N}'}{dt}$, $\frac{d\mathcal{N}''}{dt}$, correspond to the plane of the ring, and to its principal axes;* whereas the preceding values [3628—3630], relative to the action of Saturn, correspond to the plane of the equator of Saturn. Therefore, when substituting the preceding expressions [3657]

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* (2323) In the values [3628—3630], depending on the action of Saturn, the equator of the planet is taken as the plane of $x'y'$ [3614], corresponding to $\mathcal{N}$ [226]. The plane of the ring is also supposed to be inclined to the plane of $x'y'$, by an angle θ, as in [2907g], whose square is neglected [3623]; and the first principal axis of the ring forms the angle φ, with the line of intersection of the equator and ring, as in [3625a, d, 2907f]. Hence it appears, that if we substitute $\sin.\theta=\theta$, $\cos.\theta=1$, $\frac{d\mathcal{N}}{dt}=0$ [3620], in the second members of the three equations [2905—2907], and then divide them by dt, we shall obtain the following expressions, corresponding to these equations, respectively; [3657a] [3657b] [3657c]

$$-\frac{d\mathcal{N}''}{Cdt}.\theta\,; \qquad -\frac{d\mathcal{N}'}{Adt}.\sin.\varphi+\frac{d\mathcal{N}''}{Adt}.\cos.\varphi\,; \qquad -\frac{d\mathcal{N}'}{Bdt}.\cos.\varphi-\frac{d\mathcal{N}''}{Bdt}.\sin.\varphi\,;$$ [3657d]

in which we must substitute the values of $d\mathcal{N}'$, $d\mathcal{N}''$ [3629, 3630], to obtain the corresponding terms of [3660, 3662], depending on the action of Saturn; observing, that these last equations are equivalent to [2905—2907], divided by dt. [3657e]

In computing the action of the planet L, we have taken the plane of the ring [3632] for the plane of $X''Y''$, which corresponds to $d\mathcal{N}$ [3633]; X'', or x'', being the first principal axis of the ring [3632, 3625a]. Then, by proceeding in the same manner as we have done for the earth, in [2977'—2979], we shall have, for this case, $\theta=0$, $\varphi=0$, as in [3659]. These values are to be substituted in the second members of [2905—2907], after dividing by dt, to obtain the corresponding expressions of [3660—3662]; which will become, respectively, [3657f] [3657g]

$$0\,; \qquad \frac{d\mathcal{N}''}{Adt}\,; \qquad -\frac{d\mathcal{N}'}{Bdt}\,;$$ [3657h]

observing, that $d\mathcal{N}=0$ [3654]. Substituting, in [3657h], the values of $d\mathcal{N}'$, $d\mathcal{N}''$ [3655, 3656], we shall get the corresponding terms of the second members of [3660—3662], depending on the action of the body L. [3657i]

[3628—3630], relative to the action of Saturn, in [2905—2907], we
[3658] may suppose, in these equations, $\sin.\theta=\theta$, $\cos.\theta=1$; because of the smallness of θ, whose square we shall neglect. But when substituting, in the same equations [2905—2907], the preceding values of $\frac{dN'}{dt}$, $\frac{dN''}{dt}$ [3655, 3656], relative to the action of the body L; we must first suppose,
[3659] in those equations, $\sin.\theta=0$, $\cos.\theta=1$, $\sin.\varphi=0$, and $\cos.\varphi=1$. Hence we shall have, by connecting all these terms,*

[3660]
$$\frac{dp}{dt}+\left(\frac{B-A}{C}\right).qr=0;$$

General differential equations in p, q, r.

$$\frac{dq}{dt}+\left(\frac{C-B}{A}\right).rp=\frac{2\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}'^{5}}.\left(\frac{C-B}{A}\right).\theta.\cos.\varphi$$

[3661]
$$+\frac{3L}{2r''^{3}}.\left(\frac{C-B}{A}\right).(\cos.^{2}\theta'-\tfrac{1}{2}\sin.^{2}\theta').\theta.\cos.\varphi$$

$$+\frac{3L}{2r''^{3}}.\left(\frac{C-B}{A}\right).\{\sin.\theta'.\cos.\theta'.\cos.(\varphi-\psi)-\tfrac{1}{2}\theta.\sin.^{2}\theta'.\cos.(\varphi-2\psi)\};$$

$$\frac{dr}{dt}+\left(\frac{A-C}{B}\right).pq=\frac{2\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}'^{5}}.\left(\frac{A-C}{B}\right).\theta.\sin.\varphi$$

[3662]
$$+\frac{3L}{2r''^{3}}.\left(\frac{A-C}{B}\right).\{\cos.^{2}\theta'-\tfrac{1}{2}\sin.^{2}\theta'\}.\theta.\sin.\varphi$$

$$+\frac{3L}{2r''^{3}}.\left(\frac{A-C}{B}\right).\{\sin.\theta'.\cos.\theta'.\sin.(\varphi-\psi)-\tfrac{1}{2}\theta.\sin.^{2}\theta'.\sin.(\varphi-2\psi)\}.$$

[3660a] * (2324) The first members of [3660—3662] are evidently the same as those of [2905—2907], divided by dt. The second members are obtained by the substitution of the values [3657d, h]. We shall first compute the values [3657d], which admit of several reductions. *First.* As dN' [3629], is of the order θ, the term $-\frac{dN'}{Cdt}.\theta$ [3657d],
[3660b] corresponding to the first equation, becomes of the order θ^2, which may be neglected [3658]; therefore this term vanishes from the second member of [3660], and as the similar term of [3657h] is equal to nothing, the second member of [3660] will wholly
[3660c] vanish. *Second.* If we suppose, for brevity, that $e=\frac{\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}'^{5}}.(B-A).\theta$, we shall have, by neglecting, for a moment, the first term of [3630], and noticing only the
[3660d] terms of [3629, 3630], depending on 2φ, $\frac{dN'}{dt}=e.\sin.2\varphi$; $\frac{dN''}{dt}=-e.\cos.2\varphi$.

r and q being very small,* we may neglect the product $\left(\frac{B-A}{C}\right).rq$, [3663]
and we shall get $\frac{dp}{dt}=0$, or p constant. We shall then put, as in [3487],

$$s=\theta.\sin.\varphi; \qquad s'=\theta.\cos.\varphi; \qquad [3664]$$

hence we shall get, as in [3492],

$$r=\frac{ds}{dt}-ps'; \qquad q=-\frac{ds'}{dt}-ps. \qquad [3665]$$

Substituting these in the second and third of the expressions [3657d], and then reducing, by means of [24, 32] Int., we get, respectively,

$$-\frac{dN'}{Adt}.\sin.\varphi+\frac{dN''}{Adt}.\cos.\varphi=-\frac{e}{A}.\{\sin.\varphi.\sin.2\varphi+\cos.\varphi.\cos.2\varphi\}$$

$$=-\frac{e}{A}.\cos.\varphi=-\frac{\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}'^{5}}.\left(\frac{B-A}{A}\right).\theta.\cos.\varphi \qquad [3660e]$$

$$-\frac{dN'}{Bdt}.\cos.\varphi-\frac{dN''}{Bdt}.\sin.\varphi=-\frac{e}{B}.\{\cos.\varphi.\sin.2\varphi-\sin.\varphi.\cos.2\varphi\}$$

$$=-\frac{e}{B}.\sin.\varphi=-\frac{\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}'^{5}}.\left(\frac{B-A}{B}\right).\theta.\sin.\varphi. \qquad [3660f]$$

The first term of [3630], depending on the factor $(A+B-2C)$, which was neglected in dN'' [3660d], produces in these expressions the following terms, respectively;

$$-\frac{\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}'^{5}}.\left(\frac{A+B-2C}{A}\right).\theta.\cos.\varphi; \qquad \frac{\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}'^{5}}.\left(\frac{A+B-2C}{B}\right).\theta.\sin.\varphi. \qquad [3660g]$$

Connecting the first of these expressions with that in [3660e], we get the terms in the first line of the second member of [3661]; and by connecting the second term of [3660g] [3660h] with [3660f], we get the first line of the second member of [3662]. Lastly, if we substitute, in $\frac{dN''}{Adt}$ [3657h], its value, deduced from [3656], we shall get, without any reduction, the second or third lines of the second member of [3661]; and in like manner, by substituting [3655] in $-\frac{dN'}{Bdt}$ [3657h], we get the second and third lines of [3662]. [3660i]

* (2325) This may be proved as in a similar case [3013a, &c.]. If we neglect, in the equation [3660], the term $\left(\frac{B-A}{C}\right).rq$, on account of its smallness, we shall get [3663a]
$\frac{dp}{dt}=0$; or $dp=0$; whose integral is $p=$ constant.

Substituting these values in the preceding differential equations [3661, 3662], they become,*

[3666] $$\frac{dds}{dt^2}+\left(\frac{C-B-A}{B}\right).p.\frac{ds'}{dt}+\left(\frac{C-A}{B}\right).\varepsilon^2 s$$

Differential equations in s, s'.

$$=\frac{3L}{2r''^3}.\left(\frac{A-C}{B}\right).\{\sin.\theta'.\cos.\theta'.\sin.(\varphi-\psi)-\tfrac{1}{2}\theta.\sin.^2\theta'.\sin.(\varphi-2\psi)\};$$

[3667] $$\frac{dds'}{dt^2}-\left(\frac{C-B-A}{A}\right).p.\frac{ds}{dt}+\left(\frac{C-B}{A}\right).\varepsilon^2 s'$$

First form.

$$=\frac{3L}{2r''^3}.\left(\frac{B-C}{A}\right).\{\sin.\theta'.\cos.\theta'.\cos.(\varphi-\psi)-\tfrac{1}{2}\theta.\sin.^2\theta'.\cos.(\varphi-2\psi)\},$$

ε^2 being equal to the following expression;

[3668] ε^2. $$\varepsilon^2=p^2+\frac{2\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}'^5}+\frac{3L}{2r''^3}.\{\cos.^2\theta'-\tfrac{1}{2}\sin.^2\theta'\}.$$

The preceding equations become more simple, by observing, that in the
[3669] present case, we have,† $A+B=C$ [229], which reduces these equations

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* (2326) The differentials of [3665], supposing p to be constant, as in [3663], give

[3665a] $$\frac{dr}{ds}=\frac{dds}{dt^2}-p.\frac{ds'}{dt};\qquad \frac{dq}{dt}=-\frac{dds'}{dt^2}-p.\frac{ds}{dt}.$$

The two first lines of the second member of [3661], are equal to ε^2-p^2, [3668], multiplied by $\left(\frac{C-B}{A}\right).\theta.\cos.\varphi$; or $\left(\frac{C-B}{A}\right).s'$ [3664]; so that they may be
[3665b] represented by $(\varepsilon^2-p^2).\left(\frac{C-B}{A}\right).s'$; in like manner, the two first lines of the second
[3665c] member of [3662], are equal to $(\varepsilon^2-p^2).\left(\frac{A-C}{B}\right).s$; substituting these expressions, and [3665a, 3665], in [3661, 3662]; putting also T, T', for the third lines of the second members of these last equations, respectively, we shall get, without any reduction,

[3665d] $$-\frac{dds'}{dt^2}-p.\frac{ds}{dt}+\left(\frac{C-B}{A}\right).p.\left\{\frac{ds}{dt}-ps'\right\}=(\varepsilon^2-p^2).\left(\frac{C-B}{A}\right).s'+T;$$

[3665e] $$\frac{dds}{dt^2}-p.\frac{ds'}{dt}+\left(\frac{A-C}{B}\right).p.\left\{-\frac{ds'}{dt}-ps\right\}=(\varepsilon^2-p^2).\left(\frac{A-C}{B}\right).s+T'.$$

The terms of these equations, multiplied by p^2, destroy each other; then transposing the
[3665f] terms depending on ε^2, and making some slight reductions, we shall find, that [3665e] is the same as [3666]; and [3665d], changing its signs, is the same as [3667].

† (2327) The ring is supposed to be void of thickness [3611], therefore $z''=0$;
[3669a] hence we get, from [229], $S.y''^2.dm=A$; $S.x''^2.dm=B$; $S.(x''^2+y''^2).dm=C$;

to the forms [3670, 3671], neglecting, in their second members, the terms multiplied by θ;

$$\frac{dds}{dt^2}+\varepsilon^2 s=-\frac{3L}{2r''^3}\cdot\sin.\theta'.\cos.\theta'.\sin.(\varphi-\psi);$$ [3670]

Second form.

$$\frac{dds'}{dt^2}+\varepsilon^2 s'=-\frac{3L}{2r''^3}\cdot\sin.\theta'.\cos.\theta'.\cos.(\varphi-\psi).$$ [3671]

If we consider the orbit of L as invariable, we shall obtain, from [3029],* [3671′]

$$d\varphi-d\psi=p\,dt;$$ [3672]

consequently,

$$\varphi-\psi=pt+I,$$ [3673]

I being an arbitrary constant quantity. The differential equations in s, s', will then give, by integration,†

$$s=M.\sin.(\varepsilon t+E)-\frac{\frac{3L}{2r''^3}\cdot\sin.\theta'.\cos.\theta'}{\varepsilon^2-p^2}\cdot\sin.(\varphi-\psi);$$ [3674]

Values of s, s'.

$$s'=M'.\cos.(\varepsilon t+E')-\frac{\frac{3L}{2r''^3}\cdot\sin.\theta'.\cos.\theta'}{\varepsilon^2-p^2}\cdot\cos.(\varphi-\psi);$$ [3675]

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consequently the sum of the two first is equal to the last, or $A+B=C$. From this equation, we obtain $\frac{C-B-A}{A}=0$; $\frac{C-A}{B}=1$; $\frac{A-C}{B}=-1$; $\frac{C-B}{A}=1$; $\frac{B-C}{A}=-1$. [3669b] Substituting these in [3666, 3667], and neglecting θ, on account of its smallness, we obtain [3670, 3671].

* (2328) Neglecting θ^2, we may put [3029] under the form [3672], and as p is nearly constant, its integral will be as in [3673]. [3672a]

† (2329) The orbit of L being considered as invariable [3671′], and the angle θ' [3646] as constant, we shall have, from [3670], by putting $\frac{3L}{2r''^3}\cdot\sin.\theta'.\cos.\theta'=\alpha K$, and [3674a] using [3673], $\frac{dds}{dt^2}+\varepsilon^2 s+\alpha K.\sin.(pt+I)=0$. This is of the form [865a], and its integral [871, 865b], gives $s=b.\sin.(\varepsilon t+\varphi)-\frac{\alpha K}{\varepsilon^2-p^2}\cdot\sin.(pt+I)$, as in [3674b] [3674], changing the arbitrary constant quantities b, φ, into M, E, respectively. In like manner, from [3671], we get [3675].

M, E, M', E', being four arbitrary constant quantities. The inclination of the plane of the ring to that of the equator of Saturn, is equal to
[3676] $\sqrt{s^2+s'^2}$;* it is therefore necessary, in order that this inclination should always be small, that M and M' should be small; moreover, the
[3677] coefficient $\frac{\frac{3L}{2r''^3}\cdot\sin.\theta'.\cos.\theta'.}{\varepsilon^2-p^2}$ must also be small. Now this cannot be the case, if Saturn be perfectly spherical; for then we shall have,†

[3678] $$\varepsilon^2-p^2=\frac{3L}{2r''^3}\cdot\{\cos.^2\theta'-\tfrac{1}{2}\sin.^2\theta'\};$$

[3679] and the preceding coefficient will become $\frac{\sin.\theta'.\cos.\theta'}{\cos.\theta'^2-\frac{1}{2}\sin.\theta'^2}$, which will be very sensible.

If Saturn be flattened, by means of its rotatory motion, this coefficient will become,‡

[3680] $$\frac{\frac{3L}{2r''^3}\cdot\sin.\theta'.\cos.\theta'}{\frac{2\alpha.(h-\frac{1}{2}\varphi')}{r_{\prime}^{\prime 5}}+\frac{3L}{2r''^3}\cdot\{\cos.^2\theta'-\frac{1}{2}\sin.^2\theta'}.$$

[3681] Supposing L to be the sun; $r_{\prime}$ the distance from the centre of Saturn to
[3682] its outer satellite; T the time of a sideral revolution of Saturn; T' the
[3683] time of a sideral revolution of this satellite; then the mass of Saturn

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[3676a] * (2330) From [3664], we get $\theta=\sqrt{(s^2+s'^2)}$, representing the inclination of the plane of the ring to the plane of the equator [3623].

[3677a] † (2331) Saturn being supposed spherical, its radius [3597—3601] must be constant, and we shall have $Y^{(2)}=0$; hence the ellipticity [3603—3606'], is equal to nothing, and the centrifugal force $\alpha\varphi'=0$ [3598, 1731], consequently the factor $(h-\frac{1}{2}\varphi')=0$.
[3677b] Hence the value of ε^2 [3668], becomes $\varepsilon^2=p^2+\frac{3L}{2r''^3}\cdot\{\cos.^2\theta'-\frac{1}{2}\sin.^2\theta'\}$, from which we get ε^2-p^2, as in [3678], and by substituting this in the coefficient [3677], which occurs in [3674, 3675], it becomes as in [3679]. This may be of considerable magnitude because $\theta'=33^{\circ}$ [3686].

[3680a] ‡ (2332) This is found by substituting, in the coefficient [3677], the value of ε^2-p^2, deduced from [3668].

being taken for unity, we shall have, as in [709],*

$$\frac{L}{r''^3}=\frac{1}{r_{\prime}^3}\cdot\left(\frac{T'}{T}\right)^2. \qquad [3684]$$

Hence the coefficient of [3680], becomes

$$\frac{\frac{3\,r_{\prime}'^5}{4\,r_{\prime}^3}\cdot\left(\frac{T'}{T}\right)^2\cdot\sin.\theta'.\cos.\theta'}{\alpha.(h-\tfrac{1}{2}\varphi')+\frac{3\,r_{\prime}'^5}{4\,r_{\prime}^3}\cdot\left(\frac{T'}{T}\right)^2\cdot\{\cos.^2\theta'-\tfrac{1}{2}\sin.^2\theta'\}}. \qquad [3685]$$

Taking the semi-diameter of Saturn for unity, we have, by observation,†

$$\begin{aligned} T &= 10759^{\text{days}},08\,; \\ T' &= 79^{\text{days}},3296\,; \\ r_{\prime} &= 59,154\,; \\ \theta' &= 33^{\circ}. \end{aligned} \qquad [3686]$$

We shall then suppose $r_{\prime}'=2$,‡ which differs but little from the true value; then we shall have the proposed coefficient, reduced to seconds, equal to§ [3687]

$$\frac{0'',001727}{\alpha.(h-\tfrac{1}{2}\varphi')+0,0000000039824}. \qquad [3688]$$

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* (2333) Neglecting the mass of the satellite p [709], and then changing $m', 'M, h, a$, T, into 1, L, $r_{\prime}$, r'', T', respectively, we get $\frac{1}{L}=\frac{r_{\prime}^3}{r''^3}\cdot\left(\frac{T}{T'}\right)^2$, which gives for $\frac{L}{r''^3}$, the value [3684]. Substituting this in [3680], and then multiplying the numerator and denominator by $\frac{1}{2}\,r_{\prime}'^5$, it becomes as in [3685]. [3685a]

† (2334) These values correspond nearly with those in §36, Book VIII [7669, 7670, 7676]. The values $r_{\prime}$, θ', being there called a, A, respectively, A being put, in [7638], equal to the inclination of the equator of Saturn to the orbit, and in [7682], $A=33^{\circ},3333$. Also T, T', of that article, are here called T', T, respectively. [3686a]

‡ (2335) This corresponds with [2109], $\frac{a}{R}=2$, R being, by [2104], the radius of Saturn, which is here taken for unity, and a [2073'] the radius of the ring, here called $r_{\prime}'$. [3687a]

§ (2336) Substituting the values [3686, 3687] in [3685], and multiplying it by the radius in seconds, 636620″, it becomes as in [3688] nearly. [3688c]

The action of Saturn retains the rings in the plane of its equator.

Hence we see, that this coefficient will be very great, if $\alpha.(h-\frac{1}{2}\varphi')$ vanish,* and that it will become very small and insensible, if this quantity
[3689] have a sensible value. *Thus the action of Saturn, arising from the oblateness of its form, constantly retains the rings, so as to keep them nearly in the plane of its equator; and the different rings of Saturn are by this cause retained in*
[3690] *the same plane. This is therefore the cause of the phenomena in question; from which the author inferred, that Saturn revolved about its axis, before it had been discovered by observation of the spots upon its surface.*

The disturbing forces of the satellites cannot prevent the rings from remaining in the same plane.

22. It is evident, from the preceding analysis, that the action of the fifth
[3690′] satellite of Saturn cannot, in a perceptible manner, prevent the different rings of this planet from being in the same plane.† As it regards the mutual attraction of the rings, and the attraction of the other satellites of Saturn, which move nearly in the plane of the rings, it is evident,
[3691] that their mutual action upon each other cannot alter the coincidence of the rings.

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* (2337) Putting $\alpha.(h-\frac{1}{2}\varphi')$ equal to nothing, the coefficient [3688] becomes
[3689a] about 43°. Now in Book VIII, § 1 [6044], the ellipticity of Saturn is put $=\rho$, and its centrifugal force at the equator φ; which quantities are here called αh, $\alpha\varphi'$ [3606′, 3598]. If we also put, as in Book VIII, § 36 [7676], $t=0^{\mathrm{days}},428=$ the time of rotation of
[3689b] Saturn, and change the distance of the outer satellite from a into $r_{,}$, also T into T', to conform to the present notation, the value of $\rho-\frac{1}{2}\varphi$ [7680], will be

[3689c]
$$\alpha.(h-\tfrac{1}{2}\varphi')=\frac{243}{670}\cdot\frac{T'^2}{t^2.r_{,}^3};$$

which, by using the above values of T, $r_{,}$, t, becomes $\alpha.(h-\frac{1}{2}\varphi)=0,0602$.
[3689d] Substituting this in the coefficient [3688], it becomes $0'',029$; so that the attraction of Saturn's equator reduces this coefficient from 43 degrees to about $\frac{1}{30}$ of a second.

† (2338) To estimate the effect of the attraction of the fifth satellite of Saturn, we
[3691a] shall suppose its mass to bear the same proportion to that of Saturn, as Jupiter's outer satellite does to Jupiter, so that we may put L equal to the value of m''', Book VIII, § 21
[3691b] [6831], or nearly $L=0,00006$. Substituting this, and the values of $r_{,}$, $r'_{,}$ [3686, 3687], in the coefficient [3677], which occurs in the values of s, s' [3674, 3675], it becomes nearly $\frac{3\times 0,00006}{2.(59,154)^3}\cdot\sin.\theta'.\cos.\theta'$, divided by $\frac{2\alpha.(h-\frac{1}{2}\varphi')}{32}+\frac{3\times 0,00006}{2.(59,154)^3}.(\cos.^2\theta'-\frac{1}{2}\sin.^2\theta')$; and if we suppose $\theta'=33°$, this will become nearly $\frac{0,000000003}{\alpha.(h-\frac{1}{2}\varphi)+0,000000004}$, which is
[3691c] insensible; consequently the inclination θ [3676a], will not be affected sensibly by terms of this order in s, s'.

A ring may be considered as a mass of satellites collected in that form. [3691′]
We can easily imagine, that the action of the equator of Saturn, which retains the different rings in the same plane, must, for the same reason, retain in this plane the orbits of the satellites, which were primitively situated in the plane. Reciprocally, *if the different satellites of a planet* [3692]
move in the same plane, which is considerably inclined to the plane of the orbit of the planet; we may conclude, that they are retained in this plane by the action of its equator; consequently the *planet must have a rotatory motion about an axis, which is nearly perpendicular to the plane of the orbits of the* [3693]
satellites. Therefore we may affirm, that the planet Uranus, whose satellites move in a plane nearly perpendicular to the ecliptic, revolves about an axis, [3693′]
which is very slightly inclined to the ecliptic.

The terms of the expression of θ, which depend on the actions of the [3694]
sun and the outer satellite of Saturn, being insensible,* and the dimensions of the ring not entering into the other terms; it is evident, that if several concentrical rings are firmly attached together, and move nearly in the [3695]
equator of Saturn, the action of the sun and the outer satellite will not [3695′]
draw them sensibly from it; thus the *result which we have found for one ring, neglecting its width, is equally true for a ring of any width whatever.* [3696]

The only part of the expression of θ, which could be perceptible, depends on the arbitrary coefficients, and this part is independent of the [3697]
position of the equator of Saturn, relatively to its orbit, or to the orbit of its outer satellite. Hence it follows, that this equator, while in motion, by means of the small action impressed on it by the sun, and that satellite, draws with it the planes of the rings, and the planes of the orbits of the [3698]
satellites which were primitively situated in the plane of the equator. In the same manner, we have seen, in §17, that the plane of the ecliptic, by

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* (2339) This is shown in the preceding note, and by neglecting the terms depending on the sun and satellites, in [3674, 3675], they become

$$s = M \,.\, \text{sin.}\,(\varepsilon t + E)\,; \qquad s' = M' .\, \text{sin.}\,(\varepsilon t + E')\,; \qquad [3694a]$$

in which M, M', E, E', are arbitrary constant quantities, and ε [3668] is independent of the dimensions of the rings; so that they will be the same for another ring contiguous to that one we have just considered; hence we perceive the correctness of the remarks of the [3694b]
author in [3695′].

its secular motion, draws with it the planes of the lunar equator and orbit, so as to render the mutual inclinations of these three planes constant, and make their mutual intersections coincide with each other.*

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* (2340) We have seen, in the preceding note, that the only sensible terms of the values
[3697a] of s, s', are $s = M.\sin.(\varepsilon t + E)$, $s' = M'.\sin.(\varepsilon t + E')$. These expressions are independent of the elements θ', ψ, which determine the position of the orbit of the body L [3646]; whether this body be the sun or the outer satellite; therefore $\theta = \sqrt{(s^2 + s'^2)}$ [3676a], must be independent of the position of the orbit of Saturn, or of its outer satellite, as is observed in [3697]. Now the equator of Saturn has a motion corresponding to the
[3697b] *precession* and *nutation* of the earth's equator, and this motion must depend on the situation
[3697c] of the orbits of Saturn, and of its satellite, relatively to the equator of Saturn; but the effect of these motions is not perceived in the value of θ; therefore we must infer, as
[3697d] in [3698], that the protuberant matter in Saturn's equator draws with it the planes of the rings, and the planes of the orbits of the satellites which were originally situated in the plane of the equator.

Ira Berry and Robert W. Macnair, compositors.

ERRATA IN VOLUME I.

Page 215, lines 11, 14 [327d, e], *for* $\frac{\delta p}{p}$, *read* $\frac{\delta p}{\rho}$.

Page 253, line 17, *for* relative, *read* relative orbit; line 3 from the bottom, *for* a'^3, *read* $2a'^3$.

Page 270, lines 1, 8 from the bottom, dx is to be changed into dy, in the first term.

Page 271, line 2 from the bottom, *for* x, *read* y.

Page 286, line 11, *for* $(x^2+y^2+x^2)$, *read* $(x^2+y^2+z^2)$.

Page 292, line 4 from the bottom, *for* 63, *read* 62.

Page 296, [480a]. The signs of u, in the first, third, and fourth terms, are to be changed.

IN VOLUME II.

Page 233, line 6 from the bottom, *for* of gravity, *read* of the centrifugal force.